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Short Contribution

Gravity Torques for Surface Waves

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The effects of the gravity torques acting on the angular momentum of surface gravity waves are calculated theoretically. For short crested waves the gravity torque is caused by the force of gravity on the orbiting fluid particles acting down the slopes of the crests and troughs and in the direction parallel to the crests and troughs. The gravity torque tries to rotate the angular momentum vectors, and thus the waves themselves, counterclockwise in the horizontal plane, as viewed from above, in both hemispheres. The amount of rotation per unit time is computed to be significant assuming reasonable values for the along-crest and trough slopes for waves in a storm area. The gravity torque has a frequency which is double the frequency of the waves. For long crested waves the gravity torque acts in the vertical plane of the orbit and tries to decelerate the particles when they rise and accelerate them when they fall. By disrupting the horizontal cyclostrophic balance of forces on the fluid particles (centrifugal force versus pressure force) the gravity torque accounts qualitatively for the three characteristics of breaking waves: that they break at the surface, that they break at the crest, and that the crest breaks in the direction of wave propagation.

Keywords:
· Surface waves,
· angular momentum,
· gravity torques,
· wave breaking,
· wave refraction.

1. Introduction

Surface gravity waves have orbital angular momentum. Each fluid particle has angular momentum with respect to the mean position of its orbit. For small amplitude progressive waves in deep water the particle orbits are all circular, so the magnitude of the angular momentum per unit volume of a surface particle is, neglecting the constant density, easily computed to be the angular frequency of the wave times the square of the radius of the particle, or the square of the wave amplitude, because for surface particles the orbital radius equals the amplitude (see Eq. (3)).

Since surface waves have angular momentum, then changes in the angular momentum must be caused by a torque. Angular momentum can be changed by a torque either in direction or in magnitude or in both. Wave refraction by wave-current interaction is a process in which a shear torque changes the direction of the angular momentum without changing its magnitude (Kenyon and Sheres, 1996). When wave refraction occurs in decreasing mean water depth, i.e. bottom refraction, there is also a torque, involving the bottom depth and slope, that changes only the direction of the angular momentum.

Do any other torques exist that can change the angular momentum of surface gravity waves, and if so how do these changes manifest themselves? What is needed is a list of all possible forces that can act on an orbiting fluid particle as surface waves pass by. The force of friction, for example, can change the magnitude of the angular momentum, leaving its direction constant, in either of two ways. Internal friction within the medium that transmits the waves produces a torque which always decreases the angular momentum. A wind torque at the air-water surface, on the other hand, can increase the angular momentum of the surface particles, through a frictional interaction between the air and water, under the right conditions (the wind must blow faster than the waves propagate, for example).

If the Coriolis force that acts on the particle velocity during surface wave propagation is unbalanced by any other force, it therefore causes a torque on the waves. However, it is clear that the Coriolis force (and therefore the Coriolis torque) cannot change the magnitude of the angular momentum because it acts always at right angles to the velocity measured relative to the earth's surface.

That leaves the force of gravity. A gravity torque can exist, but what are its effects? This is the subject of the following brief report, and to my knowledge it has never been discussed before.

2. Short Crested Waves

Consider a surface gravity wave with an amplitude that monotonically increases to the right, facing in the direction of wave propagation, parallel to the crests and troughs (see Fig. 1). At the crest there will be a gravitational force $\vec{F}$ on a fluid particle acting to the left, down the sloping surface, with magnitude $F = g\sin\theta$, where g is the acceleration of gravity and θ is the angle of the wave surface from the horizontal. The gravity torque $\vec{t}_g$ on the surface particle has magnitude

$$t_g = \frac{1}{2} ag \sin 2\theta \quad (1)$$

where a is the magnitude of the radius $\vec{r}$ of the particle, and it is also the wave amplitude since the particle is at the surface. For convenience the constant density of the fluid particle has been omitted from (1). The direction of the gravity torque is opposite to that of the direction of wave propagation because of the relation $\vec{t}_g = \vec{r} \times \vec{F}$ and the fact that $\vec{r}$ points up and $\vec{F}$ points mainly to the left and also a little bit down.

