Comments on "Turbulence Closure, Steady State, and Collapse into Waves"

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ABSTRACT

Two-equation models are being increasingly used to model turbulence in geophysical flows. A salient aspect of these flows is the stable gravitational stratification, which implies that turbulent fluctuations can generate internal waves that drain energy from turbulent eddies. This energy is not available for mixing, and therefore this transfer of energy from turbulence to internal waves has strong implications to mixing in the atmospheric boundary layer and the oceanic mixed layer. How to parameterize energy leakage to internal waves in turbulence models has been the subject of many studies, most recently by Baumert and Peters. This comment is an attempt to critique their work and to explore alternative options.

1. Traditional models

We start, as Baumert and Peters (2004, henceforth BP04) do, from a set of equations for the turbulence kinetic energy (TKE) and its dissipation rate, the so-called k–ε model. Ignoring diffusion terms, the governing equations can be written as

$$\dot{k} = P + B - \varepsilon \quad \text{and}$$
$$\dot{\varepsilon} = (C_{\varepsilon 1}P + C_{\varepsilon 3}B - C_{\varepsilon 2}\varepsilon)(\varepsilon/k), \tag{1}$$

where $P = \nu_t S^2$ is the shear production, $B = -\nu_t' N^2$ is the buoyancy production, and ε is the dissipation rate of TKE, S being the shear frequency and N being the buoyancy frequency. The turbulent eddy viscosities ν_t and ν_t' are given by $\nu_t = C_\mu(k^2/\varepsilon)$ and $\nu_t' = \nu_t/\sigma$, where σ is the Prandtl number. The values of the constants are $C_{\varepsilon 1} = 3/2$, $C_{\varepsilon 2} = 2$, and $C_\mu = \pi^{-2}$; the value of $C_{\varepsilon 3}$ is uncertain.

If we define characteristic turbulence time and length scales as

$$\tau = 2k/\varepsilon \quad \text{and} \quad L = \pi^{-3/2}k^{3/2}/\varepsilon, \tag{2}$$

Eq. (1) can be recast in the form

$$\dot{\tau} = 2\left[(C_{\varepsilon 2} - 1) + (1 - C_{\varepsilon 1})\frac{P}{\varepsilon} + (1 - C_{\varepsilon 3})\frac{B}{\varepsilon}\right] \quad \text{and}$$
$$\frac{\dot{L}}{L}\tau = (2C_{\varepsilon 2} - 3) + (3 - 2C_{\varepsilon 1})\frac{P}{\varepsilon} + (3 - 2C_{\varepsilon 3})\frac{B}{\varepsilon}. \tag{3}$$

If we define

$$\tau_\infty = 2\pi(C_{\varepsilon 1} - 1)^{-1/2}/S \quad \text{and because} \tag{4}$$
$$\frac{B}{\varepsilon} = -\frac{1}{\sigma}\left(\frac{\tau}{T}\right)^2, \tag{5}$$

Eq. (3) can be rewritten as

$$\dot{\tau} = 2\left[(C_{\varepsilon 2} - 1) - \left(\frac{\tau}{\tau_\infty}\right)^2 - \frac{(1 - C_{\varepsilon 3})}{\sigma}\left(\frac{\tau}{T}\right)^2\right] \quad \text{and}$$
$$\frac{\dot{L}}{L}\tau = (2C_{\varepsilon 2} - 3) - \frac{(3 - 2C_{\varepsilon 1})}{(1 - C_{\varepsilon 1})}\left(\frac{\tau}{\tau_\infty}\right)^2 + \frac{(3 - 2C_{\varepsilon 3})}{\sigma}\left(\frac{\tau}{T}\right)^2, \tag{6}$$

where $T = 2\pi/N$ is the buoyancy period and

$$\frac{\tau}{\tau_\infty} = \frac{T}{\tau_\infty}\left(\frac{\tau}{T}\right) = \sqrt{\frac{C_{\varepsilon 1} - 1}{R_g}}\left(\frac{\tau}{T}\right); \tag{7}$$

$R_g = N^2/S^2$ is the gradient Richardson number. Note that

$$\frac{\tau}{T} = \left(\frac{L}{L_O}\right)^{3/2}, \tag{8}$$

where L_O is the Ozmidov length scale $\sqrt{\varepsilon/N^3}$.

