European Journal of Mechanics B/Fluids 55 (2016) 157–161

Contents lists available at ScienceDirect

European Journal of Mechanics B/Fluids

journal homepage: www.elsevier.com/locate/ejmflu

Triad resonance between a surface-gravity wave and two high frequency hydro-acoustic waves

Usama Kadri

Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

ARTICLE INFO

Article history:
Received 7 November 2014
Received in revised form
3 July 2015
Accepted 19 September 2015
Available online 30 September 2015

Keywords:
Nonlinear wave interaction
Hydro-acoustic waves
Gravity waves
Triad resonance

ABSTRACT

A new class of resonant triad in the family of gravity–acoustic waves has been found. I show that a hydro-acoustic wave interacting with a surface-gravity wave may generate a second hydro-acoustic wave. Interestingly, the two acoustic waves propagate in the same direction with similar wavelengths, that are almost double of that of the gravity wave. The evolution of the wave triad amplitudes is periodic and it is derived analytically, in terms of *Jacobian elliptic* functions and *elliptic integrals*. The physical importance of this type of triad interactions is the modulation of pertinent hydro-acoustic signals, leading to inaccurate signal perceptions.

1. Introduction

Resonance interactions of waves play a prominent role in energy share among the different wave types involved. Such interactions may significantly contribute, among others, to the evolution of the ocean energy spectrum by exchanging energy between surface-gravity waves [1,2]; surface and internal gravity waves [3–5]; or even surface and compression-type waves [6–8], that can transfer energy from the upper ocean through the whole water column reaching down to the seafloor [9–11].

A resonant triad occurs among a triplet of waves, usually involving interaction of nonlinear terms of second order perturbed equations. Until recently, it has been believed that in a homogeneous fluid a resonant triad is possible only when tension forces are included, or at the limit of a shallow water [12]. Moreover, [9] argued that when the compressibility of water is considered, no resonant triads can occur within the family of gravity–acoustic waves. However, [7] proved that, under some circumstances, resonant triads comprising two opposing surface-gravity waves of similar periods (though not identical) and a much longer acoustic–gravity[1] wave, of almost double the frequency, exist.

In this paper I report on a new resonant triad involving a surface-gravity wave and two hydro-acoustic waves of almost double the length. Since the lengths of the gravity and acoustic waves are comparable, the present resonance is relevant to hydro-acoustic waves of relatively high frequency. This resonance is a second of its kind in the family of gravity–acoustic waves, and it has a significantly different characteristics compared to other resonant triads. Here, even though the interaction of gravity and acoustic modes does not concern short and long waves, the corresponding frequencies are disparate.

2. Background

2.1. Governing equations

Consider a two dimensional Cartesian coordinate system (x, z) with the origin in the undisturbed free surface, and the z-axis vertically upwards. Let $z = \eta$ be the equation of the free surface, and $z = -h$ the equation of the rigid flat bottom. Assume that the density is a function of pressure alone, the viscosity is negligible, and the velocity $\mathbf{u}$ is irrotational, so that $\mathbf{u} = \nabla\varphi$. Approximate to quadratic terms, the equations of motion can then be integrated to obtain the field equation [6]

$$\varphi_{tt} - c^2\nabla^2\varphi + g\varphi_z = -2\varphi_x\varphi_{xt} - 2\varphi_z\varphi_{zt} \quad (-h \le z \le 0), \tag{1}$$

where c is the speed of sound in the fluid, g is the acceleration due to gravity, and t is the time. The boundary condition at the bottom is

$$\varphi_z = 0 \quad (z = -h). \tag{2}$$

From the continuity equation we know that $D\rho/Dt - \rho\nabla^2\varphi = 0$, where (D/Dt) is the differentiation following motion, and ρ

E-mail address: ukadri@mit.edu.

