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Gravity waves on ice-covered water

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Abstract. Gravity waves propagating on the surface of ice-covered water of finite depth are considered. The ice layer is viewed as a suspension, with an effective viscosity much greater than that of water and a density slightly less than that of water. It is treated as a viscous liquid, and the water beneath it is treated as an inviscid liquid. The linearized motion of gravity waves is analyzed for this two-layer model, and the dispersion equation is obtained. It is solved numerically for waves of any length. It is also simplified for waves short compared to the layer thickness and for waves long compared to the layer thickness. This equation yields dispersion and strong attenuation, both of which depend upon the effective viscosity of the suspension.

1. Introduction

Broken ice modifies the speed of surface gravity waves on water and attenuates them. To analyze these effects, we consider the ice-containing upper layer of water to be a suspension of solid bodies in water. The viscosity $\mu(c)$ of such a suspension is very large when its concentration c is high, despite the fact that the viscosity of water is small. Therefore we treat the upper layer as a viscous incompressible liquid and the water beneath it as an inviscid incompressible liquid. In this way we convert the system into one consisting of a viscous upper layer of liquid and an inviscid lower layer of liquid. Then we solve this two-layer system for waves of small amplitude, using linear theory, and we obtain the exact dispersion equation. We solve it numerically for waves of any length. In addition, we simplify it for waves short compared to the layer thickness and also for waves long compared to the layer thickness.

The resulting dispersion and attenuation depend upon the effective viscosity coefficient $\mu(c)$ of the ice-water suspension. It can be very large for large c since, for a periodic suspension of spheres at high concentration, $\mu(c)$ becomes infinite when the spheres touch one another [*Nunan and Keller*, 1984] The values of $\mu(c)$ for frazil, brash, and pancake ice can differ from one another because $\mu(c)$ depends upon both particle shape and concentration.

There have been previous studies of surface gravity waves in water covered by floating ice. For example, *Peters* [1950] and *Weitz and Keller* [1950, 1953] studied the reflection of waves from floating ice covering half the surface in water of infinite depth and in water of finite depth, respectively. The ice was treated as mass loading of the surface. *Goldstein and Keller* [1953] treated the same reflection problem in shallow water and also considered wave reflection from a floating mat with surface tension. *Wadhams* [1974] assumed that in regions of continuous ice cover the ice-water layer has bending stiffness like an elastic plate. For shallow water his dispersion equation is the same as that of *Goldstein and Keller* [1953, equation (14)], with the surface tension T related to the Young's modulus E by $T = k^2h^3\,E/12(1-\mu)$ where μ is the Poisson ratio of the plate, h is its thickness, and k is the wavenumber. *Squire* [1993] has compared the mass-loading and elastic plate models. *Weber* [1987] considered the same two-layer model which we consider here, solved the equations approximately, and examined some properties of the solution.

The use of synthetic aperture radar has made it possible to measure the amplitudes and wavelengths of waves in ice fields. *Wadhams and Holt* [1991] and *Wadhams et al.* [1996] have made such measurements and used the mass loading model to analyze the results. *Liu et al.* [1991a, 1991b] have also made such measurements and compared the results with the predictions of a different model which involves viscosity. *Martin and Kauffman* [1981] made field and laboratory studies of wave damping by grease ice.

Recently, *Newyear and Martin* [1995] made a laboratory study of wave propagation in ice-covered water. For all of their parameter values the wavenumber was smaller than that without ice, and there was strong damping. The mass-loading model predicts an increase in wavenumber due to ice and yields no damping, so it is not applicable for those parameter values. The present theory was devised to explain the observed results. Newyear and Martin have already compared their measurements with the theoretical results for waves on a viscous liquid of infinite depth and obtained fairly good

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agreement. Since the present theory takes account of the finite depth of the ice layer and the finite depth of the water layer beneath it, its results should agree even better with the data.

2. Formulation

We consider a suspension of ice in water occupying the horizontal layer $0 < y < -h$. It has the constant concentration c, density $\rho(c)$, and effective viscosity coefficient $\mu(c)$, and its motion is governed by the Navier-Stokes equations for an incompressible liquid. Beneath it, in the layer $-h < y < -H-h$, is water of density ρ_0 and viscosity zero, governed by the Euler equations for an inviscid, incompressible liquid. Above, the suspension is air at constant pressure, and below, the water is a rigid bottom. In the basic state of rest the pressure is hydrostatic, while the free surface $y = 0$ and the interface $y = -h$ are horizontal.