For $\dot{\tau} = 0$, the so-called structural equilibrium (Baumert and Peters 2000; BP04), Eq. (6) yields

$$\left(\frac{\tau}{T}\right)^2 = \frac{R_g}{(C_{\varepsilon 1} - 1) + (1 - C_{\varepsilon 3})(R_g/\sigma)}, \tag{9}$$

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which, using Eq. (8), leads to

$$\frac{L}{L_O} = \left[\frac{R_g}{(C_{\varepsilon 1} - 1) + (1 - C_{\varepsilon 3})(R_g/\sigma)}\right]^{3/4}. \quad (10)$$

Assuming the Ellison scale $L_E = cL$, we get the most salient result of this manipulation of the governing equations:

$$\frac{L_E}{L_O} = c\left[\frac{R_g}{(C_{\varepsilon 1} - 1) + (1 - C_{\varepsilon 3})(R_g/\sigma)}\right]^{3/4}. \quad (11)$$

If we put $C_{\varepsilon 3} = 1$, as BP04 suggest, the Prandtl number drops out, and, using $C_{\varepsilon 1} = 3/2$, we get

$$\frac{L_E}{L_O} = c(2R_g)^{3/4}. \quad (12)$$

If we assume $c = 2$ (the value is not accurately known) as they do, we get exactly the relationship they drive, which they also claim shows good agreement with data because it shows the characteristic $R_g^{3/4}$ variation of L_E/L_O displayed by observational data. However, Eq. (12) is a severe underestimate. Note also that Eq. (12) was derived from the traditional two-equation model, without the explicit internal-wave (IW) energy drain term as in BP04.

If, in addition, one assumes $\dot{L} = 0$, the so-called steady-state condition, since the second term on the rhs of the $\dot{L}$ equation vanishes because $C_{\varepsilon 1} = 3/2$ and since $C_{\varepsilon 2} = 2$, we get

$$\left(\frac{\tau}{T}\right)^2 = \frac{\sigma}{(2 - C_{\varepsilon 3})}. \quad (13)$$

Equating Eqs. (9) and (13),

$$R_g^{\text{ss}} = \frac{(C_{\varepsilon 1} - 1)\sigma}{(2 - C_{\varepsilon 3})} = \frac{\sigma/2}{(2 - C_{\varepsilon 3})} \quad (14)$$

is the value of the steady-state Richardson number. Note that this differs from the BP04 expression in that σ has replaced σ_0, its value under neutral stratification in Eq. (14).

In traditional second-moment closure models that use only the TKE equation, algebraic equations are written for the second moments (e.g., Mellor and Yamada 1982). Using Kantha (2003; see also Kantha and Clayson 1994) expressions for the second moments, one can derive the following expression for the Prandtl number σ:

$$\left(\frac{A_2}{A_1}B_1\gamma_1\right)\sigma^2 - \left\{\frac{A_2}{A_1}[B_1(\gamma_1 + \gamma_2) - 3A_1]R_g + B_1(\gamma_1 - C_1)\right\}\sigma + [B_1(\gamma_1 - C_1) + 6A_1 + 3A_2(1 - C_2)]R_g = 0, \quad (15)$$

the neutral value of the Prandtl number being

$$\sigma_0 = \frac{A_1}{A_2}\left(\frac{\gamma_1 - C_1}{\gamma_1}\right).$$

The values of the primary closure constants are $A_1 = 0.58$, $B_1 = 16.6$, $C_1 = 0.038$, $A_2 = 0.62$, $B_2 = 9.63$, $C_2 = 0.43$, and $C_3 = 0.2$, so that $\gamma_1 = 0.26$ and $\gamma_2 = 0.79$ (see Kantha 2003). Using the value of σ given by Eq. (15) in Eq. (11), one can deduce the variation of L_E/L_O with R_g.

