J. Fluid Mech. (2013), *vol.* 735, R6, doi:10.1017/jfm.2013.539

JFM RAPIDS
journals.cambridge.org/rapids

Generation of an acoustic-gravity wave by two gravity waves, and their subsequent mutual interaction

Usama Kadri[1],† and Michael Stiassnie[2],‡

[1]Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

[2]Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090, Vienna, Austria

(Received 12 August 2013; revised 1 September 2013; accepted 7 October 2013; first published online 29 October 2013)

The nonlinear triad interaction of two opposing gravity waves with almost identical frequencies and one much longer acoustic-gravity wave is studied for non-resonance, as well as for exact resonance conditions. For non-resonance conditions the previously known results for a 'bound' acoustic-gravity wave are recovered. For resonance, or near-resonance conditions, where all three waves are 'free waves', the interaction is recurrent and the amplitude of the free acoustic-gravity wave turns out to be much larger than that known for the bound wave. The results for the recurrent evolution are given analytically, in terms of *Jacobian elliptic* functions and *elliptic integrals*.

Key words: acoustics, compressible flows, waves/free-surface flows

1. Introduction

The importance of the nonlinear triad interaction of gravity waves and acoustic-gravity waves in an ocean of finite depth, for the generation of microseisms in the solid earth, as well as in generating atmospheric noise, has recently been explained in a comprehensive paper by Ardhuin & Herbers (2013), which also includes a long list of related references. Among the most significant earlier works on this topic are the contributions of Longuet-Higgins (1950), Hasselmann (1962) and Kibblewhite & Wu (1991).

It is our understanding that most of the work published so far treats the acoustic-gravity wave as 'bound' to the 'free' gravity waves, and thus avoids the more complicated case of exact resonance, for which the acoustic-gravity wave is also 'free'. It is hoped that the contribution of this paper will close this gap.

† Email address for correspondence: usama.kadri@gmail.com

‡ Permanent address: Faculty of Civil and Environmental Engineering, Technion, Haifa 32000, Israel

The general equations and their linear solutions are presented in §§ 2 and 3, respectively; whereas the existence of resonating triads is discussed in § 4. In § 5 we develop the appropriate second-order interaction equations, and discuss the difference between bound waves and free waves. In §§ 6 and 7 we include the solutions for the bound acoustic-gravity wave, and for the resonating triads (i.e. free acoustic-gravity wave), respectively. The newly found results are discussed in § 8. Interesting background material on the general theory of nonlinear interaction can be found in a recent book by Kartashova (2010).

2. General equations

Take Cartesian axes (x, z) with the origin in the undisturbed free surface, and the z-axis vertically upwards. Let $z=-h$ be the equation of the rigid flat bottom and $z=\eta$ the equation of the free surface. We shall assume that viscosity is negligible and that the velocity $\bar{u}$ is irrotational, so that $\bar{u}=\operatorname{grad}\varphi$. We assume also that the density is a function of the pressure alone. Then the equations of motion can be integrated and, according to Longuet-Higgins (1950), produce the field equations

$$\frac{\partial^2\varphi}{\partial t^2}-c^2\nabla^2\varphi+g\frac{\partial\varphi}{\partial z}=-\frac{\partial}{\partial t}(\bar{u}^2), \quad -h\leqslant z\leqslant \eta, \tag{2.1}$$

where c is the speed of sound in the medium, g is the acceleration due to gravity, and t is the time. Note that in (2.1) and below we keep linear and quadratic terms only. From the continuity equation we know that $\mathrm{D}\rho/\mathrm{D}t-\rho\nabla^2\varphi=0$, where $(\mathrm{D}/\mathrm{D}t)$ is the differentiation following motion, and ρ is the fluid density. Since the flow is barotropic and a particle at the free surface remains at the free surface, where the pressure is atmospheric, the kinematic boundary condition is reduced to the following (see e.g. Longuet-Higgins 1950):

$$\nabla^2\varphi=0, \quad z=\eta. \tag{2.2}$$

On the other hand, the dynamic free-surface boundary condition is obtained from the equation of motion, giving

$$g\eta=-\frac{\partial\varphi}{\partial t}+\frac{1}{2}\bar{u}^2, \quad z=\eta. \tag{2.3}$$