We seek a solution of the equations of motion, the free surface conditions, the interface conditions, and the bottom condition by linearizing about the basic state. We assume that the motion is in the x, y plane, is independent of z, and is proportional to $e^{i(kx-\omega t)}$. Thus we write the horizontal velocity U, the vertical velocity V, the pressure P, the displacement η of the free surface, and the displacement $\tilde{\eta}$ of the interface in the forms

$$(U, V, P, \eta, \tilde{\eta}) = [u(y), v(y), p(y), a, b]e^{i(kx-\omega t)} \tag{1}$$

The linear Navier-Stokes equations in the upper layer lead to ordinary differential equations which have the general solution

$$\begin{aligned} u(y) &= -ik\left(Ae^{-ky} + Be^{ky}\right) - \alpha\left(Ce^{-\alpha y} - De^{\alpha y}\right) \\ &\qquad 0 > y > -h \\ v(y) &= -k\left(-Ae^{-ky} + Be^{ky}\right) \\ &\qquad -ik\left(Ce^{-\alpha y} + De^{\alpha y}\right) \\ p(y) &= -i\omega\rho\left(Ae^{-ky} + Be^{ky}\right) \end{aligned} \tag{2}$$

Here A, B, C, and D are arbitrary constants, while α is defined by

$$\alpha = \left(k^2 - i\omega/\nu\right)^{1/2} \tag{3}$$

with $\nu = \mu(c)/\rho(c)$ and $Re\ \alpha > 0$.

The linear Euler equations in the lower layer also lead to ordinary differential equations. The general solution contains two constants, but we eliminate one of them by requiring that $v(-H-h) = 0$. Then the solution becomes, with E arbitrary,

$$\begin{aligned} u(y) &= +ikE\cosh k(y+H+h) \\ &\qquad -h > y > -(H+h) \\ v(y) &= +kE\sinh k(y+H+h) \\ p(y) &= i\omega\rho_0 E\cosh k(y+H+h) \end{aligned} \tag{4}$$

The boundary conditions at the free surface are the kinematic condition $V = \eta_t$ and the vanishing of the two components of normal stress. They become

$$v(0) = -i\omega a \tag{5}$$

$$u_y(0) + ik\,v(0) = 0 \tag{6}$$

$$2\nu v_y(0) - \rho^{-1}p(0) + ga = 0 \tag{7}$$

Here g is the acceleration of gravity.

The conditions at the interface are the two kinematic conditions $V^+ = V^- = \tilde{\eta}_t$ and the continuity of the normal and tangential components of stress. In terms of the quantities defined in (1) the conditions at the interface become

$$v^+(-h) = -i\omega b \tag{8}$$

$$v^+(-h) = v^-(-h) \tag{9}$$

$$u_y^+(-h) + ikv^+(-h) = 0 \tag{10}$$

$$2\nu v_y^+(-h) - \rho^{-1}p^+(-h) + gb = -\rho^{-1}p^-(-h) + \rho^{-1}\rho_0 gb \tag{11}$$

Here the superscript plus designates the solution (2), and the superscript minus designates the solution (4). The tangential velocity u is not required to be continuous because the fluid in the lower layer is inviscid.

3. Dispersion Equation

When (2) and (4) are used in (5)–(11), a system of seven linear homogeneous equations is obtained. The seven unknowns are A, B, C, D, E, a, and b. Before writing them we solve (5) for a and then eliminate a from (7) to get

$$2\nu v_y(0) - \rho^{-1}p(0) - \frac{g}{i\omega}v(0) = 0 \tag{12}$$

Similarly, we solve (8) for b and eliminate b from (11):

$$2\nu v_y^+(-h) - \rho^{-1}p^+(-h) = -\rho^{-1}p^-(-h) - \frac{g(\rho_0 - \rho)}{i\omega\rho}v^+(-h) \tag{13}$$

Then we can eliminate E by noting from (4) that $p^-(-h) = [i\omega\rho_0/k\tanh kH]v^-(-h)$. We use this for $p^-(-h)$ in (13), and use (9) to replace $v^-(-h)$ by $v^+(-h)$. Then we get from (13)

$$\begin{aligned} &2\nu v_y^+(-h) - \rho^{-1}p^+(-h) \\ &\quad = \left[\frac{-i\omega\rho_0}{k\rho\tanh kH} - \frac{g(\rho_0-\rho)}{i\omega\rho}\right]v^+(-h) \end{aligned} \tag{14}$$

We now substitute the solution (2) into the four equations (6), (10), (12) and (14). This yields four equations for A, B, C, and D:

$$2ik^2(A - B) + (\alpha^2 + k^2)(C + D) = 0 \tag{15}$$

$$\begin{aligned} &2ik^2(Ae^{kh} - Be^{-kh}) \\ &\quad +(\alpha^2 + k^2)(Ce^{\alpha h} + De^{-\alpha h}) = 0 \end{aligned} \tag{16}$$

$$(2\nu k^2 - i\omega)(A + B) - 2\nu ik\alpha(C - D)$$

$$-\frac{gk}{i\omega}(-A+B+iC+iD)=0 \tag{17}$$

$$\begin{aligned}&(2\nu k^2-i\omega)(Ae^{kh}+Be^{-kh})\\&-2\nu ik\alpha(Ce^{\alpha h}-De^{-\alpha h})\\&+\left[\frac{i\omega\rho_0}{\rho\tanh kH}+\frac{g(\rho_0-\rho)k}{i\omega\rho}\right]\\&\cdot\left(-Ae^{kh}+Be^{-kh}+iCe^{\alpha h}+iDe^{-\alpha h}\right)=0\end{aligned} \tag{18}$$

