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A note on Tennekes hypothesis and its impact on second moment closure models

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Abstract

Tennekes [Lecture Notes on Turbulence, World Scientific (1989) 32] postulated that the shear production term should drop out of the equation for the rate of change of the turbulence macroscale. This hypothesis has been invoked by some second moment closure modelers, to conveniently fix the value for one of the constants in the equation for the turbulence length scale. In this note, we examine Tennekes hypothesis and its impact on second moment turbulence closure models, which are a part of present-day ocean circulation models.

Keywords: Second moment closure; Turbulence models

1. Introduction

Ever since Baumert and Peters (2000) invoked Tennekes hypothesis (Tennekes, 1989) to determine one of the closure constants in the equation for the turbulence macroscale, some second moment closure modelers have embraced this idea wholeheartedly, simply because of its appealing simplicity (for example, Umlauf and Burchard, 2003; Kantha, 2004). Another reason is that the constant determined in this fashion is fairly close in value to that determined by other means.

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However, Tennekes hypothesis has not been examined critically as to its validity and impact. In this note we correct this oversight.

For the sake of uniformity, we will follow the notation of Umlauf and Burchard (2003, UB henceforth) and Kantha and Carniel (2003, KC henceforth), instead of that used by Kantha (2004). Following KC, it is possible to write a conservation equation for a generic quantity $\psi = (c_\mu^0)^p k^m \ell^n$ involving the macro-length scale of turbulence ℓ, which is of the form

$$\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x_k}(U_k \psi) = \frac{\partial}{\partial z}\left(\frac{v_t}{\sigma_\psi}\frac{\partial \psi}{\partial z}\right) + \frac{v_t}{\sigma_\psi}\frac{\xi}{\psi}\left(\frac{\partial \psi}{\partial z}\right)^2 + \frac{\psi}{k}\left(c_{\psi 1}P + c_{\psi 3}G - c_{\psi 2}\varepsilon\right) \tag{1}$$

where the first two terms on the RHS denote turbulent diffusion of ψ; P is the shear production, G is the buoyancy production/destruction, and ε is the dissipation of turbulence kinetic energy (TKE). The quantities $\sigma_\psi, c_\mu^0, c_{\psi 1}, c_{\psi 2}$ and $c_{\psi 3}$ are closure constants and v_t is the turbulent viscosity.

The quantity k, the TKE, is given by the conservation equation:

$$\frac{\partial k}{\partial t} + \frac{\partial}{\partial x_k}(U_k k) = \frac{\partial}{\partial z}\left(\frac{v_t}{\sigma_k^\psi}\frac{\partial k}{\partial z}\right) + (P + G - \varepsilon) \tag{2}$$

where once again, the first term on the RHS is the turbulent diffusion of k. The quantity σ_k^ψ is another closure constant. σ_ψ and σ_k^ψ are Schmidt numbers. Along with the stability functions that control the value of turbulent viscosity, Eqs. (1) and (2) constitute a two-equation model of turbulence (see KC).

The exponents p, m and n take particular values for the different turbulence length scale equations that have been used in the past: (i) $k-\varepsilon$ model: $p = 3$, $m = 3/2$, $n = -1$; (ii) $k-\omega$ model: $p = -1$, $m = 1/2$, $n = -1$; (iii) $k - k\tau$ model: $p = -3$, $m = 1/2$, $n = 1$; (iv) $k - k\ell$ model: $p = 0$, $m = 1$, $n = 1$; (v) $k - \ell$ model: $p = 0$, $m = 0$, $n = 1$; and (vi) $k - \tau$ model: $p = 0$, $m = -1/2$, $n = 1$. Quantity ω is the turbulence frequency and $\tau = k/\varepsilon$ is the turbulence time scale. The closure constants are determined by the equations:

$$\begin{aligned}
&(c_\mu^0)^2 = 0.3, \\
&\sigma_\psi = \frac{n^2 k^2 (1+\xi)}{(c_\mu^0)^2 (c_{\psi 2} - c_{\psi 1})} \\
&c_{\psi 2} = m + n\left(\frac{1}{2} + \frac{1}{d}\right) \\
&(\alpha L)^2 = \frac{2}{3}(c_\mu^0)^2 R \sigma_k^\psi \\
&(c_\mu^0)^2 R \sigma_\psi c_{\psi 2} = (m\alpha + n)\left[\left(m + \frac{1}{2}\right)\alpha + n + \xi(m\alpha + n)\right] L^2
\end{aligned} \tag{3}$$

