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Short Contribution

Cyclostrophic Balance in Surface Gravity Waves*

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Abstract: The cyclostrophic balance (pressure force *vs.* force centrifugal force) is shown to be satisfied for all fluid particles in surface gravity waves with sinusoidal form and circular particle orbits. Consequences of the cyclostrophic balance are 1) that the normal dispersion relation for deep water hold and 2) that the orbital radius decrease with increasing depth at the usual exponential rate, from which it follows that the wave pressure and particle speed also decrease with depth exponentially. In addition, the cyclostrophic and hydrostatic balances together predict wave breaking at the crests for amplitudes exceeding one divided by the wave number. In contrast to the traditional perturbation method, based on irrotational flow, the cyclostrophic method does not demand that the amplitude be much less than a wave length and does not require an infinite wave train.

1. Introduction

A balance between pressure and centrifugal forces, or the cyclostrophic balance (*e.g.* Neumann and Pierson, 1966, p. 168), is applied here to individual fluid particles in order to provide a simple theoretical understanding of the orbital motion in deep water surface gravity waves. The force balance method has several advantages over the traditional perturbation method (*e.g.* Lamb, 1932, p. 367), which is based on the assumption of irrotational flow, as will be discussed later.

Experimental characteristics of the particle motion are as follows. When surface gravity waves pass by a stationary observer from left to right, the sense of the orbital motion is clockwise and the phase is defined by the vertical velocity of the particles being upward where the surface slopes downward (maximum vertical velocity corresponding to maximum slope). The shape of the orbits is circular (neglecting any drift) for deep water waves and the orbital radius, which equals the wave amplitude at the surface, decreases with increasing depth, becoming vanishingly small at depths comparable to a wave length.

The circular particle motion is a remarkable feature which distinguishes these fluid waves from all the other wellknown waves in physics (*e.g.* Feynman *et al.*, 1963, p. 51-4). Visualization of the particle movement is aided approximately by the use of small neutrally buoyant particles (*e.g.* see the streak photographs in Sommerfeld, 1964, p. 181, and Wiegel, 1964, pp. 19, 52).

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2. Surface horizontal balance

Consider sinusoidal surface gravity waves propagating in deep water in the positive x-direction (5) and a circularly orbiting surface particle moving vertically upward where the downward surface slope is a maximum. The horizontal centrifugal force $(c.f.)_{\mathrm{horiz.}}$ on this fluid particle has magnitude

$$(c.f.)_{\mathrm{horiz.}} = \rho\omega^2 a\,, \qquad (1)$$

where ρ is the constant fluid density, ω the wave frequency, and a is the orbital radius (as well as the wave amplitude).

To compute the horizontal pressure force on this same fluid particle, which is at height $z=0$, look in the negative x-direction a distance Δx where the surface elevation is $\Delta z>0$ (Δx and Δz are both infinitesimal displacements). The change in pressure Δp over the horizontal distance Δx is due to the hydrostatic pressure over the vertical distance Δz

$$\Delta p = \rho g \Delta z. \qquad (2)$$

The atmospheric pressure is assumed constant and therefore does not appear in Eq. (2). Since $\Delta p>0$ the pressure force

$$-\frac{\Delta p}{\Delta x}=-\rho g\frac{\Delta z}{\Delta x}\rightarrow-\rho g\frac{\partial\zeta}{\partial x}=\rho gka \quad (3)$$

acts in the positive x-direction, where k is the wave number and the maximum slope of the sine wave (5) was used in Eq. (3).

If the two force components (1) and (3) are to be oppositely directed, then the centrifugal force (1) must act in the negative x-direction, implying that the sense of the circular motion is clockwise (in agreement with observations). Then if the magnitudes of the two oppositely directed force components (1) and (3) are to be equal, the requirement is

$$\omega^2=gk\,, \quad (4)$$

which is the well-known dispersion relation connecting frequency and wave number for deep water waves.

The two horizontal forces can be balanced for all positions of the surface particle around its circular orbit, assuming (4) holds, because in general the right sides of (1) and (3) are multiplied by $\cos(kx-\omega t)$ for the surface elevation ζ

$$\zeta=a\sin(kx-\omega t)\,. \quad (5)$$

The horizontal balance of forces implied by Eq. (4) is not restricted to wave amplitudes that are small compared to a wave length; the amplitude cancels when Eqs. (1) and (3) are equated. Also despite Eq. (5), the horizontal balance of forces does not require the wave train to be infinite in the x-direction, because the horizontal pressure force on a surface particle only depends on the local surface slope at the position of that particle, as Eqs. (2) and (3) show. In the traditional method of deriving Eq. (4), however, the small amplitude assumption $ak\ll1$ must be made, because of the perturbation analysis involved, and the wave train is assumed infinite to solve Laplace's equation, which is a consequence of the irrotational assumption (and incompressibility).