The necessary and sufficient condition for the equations (15) – (18) to have a nontrivial solution is the vanishing of the determinant of the coefficient matrix. This condition is worked out in Appendix A, and it is given by

$$\begin{aligned}&\left\{(2\nu k^2-i\omega)-\frac{4\nu k^3\alpha}{(\alpha^2+k^2)\sinh\alpha h}(e^{kh}-e^{\alpha h})\right.\\&\left.+\left[\frac{2k^3(-2\nu\alpha\omega+ig)}{\omega(\alpha^2+k^2)}+\frac{gk}{i\omega}\right]\right\}\cdot\left\{(2\nu k^2-i\omega)e^{-kh}\right.\\&+\left[\frac{i\omega\rho_0}{\rho\tanh kH}+\frac{g(\rho_0-\rho)k}{i\omega\rho}\right]\left[e^{-kh}-\frac{2k^2}{\alpha^2+k^2}e^{\alpha h}\right]\\&\left.+\frac{4\nu k^3\alpha}{(\alpha^2+k^2)}e^{\alpha h}+m(e^{\alpha h}-e^{-kh})\right\}\\&-\left\{2\nu k^2-i\omega-\frac{4\nu k^3\alpha}{(\alpha^2+k^2)\sinh\alpha h}(e^{\alpha h}-e^{-kh})\right.\\&\left.-\left[\frac{2k^3(-2\nu\alpha\omega+ig)}{\omega(\alpha^2+k^2)}+\frac{gk}{i\omega}\right]\right\}\cdot\left\{(2\nu k^2-i\omega)e^{kh}\right.\\&-\left[\frac{i\omega\rho_0}{\rho\tanh kH}+\frac{g(\rho_0-\rho)k}{i\omega\rho}\right]\left[e^{kh}-\frac{2k^2}{\alpha^2+k^2}e^{\alpha h}\right]\\&\left.-\frac{4\nu k^3\alpha}{\alpha^2+k^2}e^{\alpha h}+m(e^{kh}-e^{\alpha h})\right\}=0\end{aligned} \tag{19}$$

Equation (19) is the dispersion equation relating k and ω.

In order to compute k as a function of ω and the other parameters from (3.8), it is convenient to introduce dimensionless variables. One choice of such variables, based upon H as the length scale and $(gH)^{1/2}$ as the velocity scale, is

$$\begin{aligned}&\hat{k}=kH\quad\hat{\alpha}=\alpha H=(\hat{k}^2-i\hat{\omega}R)^{1/2}\quad\hat{\omega}=\omega(H/g)^{1/2}\\&R=(gH^3)^{1/2}/\nu\quad\hat{h}=h/H\quad\hat{\rho}_0=\rho_0/\rho\\&\hat{q}=i\hat{\omega}\hat{\rho}_0/\tanh\hat{k}+(\hat{\rho}_0-1)\hat{k}/i\hat{\omega}\end{aligned} \tag{20}$$

With these variables (19) is equivalent to the vanishing of the following determinant of four rows and four columns:

$$\begin{vmatrix}2i\hat{k}^2 & -2i\hat{k}^2 & \hat{\alpha}^2+\hat{k}^2 & \hat{\alpha}^2+\hat{k}^2\\ 2i\hat{k}^2e^{\hat{k}\hat{h}} & -2i\hat{k}^2e^{-\hat{k}\hat{h}} & (\hat{\alpha}^2+\hat{k}^2)e^{\hat{\alpha}\hat{h}} & (\hat{\alpha}^2+\hat{k}^2)e^{-\hat{\alpha}\hat{h}}\\ 2\hat{k}^2-i\hat{\omega}R+R\hat{k}/i\hat{\omega} & 2\hat{k}^2-i\hat{\omega}R-R\hat{k}/i\hat{\omega} & -2i\hat{k}\hat{\alpha}-R\hat{k}/\hat{\omega} & 2i\hat{k}\hat{\alpha}-R\hat{k}/\hat{\omega}\\ (2\hat{k}^2-i\hat{\omega}R-\hat{q}R)e^{\hat{k}\hat{h}} & (2\hat{k}^2-i\hat{\omega}R+\hat{q}R)e^{-\hat{k}\hat{h}} & (-2i\hat{k}\hat{\alpha}+i\hat{q}R)e^{\hat{\alpha}\hat{h}} & (2i\hat{k}\hat{\alpha}+i\hat{q}R)e^{-\hat{\alpha}\hat{h}}\end{vmatrix}=0 \tag{21}$$

In using (21) we have set $\hat{\rho}_0=1$ and chosen two values for the dimensionless Reynolds number R ($R=100$ and 1000) and two values for $\hat{h}$ ($\hat{h}=.1$ and .4). For each combination of the parameters we chose a value of $\hat{\omega}$ in the range $0\le\hat{\omega}\le 5$. Then we calculated the roots $\hat{k}$ of (19) which lie in some large rectangle in the first quadrant of the complex $\hat{k}$ plane with the origin at one corner. We selected the root with the smallest imaginary part, which represents the least damped mode, and calculated it for different values of $\hat{\omega}$. The results for Re $\hat{k}$ and Im $\hat{k}$ are shown in Figure 1 and Figure 2. In Figure 3, the curves from Figure 1 are replotted with the ordinates replaced by Re $\hat{k}/\hat{k}_0$. Here $\hat{k}_0$ is the dimensionless wavenumber of a wave of dimensionless frequency $\hat{\omega}$ in water of depth $H+h$, so it satisfies $\hat{\omega}^2=\hat{k}_0\tanh[\hat{k}_0(1+\hat{h})]$. In Figure 4 the curves from Figure 2 are replotted with a logarithmic scale for the ordinate.

