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Shoaling Surface Gravity Waves Cause a Force and a Torque on the Bottom

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Freely propagating surface gravity waves are observed to slow down and to stop at a beach when the bottom has a relatively gentle upward slope toward the shore and the frequency range of the waves covers the most energetic wind waves (sea and swell). Essentially no wave reflection can be seen and the measured reflected energy is very small compared to that transmitted shoreward. One consequence of this is that the flux of the wave's linear momentum decreases in the direction of wave propagation, which is equivalent to a time rate of change of the momentum. It takes a force to cause the time rate of change of the momentum. Therefore, the bottom exerts a force on the waves in order to decrease the momentum flux. By Newton's third law (action equals reaction) the waves then impart an equal but opposite force to the bottom. In shallow (but finite) water depths the wave force per unit bottom area is calculated, for normal angle of incidence to the beach, to be directly proportional to the square of the wave amplitude and to the bottom slope and inversely proportional to the mean depth; it is independent of the wave frequency. Constants of proportionality are: 1/4, the fluid density and the acceleration of gravity. Swell attenuation near coasts and some characteristics of sand movement in the near-shore region are not inconsistent with the algebraic structure of the wave force formula. Since the force has a depth variation which is significantly faster than that of the dimensions of the particle orbits in the vertical direction, the bottom induces a torque on the fluid particles that decreases the angular momentum flux of the waves. By an extension of Newton's third law, the waves also exert an equal but opposite torque on the bottom. And because the bottom force on the waves exists over a horizontal distance, it does work on the waves and decreases their energy flux. Thus, theoretically, the fluxes of energy, angular and linear momentum are not conserved for shoaling surface gravity waves. Mass flux, associated with the Stokes drift, is assumed to be conserved, and the wave frequency is constant for a steady medium.

Keywords:
· Shoaling waves,
· wave force,
· wave torque.

1. Introduction

A framework is erected in this paper for understanding certain characteristics of surface gravity waves that travel into regions of continuously decreasing mean depths of water. The framework is based upon two constants of the motion: wave frequency and mass flux.

There has never been any question that for a steady medium the wave frequency must be constant, and so this principle is applied without controversy to shoaling waves when the initial wave amplitude is constant and there are no time variable currents, winds, etc. For example, from the constancy of the frequency and the given dispersion relation of the waves it follows that the wavelength decreases as the mean depth decreases, which, together with the depth decrease itself, leads to the decrease in the phase speed. Formally, the constancy of the wave frequency is derived from the ray equations, which are based on the central assumption that variations in the environment (i.e. the depth) change slowly in one wavelength. Therefore, we adopt this fundamental assumption from the start.

Neither can mass flux conservation be questioned per se, but to place this law above all others, such as the traditional energy flux conservation law, is to break away from the classical analytical methods used in nearshore circulation studies. But by doing so we achieve a consistency with measurements that show a rather dramatic at-

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tenuation of swell amplitude near coasts, accumulated over a thirty-year period, because the wave amplitude is predicted to decrease monotonically toward the beach, in qualitative agreement with the data. Theories which include frictional damping have not so far been able to adequately explain the shoreward decrease of the swell amplitude.

Arriving at the present starting point was the result of an intellectual dilemma. First, energy flux conservation has maintained a firm grip on the minds of physical oceanographers for a considerable period of time. After all, it is difficult to imagine any physical process that could result in the energy flux not being conserved during pure wave-bottom refraction, assuming that bottom and internal friction, reflection, diffraction, nonlinear wave-wave and wave-bottom interactions and wave breaking have negligible effects on the energy budget. Moreover no external forces (wind stress) and no currents (tides) are postulated to be present to complicate the issue. These assumptions have always seemed very reasonable until the past couple of years, when ocean observations of swell attenuation near beaches have begun to exert their influence (e.g. Ardhuin *et al.*, 2003). By taking the energy flux to be a constant, the evolution of the amplitude of shoaling waves is computed straightforwardly, and this has repeatedly been done in the literature, starting perhaps with Rayleigh (e.g. Ippen, 1966).