[1] Acoustic–gravity wave is a very long hydro-acoustic wave that is affected by the force of gravity.

http://dx.doi.org/10.1016/j.euromechflu.2015.09.008

is the fluid density. Since the flow is barotropic and a particle at the free surface forever remains there, where the pressure is atmospheric, the kinematic boundary condition is reduced to $\nabla^2\varphi = 0$ (e.g. see [6]). On the other hand, the dynamic free surface boundary condition, $\varphi_t+(\varphi_x^2+\varphi_z^2)/2+g\eta = 0$, is obtained from the equation of motion. Expanding the kinematic condition in a Taylor series around $z = 0$ and making use of the dynamic conditions, [7] obtained the combined condition at the free surface

$$g\nabla^2\varphi = \varphi_t\varphi_{xxz} + \varphi_t\varphi_{zzz} \quad (z = 0). \tag{3}$$

An alternative formulation of the free surface boundary condition can be obtained after some simple manipulation of Eqs. (1) and (3) (see [7])

$$\varphi_{tt} + g\varphi_z = -2\varphi_x\varphi_{xt} - 2\varphi_z\varphi_{zt} + g^{-1}\varphi_t\varphi_{ttz} + \varphi_t\varphi_{zz} \quad (z = 0), \tag{4}$$

where, to quadratic order, (1) together with (3), are equivalent to (1) with (4).

2.2. Linear solution

Neglecting nonlinear terms in (1) and (3), and for practical purposes neglecting the gravity term in (1) and seeking a progressive-wave solution with frequency ω, the linear solution is obtained [13]

$$\varphi = \frac{giA}{2\omega}\frac{\cosh\left[\mu\,(h+z)\right]}{\cosh(\mu h)}\mathrm{e}^{\mathrm{i}(kx-\omega t)} + \text{c.c.}, \tag{5}$$

where c.c. denotes complex conjugates, $k^2 = \mu^2 + \omega^2/c^2$, and the dispersion relation is given by

$$\omega^2 = g\mu\tanh(\mu h). \tag{6}$$

The leading root in the dispersion relation (6) is real with a corresponding real wavenumber resembling the surface-gravity mode. On the other hand, the remaining infinity of roots are all imaginary. At any prescribed frequency ω, and water depth $h > h_{cr} \equiv \pi c/2\omega$ there is at least one propagation mode, with imaginary root but real wavenumber. Such modes are referred to as hydro-acoustic. All remaining modes having imaginary wavenumbers and are known as evanescent modes. Thus, for the acoustic modes, provided that $\lambda = \mathrm{i}\mu$ is real, we can exclusively write

$$\omega^2 = -g\lambda\tan(\lambda h) \tag{7}$$

and

$$k^2 = \frac{\omega^2}{c^2} - \lambda^2 > 0. \tag{8}$$

Note that for a gravity wave of frequency $\omega = \sigma$, travelling in deep water, $\mu \simeq k$, the dispersion relation reduces to

$$\sigma^2 = gk. \tag{9}$$

3. Resonant triads

Concerning a triad involving two acoustic modes of frequencies ω_1 and ω_2, and wavenumbers q_1 and q_2, and a single gravity mode of frequency σ and wavenumber k, we are seeking to satisfy the resonance conditions

$$\sigma = \omega_1 - \omega_2, \qquad k = q_1 + q_2, \tag{10}$$

and the dispersion relations (7) and (9) for the acoustic and gravity modes, respectively. From the gravity mode dispersion relation (9) we can combine the resonance conditions,

$$(\omega_1 - \omega_2)^2 = g\,(q_1 + q_2)\,. \tag{11}$$

On the other hand, for the hydro-acoustic waves it is known that $\lambda h = \pi/2+\Delta$, for the first mode, where $\Delta \ll 1$. Upon substitution in (7), the dispersion relation of the first acoustic mode reduces to

$$\omega^2 = \frac{g\lambda}{\lambda h - \pi/2}, \tag{12}$$

to leading order in Δ. Isolating λ and substituting in (8) and (11) we obtain a relation between ω_1 and ω_2,

$$(\omega_1 - \omega_2)^2 = g\left(\sqrt{\frac{\omega_1^2}{c^2} - \frac{\omega_1^4\pi^2/4}{\left(\omega_1^2 h - g\right)^2}} + \sqrt{\frac{\omega_2^2}{c^2} - \frac{\omega_2^4\pi^2/4}{\left(\omega_2^2 h - g\right)^2}}\right). \tag{13}$$

By requiring wavenumbers to be real, it is easy to show that the acoustic cut-off frequency is $\omega_{cr} \simeq \pi c/2h$, which corresponds to a Longuet-Higgins resonance (see [8,6]). For any $\omega_2 > \omega_{cr}$ one can always find ω_1, from (13), and σ, from (11), that satisfy the dispersion relations and resonance conditions.