However, the values of both c and $C_{\varepsilon 3}$ are uncertain. Note that for the value of $C_{\varepsilon 3} = 1$ chosen by BP04, σ drops out and therefore Eq. (15) becomes irrelevant. Thus the traditional model yields the *same* result as the BP04 model! Also, a value higher than $c = 2$ is preferable if one is to match the experimental data reasonably well. If we choose $c = 3$, we get the blue curve in Fig. 1a for $C_{\varepsilon 3} = 1$. BP04 curve ($c = 2$) is shown in green. It is clear that $c = 3$ yields a better agreement with data if $C_{\varepsilon 3}$ is kept at unity.

However, $C_{\varepsilon 3}$ does not have to be unity. If we put $C_{\varepsilon 3} = -0.5$, we get the red line in Fig. 1a. Note that because experiments indicate that the upper bound on L_E/L_O is about 2.0, the different curves should be terminated when this value is reached. It is clear that $C_{\varepsilon 3} = -0.5$ and $c = 3$ yield a better agreement with observational data than do the original BP04 values of $C_{\varepsilon 3} = 1$ and $c = 2$.

2. BP04 model

BP04 add an additional term on the rhs of Eq. (1) to allow for leakage of TKE into internal waves so that the governing equations become

$$\dot{k} = P + (B + W) - \varepsilon \quad \text{and}$$

$$\dot{\varepsilon} = [C_{\varepsilon 1}P + C_{\varepsilon 3}(B + W) - C_{\varepsilon 2}\varepsilon](\varepsilon/k), \quad (16)$$

where $B + W$ is arbitrarily put equal to $-\nu_t N^2/\sigma_0$ so that the IW leakage term becomes

$$W = -(\nu_t N^2/\sigma_0) + \nu_t' N^2 = (-\nu_t N^2/\sigma_0)\left(1 - \frac{\sigma_0}{\sigma}\right). \quad (17)$$

It can be shown that Eqs. (16) and (17) reduce to

$$\dot{\tau} = 2\left[(C_{\varepsilon 2} - 1) - \left(\frac{\tau}{\tau_\infty}\right)^2 - \frac{(1 - C_{\varepsilon 3})}{\sigma_0}\left(\frac{\tau}{T}\right)^2\right] \quad \text{and}$$

$$\frac{\dot{L}}{L}\tau = (2C_{\varepsilon 2} - 3) - \frac{(3 - 2C_{\varepsilon 1})}{(1 - C_{\varepsilon 1})}\left(\frac{\tau}{\tau_\infty}\right)^2 + \frac{(3 - 2C_{\varepsilon 3})}{\sigma_0}\left(\frac{\tau}{T}\right)^2, \quad (18)$$