Second, surface gravity waves transport other properties besides energy. They obviously have angular momentum, for example, that is related to the orbital motion of the fluid particles, whether those orbits are circles in deep water or ellipses in intermediate to shallow water. Unfortunately, angular momentum has not been discussed very much for surface waves, particularly not for shoaling waves, and only relatively recently have a few investigators begun to explore its interesting consequences (e.g. Kenyon and Sheres, 1996). Equally hard to imagine is why the flux of the magnitude of the angular momentum should not be conserved as well as energy flux when surface waves shoal. What are the torques that would cause the magnitude of the angular momentum to change during shoaling? No candidates have so far emerged.

For oblique angles of incidence it is well known that the crests and troughs swing around to become ever more parallel to the beach as a consequence of wave-bottom refraction, in which case the directions of the angular momentum vectors rotate in the horizontal plane. This implies that there must be a torque to change the direction of the angular momentum, presumably supplied by the bottom, but this topic lies outside the scope of the present discussion.

However, the dilemma we experienced is: if energy and angular momentum flux are both conserved simultaneously for propagation in intermediate to shallow mean depths of water, we immediately reach a contradiction. The wave amplitude function predicted by energy flux conservation does not agree at all (qualitatively or quantitatively) with that predicted by angular momentum flux conservation during shoaling. Inadvertently we have apparently overspecified the problem. In that case, which of the two conservation principles do we want to give up, energy or angular momentum? According to the conceptual arguments appearing next, it turns out that we must abandon both of them.

At any rate, the traditional conservation of energy flux forecasts that the wave amplitude should initially decrease to a minimum and then continuously increase thereafter. It is the shoreward increase in amplitude with decrease in distance to the beach that is contradicted by the results of the swell attenuation studies, but this by itself is probably enough to overthrow the long-cherished belief in the conservation of energy flux, at least in its purest form (without friction).

When surface waves shoal, they slow down. We know this from observations and we know it theoretically from the functional form of the phase speed, which decreases monotonically as the mean depth decreases. This decrease in phase speed can be seen most quickly in the shallow water limit, where $c = \sqrt{gh}$ (c = phase speed, h = mean depth), in which the wavelength drops out of the problem. Thus decreasing depth produces decreasing phase speed.

Not only do the waves slow down during shoaling, they also come to a complete stop at the beach. This is observed to happen whether the waves break first or not. Also, according to the centuries old formula, $c = \sqrt{gh}$, a surface gravity wave should simply come to a stop at the shoreline ($h = 0$). Why has it taken so long for this idea to sink in? In addition, measurements have shown that for the bottom slopes normally found around the rims of the world's oceans, the amount of reflected energy in the frequency band of the most energetic wind waves is insignificant compared to that transmitted shoreward; the lower frequency waves are reflected (Munk *et al.*, 1963; Elgar *et al.*, 1994). Aditionally, casual observations confirm that at most beaches forming a ring around a large ocean (e.g. the North Pacific), most of the time, waves can be seen coming into shore but none are noticed returning to sea (excluding special cases of strong offshore winds and vertical cliffs).

Now, surface gravity waves also carry linear momentum, just as light and sound waves do, and they cause a force on solid objects upon reflection from them. In the case of surface waves the linear momentum is directly proportional to the Stokes drift (velocity) through the particle (fluid) density (e.g. Barnett and Kenyon, 1975). Linear momentum travels with the wave speed, which is the group speed in deep water and the phase speed in shallow water. The slowing down and stopping of the waves strongly suggests that the rate at which the momentum is carried shoreward by the waves decreases. In other words, the momentum flux is not constant but diminishes toward the shoreline. When the wave comes to a halt at the beach, the linear momentum of the waves must be entirely absorbed by the bottom.

A potentially confusing idea can be anticipated and dispelled at this point. Traditionally, energy—and therefore also by analogy linear and angular momentum—is transmitted with the group speed. And it is known that the group speed has a weak maximum at one particular intermediate depth for a given wavelength. From this analytic feature one might conclude that the transmission of linear momentum would actually speed up slightly around this one mean depth. Whether or not observations will eventually confirm this theoretical fine point is left for the future. But the conclusion remains valid that the overall momentum flux must decrease from deep water to the beach where the waves slow down and eventually stop.