4. Amplitude evolution equations

For the resonant triad case, we assume that the complex amplitudes of the gravity mode $S(\tau)$, and the two acoustic modes $A_1(\tau)$, and $A_2(\tau)$, are all slow variables in time τ. The first order velocity potential of the triad is given by

$$\phi^{(1)} = S(\tau)\mathrm{e}^{kz}\mathrm{e}^{\mathrm{i}(kx-\sigma t)} + A_1(\tau)\cos[\lambda_1(z+h)]\mathrm{e}^{-\mathrm{i}(q_1x+\omega_1 t)} + A_2(\tau)\cos[\lambda_2(z+h)]\mathrm{e}^{\mathrm{i}[(k-q_1)x-(\sigma+\omega_1)t]} + \text{c.c.} \tag{14}$$

The governing equations for the second order potential are the field equation

$$\phi_{tt}^{(2)} - c^2\nabla^2\phi^{(2)} + g\phi_z^{(2)} = -2\phi_{\tau t}^{(1)} - 2\phi_x^{(1)}\phi_{xt}^{(1)} - 2\phi_z^{(1)}\phi_{zt}^{(1)} \quad (-h < z < 0), \tag{15}$$

the bottom boundary condition

$$\phi_z^{(2)} = 0 \quad (z = -h), \tag{16}$$

and the combined surface condition

$$\phi_{tt}^{(2)} + g\phi_z^{(2)} = -2\phi_{\tau t}^{(1)} - 2\phi_x^{(1)}\phi_{xt}^{(1)} + 2g^{-1}\phi_{tt}^{(1)}\phi_{zt}^{(1)} + 2g^{-1}\phi_{ttt}^{(1)}\phi_z^{(1)} - g^{-2}\phi_{tttt}^{(1)}\phi_t^{(1)} \quad (z = 0). \tag{17}$$

We define the second order potential with amplitudes changing slowly in time

$$\phi^{(2)} = F_S(z,\tau)\mathrm{e}^{\mathrm{i}(kx-\sigma t)} + F_{A_1}(z,\tau)\mathrm{e}^{-\mathrm{i}(q_1x+\omega_1 t)} + F_{A_2}(z,\tau)\mathrm{e}^{\mathrm{i}[(k-q_1)x-(\sigma+\omega_1)t]} + \text{c.c.} \tag{18}$$

4.1. Derivation of the evolution equations

Substituting (18) in the field equation, we write for the gravity wave

$$F_{S,zz} - k^2F_S = -\frac{2\mathrm{i}\sigma}{c^2}\mathrm{e}^{kz}S_\tau + \frac{\mathrm{i}\sigma}{c^2}\Big\{\left(-q_1^2 - \lambda_1\lambda_2 + q_1k\right) \times \cos\left[\left(\lambda_1-\lambda_2\right)(z+h)\right] + \left(-q_1^2 + \lambda_1\lambda_2 + q_1k\right) \times \cos\left[\left(\lambda_1+\lambda_2\right)(z+h)\right]\Big\}A_1^*A_2. \tag{19}$$

Multiply both sides by the vertical gravity mode shape $\exp(kz)$ and integrate over the water depth

$$\int_{-h}^{0} e^{kz} F_{S,zz}\,dz - k^2 \int_{-h}^{0} e^{kz} F_S\,dz = -\frac{2i\sigma}{c^2} S_\tau \int_{-h}^{0} e^{2kz}\,dz + \frac{i\sigma}{c^2} A_1^* A_2 \times \int_{-h}^{0} e^{kz} \Big\{ \left(-q_1^2 - \lambda_1\lambda_2 + q_1 k\right) \cos\left[(\lambda_1 - \lambda_2)(z+h)\right] + \left(-q_1^2 + \lambda_1\lambda_2 + q_1 k\right) \cos\left[(\lambda_1 + \lambda_2)(z+h)\right] \Big\}\,dz, \quad (20)$$

where the first term of the left-hand side of Eq. (20) is evaluated by integration by parts,

$$\int_{-h}^{0} e^{kz} F_{S,zz}\,dz = e^{kz} F_{S,z}\Big|_{-h}^{0} - k \int_{-h}^{0} e^{kz} F_{S,z}\,dz = e^{kz} F_{S,z}\Big|_{-h}^{0} - k e^{kz} F_S + k^2 \int_{-h}^{0} e^{kz} F_S\,dz. \quad (21)$$