At the trough the magnitude and direction of the torque are the same as they are at the crest because the signs of both the gravity force and particle radius are reversed (the gravity force always acts downhill). The maximum value of the gravity torque is given by (1). In between crest and trough, where the wave surface is level, the gravity torque in (1) vanishes. Therefore, the frequency of the gravity torque is double the frequency of the wave.

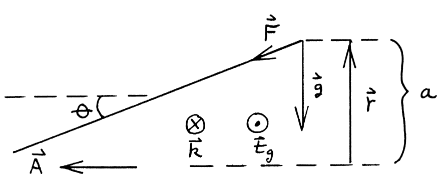

Fig. 1. A surface gravity propagates into the paper parallel to the wave number $\vec{k}$ and has a crest sloping up to the right, making an angle θ with the horizontal. The angular momentum vector $\vec{A}$ points to the left. There is a gravity force $\vec{F} = g\sin\theta$ acting along the crest, where g is the acceleration of gravity. The radius vector of a surface particle is $\vec{r}$ which points up and has magnitude a, i.e. the wave amplitude. The gravity torque $t_{\vec{g}} = \vec{r} \times \vec{F}$ points out of the paper or antiparallel to the direction of wave propagation.

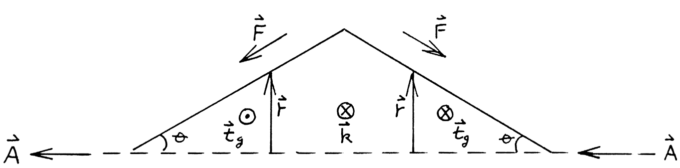

Fig. 2. A short crested wave, or a hump, that propagates into the paper. The gravity torques point in opposite directions on each side of the hump in such a way as to try to rotate the angular momentum vectors $\vec{A}$, which point to the left, counterclockwise as viewed from above.

It can be noticed from (1) that the magnitude of the gravity torque is independent of the angular momentum, and therefore of the wave frequency, unlike the shear torque for wave-current interaction (Kenyon and Sheres, 1996), for example. Below the surface the gravity torque will decrease until it vanishes at the depth of wave influence. In deep water the rate at which the gravity torque decreases with increasing depth will be exponential, but it will decrease like e^{2kz}, where k is the wave number and $0 \le z \le -\infty$. This is due to the fact that both the particle radius and the crest slope decrease like e^{kz}.

Next consider a short crested wave, a wave hump, in which the wave amplitude starts from zero, increases to the right of the propagation direction, reaches a maximum, and then decreases to zero again (see Fig. 2). On the left side of the hump the gravity torque will be directed anti-parallel to the direction of wave propagation, as deduced above. To the right of the hump the gravity torque will point in the direction of wave propagation, because the sign of the gravity force is reversed whereas the sign of the particle radius remains the same. The net effect of all the gravity torques would be to try to rotate the hump (and the trough) counterclockwise in the horizontal plane, as viewed from above. The counterclockwise rotational effect will be true in either hemisphere.

At first it might be thought that gravity, acting along and down the crest, would try to make the hump collapse. This does not happen, however, because the gravity force acts on the rotating fluid particle thereby producing a torque that tries to change the direction of the angular momentum vector in the horizontal plane, since the torque itself acts in the horizontal plane.