The spatial rate of change of the momentum flux is equivalent to the time rate of change of the momentum. It takes a force to cause a time rate of change in the linear momentum. Therefore, there must be a force present to make the momentum flux decrease in the direction of wave propagation, and this force can only be related to the sloping bottom, assuming that the mean sea level remains fixed. (Casual observations, at least, do not reveal any changes in mean sea level.) By Newton's third law (action equals reaction) the waves must then exert an equal but opposite force back on the bottom. Certain geological types of ocean bottom could support such a force without deformation over short time-scales (i.e. comparable to a wave period); others, such as sand, might not always be able to.

As will be demonstrated later, the force that changes the linear momentum has a vertical variation, which is much faster than that of the orbital dimensions in the vertical direction and therefore it causes a torque on the fluid particles, resulting in a change (decrease) in the orbital angular momentum flux of the waves. Conversely, the waves impart an equal but opposite torque to the bottom (this is an extension of Newton's third law for linear motion to rotary motion; e.g. see Ference *et al*., 1956). Consequently, neither the linear nor the angular momentum flux can be conserved when surface waves shoal. And since the force that decreases the linear momentum flux acts over a horizontal distance as well, work is done on the waves such that energy flux cannot be considered constant either.

What tools do we have left to analyze shoaling surface waves? Mass flux conservation. We shall see how far we can get using that one principle. The constancy of the frequency is useful but it is not an essential element in what follows.

2. Mass Flux Conservation

A more complete heading for this section would be: conservation of mass flux and non-conservation of the fluxes of energy, linear and angular momentum. It is simpler for most purposes to use the shallow water limit throughout because otherwise the group velocity is such a cumbersome function of the depth and wave number in the intermediate depth range that the presentation of the main points would be obscured by the algebraic complexity. Also for convenience we assume that the waves progress normal to a straight shoreline and that the depth contours are straight and parallel to the beach. Friction is neglected throughout. To be definite, the waves travel in the positive x-direction, which points toward shore, and the z-axis points vertically up away from the mean free surface.

We envision a three-dimensional mass circulation in the near-shore zone. A two-dimensional circulation would be more appropriate for a narrow wave channel in the laboratory. Of all the conceivable flow configurations that the mass could use to return to sea from the shore (compensation flow, undertow, etc.) it appears that much of the time the ocean chooses the three-dimensional solution for reasons that are still not completely understood. The waves bring mass into the beach by means of the Stokes drift, and we are not considering any other ways that mass can be transported shoreward. We find it convenient to assume that mass returns seaward via narrow, swift and well-spaced rip currents (Shepard, 1959). Rip currents are fed by longshore currents, but the generation and maintenance of both these currents are outside our present interest. Fortunately, the main results to follow do not depend on knowing how longshore and rip currents are caused. In the broad regions between adjacent rip currents the mass flux must be constant through all vertical cross-sections.

Mass flux conservation between rip currents is given by

$$\rho \int_{-h}^{0} U dz = const = \overline{M} \tag{1}$$

where ρ is the constant fluid density, $z = -h$ is the bottom depth, $z = 0$ is the mean free surface, the *const* means independent of x, and U is the Stokes drift, which for finite constant mean depth is

$$U = c(ak)^2 \frac{\cosh 2k(z+h)}{2\sinh^2 kh} \tag{2}$$

as originally derived by Stokes (1847). In (2) c is the phase speed, k the wave number, a the wave amplitude, and z is an arbitrary mean depth of a fluid particle.

Putting (2) into (1) and performing the depth integration gives

$$\frac{\rho g a^2}{2c} = \frac{\overline{E}}{c} = \overline{M} = const \tag{3}$$

where

$$\overline{E} = \frac{1}{2}\rho g a^2 \tag{4}$$

is the total mean energy per unit horizontal area of a surface wave valid in any arbitrary mean depth. In this section Eqs. (6)–(8) and (12) all depend on (3) as the primary assumption.

In the shallow water limit the phase speed is

$$c^2 = gh \tag{5}$$

which is a classical result in fluid dynamics (Lamb, 1932). Equation (5) shows that the phase speed decreases as the depth decreases and at a rate proportional to the square root of the mean depth. We assume that the mean depth h varies slowly in one wavelength so that in (2) and (5) and elsewhere the depth can be taken as approximately constant locally.