3. Surface vertical balance

Consider now the clockwise orbiting fluid particle at the crest of Eq. (5), where it moves horizontally in the positive x-direction. The vertical centrifugal force $(c.f.)_{\text{vert.}}$ on the surface particle is directed upward and has the same magnitude as Eq. (1)

$$(c.f.)_{\text{vert.}}=\rho\omega^2a\,. \quad (6)$$

An equal and opposite force has to be found to balance Eq. (6), and a downward pressure force, or the corresponding vertical pressure gradient, is the only possibility. Therefore, the vertical pressure gradient at the surface must be

$$\partial p/\partial z=\rho\omega^2a\,, \quad (7)$$

where the pressure decreases with increasing vertical distance downward at the crest (the positive z-direction is up). The vertical acceleration of the surface at the crest is downward, because $\partial^2\zeta/\partial t^2=-\omega^2a$ from Eq. (5). By Newton's second law a downward force must exist to cause the downward acceleration of a particle at the crest, and Eq. (7) is the required force.

Equation (7), which exists only in the presence of waves, is to be distinguished from the normally much larger hydrostatic pressure gradient, $\partial p/\partial z=-\rho g$, which exists with or without waves. When the two vertical pressure gradients have equal magnitudes at the surface, $\omega^2a=g$, or $ka=1$ from Eq. (4), and this places an upper limit on the size of the wave amplitude for a given wave number, because for amplitudes a larger than $1/k$ the tips of the crests would become detached from the wave. The amplitude restriction $ka=1$ is very mild.

For a general position of the particle in its circular orbit the vertical force balance at the surface is still accomplished because the right sides of Eqs. (6) and (7) are each multiplied by $\sin(kx-\omega t)$.

4. Force balances at depth

Suppose that $r(z)$ is the radius of the circular orbits at mean depth z, and at the still water level ($z=0$), $r(0)=a$. The centrifugal force components (1) and (6) become then

$$(c.f.)_{\text{vert.}}=(c.f.)_{\text{horiz.}}=\rho\omega^2r(z)\,. \quad (8)$$

By Newton's second law the vertical pressure force required to balance $(c.f.)_{\text{vert.}}$ in Eq. (8) below the crest is

$$\partial p/\partial z=\rho\omega^2r(z) \quad (9)$$

as adapted from Eq. (7), and for an arbitrary position (9) becomes

$$\partial p/\partial z=\rho\omega^2 r(z)\sin(kx-\omega t). \quad (10)$$

Where the orbiting particle moves vertically upward $\partial p/\partial z=0$, and so the wave pressure is constant with depth (or zero neglecting atmospheric pressure). At a distance Δx toward the negative x-direction (10) gives

$$\partial p/\partial z=\rho\omega^2 r(z)k\Delta x \quad (11)$$

and the total pressure at depth z from Eqs. (11) and (2) is

$$p=\rho g\Delta z-\int_0^z \rho\omega^2 r(z)k\Delta x dz=\Delta p. \quad (12)$$

Therefore $p=\Delta p$, where Δp is the total change in pressure over Δx at z, since the atmospheric pressure and the hydrostatic pressure between 0 and z cancel out of Δp.

The pressure force in the positive x-direction from Eq. (12) is

$$-\frac{\Delta p}{\Delta x}\to-\partial p/\partial x=\rho\omega^2\left[a+k\int_0^z r(z)dz\right] \quad (13)$$

using Eqs. (3) and (4). Equating (13) and the oppositely directed $(c.f.)_{\text{horiz.}}$ in Eq. (8) for the cyclostrophic balance gives

$$a+k\int_0^z r(z)dz=r(z), \quad (14)$$

the solution to which, incorporating the condition $r(0)=a$, is

$$r(z)=ae^{kz}, \quad (15)$$

i.e. the traditional exponential depth decay of the orbital radius.

As before it is easy to show that the cyclostrophic balance holds for any position of the fluid particle around its orbit at mean depth z. It is obvious from the development leading to Eq. (15) that the exponential decay of the orbital radius does not depend on the sinusoidal surface elevation being horizontally infinite.

5. Discussion

The cyclostrophic balance on individual fluid particles is fulfilled at all depths for sinusoidal surface gravity waves with circular particle orbits. Requirements of the cyclostrophic balance are that the normal dispersion relation for deep water waves hold and that the orbital radius decrease with increasing depth at the usual exponential rate. The cyclostropic and hydrostatic balances together predict wave breaking at the crests when the amplitude times the wave number exceeds one. No other restrictions are placed on the amplitude by the cyclostrophic balance, nor is it necessary that the wave train be infinite horizontally. However, the traditional method, which begins with a perturbed irrotational flow, demands infinitesimal amplitudes and assumes an infinite wave train.

The wave pressure, from Eq. (10), and the orbital particle speed, which equals ωr, are both proportional to the orbit radius and therefore decrease exponentially with depth.

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表面重力波における旋衡流的平衡

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要旨: 正弦波的な波形で粒子軌道が円をなす表面重力波のすべての流体粒子について，圧力傾度力と遠心力とによる旋衡流的 (cyclostrophic) 平衡が満たされていることを示す．旋衡流的平衡の結果として， 1) 深海波についての通常の分散関係が成り立つ， 2) 軌道半径は深さとともに指数関数的に減少する，したがって，波の圧力と粒子の運動の速さも深さとともに指数関数的に減少する．さらに，旋衡流的平衡と静水圧的平衡の双方とから，振幅が波数の逆数を超えると波の峰で砕波することが予測される．非回転運動の仮定に基づく旧来の摂動法に比べると，旋衡流的方法は，振幅が波長よりも著しく小さいことを必要とせず，波の列が無限に続く必要もない．

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