Expanding (2.2) in a Taylor series around $z=0$ and using (2.3), we obtain

$$\nabla^2\varphi=\frac{1}{g}\varphi_t\varphi_{xxz}+\frac{1}{g}\varphi_t\varphi_{zzz}, \quad z=0. \tag{2.4}$$

The boundary condition on the bottom is

$$\varphi_z=0, \quad z=-h. \tag{2.5}$$

Equation (2.1) is rewritten in terms of the potential

$$\varphi_{tt}-c^2\nabla^2\varphi+g\varphi_z=-2\varphi_x\varphi_{xt}-2\varphi_z\varphi_{zt}, \quad -h\leqslant z\leqslant 0. \tag{2.6}$$

The field (2.6) together with the boundary conditions (2.4) and (2.5) define the problem for the potential $\varphi(x, z, t)$ within the region $-h\leqslant z\leqslant 0$. Some simple manipulations with (2.4) and (2.6) lead to an alternative formulation of the free-surface boundary condition:

$$\varphi_{tt}+g\varphi_z=-2\varphi_x\varphi_{xt}-2\varphi_z\varphi_{zt}+\frac{1}{g}\varphi_t\varphi_{ttz}+\varphi_t\varphi_{zz}, \quad z=0. \tag{2.7}$$

Note that to quadratic order, (2.4) together with (2.6) is equivalent to (2.7) with (2.6).

3. Linear solutions

Ignoring the right-hand side of (2.4) and (2.6), and seeking a progressive-wave solution with frequency ω, one obtains

$$\varphi = \frac{giA}{2\omega}\frac{\lambda\cosh[\lambda(h+z)] - \gamma\sinh[\lambda(h+z)]}{\lambda\cosh(\lambda h) - \gamma\sinh(\lambda h)}\,\mathrm{e}^{\gamma z}\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} + \text{c.c.}, \tag{3.1}$$

where $\gamma = g/2c^2$, $k^2 = \lambda^2 + \omega^2/c^2 - \gamma^2$, and the dispersion relation is given by

$$\omega^2 = g(\lambda^2 - \gamma^2)/[\lambda\coth(\lambda h) - \gamma]. \tag{3.2}$$

Here A is the freely chosen wave amplitude, so the free-surface elevation is

$$\eta = A\cos(kx - \omega t). \tag{3.3}$$

Details of the above derivation can be found in Dalrymple & Rogers (2006).

For all practical purposes one can replace γ by zero in the above equations, which yields

$$\varphi = \frac{giA}{2\omega}\frac{\cosh[\lambda(h+z)]}{\cosh(\lambda h)}\mathrm{e}^{\mathrm{i}(kx-\omega t)} + \text{c.c.}, \tag{3.4}$$

$$\omega^2 = g\lambda\tanh(\lambda h), \tag{3.5}$$

$$k^2 = \lambda^2 + \omega^2/c^2. \tag{3.6}$$

The dispersion relation (3.5) always has one real root λ_0, and infinitely many imaginary roots λ_n, $n = 1, 2, \ldots$. According to (3.6), λ_0 always produces a real wavenumber k_0. Hereafter we shall call this mode a 'gravity wave'. For a given frequency ω, and for a water depth $h < h_{cr} \equiv \pi c/2\omega$, all λ_n, $n = 1, 2, \ldots$ produce imaginary k_n, i.e. evanescent modes. However, for water depth $h > h_{cr}$ at least λ_1 produces a real k_1, which we will refer to as an 'acoustic-gravity wave'. As an example, taking $\omega = 2\ \mathrm{s}^{-1}$, $c = 1500\ \mathrm{m\ s}^{-1}$, $g = 9.81\ \mathrm{m\ s}^{-2}$, and $h = 4000$ m gives $h_{cr} = 1178$ m, $k_0 = 0.41\ \mathrm{m}^{-1}$, and $k_1 = 0.00062\ \mathrm{m}^{-1}$. The corresponding wavelengths are $\lambda_0 = 15$ m for the gravity wave, and $\lambda_1 = 10$ km for the acoustic-gravity wave.