From (5) and (3) we see that

$$a \propto h^{1/4} \tag{6}$$

which shows that the amplitude decreases as the depth decreases because it is proportional to the fourth root of the depth. At first sight a decreasing amplitude accompanying a decreasing mean depth might tend to contradict experience. However, this result is qualitatively consistent with recent swell attenuation observations (Ardhuin *et al.*, 2003), as discussed later. Only physical and numerical constants (ρ, g, 1/2, etc.) are not explicitly shown in (6) and some of the following equations ((7), (8), (12)).

Consider next the standard expression for the energy flux

$$\overline{E}c_g \approx \overline{E}c \propto h \tag{7}$$

where in the shallow water limit the group and phase speeds are equivalent. Equation (7) has been evaluated with (4)–(6). The energy flux decreases as the depth decreases, according to (7). In other words, the energy flux is not conserved during shoaling, which is a direct consequence of the fact that mass flux (1) is conserved. Physical reasoning for why energy flux is not conserved is tied to the non-conservation of linear momentum as discussed in the next section. Concluding that energy flux is not conserved as the waves shoal disagrees with some current and most past thinking on the subject.

For the flux of wave linear momentum we have

$$\overline{M}c_g \approx \overline{M}c = \overline{E} \propto h^{1/2} \tag{8}$$

where $\overline{M}$ is given by (1), and it has been shown before for surface waves (e.g. Barnett and Kenyon, 1975; see also (3) above) that

$$\overline{M} = \frac{\overline{E}}{c}. \tag{9}$$

Thus the momentum flux is not conserved; it decreases as the depth decreases. This result is at least not inconsistent with earlier work, in the sense that there is not an established law stating that linear wave momentum flux should be conserved (or not conserved) during shoaling, like there is for energy flux.

“Radiation stress” and “excess momentum flux” are designations for a general concept that, when applied specifically to shoaling waves, leads to a correlation between a variation in momentum flux and a variation of the mean sea level (e.g. Phillips, 1966), but so far this method has not led to a prediction of a wave force on the bottom. Furthermore, the mean sea level is forecast to first decrease toward shore before eventually rising near the beach. This is counterintuitive since one would think that to slow down and to stop the incoming wave momentum, the mean sea level should only rise monotonically toward the coast.

From a theoretical point of view, as discussed later, a variation in mean sea level cannot account for a decrease in angular momentum flux during shoaling because it would produce a barotropic offshore pressure gradient that is incapable of causing a torque on the orbiting fluid particles. Of all the properties brought toward the coast by the waves (mass, energy, linear and angular momentum), the only one that does not return seaward is the orbital angular momentum. Rip currents carry mass, energy, and linear momentum, but they do not transport angular momentum about horizontal axes parallel to the shore. (Longshore currents have no orbital angular momentum either.) Therefore, a torque must exist to damp down the shoreward moving angular momentum, and a barotropic offshore pressure gradient, related to a shoreward increase in mean sea level, cannot possibly do the job.

Similarly, it can be shown relatively easily that the angular momentum flux is not conserved during shoaling. Start with the angular momentum A of a fluid particle at an arbitrary mean depth where the total depth is finite

$$A = \frac{\rho\omega a^2}{2} \frac{\sinh 2k(z+h)}{\sinh^2 kh}. \quad (10)$$

Equation (10) has probably never been published before but it originates directly from $A = \rho|\vec{r} \times \vec{v}| = \rho|(xw - zu)|$, where $\vec{r}$ is the position vector and $\vec{v}$ is the tangential velocity of the particle, and the Lagrangian coordinates and velocity components of a fluid particle in a progressive surface gravity wave as given in the textbooks (e.g. Lamb, 1932). One interesting property is that the angular momentum of all fluid particles is independent of time as they move around their elliptical orbits.

When the angular momentum per unit volume in (10) is integrated over the water column to get the angular momentum per unit horizontal area, the result is

$$\overline{A} = \int_{-h}^{0} A dz = \frac{\rho c a^2}{2}. \quad (11)$$

Now, for the flux of the angular momentum per unit horizontal area we have

$$\overline{A} c_g \approx \overline{A} c \propto h^{3/2} \quad (12)$$

in the shallow water limit. Consequently, the angular momentum flux is not conserved since it decreases as the depth decreases. This result does not violate any established precepts, to the best of my knowledge.