Before making any speculations as to possible applications of the gravity torque to surface waves in the ocean, however, the magnitude of the gravity torque should be estimated and compared with that of other torques that are already available. For example the magnitude of the shear torque, t_s, for the wave-current interaction problem (Kenyon and Sheres, 1996), which has been known to be an important process for at least 25 years (Kenyon, 1971), is for surface particles

$$t_s = As = \left(\frac{2\pi a^2}{T} \right) s \quad (2)$$

where A is the magnitude of the angular momentum, T is the wave period and s is the magnitude of the current shear. For the following numerical values: $a = 1m$, $T = 6$ sec, $g = 10m/$

sec^2, and $s = 10^{-4}$/sec (Sheres *et al.*, 1985), it can be seen from (1) and (2) that the gravity torque will have approximately the same value as the shear torque when the surface slope is given by $\sin\theta = 10^{-5}$, which is quite a small slope, i.e. about 1 m rise in a horizontal distance of 100 km. It would seem that most short crested waves have slopes considerably larger than this, although very few measurements of crest slopes exist, probably. Therefore, the gravity torque is significant in magnitude compared to the shear torque and warrants a little more discussion.

Unlike the shear torque of the wave-current interaction problem, the gravity torque acts *in situ* and not following a wave group. If a long wave train passes by a fixed position, then the gravity torque will have time to rotate the angular momentum locally in the counterclockwise direction. The first wave in a wave train will experience very little effect from the gravity torque, but waves further back in the train will be affected more.

An estimate can be made of the amount of counterclockwise rotation, per unit time, in the horizontal plane of the angular momentum of short crested waves as follows. From the angular momentum balance and the definition of the angular momentum of a surface particle in deep water we have

$$\frac{dA}{dt} = \Omega A = t_g \qquad A = \omega a^2 \tag{3}$$

where $\Omega = d\alpha/dt$ is the angular rate of rotation in the horizontal plane and the angle α is measured counterclockwise; $\omega = 2\pi/T$.

From (1) and (3) we can estimate the amount of rotation, $\Delta\alpha$, of the angular momentum vector in the time interval Δt

$$\Delta\alpha = \frac{g \sin 2\theta T \Delta t}{4\pi a} \tag{4}$$

where a is the maximum amplitude of the hump. Using the same numerical values as before, $a = 1m$, $T = 6$ sec, $g = 10m/$sec^2, $\sin\theta = 10^{-5}$, then from (4) $\Delta\alpha \approx 6 \times 10^{-4}$ when $\Delta t = 6$ sec, i.e. one wave period. After a time interval of 10 wave periods the angular rotation would be 10 times larger, or $\Delta\alpha \approx 6 \times 10^{-3}$, which is about 0.3 degrees. This amount of rotation is small and probably insignificant. However for crest slopes 10 to 100 times bigger ($0^{-4} \le \sin\theta \le 10^{-3}$), which could easily be imagined within a storm, then the angle of rotation becomes potentially much more important, even within a single wave period.

The theoretical features of the gravity torque on short crested waves, that the waves rotate counterclockwise in either hemisphere, as viewed from above, is qualitatively consistent with observations by Munk *et al*. (1963) and Snodgrass *et al*. (1966) on swell propagating from the South to the North Pacific. When projected back from California along the direction of propagation, assuming no torques of any kind including that for wave-current interaction, the source of the waves was found to be consistently to the left of where the actual storms were by a few degrees or a few hundred kilometers, and sometimes the projected source was on land (Antarctica). The gravity torque has the right sense and possibly the right magnitude to account for the calculated displacement of the wave sources.

However the discrepancy in projected source direction was found by Munk *et al*. (1963) to be larger for higher frequency waves, which is inconsistent with the gravity torque mechanism, as applied to swell, because the amount of angular rotation of the angular momentum vectors in (4) increases as the wave period increases. But both the bending of the rays and its frequency dependence have already been shown to be consistent with the wave-current interaction process (Kenyon, 1971).

On the other hand, if wind waves in or near a storm are considered, then it is generally accepted that there is a relationship between the significant wave height and the significant wave period. For example, Toba (1972) proposed that the significant wave height is proportional to the three halves power of the significant wave period. Accepting this proportionality, and inserting it into (4), the result emerges that the amount of rotation per unit time of the short crested waves decreases with increasing wave period. Now some consistency between the gravity torque mechanism and the observations of Munk *et al*. (1963) has been obtained, at least for those recording stations that were sufficiently close to the storm that the wind wave model is more valid than the swell model.