4. Existence of resonating triads

In this section we show that if the water is deep enough, for a given gravity wave (with frequency ω_a and wavenumber k_a) there exists another opposing gravity wave (with ω_b and k_b) of nearly the same wavelength, and an acoustic-gravity wave (with ω_c and k_c), where the three waves satisfy the conditions

$$k_a + k_b = k_c, \quad \omega_a + \omega_b = \omega_c, \tag{4.1}$$

and obey the dispersion relation (3.2).

From the example in the previous section it is clear that the gravity waves are in so-called deep water, for which (3.5) simplifies to

$$\omega_j^2 = g|k_j|, \quad j = a, b. \tag{4.2}$$

Substituting (4.2) into (4.1) finally yields

$$\omega_a = \frac{1}{2}\left(\omega_c + g\frac{k_c}{\omega_c}\right) \quad \text{and} \quad \omega_b = \frac{1}{2}\left(\omega_c - g\frac{k_c}{\omega_c}\right). \tag{4.3}$$

Note that in order to obtain accurate triad solutions, we first choose λ_c. Substituting in (3.2) and then in $k^2 = \lambda^2 + \omega^2/c^2 - \gamma^2$, we obtain ω_c and k_c. Substituting these in (4.3), we obtain the frequencies ω_a and ω_b. Now, substituting ω_j $(j = a, b)$ in the dispersion relation (4.2), we finally obtain the wavenumbers k_a and k_b, accurately.

For the values of c, g, and h as in § 3, the following example of a wave triad (with two opposing gravity waves and one acoustic-gravity wave) has been calculated:

$$k_a = 0.102249\ \mathrm{m}^{-1}, \quad \omega_a = 1.00153\ \mathrm{s}^{-1}, \tag{4.4a}$$

$$k_b = -0.101625\ \mathrm{m}^{-1}, \quad \omega_b = 0.99847\ \mathrm{s}^{-1}, \tag{4.4b}$$

$$k_c = 0.000624\ \mathrm{m}^{-1}, \quad \omega_c = 2.00000\ \mathrm{s}^{-1}. \tag{4.4c}$$

The corresponding wavelengths are ∼60 m for the gravity waves, and 10 km for the acoustic-gravity wave. In order to examine the accuracy of the solution, we take k_j and ω_j (for $j = a, b$) from (4.4) and substitute into $\lambda^2 = k^2 - \omega^2/c^2 + \gamma^2$ to obtain λ_j; then we substitute these into (3.2), which yields

$$\omega_a = 1.00153\ \mathrm{s}^{-1}, \quad \omega_b = 0.99847\ \mathrm{s}^{-1}. \tag{4.5}$$

These values are identical to those in (4.4) up to the 16th significant digit (from which we present only six). Note that for the sake of simplicity we have limited the discussion to collinear triads only. Non-collinear resonating triads, for which (4.1) must be replaced by

$$\bar{k}_a + \bar{k}_b = \bar{k}_c, \quad \omega_a + \omega_b = \omega_c, \tag{4.6}$$

do exist, and exhibit very similar behaviour to that of the collinear case.

5. Second-order interaction equations

5.1. *Gravity waves (a) and (b)*

For the gravity waves, (2.6) reduces to

$$\nabla^2 \varphi^{(a,b)} = 0, \quad z \leqslant 0. \tag{5.1}$$

Equation (2.7), together with the fact that to leading order $\varphi_z^{(c)} = -g^{-1}\varphi_{tt}$, at $z = 0$ (see (2.4) and (2.6)), yields

$$\varphi_{tt}^{(a,b)} + g\varphi_z^{(a,b)} = 2g^{-1}\varphi_{tt}^{(c)}\varphi_{zt}^{(b,a)} + 2g^{-1}\varphi_{ttt}^{(c)}\varphi_z^{(b,a)} - g^{-2}\varphi_{tttt}^{(c)}\varphi_t^{(b,a)}, \quad z = 0. \tag{5.2}$$

Note that terms with first x-derivatives of $\varphi^{(c)}$ at $z = 0$ have been neglected, due to their smallness in comparison to terms with first z-derivatives.