To sum up, if mass flux is conserved, then neither energy, nor angular or linear momentum are conserved; they all decrease as the depth decreases.

3. Wave Force and Torque on the Bottom

Wave force equals time rate of change of linear momentum, which equals the spatial rate of change of the momentum flux. Thus

$$\frac{\partial \overline{M}}{\partial t} = \frac{\partial}{\partial x}\left(M c_g\right) = \overline{M} \frac{\partial c_g}{\partial x} = \overline{F} \quad (13)$$

since $\overline{M}$ = *const* (independent of x) by assumption from (1), and $\overline{F}$ is the force per unit "horizontal" area. We have assumed a small bottom slope, so the force per unit horizontal area can be thought of as applying to the bottom even though it is not exactly horizontal. To obtain the total force on a section of the bottom, the force (13) needs to be multiplied by the area of the section. If the wave amplitude is independent of time, then the force on the bottom in (13) is in a steady state. In (13) the force is represented as a scalar, but we know it points parallel to the x-axis: the wave force on the bottom is in the positive x-direction (toward the shore), the equal but opposite bottom force on the waves is in the negative x-direction (toward the sea).

It is well known (e.g. from the earlier usage of the energy flux conservation principle) that the group speed has a maximum, although a very weak one, at intermediate depths. For a given wave number there is one mean depth for which the derivative of the group speed vanishes. So at this particular depth the force per unit area will be zero by (13). Both seaward and shoreward of this "critical" depth the force will exist, though. But the more important region for the wave force is between the critical depth and the shoreline because of the depth dependence of the force, which increases as the depth decreases in shallow water, as shown in (14).

In the shallow water limit (13) reduces to

$$\overline{F} = \frac{1}{2}\left(\frac{g}{h}\right)^{1/2} \overline{M} s = \frac{1}{4} \frac{\rho g a^2 s}{h} = \frac{1}{2} \frac{\overline{E} s}{h} \quad (14)$$

where $s = dh/dx$ is the bottom slope. By (14) the force decreases as the slope decreases and it vanishes when the bottom is flat, a physically reasonable result in the absence of friction. In any case, the analysis is restricted to small slopes in order to meet the requirement that the depth change slowly in one wavelength. The force in (14) also increases as the mean depth decreases. Since the depth decreases in the positive x-direction, the bottom slope is negative and the momentum decreases with time, as it should. Notice that the force in (14) is independent of the frequency of the waves, so long as the waves are within the frequency band for which complete absorption takes place (i.e. no reflection). Thus the total force for a superposition of different frequencies is proportional to the sum of the squares of the amplitudes of the individual wave components.

As a relatively minor point, seaward of the critical depth the force changes sign according to (13) because of the maximum in the group speed. This may be counter-intuitive, but in any case the wave force is very weak in this depth range and it goes to zero in the deep-water limit.

Another minor point is the apparent mathematical singularity in the force (14) in the shallow water limit at the shoreline ($h = 0$). Physically, the force would not blow up there. Taking the mass flux $\overline{M}$ constant right up to the shoreline is unrealistic. Before the shoreline is reached mass will begin to move sideways in longshore currents parallel to the beach, which then supply the rip currents. Therefore, $\overline{M}$ goes to zero as h goes to zero and this will erase the singularity in the denominator of (14). There is probably still a certain depth interval in which the trend is realistic that the force increases as the depth decreases and right near the shoreline the force may reach a maximum.

Wave torque equals the time rate of change of the angular momentum, which is equal to the spatial rate of change of the angular momentum flux. Therefore

$$\frac{\partial \bar{A}}{\partial t} = \frac{\partial}{\partial x}\left(\bar{A} c_g\right) = \bar{T} \quad (15)$$

since $\bar{A}$ is not constant (it depends on x), and $\bar{T}$ is the torque per unit horizontal area. Again we interpret the torque as being applied to the gently sloping bottom. With the constraint (3) the shallow water limit reduces (15) to

$$\bar{T} \propto h^{1/2} s \quad (16)$$

showing that the torque, like the force, decreases as the slope decreases, but unlike the force the torque also decreases as the depth decreases. In addition, the angular momentum decreases with time because the slope is negative.