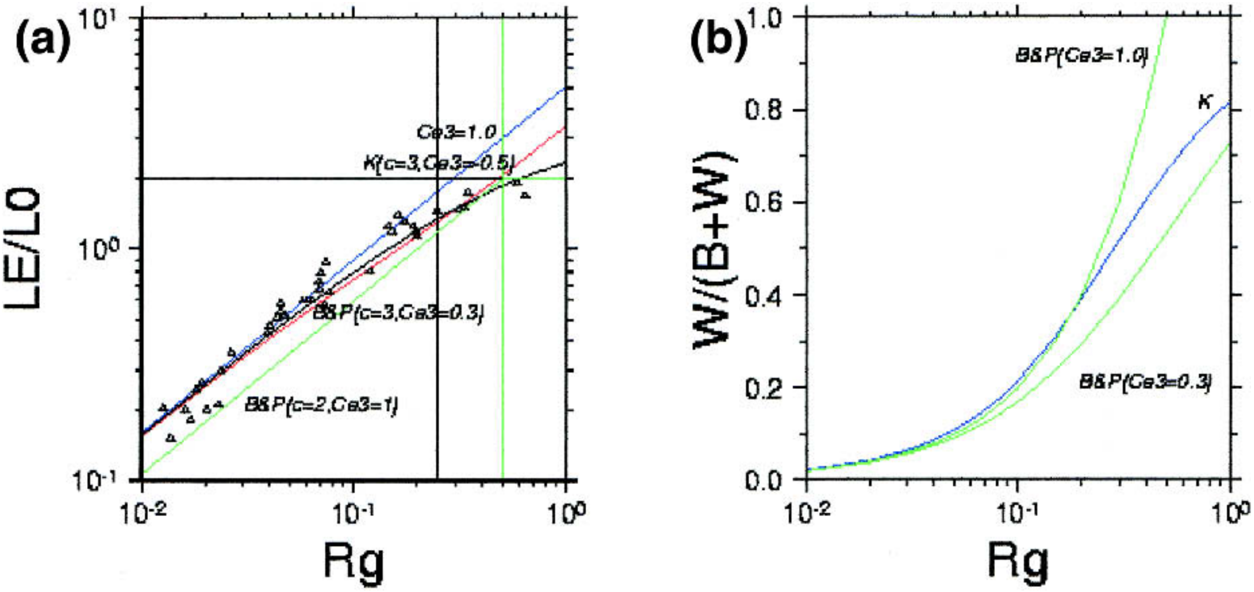

FIG. 1. Plot of (a) the ratio of Ellison scale to Ozmidov scale L_E/L_O and (b) the ratio of IW energy flux to the sum of the IW flux and the buoyancy flux $W/(B + W)$ vs gradient Richardson number R_g. The original BP04 model result is shown in green in (a). The blue line in (a) corresponds to $c = 3$ and holds for both the traditional and BP04 models. The red line in (a) is the traditional model result for $c = 3$ and $C_{\varepsilon 3} = -0.5$, and the black curve is for the BP04 model for $c = 3$ and $C_{\varepsilon 3} = 0.3$. The green curves in (b) are for the BP04 postulate for Prandtl number σ [Eq. (22)] for $C_{\varepsilon 3} = 0.3$ and 1.0, whereas the blue curve uses the second-moment closure to compute σ in the BP04 model and is independent of the value of $C_{\varepsilon 3}$. Triangles denote observational data points from BP04.

so that

$$\frac{L_E}{L_O} = c\left[\frac{R_g}{(C_{\varepsilon 1} - 1) + (1 - C_{\varepsilon 3})(R_g/\sigma_0)}\right]^{3/4} \quad \text{and} \tag{19}$$

$$R_g^{\mathrm{ss}} = \frac{(C_{\varepsilon 1} - 1)\sigma_0}{(2 - C_{\varepsilon 3})} = \frac{\sigma_0/2}{(2 - C_{\varepsilon 3})}. \tag{20}$$

Equations (18)–(20) differ from the corresponding equations for the traditional model Eqs. (6), (11), and (14) *only* in that σ_0 has replaced σ. Note that for $C_{\varepsilon 3} = 1$, Eq. (19) is *exactly* the same as Eq. (11) and hence both the traditional and BP04 models yield the same result for the variation of L_E/L_O with R_g!

Figure 1a shows the variation of L_E/L_O with R_g (black curve) for the BP04 model [Eq. (19)] for $c = 3$ and $C_{\varepsilon 3} = 0.3$ (the optimum value for agreement with data). This is, however, roughly equivalent to the results of the traditional model [Eq. (11)] for $c = 3$ and $C_{\varepsilon 3} = -0.5$ (red curve).