5.2. *Acoustic-gravity wave (c)*

For the acoustic-gravity wave, (2.6) reduces to

$$\varphi_{tt}^{(c)} - c^2\nabla^2\varphi^{(c)} + g\varphi_z^{(c)} = -2\varphi_x^{(a)}\varphi_{xt}^{(b)} - 2\varphi_x^{(b)}\varphi_{xt}^{(a)} - 2\varphi_z^{(a)}\varphi_{zt}^{(b)} - 2\varphi_z^{(b)}\varphi_{zt}^{(a)}, \quad z \leqslant 0 \tag{5.3}$$

whereas (2.4) together with (5.1) give

$$\nabla^2\varphi^{(c)} = 0, \quad z = 0. \tag{5.4}$$

Due to the very different vertical structure of the acoustic-gravity wave (c) in comparison to the gravity waves (a) and (b), (5.3) is rewritten as

$$\varphi_{tt}^{(c)} - c^2\nabla^2\varphi^{(c)} + g\varphi_z^{(c)} = \begin{cases} -2\varphi_x^{(a)}\varphi_{xt}^{(b)} - 2\varphi_x^{(b)}\varphi_{xt}^{(a)} - 2\varphi_z^{(a)}\varphi_{zt}^{(b)} - 2\varphi_z^{(b)}\varphi_{zt}^{(a)}, & z=0, \\ 0, & z<0. \end{cases} \tag{5.5}$$

Substituting (5.4) into the upper line of (5.5) gives

$$\varphi_{tt}^{(c)} + g\varphi_z^{(c)} = -2\varphi_x^{(a)}\varphi_{xt}^{(b)} - 2\varphi_x^{(b)}\varphi_{xt}^{(a)} - 2\varphi_z^{(a)}\varphi_{zt}^{(b)} - 2\varphi_z^{(b)}\varphi_{zt}^{(a)}, \quad z=0. \tag{5.6}$$

The lower line of (5.5) gives the field equation

$$\varphi_{tt}^{(c)} - c^2\nabla^2\varphi^{(c)} + g\varphi_z^{(c)} = 0, \quad z<0. \tag{5.7}$$

Additionally, $\varphi^{(c)}$ has to satisfy the bottom boundary condition

$$\varphi_z^{(c)} = 0, \quad z=-h. \tag{5.8}$$

5.3. *The distinction between bound waves and free waves*

To quadratic order, any two gravity waves, similar to (a) and (b) of § 4, will generate a bound acoustic-gravity wave to be denoted by (d). The bound wave (d) will have a similar structure to (3.4), it will satisfy (3.6) and (4.1), but not the dispersion relation (3.5). The wave amplitude of the bound wave $A^{(d)}$ depends on the wave amplitudes of the gravity waves: $A^{(d)} \propto A^{(a)} \times A^{(b)}$, and all three amplitudes are constants. On the other hand, in the case of resonating triads the wave amplitudes $A^{(a)}$, $A^{(b)}$, and $A^{(c)}$ are interdependent through a system of ordinary differential equations, and are evolving with time. The bound wave solution and the resonating triad solution are presented in §§ 6 and 7, respectively.

6. Solution for the bound wave (d)

For gravity waves in deep water, (3.4) yields

$$\varphi^{(j)} = \frac{giA^{(j)}}{2\omega_j} e^{|k_j|z} e^{i(k_j x - \omega_j t)} + \text{c.c.}, \quad j=a,b. \tag{6.1}$$

For the acoustic-gravity wave (d), (3.4) gives

$$\varphi^{(d)} = \frac{giA^{(d)}}{2\omega_d} \frac{\cosh[\lambda_d(h+z)]}{\cosh(\lambda_d h)} e^{i(k_d x - \omega_d t)} + \text{c.c.} \tag{6.2}$$

Equation (6.2) satisfies the bottom boundary condition (5.8). It also satisfies the field (5.7), provided that $k_d^2 = \lambda_d^2 + \omega_d^2/c^2$.