The roots of (21) other than the least damped one correspond to other waves. Which ones occur, and their amplitudes, depends upon the way in which they are excited. It would be useful to determine those which are excited near the edge of an ice field by a wave from the open ocean.

4. Short Waves and Long Waves

Waves are short compared to the layer thickness h when $\omega^2h/g\gg 1$. Then both Re kh and Re αh are

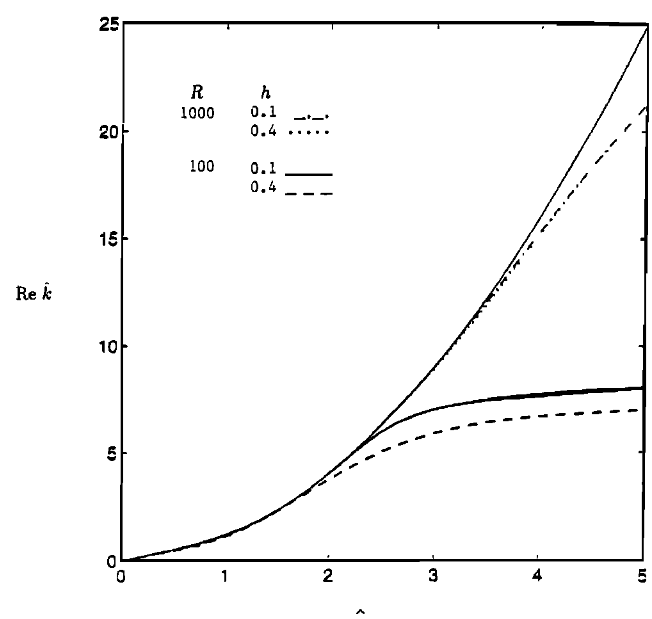

Figure 1. Re$\hat{k}$, the real part of the dimensionless wavenumber $\hat{k}$, is a function of the dimensionless frequency $\hat{\omega}$ based upon the dispersion equation (21). The upper solid line for $\hat{h}=0$ corresponds to no ice. The next two curves below it are for $R=1000$ with $\hat{h}=0.1$ (dot dash) and $\hat{h}=0.4$ (dotted). The bottom two curves are for $R=100$ with $\hat{h}=0.1$ (solid) and $\hat{h}=0.4$ (dashed). The curves for $R=1000, \hat{h}=0.4$ and $R=100, \hat{h}=0.1$ are nearly coincident.

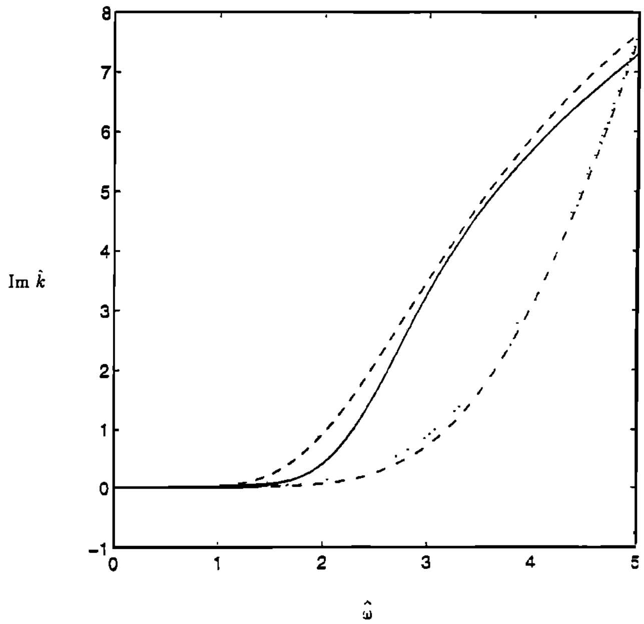

Figure 2. Im$\hat{k}$, the imaginary part of the dimensionless wavenumber $\hat{k}$, as a function of the dimensionless frequency $\hat{\omega}$ based upon (21). There is no curve for $\hat{h} = 0$ since then Im$\hat{k} = 0$. The other curves are labelled as in Figure 1. Thus the bottom two curves are for $R = 1000$ with $\hat{h} = 0.1$ (dot dash) and $\hat{h} = 0.4$ (dashed). The top two curves are for $R = 100$ with $\hat{h} = 0.1$ (solid) and $\hat{h} = 0.4$ (dashed).