3. Energy flux to internal waves

The ratio of the IW energy flux to the sum of the IW energy flux and buoyancy flux in the BP04 model is

$$\frac{W}{B + W} = 1 - \frac{\sigma_0}{\sigma}. \tag{21}$$

The value of σ can be computed from Eq. (15). However, BP04 once again *arbitrarily* parameterize σ as

$$\frac{\sigma_0}{\sigma} = 1 - \left(\frac{\tau}{T}\right)^2 \text{ so that } \frac{W}{B + W} = \left(\frac{\tau}{T}\right)^2. \tag{22}$$

When $\dot{\tau} = 0$—that is, the so-called structural equilibrium—Eq. (22) becomes

$$\frac{W}{B + W} = \frac{R_g}{(C_{\varepsilon 1} - 1) + (1 - C_{\varepsilon 3})R_g/\sigma_0}. \tag{23}$$

If one assumes $C_{\varepsilon 3} = 1$ as BP04 did, then

$$\frac{W}{B + W} = 2R_g. \tag{24}$$

Figure 1b shows $W/(B + W)$ as a function of the gradient Richardson number R_g. The blue curve corresponds to Eq. (19), where σ is computed from Eq. (15), and the green curves correspond to Eq. (23). It is clear that the arbitrary postulate Eq. (22) is not an indispensable part of the BP04 model.

4. Richardson numbers

By putting $C_{\varepsilon 3} = 1$ in Eq. (18), BP04 show that $\dot{\tau} = 0$ can be reached only when $\tau = \tau_\infty$. Because by definition $(\tau_\infty/T)^2 = 2R_g$, this also means that R_g^{cr}, the value of R_g beyond which turbulence is quenched, becomes 1/2. However, this is artificial because it requires $C_{\varepsilon 3} = 1$. The same result is not obtainable for other values of

$C_{\varepsilon 3}$. The value of R_g^{cr} can, however, be determined by imposing an upper bound on L_E/L_O of about 2.0. This yields, for the traditional model ($c = 3$, $C_{\varepsilon 3} = -0.5$), $R_g^{cr} \sim 0.48$, fairly close to the original BP04 value of 0.5 and consistent with observational data on R_g^{cr}. The BP04 model value for R_g^{cr} is about 0.6 for $c = 3$ and $C_{\varepsilon 3} = 0.3$.

In a similar way, the steady-state value of R_g, R_g^{ss}, becomes 1/4 only if BP04 model [Eq. (20)] is used and the value of σ_0 is assumed to be 0.5, a value that is unrealistic and contrary to most observational data. If instead one uses the traditional model, Eq. (14), the value of R_g^{ss} depends now on the Prandtl number and is not independent of the flow conditions.

5. Conclusions

The BP04 model with its explicit IW energy drain term, which is, however, modeled in an ad hoc fashion, is not an improvement over traditional models. With their insistence that $C_{\varepsilon 3} = 1$ and $c = 2$, the agreement with data is actually worse than that of traditional models. Choosing $C_{\varepsilon 3} = 0.3$ and $c = 3$, it can be made to agree well with data on Ellison length scale. However, similar results can be obtained with the traditional model if $C_{\varepsilon 3} = -0.5$ and $c = 3$ are chosen. There is no need for an explicit parameterization of the energy drain by internal waves.

Helmut Baumert and H. Peters (2004, personal communication; see also Baumert et al. 2005) have applied the BP04 model to explain the Dickey and Mellor (1980) observations of the decay of grid-generated turbulence in a stably stratified fluid. However, an alternative explanation is possible, using the conventional model, *without* invoking the leakage of TKE into internal waves during the decay process, if the internal waves are postulated to be generated during the initial passage of the grid. Until measurements that distinguish between the turbulence and internal-wave fields are made and the amount of TKE going into internal waves is quantified, it would be difficult to justify explicit parameterization of energy drain by internal waves in the TKE equation.

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