Substituting (6.1) and (6.2) into the free-surface boundary condition (5.6) and requiring that

$$k_a + k_b = k_d, \quad \omega_a + \omega_b = \omega_d, \tag{6.3}$$

leads to

$$A^{(d)} = -\frac{g\omega_d}{\omega_d^2 - g\lambda_d \tanh(\lambda_d h)} \left(\frac{k_a k_b}{\omega_a} + \frac{k_a k_b}{\omega_b} - \frac{|k_a||k_b|}{\omega_a} - \frac{|k_a||k_b|}{\omega_b} \right) A^{(a)} A^{(b)}. \tag{6.4}$$

For opposing waves $|k_a| = k_a$ and $|k_b| = -k_b$, (6.4) reduces to

$$A^{(d)} = -\frac{2g\omega_d^2|k_a||k_b|}{\omega_d^2 - g\lambda_d \tanh(\lambda_d h)} \frac{A^{(a)}A^{(b)}}{\omega_a\omega_b} = -\frac{2\omega_a\omega_b\omega_d^2 A^{(a)}A^{(b)}}{g[\omega_d^2 - g\lambda_d \tanh(\lambda_d h)]}. \tag{6.5}$$

It is reassuring that for $\omega_b = \omega_a$, $k_d = 0$ and (6.2) with (6.5) agree with the third line in equation (172) of Longuet-Higgins (1950). Note also that (6.5) tends to infinity at exact resonance, since the denominator contains the dispersion relation (3.5).

7. Resonating triad solution: free wave (c)

For resonating triads, (6.5) gives infinite values. To overcome this difficulty we assume that $A^{(a)}$, $A^{(b)}$ and $A^{(c)}$ are slowly varying functions of the variable

$$\zeta = (t - \omega_c x / k_c c^2), \tag{7.1}$$

which guarantees that (5.7) is satisfied to appropriate order. Substituting (6.1) and (6.2), with (d) replaced by (c), into (5.2) and (5.5) produces the following system of ordinary differential equations:

$$A_\zeta^{(a)} = (\mathrm{i}\omega_c/4g)(-2\omega_b^2 + 2\omega_b\omega_c - \omega_c^2)(A^{(b)})^* A^{(c)}, \tag{7.2a}$$

$$A_\zeta^{(b)} = (\mathrm{i}\omega_c/4g)(-2\omega_a^2 + 2\omega_a\omega_c - \omega_c^2)(A^{(a)})^* A^{(c)}, \tag{7.2b}$$

$$A_\zeta^{(c)} = (-\mathrm{i}\omega_a\omega_b/g)(\omega_a + \omega_b) A^{(a)} A^{(b)}. \tag{7.2c}$$

According to (4.4), $\omega_b \simeq \omega_a \equiv \omega$ and $\omega_c \simeq 2\omega_a = 2\omega$. Thus the system (7.2) simplifies to

$$A_\zeta^{(a)} = -\mathrm{i}(\omega^3/g)(A^{(b)})^* A^{(c)}, \tag{7.3a}$$

$$A_\zeta^{(b)} = -\mathrm{i}(\omega^3/g)(A^{(a)})^* A^{(c)}, \tag{7.3b}$$

$$A_\zeta^{(c)} = -2\mathrm{i}(\omega^3/g) A^{(a)} A^{(b)}. \tag{7.3c}$$

By standard techniques (see Shemer & Stiassnie (1985)), one can show that

$$|A^{(a)}|^2 = (2\omega^3/g)Z + |A_0^{(a)}|^2, \tag{7.4a}$$

$$|A^{(b)}|^2 = (2\omega^3/g)Z + |A_0^{(b)}|^2, \tag{7.4b}$$

$$|A^{(c)}|^2 = -(4\omega^3/g)Z + |A_0^{(c)}|^2, \tag{7.4c}$$

where Z is governed by

$$\begin{aligned} &[(2\omega^3/g)Z + |A_0^{(a)}|^2][(2\omega^3/g)Z + |A_0^{(b)}|^2][(-4\omega^3/g)Z + |A_0^{(c)}|^2] \\ &\quad = Z_\zeta^2 + [\mathrm{Re}\{A_0^{(a)} A_0^{(b)} (A_0^{(c)})^*\}]^2 \end{aligned} \tag{7.5}$$

and $A_0^{(j)} = A^{(j)}(\zeta = 0), j = a, b, c$.