large. Therefore e^{-kh} and $e^{-\alpha h}$ are negligible in (16) and (18), and then those equations yield $A = 0$ and $C = 0$. With $A = C = 0$, (15) and (17) become two equations for B and D. Equating to zero the determinant of the matrix of coefficients of B and D in (15) and (17) yields

$$\left(2\nu k^2 - i\omega - \frac{gk}{i\omega}\right)(\alpha^2+k^2) + 2ik^2\left(2\nu ik\alpha - \frac{gk}{\omega}\right) = 0 \tag{22}$$

By setting $\alpha = (k^2 - i\omega/\nu)^{1/2}$ in (22) and rearranging we obtain

$$gk = -(2\nu k^2 - i\omega)^2 + 4\nu^2 k^3 (k^2 - i\omega/\nu)^{1/2} \tag{23}$$

This is the dispersion equation for short waves. It is the same as the dispersion equation for waves on a viscous liquid of infinite depth given by *Lamb* [1945, p. 627, equation (14)]. Since we have assumed that Re $\alpha > 0$, the square root in (23) is that with a positive real part.

We shall now solve (23) for k with ω real. First, we introduce the dimensionless wavenumber $\tilde{k}$ and the dimensionless frequency $\tilde{\omega}$ defined by

$$\tilde{k} = (\nu^2/g)^{1/3} k \qquad \tilde{\omega} = (\nu/g^2)^{1/3}\omega \tag{24}$$

Then we rewrite (23) in the form

$$\tilde{k} = \tilde{\omega}^2 + 4i\tilde{\omega}\tilde{k}^2 + 4\tilde{k}^3\left(\tilde{k}^2 - i\tilde{\omega}\right)^{1/2} - 4\tilde{k}^4 \tag{25}$$

For $\tilde{\omega} \ll 1$ we solve (25) for $\tilde{k}$ in powers of $\tilde{\omega}$, and we get

$$\begin{aligned} \tilde{k} &\sim \tilde{\omega}^2 + 4i\tilde{\omega}^5 + 4e^{-i\pi/4}\,\tilde{\omega}^{13/2} - 36\tilde{\omega}^8 \\ \tilde{k} &\sim \left[\tilde{\omega}^2 + 2^{3/2}\,\tilde{\omega}^{13/2} - 36\tilde{\omega}^8\right] + i\left[4\tilde{\omega}^5 - 2^{3/2}\,\tilde{\omega}^{13/2}\right] \end{aligned} \tag{26}$$

For $\tilde{\omega} \gg 1$ we solve (36) for $\tilde{k}$ in power of $\tilde{\omega}^{-1}$, and we obtain

$$\begin{aligned} \tilde{k} &= x\tilde{\omega}^{1/2} + O(\tilde{\omega}^{-1}) \\ \tilde{k} &= (0.236 + 0.428i)\,\tilde{\omega}^{1/2} + O(\tilde{\omega}^{-1}) \end{aligned} \tag{27}$$

Here x with Re $x > 0$ and Im $x > 0$ is a root of the equation

$$1 + 4ix^2 + 4x^3(x^2 - i)^{1/2} - 4x^4 = 0 \tag{28}$$

We have also solved (25) numerically. In Figure 5, we show Re $\tilde{k}$ and Im $\tilde{k}$ as functions of $\tilde{\omega}$ on the basis of the numerical solution, and we also show the expansions (26) and (27).

Waves are long compared to the layer thickness h when $\omega^2 h/g \ll 1$. If the Reynolds number $\omega h^2/\nu$ is small also ($\omega h^2/\nu \ll 1$), then both kh and αh are small, and we can simplify the dispersion equation to the form

$$\frac{k\tanh kH}{\omega^2/g} = \frac{1 - \dfrac{k^2 gh(\omega^2 - 4ik^2\omega\nu)}{\omega^4 + 16k^4\omega^2\nu^2}}{1 - \dfrac{\omega^2 h\rho}{g\rho_0} + \dfrac{k^2 gh\left(\dfrac{\rho-\rho_0}{\rho_0}\right)(\omega^2 - 4ik^2\omega\nu)}{\omega^4 + 16k^4\omega^2\nu^2}} \tag{29}$$