To better appreciate how a torque can arise in the first place, consider the linear momentum per unit volume m

$$m = \rho U = \rho c (ak)^{1/2} \left[\frac{\cosh 2k(z+h)}{2\sinh^2 kh}\right] \quad (17)$$

from (2). The time rate of change of the momentum per unit volume gives the force per unit volume. From the structure of (17), in particular the quantity in the square brackets, it can be seen that, for a given wave number, the depth variation of the force per unit volume will be much more rapid in intermediate to shallow mean depths than the depth variation of orbital dimension in the vertical direction, which is proportional to $\sinh k(z+h)/\sinh kh$ according to the textbooks (Lamb, 1932). In other words, the seaward force at the top of the orbit will be significantly larger than that at the bottom of the orbit, which produces a net torque on the particle that will reduce its orbital rotation.

For a horizontal variation in mean sea level, for example a rise in sea level at the coast, the offshore pressure gradient will be independent of depth for a homogeneous fluid. In shallow water with breaking waves, mixing the water column from top to bottom, we do not expect any significant vertical stratification of the density. Thus the horizontal pressure gradient due to mean sea level changes, if there is any, must be barotropic, and a depth independent pressure gradient is incapable of causing a torque on the orbiting particles (the force is the same at the top and bottom of the orbits of the particles). In other words, a variation in mean sea level cannot decrease the incoming angular momentum flux, which we know must happen during shoaling.

4. Discussion

As mentioned above, swell attenuation studies have been carried out recently near the East Coast of the US (Ardhuin *et al.*, 2003). Furthermore, 30 years before that similar experiments were done in the North Sea (Hasselmann *et al.*, 1973). The algebraic form of the wave force on the bottom in (14) is qualitatively consistent with a few characteristics of these observations. For example, in the North Sea, no completely satisfactory explanation for the shoreward decay of the swell amplitude, in terms of generally accepted friction laws, was found. However, two characteristics of the decay rate were discovered: an increase of the decay rate with an increase in initial swell energy and with a decrease in mean depth. Both of these features can be understood in terms of the far righthand side of (14) because the bottom force on the waves increases with increases in the energy and with decreases in the depth.

One further point with respect to amplitude that needs mentioning is the common observation that shoaling waves normally "peak up" before they break. In other words, although the amplitude might initially decrease shoreward, as is usually believed, the amplitude would then appear to increase until the breaking point is reached. However, the peaking up phenomenon may actually involve more of a change in wave shape than an increase in wave height. An explanation of the peaking up process already exists, as related next.

Kenyon and Sheres (1991) used a shallow water model to show theoretically that at finite height the wave profile takes on an asymmetric shape if the stability of the wave is to be maintained. The asymmetry is characterized by higher narrower crests and shallower broader troughs. This is consistent with the "peaking up" observed by beach goers in general and by surfers in particular. Overall wave height increases are not so dramatic if the troughs rise up at the same time as the crests become higher. At the heart of the model is Einstein's (1916) method of balancing static and dynamic pressure differences between crest and trough along the surface streamline in the reference frame that makes the waves appear steady. In a straightforward way the method is adapted to finite constant mean depth and finite wave heights can be investigated under certain assumptions. For gradually decreasing mean depth the model predicts increasing asymmetry in the wave profile, which agrees with the observed peaking up of the waves.

Nearshore sediment transport is a very complicated subject, involving several empirical attempts to relate sand movement to wave energetics, etc. It was not our original intention to try to step in here from the sidelines and offer any suggestions. But there are a few observations that are not inconsistent with the functional form of the wave force on the bottom in Eq. (14), and we are prompted to at least point them out. One observed fact is the characteristic concave shape to the bottom profile and another one is the positive correlation between sediment size and bottom slope (e.g. Komar, 1998).

A concave bottom profile agrees qualitatively with (14) because for a given sediment size and a given initial constant slope the wave force on the bottom increases as the depth decreases, which would tend to produce some initial scouring followed by a piling up of sand near the mean waterline. The net result would be to turn a linear profile into a concave one. Typically, beach profiles are measured a considerable distance beyond the breaker point, whereas (14) is not expected to apply shoreward of this point.