Substituting in (20), evaluating the right-hand side, and making use of the bottom boundary condition (16) yields

$$F_{S,z}(0) - kF_S(0) = -\frac{i\sigma}{c^2 k} S_\tau + \frac{i\sigma \left\{ \Gamma_s^+ + \Gamma_s^- + \Gamma_c^+ + \Gamma_c^- \right\} A_1^* A_2}{c^2 \left(k^4 + 2k^2\lambda_2^2 + 2k^2\lambda_1^2 + \lambda_2^4 - 2\lambda_1^2\lambda_2^2 + \lambda_1^4\right)} \quad (22)$$

where the coefficients Γ are given in Appendix A. On the other hand substituting F_j in the combined surface condition (17) gives

$$gF_{j,z}(0) - \omega^2 F_j(0) = 2i\sigma S_\tau + i\sigma \Big\{ \left(-\lambda_1\lambda_2 + q_1^2 - q_1 k\right) \cos[(\lambda_1 + \lambda_2)h] + \left(\lambda_1\lambda_2 + q_1^2 - q_1 k\right) \cos[(\lambda_1 - \lambda_2)h] \Big\} A_1^* A_2. \quad (23)$$

Recall that from the gravity dispersion relation $\sigma^2 g^{-1} = k$; substituting (23) into (22), and after neglecting small quantities, the evolution equations for the gravity wave is obtained. Similarly, one can repeat the same steps to derive the evolution equations of the acoustic modes. Finally, we can write the three evolution equations,

$$\frac{dS}{d\tau} = \beta A_1^* A_2, \qquad \frac{dA_1}{d\tau} = \alpha_1 A_2 S^*, \qquad \frac{dA_2}{d\tau} = -\alpha_2 A_1 S, \quad (24)$$

where the coefficients β, α_1, and α_2 are given in Appendix A.

4.2. Solution of the evolution equations

An analytical solution of the evolution equations is given in a form of Jacobi elliptic functions (see Appendix B),

$$|S|^2 = |S_0|^2 - \frac{\beta}{\alpha_1} |A_{10}|^2 \mathrm{sn}^2(u, \theta) \quad (25)$$

$$|A_1|^2 = |A_{10}|^2 \mathrm{cn}^2(u, \theta), \qquad |A_2|^2 = \frac{\alpha_2}{\alpha_1} |A_{10}|^2 \mathrm{sn}^2(u, \theta) \quad (26)$$

of argument $u = \sqrt{\alpha_1\alpha_2}|S_0|\tau$ and modulus $\theta = |A_{10}|/|S_0|\sqrt{\beta/\alpha_1}$, where $|S_0|$ and $|A_{10}|$ are the initial amplitudes of the gravity and acoustic waves, respectively. The evolution period is given by $T = 2K(\theta)/\sqrt{\alpha_1\alpha_2}|S_0|$, where K is the complete elliptic integral of the first kind. For application purposes, it is likely that $\theta \ll 1$, so that the elliptic functions can be approximated by trigonometric functions that are given by equation 127.01 on page 24 of [14].

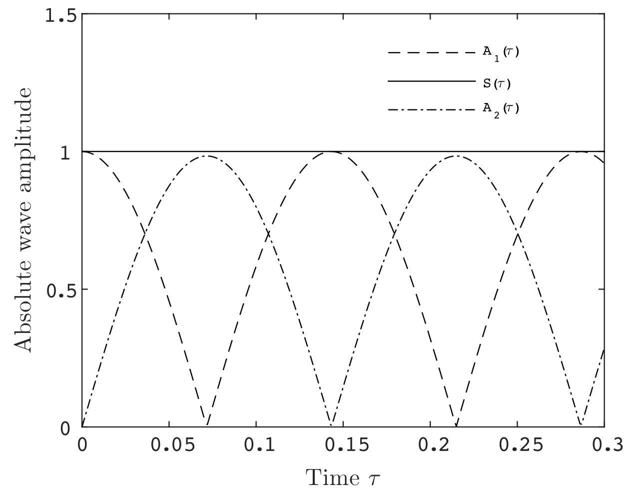

Fig. 1. Evolution of the wave triad amplitudes. $h = 4000$ m, $c = 1500$ m/s, $g = 9.81$ m/s^2.