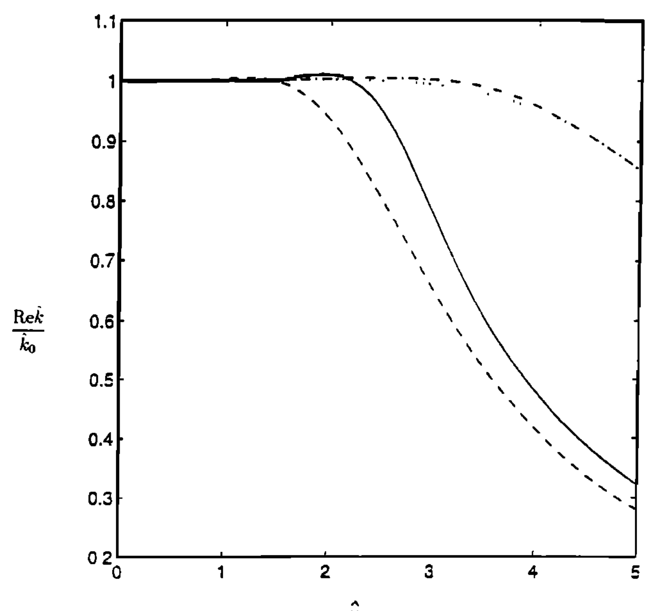

Figure 3. Re$\hat{k}/\hat{k}_0$ versus $\hat{\omega}$ based upon (21). Here $\hat{k}_0$ is the wavenumber at frequency $\hat{\omega}$ in water of depth $H + h$ without ice. The curves correspond to those in Figure 1 and are labeled in the same way: the top two for $R = 1000$ with $\hat{h} = 0.1$ (dot dash) and $\hat{h} = 0.4$ (dotted); the bottom two for $R = 100$ with $\hat{h} = 0.1$ (solid) and $\hat{h} = 0.4$ (dashed).

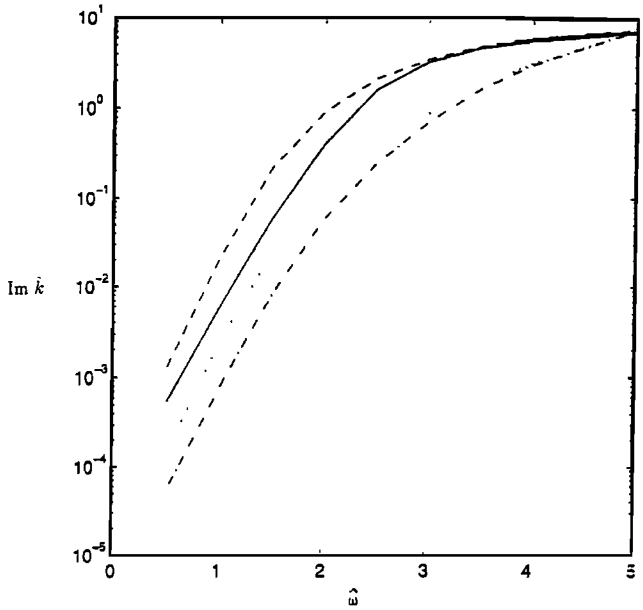

Figure 4. Im$\hat{k}$ versus $\hat{\omega}$ based upon (21) with a logarithmic scale for the ordinate. The curves correspond to those in Figure 2. The bottom two are for $R = 1000$ with $\hat{h} = 0.1$ (dot dash) and $\hat{h} = 0.4$ (dotted); the top two are for $R = 100$ with $\hat{h} = 0.1$ (solid) and $\hat{h} = 0.4$ (dashed).

This result, derived in Appendix B, is the dispersion equation for long waves and small Reynolds numbers. When $h = 0$, it is just the usual dispersion equation for waves on water of depth H.

To put (29) in dimensionless form, we use (24) and the definitions

$$\widetilde{H} = (g/\nu^2)^{1/3} H \quad \tilde{h} = (g/\nu^2)^{1/3} h \quad \tilde{\rho} = \rho/\rho_0 \tag{30}$$

Then we can write (29) as

$$\frac{\tilde{k}}{\tilde{\omega}^2} \tanh \tilde{k}\widetilde{H} = \frac{1 - \tilde{k}^2\tilde{h}(\tilde{\omega} - 4i\tilde{k}^2)(\tilde{\omega}^3 + 16\tilde{k}^4\tilde{\omega})^{-1}}{1 - \tilde{\rho}\tilde{\omega}^2\tilde{h} + (\tilde{\rho} - 1)\tilde{k}^2\tilde{h}(\tilde{\omega} - 4i\tilde{k}^2)(\tilde{\omega}^3 + 16\tilde{k}^4\tilde{\omega})^{-1}} \tag{31}$$

If Re $\tilde{k}\widetilde{H} \gg 1$, then $\tanh \tilde{k}\widetilde{H} \sim 1$, and if $\tilde{\rho} - 1 \ll 1$, then the $\tilde{\rho} - 1$ term in the denominator of (31) may be negligible. When both of these simplifications are valid, (31) becomes

$$\tilde{k}\tilde{\omega}^{-2}(1 - \tilde{\omega}^2\tilde{h}) = 1 - \tilde{k}^2\tilde{h}(\tilde{\omega} - 4i\tilde{k}^2)(\tilde{\omega}^3 + 16\tilde{k}^4\tilde{\omega})^{-1} \tag{32}$$

The solution of (32) for $\tilde{k}$ when $\tilde{\omega} \ll 1$ is

$$\tilde{k} \sim (\tilde{\omega}^2 + 16\tilde{h}\tilde{\omega}^{10} - 80\tilde{h}^2\tilde{\omega}^{12}) + i(4\tilde{h}\tilde{\omega}^7 - 4\tilde{h}^2\tilde{\omega}^9) \tag{33}$$

When $\tilde{\omega} \gg 1$, the solution of (32) is

$$\tilde{k} \sim \frac{e^{-i\pi/4}}{2}\tilde{\omega}^{1/4} \tag{34}$$

If $\rho = \rho_0$ and $k^2 \ll \omega/\nu$, (29) becomes

$$\frac{gk}{\omega^2} \tanh kH = \frac{1 - k^2gh/\omega^2}{1 - \omega^2h/g} \tag{35}$$

The denominator vanishes at the resonant frequency $(g/h)^{1/2}$, which corresponds to the vertical oscillation of the upper layer. It is a high-frequency oscillation when h is small, with a period equal to the time of free fall through a distance $h/2$. When $\omega \ll (g/h)^{1/2}$ and $kH \ll 1$, (35) simplifies to

$$\frac{g}{\omega^2}k^2(H + h) = 1 \tag{36}$$

This is the dispersion equation for long waves on water of depth $H + h$.