The first two conditions are obtained by appealing to the logarithmic law of the wall, the third one to the decay of homogeneous turbulence, where d is the decay rate with time of TKE and the last two to the experiments on spatial decay of turbulence in a tank away from a stirring grid that generates the turbulence at one end of the tank. The quantity κ is the von Karman constant, with a traditional value of 0.4. Experiments show that d has a value slightly lower than -1 (for example, Batchelor and Townsend, 1948; Comte-Bellot and Corrsin, 1966; Gad-El-Hak and Corrsin, 1974), whereas theoretical considerations (Dickey and Mellor, 1980; Domardzki and Mellor, 1984) dictate that $d = -1$ be the asymptotic value for high Reynolds number turbulence and that α be -2. The quantity R is c_μ^0/c_μ with c_μ determined from the specific algebraic closure model used. L is around 0.2. See KC for more details.

It is possible to derive an equation for the length scale ℓ from Eqs. (1) and (2). Ignoring diffusion terms, we get (see UB):

$$\frac{1}{\ell}\frac{\partial \ell}{\partial t} = \frac{1}{k}\left[\frac{1}{n}(c_{\psi 1} - m)P + \frac{1}{n}(c_{\psi 3} - m)G - \frac{1}{n}(c_{\psi 2} - m)\varepsilon\right] \tag{4}$$

Tennekes (1989) hypothesized that on dimensional grounds, ℓ cannot depend on the ambient shear for a neutrally stratified $(G = 0)$ homogeneous shear flow. Since shear production P involves shear, the first term must then vanish, which in turn yields (see also Baumert and Peters, 2000, UB, KC):

$$c_{\psi 1} = m \tag{5}$$

Thus one of the constants in Eq. (1) is conveniently determined by this hypothesis and this value is close to that determined by alternative means. Consequently, Tennekes hypothesis has been used in recent years by some second moment closure modelers.

However, using the value determined by Eq. (3) for $c_{\psi 2}$, Eq. (4) becomes

$$\frac{1}{\ell}\frac{\partial \ell}{\partial t} = -\frac{1}{k}\left(\frac{1}{2} + \frac{1}{d}\right)\varepsilon \tag{6}$$

for neutrally stratified flows. Since d has a value ranging from -1 to -1.2, the RHS is positive and this leads to an unlimited exponential growth of ℓ in the idealized case of a homogeneous, infinite-extent, neutrally stratified flow. The experiment of Tavoularis and Karnuk (1989) confirms this.

However, the elimination of the production term in Eq. (4) implies that this unlimited growth occurs for *all* neutrally stratified flows, if the diffusion terms are negligible. This may be undesirable. Moreover, as shown below, a choice for a slightly different value for $c_{\psi 1}$ leads to a better agreement of the model P/ε with the experimental value from Tavoularis and Karnuk (1989) for the homogeneous, neutral flow.

For an equilibrium solution to be possible in the absence of diffusion then,

$$c_{\psi 1} > m \ (\text{for } n < 0) \quad \text{and} \quad c_{\psi 1} < m \quad (\text{for } n > 0) \tag{7}$$

An equation can also be derived for the turbulence time scale τ from Eqs. (1) and (2). Once again, ignoring diffusion terms (see UB):

$$\frac{1}{\tau}\frac{\partial \tau}{\partial t} = \frac{1}{k}\left[\frac{1}{n}\left(c_{\psi 1} - m - \frac{n}{2}\right)P + \frac{1}{n}\left(c_{\psi 3} - m - \frac{n}{2}\right)G - \frac{1}{n}\left(c_{\psi 2} - m - \frac{n}{2}\right)\varepsilon\right] \tag{8}$$

Using the value determined by Eq. (3) for $c_{\psi 2}$, for neutrally stratified flows, Eq. (8) becomes

$$\frac{1}{\tau}\frac{\partial \tau}{\partial t} = \frac{1}{k}\left[\frac{1}{n}\left(c_{\psi 1} - m - \frac{n}{2}\right)P - \frac{\varepsilon}{d}\right] \tag{9}$$

Once again, for equilibrium state to be possible in the absence of diffusion,

$$c_{\psi 1} > [m + (n/2)] \text{ (for } n < 0) \quad \text{and} \quad c_{\psi 1} < [m + (n/2)] \text{ (for } n > 0) \tag{10}$$

Eqs. (7) and (10) are constraints that the closure constant $c_{\psi 1}$ must satisfy.