3. Long Crested Waves

For long crested surface gravity waves, or waves whose amplitude is constant in the direction parallel to the crests and troughs, there still exists a gravity torque, which is due to the acceleration of gravity acting in the vertical plane of the particle orbits. This torque is directed parallel to the angular momentum and so has the capability of accelerating or decelerating the orbital velocity of the fluid particles, i.e. changing the magnitude of the angular momentum.

Look in the direction of the angular momentum and see the clockwise motion of the particles as a surface gravity wave propagates from left to right. For the moment consider only particles at the surface. At the position where the vertical velocity of a surface particle is up, gravity acts down and causes a torque that tries to decelerate the particle. The opposite happens where the particle moves down, the gravity torque tries to accelerate the particle.

Over a complete wave cycle it is presumed that the net effect of this gravity torque is zero, because the amount of deceleration on the rising particle is made up when the particle accelerates on the way down. However, the gravity torque causes an asymmetry with respect to the center of the orbit, which may turn out to be important. The asymmetry is expected to increase as the wave amplitude increases, for a given wave period, and also to increase as the period increases, for a given amplitude. The reason is that the accelerating and decelerating effect of gravity on the fluid particle has a longer time to operate under these conditions.

Now surface gravity waves break in an asymmetric way. They break at the surface and not below the surface. The crest breaks but not the trough. The crest breaks forward, in the direction of wave propagation, but not backward, assuming contrary winds are not blowing. What does the gravity torque have to say about the way surface waves might break?

First of all, for small amplitude waves progressing in deep water, for which the particle orbits are circular, there is a balance of forces on each fluid particle at all positions around the orbit. It is called the cyclostrophic balance (Kenyon, 1991), and it involves the outward centrifugal force and the inward pressure gradient.

At the front face of the wave the fluid particles are rising and also decelerating because of gravity. For small amplitudes and high frequencies this deceleration will be negligible and not enough to destroy the cyclostrophic balance of forces. However, for large amplitudes the deceleration will decrease the vertical velocity to the point that the outward centrifugal force becomes less than the inward pressure force. In other words the inward pressure force overbalances the centrifugal force and causes the fluid particles to accelerate in the direction of wave propagation at the wave crest. By analogous reasoning, on the back side of the crest the fluid particle has accelerated to the point that the outward centrifugal force overbalances the pressure force, and the net result is again to accelerate the fluid particles at the crest forward in the direction of wave propagation.

In summary, the gravity torque can account qualitatively for all three observed characteristics of breaking surface gravity waves, that they break at the surface, that they break at the crest, and that the crest breaks forward in the direction of wave propagation. The fact that waves break at the surface and not below the surface is due to the the radius of the particles orbits being largest at the surface, no matter what the depth of water is, and to the vertical accelerating and decelerating effect of gravity on the particles in the plane of the orbit. That the crests break and break forward is due to the horizontal inbalance in the cyclostrophic pair of forces, which is caused in turn by the changes in the vertical velocity of the particles by gravity.

When surface waves shoal, that is enter water of gradually decreasing mean depth, it is theoretically predicted and also observed that the wave amplitude increases while the wave period remains constant. Therefore the shoaling process is a mechanism for increasing the wave amplitude to the point that the breaking mechanism mentioned above can take over.

It must be noted that the cyclostrophic balance of forces still takes place in shallow water even when the particle orbits are ellipses (Kenyon, 1995). So the transformation from circular to elliptical shape orbits during shoaling is not enough by itself to cause the waves to break. However, the same reasoning discussed above applies equally well to the elliptical orbit. That is when the amplitude becomes sufficiently large, the gravity torque, acting in the vertical plane of the elliptical orbit, makes the wave break by disrupting the horizontal cyclostrophic balance which causes the crest to accelerate forward.

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