Without loss of generality we take $A_0^{(c)} = 0$, for which (7.5) simplifies to

$$Z_\zeta^2 = (-4\omega^3/g)Z[(2\omega^3/g)Z + |A_0^{(a)}|^2][(2\omega^3/g)Z + |A_0^{(b)}|^2]. \tag{7.6}$$

Using (236.00) on p. 79 of Byrd & Friedman (1971), assuming $|A_0^{(b)}| < |A_0^{(a)}|$, and inverting, we obtain

$$Z = (-g/2\omega^3)|A_0^{(b)}|^2 \mathrm{sn}^2[-(2^{1/2}\omega^3/g)|A_0^{(a)}|\zeta, |A_0^{(b)}|/|A_0^{(a)}|], \tag{7.7}$$

where $\mathrm{sn}(u, \theta)$ is the *Jacobian elliptic function* of argument u and modulus θ. Substituting (7.7) into (7.4) yields

$$|A^{(a)}|^2 = |A_0^{(a)}|^2 - |A_0^{(b)}|^2 \mathrm{sn}^2[-(2^{1/2}\omega^3/g)|A_0^{(a)}|\zeta, |A_0^{(b)}|/|A_0^{(a)}|], \tag{7.8a}$$

$$|A^{(b)}|^2 = |A_0^{(b)}|^2 \mathrm{cn}^2[-(2^{1/2}\omega^3/g)|A_0^{(a)}|\zeta, |A_0^{(b)}|/|A_0^{(a)}|], \tag{7.8b}$$

$$|A^{(c)}|^2 = 2|A_0^{(b)}|^2 \mathrm{sn}^2[-(2^{1/2}\omega^3/g)|A_0^{(a)}|\zeta, |A_0^{(b)}|/|A_0^{(a)}|]. \tag{7.8c}$$

From (7.1) and (7.8) it now becomes clear that the amplitudes $A^{(a)}$, $A^{(b)}$, and $A^{(c)}$ are functions of the slow variable $(2^{1/2}\omega^2 A_0^{(a)}/g)(\omega t - 2\omega^2 x/k_c c^2)$, where $\omega^2 A_0^{(a)}/g$ is the initial steepness of the gravity wave (a). Note that the maximum value of $|A^{(c)}|$ is

$$\max|A^{(c)}| = 2^{1/2}|A_0^{(b)}|, \tag{7.9}$$

and that the evolution period T is

$$T = (2^{1/2} g/\omega^3 |A_0^{(a)}|) K(|A_0^{(b)}|/|A_0^{(a)}|), \tag{7.10}$$

where $K(v)$ is the complete elliptic integral of the first kind. For ocean waves, the amplitude of a reflected wave, say wave (b), is $|A_0^{(b)}| \ll |A_0^{(a)}|$; then $K(v) \simeq K(0) = \pi/2$ and the period reduces to

$$T = g\pi/(2^{1/2}\omega^3 |A_0^{(a)}|). \tag{7.11}$$

8. Results

The main results of this paper are: (i) a clear demonstration of the exact resonance between two almost identical opposing gravity waves and one acoustic-gravity wave; and (ii) the derivation of an analytical solution for their subsequent recurrent behaviour. The physical importance of this phenomenon is related to the generation of microseisms by acoustic-gravity waves. In this respect it is important to note that we found that the amplitude of the 'free' acoustic-gravity wave generated at resonance or near-resonance conditions is significantly larger than the amplitude of the 'bound' acoustic-gravity wave, which is generated at conditions far from resonance. As an example of a condition far from resonance, we take the Longuet-Higgins (1950) case with two identical opposing gravity waves:

$$k_a = 0.101936\ \mathrm{m}^{-1}, \quad \omega_a = 1.0\ \mathrm{s}^{-1}, \tag{8.1a}$$

$$k_b = -0.101936\ \mathrm{m}^{-1}, \quad \omega_b = 1.0\ \mathrm{s}^{-1}, \tag{8.1b}$$

$$k_d = 0\ \mathrm{m}^{-1}, \quad \omega_d = 2.0\ \mathrm{s}^{-1}. \tag{8.1c}$$

For this case (6.5) gives

$$|A^{(d)}| = 0.2|A_0^{(a)}||A_0^{(b)}| \tag{8.2}$$

where every A, here and below, is in metres.

On the other hand, for the resonating triad case (4.4) equation (7.9) gives

$$\max|A^{(c)}| = \sqrt{2}|A_0^{(b)}|. \tag{8.3}$$

The ratio between (8.2) and (8.3) gives

$$\frac{A^{(d)}}{\max|A^{(c)}|} = 0.14|A_0^{(a)}|, \tag{8.4}$$

which is indeed much smaller than one.