5. Numerical results and discussion

The main result of this paper is the clear demonstration of resonance between a single surface-gravity wave, and two hydro-acoustic waves each of almost double the length. This new resonant triad is a second of its kind to be found in the gravity–acoustic wave family.[2] The physical importance of the phenomenon addressed here is the modulation of acoustic signals in water near the surface. In this regard, the resultant acoustic wave has a maximum amplitude comparable to the prescribed acoustic wave, both acoustic waves have lengths and periods that are similar (though not identical), and interestingly, both travel in the same direction while sharing and exchanging energy. This behaviour might cause miscommunication of such acoustic signals.

As a numerical example, consider the triad interaction of a gravity and two acoustic waves, propagating with frequencies $(\sigma, \omega_1, \omega_2) = (0.51, 20, 19.49)$ rad/s, and wavenumbers $(k, q_1, q_2) = (0.0263, 0.0133, 0.0130)$ 1/m, respectively. Initially, only the gravity and a prescribed acoustic wave interact. The amplitude of the gravity wave remains almost unaltered during the interaction, whereas the amplitude of the prescribed acoustic wave drops gradually, transferring almost all of its energy to the resultant acoustic wave, as shown in Fig. 1. Then energy transfers back and the initial conditions are restored, completing one cycle of energy share and starting a new one, recurrently.

One can derive three conserved quantities of (24) in the form of the *Manley-Rowe* relations

$$\frac{d}{d\tau}\left(\beta|S|^2 - \alpha_1|A_1|^2\right) = \frac{d}{d\tau}\left(\alpha_1|A_1|^2 + \alpha_2|A_2|^2\right) = \frac{d}{d\tau}\left(-\alpha_2|A_2|^2 - \beta|S|^2\right). \quad (27)$$

Equivalently, the quantities in (27) can be replaced by the wave actions so that,

$$\frac{E_s}{\sigma} \equiv \beta|S|^2, \qquad \frac{E_1}{\omega_1} \equiv \alpha_1|A_1|^2, \qquad \frac{E_2}{\omega_2} \equiv -\alpha_2|A_2|^2 \quad (28)$$

where $E_{tot} \equiv E_s + E_1 + E_2$ is constant. It is obvious that the amplitudes remain bounded as the signs of the coefficients, or

[2] The first gravity–acoustic resonant triad involves the interaction of two gravity waves and a single acoustic wave of almost double the period (see [7]).