2. An alternative to Tennekes hypothesis

Let

$$c_{\psi 1} = m + \beta n \tag{11}$$

where β is a small negative constant. From Eq. (3):

$$\begin{aligned} c_{\psi 2} &= m + n\left(\frac{1}{2} + \frac{1}{d}\right) \\ \sigma_\psi &= \frac{8n(1+\xi)}{15(c_{\psi 2} - c_{\psi 1})} = \frac{(2m-n)[(2m-n+1)+(2m-n)\xi]}{6c_{\psi 2}}\sigma_k^\psi \end{aligned} \tag{12}$$

If we choose $L = 0.2$, then $\sigma_k^\psi = 0.8$ (irrespective of the value of m and n), the commonly used value (see UB, KC). Constant ξ is arbitrary. In addition, we need to choose the value of d and of course β.

To determine the value of β (or equivalently $c_{\psi 1}$), we appeal to measurements of homogeneous shear turbulence by Tavoularis and Karnuk (1989), which indicate a value of about 1.6 for P/ε in this flow. We rewrite Eqs. (1) and (2) for a homogeneous neutrally stratified flow as

$$\frac{d\psi}{dt} = \frac{\psi}{k}\left(c_{\psi 1}(c_\mu^0)^4 S^2 \frac{k^2}{\varepsilon} - c_{\psi 2}\varepsilon\right) \tag{13}$$

$$\frac{dk}{dt} = (c_\mu^0)^4 S^2 \frac{k^2}{\varepsilon} - \varepsilon \tag{14}$$

using the fact that the (kinematic) shear stress can be written as $-\overline{uw} = \nu_t \frac{dU}{dz} = (c_\mu^0)^4 \frac{k^2}{\varepsilon} S$, where S is the shear. Seeking solutions of the form $k = k_0 e^{\omega S t}$, $\varepsilon = \varepsilon_0 e^{\omega S t}$, $\psi = \psi_0 e^{a\omega S t}$, it is easy to show that Eqs. (13) and (14) yield

$$\frac{P}{\varepsilon} = \frac{c_{\psi 2} - a}{c_{\psi 1} - a} \tag{15}$$

where $a = m + \frac{n}{2}$, which follows from $\varepsilon = (c_\mu^0)^{\left(3+\frac{P}{n}\right)} k^{\left(\frac{3}{2}+\frac{m}{n}\right)} \psi^{-\frac{1}{n}}$. Eq. (15) helps determine the value of $c_{\psi 1}$ (or equivalently β) once $c_{\psi 2}$, which depends only on the value of d, is chosen. Substitution of (15) into (9) shows that the turbulence time scale τ is a constant, in agreement with observations.

If we choose $d = -1.2$, following UB, it is easy to show from Eq. (12) that for the $k-\varepsilon$ model, $c_{\psi 1} = 22/15$ if we choose $\xi = 0$, and $c_{\psi 1} = 3/2$, for $\xi = -1/3$ (UB value). Both these values for $c_{\psi 1}$ violate the constraint (7). Therefore d cannot be chosen as -1.2 unless we choose $\xi < -1/3$.

On the other hand, if we choose the asymptotic value of -1 for d, following Mellor and Yamada (1982) and Kantha and Clayson (1994), $c_{\psi 2} = m - \frac{n}{2}$. For the $k - k\tau$ model for which $m = 1/2$, $n = 1$, this gives $c_{\psi 2} = 0$. Any value of $d < -1$ will also work, including a value intermediate between $d = -1$ and -1.2. However, in the following analysis, we follow UB and choose $d = -1.2$. Using the experimental value of 1.6 for P/ε in Eq. (15)

$$\begin{aligned} c_{\psi 1} &= m - \frac{n}{48} \\ c_{\psi 2} &= m - \frac{n}{3} \\ \sigma_\psi &= -\frac{128}{75} n(1+\xi) = \frac{2}{5}\frac{(2m-n)[(2m-n+1)+(2m-n)\xi]}{(3m-n)} \end{aligned} \tag{16}$$