There is strong evidence to support a positive correlation between particle size and beach slope. Large sediment sizes go with large bottom slopes, and vice versa. This is qualitatively consistent with (14) too. For example, pebbles or cobblestones would be harder to move than sand because of their greater weight (in water), and it would therefore take a larger force to move them. According to (14) for a given mean depth the wave force is larger for a larger slope. It is interesting that the positive correlation between bottom slope and sediment grain size often extends into very shallow regions in which (14) is not strictly applicable due to wave breaking. At the beaches of lakes and smaller bodies of water, however, one can on occasion observe waves coming right up to the shore without breaking and without reflecting. In these cases there is no breaker point and (14) would be useful throughout the whole shoaling zone.

At the beaches in southern California, for example, there is a prominent seasonal migration of the sand. In the summer the sand is in and in the winter it is out. When the sand is out, it is not very far out but sits in deeper water on the continental shelf (some of the sand may be permanently lost to the near-shore system by falling into submarine canyons, but rivers normally bring in new sand). This seasonal signal in sand movement is not inconsistent with the wave force formula. For constant bottom slope and beach material, the bigger the amplitude of the waves (i.e. the bigger a and M are), as in winter on the average, the larger the force is by Eqs. (13) and (14). A sufficiently large force could suspend enough sand that the longshore currents and rip currents working together would then cause a net transport of sand to deeper depths. In summer a weaker force due to smaller waves might only produce a net shoreward movement of sand.

An additional feature of the wave force on the bottom has already been noted, and we have classified it as a minor feature: the force is proportional to the derivative of the group speed in (13) and therefore it vanishes at a particular intermediate mean depth for a given initial wave number. However, if we take this feature seriously for a moment, then we arrive at an elementary cause for the formation of a trough or dip in the sand. Suppose the wave force is sufficiently strong to make sand move along the bottom but not so strong as to cause a lot of sand to be suspended in the water column, where it could be advected away from the scene by longshore currents. Shoreward of the critical depth the force will scour the bottom and move sand toward the beach. Seaward of the critical depth the force has the opposite sign and will tend to move sand further away from the beach. The net result will be a local depression in the sand bottom approximately centered at the location of the critical depth. It is not too uncommon to find sand depressions while walking out into the surf in summer, and perhaps we have just found a tentative explanation for their formation. However, such an explanation is strongly dependent on the assumption that the mass flux is conserved.

In the general case in which the wind waves do not initially travel normal to the shore or to the bottom contours, the wave rays are curved and the wave number vectors change direction along the rays. Therefore, the bottom must cause an additional force on the waves, which we know will point perpendicular to the wave number. Conversely, the wave exerts an extra force on the bottom. The calculation of the magnitude of this sideways force is left for another time, but we understand that it will be zero for normal incidence and very small for near normal incidence.

Omitted from consideration above is any treatment of the thin viscous boundary layer near the seabed. What qualitative changes in the present conclusions that might result from including a bottom boundary layer are unknown at this point.

The ocean tide does not break at the coast, but then the tide is not absorbed there either (ignoring friction). Due to its low frequency the tide is at least partially reflected at a continental slopes and shelves, depending on how much of the tide propagates parallel to the coastline. Of course, upon reflection the tide must exert a force on the coast, but we cannot use Eq. (13) or (14) to estimate it. Another feature to consider is that the tide is a forced wave, not a freely propagating one, as assumed above. Unlike the wave force on the sloping bottom calculated here, which vanishes for depths exceeding a wavelength, the horizontal tide force on a continental boundary would exist from sea level all the way to the deep sea without appreciable diminution.

Although the force on a sloping bottom, due to shoaling surface waves that are completely absorbed at the beach, may not have been calculated before Eq. (13) was put forward above, which is curious, it is even more curious that the wave force on a rigid vertical wall, where complete reflection takes place, has been computed an embarrassingly large number of times. For example, as of about 50 years ago there were over 25 separate algebraic equations published for the wave force on a vertical wall. Not only did these early formulas disagree among themselves but none of them compared satisfactorily with the laboratory measurements reported by Rundgren (1958). Since that time a few more distinctly different attempts to quantify the wave force on a vertical wall have been entered into the public domain (e.g. Kenyon, 2004), and wave force data on a vertical wall have been obtained under environmental conditions (Boccotti, 2000).

As for the torque on the bottom given by (15) and (16), we are unable to relate it to any other body of knowledge because evidently no previous discussion of it has been available.

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