Note also that for the resonating triad (4.4), the ratios between the period of the amplitude evolution and the period of the acoustic-gravity wave itself,

$$\frac{T}{T_c} = \frac{7}{|A_0^{(a)}|}, \tag{8.5}$$

is much larger than one, as it should be. The actual separation between resonating triads and nearly resonating triads on one hand, and far-from-resonance triads on the other hand, is given by the condition

$$\delta = g\frac{|\omega_c^2 - g\lambda_c \tanh(\lambda_c h)|}{\omega_a^4(|A_0^{(a)}||A_0^{(b)}|)^{1/2}}, \tag{8.6}$$

which has to be much smaller than unity for the former and much larger for the latter. The condition (8.6) was obtained by studying the solution of the more general 'combined' problem, for which (7.2*c*) is replaced by

$$A_\zeta^{(c)} + \mathrm{i}[\omega_c^2 - g\lambda_c \tanh(\lambda_c h)]A^{(c)}/4\omega = -2\mathrm{i}(\omega^3/g)A^{(a)}A^{(b)} \tag{8.7}$$

whereas (7.2*a*) and (7.2*b*) are left as before. The technique is the same as that of § 7, and it was encouraging to recover the results of that section when $\delta \ll 1$.

For $\delta \gg 1$, (7.2*a*), (7.2*b*) and (8.7) give

$$\max |A^{(c)}| = 2|A^{(d)}|, \tag{8.8}$$

where $|A^{(d)}|$ is the bound wave amplitude of (6.5).

The resonance response curve, for the system with (8.7), in terms of the detuning parameter $\varDelta = \delta^2/2^7$ (where δ is given by (8.6)), has the analytical expression

$$|A_{max}^{(c)}|/|A_0^{(b)}| = \{1 + \varDelta\mu + \mu^2 - [(1 + \varDelta\mu + \mu^2)^2 - 4\mu^2]^{1/2}\}^{1/2}, \tag{8.9}$$

where $\mu = |A_0^{(a)}|/|A_0^{(b)}|$. For $\varDelta = 0$, (8.9) gives $|A_{max}^{(c)}|/|A_0^{(b)}| = \sqrt{2}$ as in (8.3), for any μ. For the example $\mu = 0.1$, (8.9) gives $|A_{max}^{(c)}|/|A_0^{(b)}| = 0.13$ and 0.01 for $\varDelta = 1$ and 10, respectively.

Acknowledgement

This research was supported by the Israel Science Foundation (grant 63/09).

References

ARDHUIN, F. & HERBERS, T. H. C. 2013 Noise generation in the solid Earth, oceans, and atmosphere, from nonlinear interacting surface gravity waves in finite depth. *J. Fluid Mech.* **716**, 316–348.

BYRD, P. F. & FRIEDMAN, M. D. 1971 *Handbook of Elliptic Integrals for Engineers and Scientists*. Springer.

DALRYMPLE, R. A. & ROGERS, B. D. 2006 A note on wave celerities on a compressible fluid. In *Proc. 30th Int. Conference on Coastal Engineering (ICCE) 2006, 2007*, pp. 3–13.

HASSELMANN, K. 1962 On the nonlinear energy transfer in a gravity wave spectrum. Part 1. General theory. *J. Fluid Mech.* **12**, 481–501.

KARTASHOVA, E. 2010 *Nonlinear Resonance Analysis, Theory, Computation, Applications*. Cambridge University Press.

KIBBLEWHITE, A. C. & WU, C. Y. 1991 The theoretical description of wave–wave interactions as a noise source in the ocean. *J. Acoust. Soc. Am.* **89**, 2241–2252.

LONGUET-HIGGINS, M. S. 1950 A theory of the origin of microseisms. *Philos. Trans. R. Soc. Lond.* A **243**, 1–35.

SHEMER, L. & STIASSNIE, M. 1985 Initial instability and long-time evolution of Stokes waves. In *The Ocean Surface: Wave Breaking, Turbulent Mixing and Radio Probing* (ed. Y. Toba & H. Mitsuyasu), pp. 51–57. D. Reidel.