Eq. (16) determines the closure constants needed in two-equation closure models. Note that the use of the experimental value of P/ε in homogeneous, neutrally stratified flow to determine the closure constant is nothing new, and $k-\varepsilon$ modelers have done so in the past (see Pope, 2000; Durbin and Pettersson Reif, 2001). Table 1 shows the values of these constants for different models. These values satisfy both constraints (7) and (10). Note that for the UB model, the value of P/ε for neutral homogeneous shear flow is 1.66, but still within the experimental uncertainty. Since Eq. (1) is equivalent to

$$\frac{\partial \varphi}{\partial t} + \frac{\partial}{\partial x_k}(U_k \varphi) = \frac{\partial}{\partial z}\left(\frac{\nu_t}{\sigma_\varphi}\frac{\partial \varphi}{\partial z}\right) + \frac{\varphi}{k}(c_{\varphi 1}P + c_{\varphi 3}G - c_{\varphi 2}\varepsilon) \tag{17}$$

where the quantity $\varphi = \psi^r = [(c_\mu^0)^p k^m \ell^n]^r = (c_\mu^0)^{\bar{p}} k^{\bar{m}} \ell^{\bar{n}}$ and $r = 1+\xi$ (see KC), the values of model constants $\bar{p}, \bar{m}, \bar{n}, r, c_{\varphi 1}$ and $c_{\varphi 2}$ are also shown in Table 1 (and Table 2 below). Note that $\sigma_\varphi = \sigma_\psi$

Table 1
Model parameters for $d = -1.2$

Model	p	m	n	$c_{\psi 1}$	$c_{\psi 2}$	σ_k^ψ	σ_ψ	ξ	r	$\bar{p}$	$\bar{m}$	$\bar{n}$	$c_{\varphi 1}$	$c_{\varphi 2}$
$k-\varepsilon$	3	3/2	−1	73/48	11/6	0.8	1.104	−0.353	0.647	1.94	0.97	−0.647	0.984	1.186
$k-\omega$	−1	1/2	−1	25/48	5/6	0.8	0.853	−0.7	0.3	−0.3	0.15	−0.3	0.156	0.250
$k-k\ell$	0	1	+1	47/48	2/3	0.8	0.179	−1.105	−0.105	0	−0.105	−0.105	−0.103	−0.07
$k-k\tau$	−3	1/2	+1	23/48	1/6	0.8	0.853	−1.5	−0.5	1.5	−0.25	−0.5	−0.240	−0.083
$k-\ell$	0	0	+1	−1/48	−1/3	0.8	0.522	−1.306	−0.306	0	0	−0.306	0.006	0.102
$k-\tau$	0	−1/2	+1	−25/48	−5/6	0.8	0.261	−1.3	−0.3	0	0.15	−0.3	0.156	0.25

Table 2
Model parameters for $d = -1$

Model	p	m	n	$c_{\psi 1}$	$c_{\psi 2}$	σ_k^ψ	σ_ψ	ξ	r	$\bar{p}$	$\bar{m}$	$\bar{n}$	$c_{\varphi 1}$	$c_{\varphi 2}$
$k-\varepsilon$	3	3/2	−1	13/8	2	0.8	1.067	−1/4	0.75	2.25	1.125	−0.75	1.219	1.5
$k-\omega$	−1	1/2	−1	5/8	1	0.8	0.711	−7/10	0.3	−0.3	0.15	−0.3	0.188	0.3
$k-k\ell$	0	1	+1	7/8	1/2	0.8	0.225	−22/19	−0.158	0	−0.158	−0.158	−0.138	−0.079
$k-k\tau$	−3	1/2	+1	3/8	0	0.8	0.711	−3/2	−0.5	1.5	−0.25	−0.5	−0.188	0
$k-\ell$	0	0	+1	−1/8	−1/2	0.8	0.328	−16/13	−0.231	0	0	−0.231	0.029	0.115
$k-\tau$	0	−1/2	+1	−5/8	−1	0.8	0.427	−13/10	−0.3	0	0.15	−0.3	0.188	0.3

and $c_{\varphi 1} = rc_{\psi 1} \ldots$ Note that for the $k - k\tau$ model, the last equation in (3) is trivially satisfied and adds no information and hence ξ is arbitrary as long as $\xi < -1$. We chose $\xi = -1.5$ for this case.