5. Conclusion

We have obtained the dispersion equation (21) for waves on ice-covered water, treating the ice-water layer as a viscous liquid. The numerical solutions of this equation for the mode with the smallest damping are shown in Figures 1–4. They are shown for two values of the dimensionless Reynolds number $R = (gH^3)^{1/2}/\nu$, namely, $R = 100$ and $R = 1000$, and two values of the dimensionless thickness $\hat{h} = h/H$ of the ice layer, namely, $\hat{h} = 0.1$ and $\hat{h} = 0.4$. The layers are of the same density, i.e., $\hat{\rho}_0 = 1$. The dimensionless frequency $\hat{\omega} = \omega(H/g)^{1/2}$ ranges from 0 to 5. In all four cases shown in Figure 3, for large $\hat{\omega}$ the real part of the dimensionless wavenumber $\hat{k} = kH$ is much less than $\hat{k}_0 = k_0H$, the dimensionless wavenumber in ice-free water of depth $H + h$. For small values of $\hat{\omega}$, Re $\hat{k}$ is slightly greater than $\hat{k}_0$. The mass-loading theory yields Re $\hat{k} > \hat{k}_0$ for all $\hat{\omega}$.

The effect of adding a layer of inviscid liquid of small depth h on top of a layer of the same liquid of depth H can be analyzed by considering how it changes the two

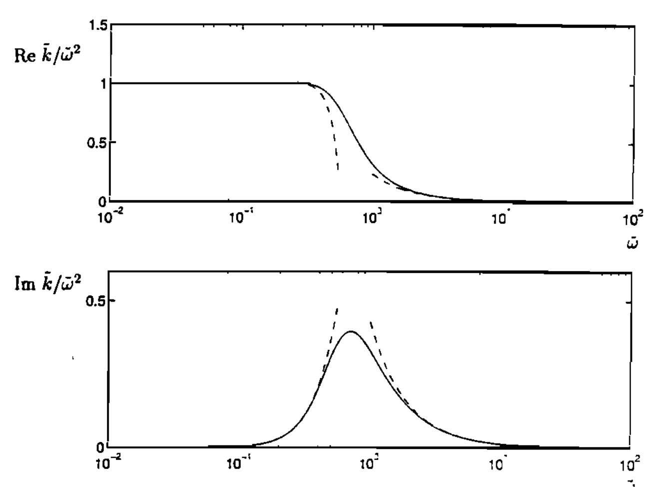

Figure 5. Re $\tilde{k}/\tilde{\omega}^2$ and Im $\tilde{k}/\tilde{\omega}^2$ as functions of $\tilde{\omega}$ based upon (4.4). The expansions (4.5) and (4.6) are shown as dashed lines.

boundary conditions at $y = H$. The dynamic condition at $y = H$ is changed from $p = \rho g \eta$ to $p = \rho g \eta + \rho h \eta_{tt}$, as in the mass-loading theory. The kinematic condition at $y = H$ is changed from $\eta_t = v$ to $\eta_t = v - h u_x$. The former change tends to increase the wavenumber, while the latter tends to decrease it, and the latter turns out to be the larger effect. Therefore the wavenumber k_0 in liquid of depth $H + h$ is less than that in liquid of depth H. In Figure 3, k/k_0 is shown, so the effect of change in depth is factored out and only the effect of viscosity in the upper layer remains.

Appendix A: Dispersion Equation

Rather than evaluating (19) directly, we first solve (15) for C:

$$C = -D - 2ik^2(\alpha^2 + k^2)^{-1}(A - B) \qquad \text{(A1)}$$

Then we use (A1) for C in (16) and solve the resulting equation for D:

$$\begin{aligned} D &= \left[-2ik^2(Ae^{kh} - Be^{-kh})(\alpha^2 + k^2)^{-1} \right. \\ &\left. +2ik^2(\alpha^2 + k^2)^{-1}(A - B)e^{\alpha h}\right] \left(e^{-\alpha h} - e^{\alpha h}\right)^{-1} \\ &= \frac{ik^2}{(\alpha^2 + k^2)\sinh \alpha h}\left[A(e^{kh} - e^{\alpha h}) \right. \\ &\left. +B(e^{\alpha h} - e^{-kh})\right] \end{aligned} \qquad \text{(A2)}$$

Now we use (A1) and (A2) to eliminate C and D from (17) and (18), and we obtain

$$\begin{aligned} &(2\nu k^2 - i\omega)(A + B) - 2\nu i k \alpha \left\{ \frac{-2ik^2}{(\alpha^2 + k^2)\sinh \alpha h} \right. \\ &\left. \cdot \left[A(e^{kh} - e^{\alpha h}) + B(e^{\alpha h} - e^{-kh})\right]\right\} \\ &+ \left[\frac{2k^3(-2\nu\alpha\omega + ig)}{\omega(\alpha^2 + k^2)} + \frac{gk}{i\omega}\right](A - B) = 0 \end{aligned} \qquad \text{(A3)}$$

$$\begin{aligned} &(2\nu k^2 - i\omega)(Ae^{kh} + Be^{-kh}) + \left[\frac{i\omega\rho_0}{\rho \tanh kH} \right. \\ &\left. + \frac{g(\rho_0 - \rho)k}{i\omega\rho}\right] \left[-Ae^{kh} + Be^{-kh} \right. \\ &\left. + \frac{2k^2}{\alpha^2 + k^2} e^{\alpha h}(A - B)\right] \\ &+ m\left[A(e^{kh} - e^{\alpha h}) + B(e^{\alpha h} - e^{-kh})\right] \\ &-2\nu i k \alpha(-2ik^2)(\alpha^2 + k^2)^{-1} e^{\alpha h}(A - B) = 0 \end{aligned} \qquad \text{(A4)}$$

Here $\alpha = (k^2 - i\omega/\nu)^{1/2}$, and m is defined by

$$\begin{aligned} m &= \left\{2\nu i k \alpha(e^{\alpha h} + e^{-\alpha h}) \right. \\ &+ \left[\frac{i\omega\rho_0}{\rho \tanh kH} + \frac{g(\rho_0 - \rho)k}{i\omega\rho}\right] \\ &\left. \cdot(-ie^{\alpha h} + ie^{-\alpha h})\right\} \frac{ik^2}{(\alpha^2 + k^2)\sinh \alpha h} \end{aligned} \qquad \text{(A5)}$$

Equating to zero the determinant of the coefficient matrix in (A3) and (A4) yields the dispersion equation, which is given in (19).

Appendix B: Long-Wave Low Reynolds Number Dispersion Equation

We shall now write out the dispersion equation to first order in kh and αh. To do so, we expand each exponential in (16) and (18) to two terms, neglecting terms of order $(kh)^2$ and $(\alpha h)^2$. The term independent of h in (16) is just the left side of (15), so it vanishes. The next term in (16) yields the equation

$$2ik^3 h(A + B) + (\alpha^2 + k^2)\alpha h(C - D) = 0 \qquad \text{(B1)}$$

Similarly, in (18), part of the term independent of h vanishes because of (17), and the remaining equation is

$$\begin{aligned} &\left[\frac{i\omega\rho_0}{\rho \tanh kH} + \frac{g\rho_0 k}{i\omega\rho}\right] \left[(-A + B + iC + iD) \right. \\ &\left. -kh(A + B) + i\alpha h(C - D)\right] \\ &+ (2\nu k^2 - i\omega)kh(A - B) - 2\nu i k \alpha^2 h(C + D) \\ &+ \frac{gk^2 h}{i\omega}(A + B) - \frac{gk\alpha h}{\omega}(C - D) = 0 \end{aligned} \qquad \text{(B2)}$$

To obtain the dispersion equation from (15), (17), (B1), and (B2) we first solve (15) for $C + D$ and (B1) for $C - D$:

$$\begin{aligned} C + D &= -2ik^2(\alpha^2 + k^2)^{-1}(A - B) & \text{(B3)} \\ C - D &= -2ik^3(\alpha^2 + k^2)^{-1}\alpha^{-1}(A + B) & \text{(B4)} \end{aligned}$$

Then we use (B3) and (B4) in (17) and solve the resulting equation for $A + B$:

$$A + B = -gk(\omega^2 + 4ik\omega\nu k^2)^{-1}(A - B) \qquad \text{(B5)}$$

Finally, we use (B3)–(B5) in (B2), and we obtain $A - B$ multiplied by a factor. In order to have a nontrivial solution the factor must vanish, which yields the dispersion equation

$$\begin{aligned} &\left[\frac{i\omega\rho_0}{\rho \tanh kH} + \frac{\rho_0 gk}{i\omega\rho}\right] \left[\frac{k^2 - \alpha^2}{k^2 + \alpha^2}\right] \left[1 - \frac{k^2 hg}{\omega^2 + 4ik^2\omega\nu}\right] \\ &+ kh\left[2\nu k^2 - i\omega - \frac{4\nu k^2\alpha^2}{\alpha^2 + k^2} \right. \\ &\left. + \frac{g^2 k^2 (k^2 - \alpha^2)}{i\omega(\omega^2 + 4ik^2\omega\nu)(k^2 + \alpha^2)}\right] = 0 \end{aligned} \qquad \text{(B6)}$$

To make (B6) look more familiar, we solve it for $k \tanh kH$, and write it in the form (29).

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