If however, we choose $d = -1$

$$\begin{aligned} c_{\psi 1} &= m - \frac{n}{8} \\ c_{\psi 2} &= m - \frac{n}{2} \\ \sigma_{\psi} &= -\frac{64}{45} n(1+\xi) = \frac{4}{15}[(2m - n + 1) + (2m - n)\xi] \end{aligned} \tag{18}$$

Table 2 shows the values of the resulting constants for different models. Note that the values of the constants $c_{\psi 1}$ and $c_{\psi 2}$ are 0.875 and 0.5 for the $k - k\ell$ model, close to the equivalent values (0.9 and 0.5) chosen by Mellor and Yamada (1982), (see also Kantha and Clayson, 1994) originally for the constants $E_1 = 1.8$ and $E_2 = 1$ in their model.

3. Concluding remarks

Since the spreading rate of free shear layers is determined by the value of $(P/\varepsilon) - 1$, which is equal to $\frac{c_{\psi 2} - c_{\psi 1}}{c_{\psi 1} - \left(m + \frac{n}{2}\right)}$, it is the difference $c_{\psi 2} - c_{\psi 1} = \frac{n}{d}\left[\frac{(P/\varepsilon) - 1}{P/\varepsilon}\right]$ that is more important than the individual values of $c_{\psi 2}$ and $c_{\psi 1}$. For the $k - \varepsilon$ model, $c_{\varepsilon 2} = 1 - \frac{1}{d}$; $c_{\varepsilon 1} = 1 - \frac{1}{d(P/\varepsilon)}$. Laboratory experiments on free shear layers indicate $(p/\varepsilon) = 1.60 \pm 0.2$ (see Tavoularis and Karnuk, 1989 for a summary). The traditional $k - \varepsilon$ model values $c_{\varepsilon 1} = 1.44$, $c_{\varepsilon 2} = 1.92$ (Durbin and Pettersson Reif, 2001) give $d = -1.087$ and $(P/\varepsilon) = 2.09$, the latter clearly well above the upper limit of experimental uncertainty. The revised values (Umlauf and Burchard, 2003) $c_{\varepsilon 1} = 1.50$, $c_{\varepsilon 2} = 1.833$ give $d = -1.2$ and $(P/\varepsilon) = 1.667$, the latter well within the experimental uncertainty. Since the value of d fixes the value of $c_{\varepsilon 2}$, it is the value of (P/ε) for free shear layers that should determine the value of $c_{\varepsilon 1}$. The possible range of values is as follows: for $d = -1.2$, $c_{\varepsilon 2} = 1.833$ and $c_{\varepsilon 1} = 1.595$, 1.521 and 1.463 for $(P/\varepsilon) = 1.4$, 1.6 and 1.8; for $d = -1$, $c_{\varepsilon 2} = 2$ and $c_{\varepsilon 1} = 1.714$, 1.625 and 1.556 for $(P/\varepsilon) = 1.4$, 1.6 and 1.8; and for $d = -1.087$, $c_{\varepsilon 2} = 1.44$ and $c_{\varepsilon 1} = 1.657$, 1.575 and 1.511 for $(P/\varepsilon) = 1.4$, 1.6 and 1.8. We have elected to choose $(P/\varepsilon) = 1.6$. Note however, that the difference in the performance of the $k - \varepsilon$ model with constants chosen from Table 1 ($\sigma_{\varepsilon} = 1.104, c_{\varepsilon 1} = 73/48 = 1.521, c_{\varepsilon 2} = 11/6 = 1.833$) or Table 2 ($\sigma_{\varepsilon} = 1.067$, $c_{\varepsilon 1} = 13/8 = 1.625$, $c_{\varepsilon 2} = 2.0$), and that of the UB generic length scale model ($\sigma_{\varepsilon} = 1.22, c_{\varepsilon 1} = 1.5, c_{\varepsilon 2} = 11/6 = 1.833$) is very small. The difference in the TKE distributions is almost imperceptible, while the difference in the eddy viscosity distributions is noticeable but very small. Consequently, invoking or abandoning Tennekes' hypothesis does not have a major impact on two-equation turbulence model results.

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