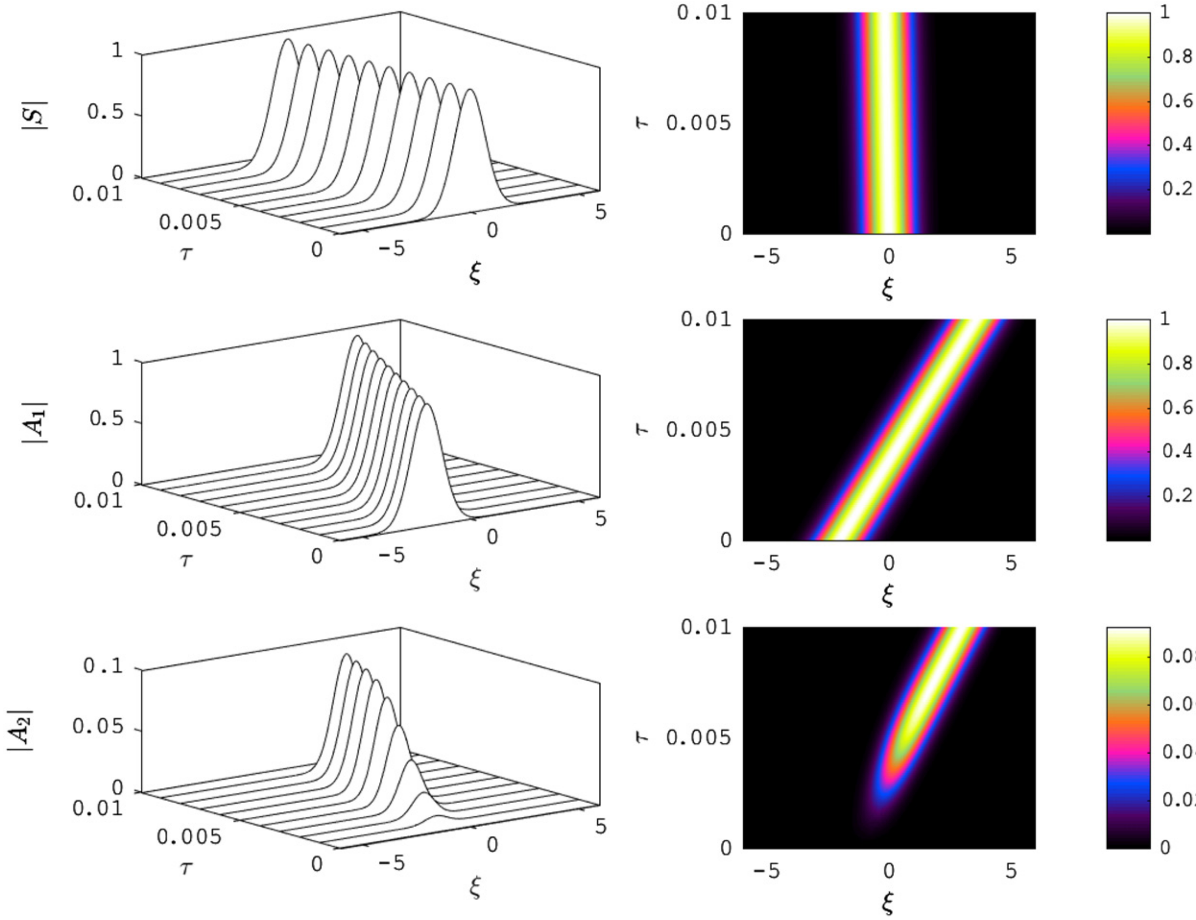

Fig. 2. Spatio-temporal evolution of the wave triad amplitudes; $h = 100$ m, $c = 1500$ m/s, $g = 9.81$ m/s^2, $(\sigma, \omega_1, \omega_2) = (0.32, 25, 24.68)$ rad/s.

energies, differ [15]. For near resonance, with small frequency mismatch $\sigma - \omega_1 + \omega_2 = \Delta\omega$, the evolution equations will have additional exponential factors of the form $\exp(-\mathrm{i}\Delta\omega\tau)$. Similarly, to the resonance solution, this case also results in solutions in the form of Jacobi elliptic functions [16], though the total energy is no longer conserved. Since for most practical purposes the amplitudes of the two acoustic waves, A_1 and A_2, are much smaller than that of the gravity mode, and each depends on τ alone, one can obtain the following linearised approximate equations,

$$|S| = |S_0|, \qquad \frac{\mathrm{d}^2|A_j|}{\mathrm{d}\tau^2} + \alpha_1\alpha_2|S_0|^2|A_j| = 0, \quad (j = 1, 2) \tag{29}$$

where A_1 and A_2 are periodic. Here, the total energy E_{tot} is periodic in frequency $\Delta\omega$.

The analysis presented above addresses the case of uniform amplitude envelopes, where the spatial dependency is neglected. In the case of nonuniform envelopes, say Gaussian, the evolution equation should also account for a slow variable in space ξ and in time τ. In this case, the evolution equations write

$$\frac{\mathrm{d}S}{\mathrm{d}\tau} - \frac{g}{2\sigma}\frac{\mathrm{d}S}{\mathrm{d}\xi} = \beta A_1^* A_2, \tag{30}$$

$$\frac{\mathrm{d}A_1}{\mathrm{d}\tau} - \frac{gq_1}{\omega_1\lambda_1\cos\lambda_1 h}\frac{\mathrm{d}A_1}{\mathrm{d}\xi} = \alpha_1 A_2 S^*, \tag{31}$$

$$\frac{\mathrm{d}A_2}{\mathrm{d}\tau} - \frac{gq_2}{\omega_2\lambda_2\cos\lambda_2 h}\frac{\mathrm{d}A_2}{\mathrm{d}\xi} = -\alpha_2 A_1 S. \tag{32}$$

As previously, the gravity wave remains almost unaltered, while the envelope slowly displaces to the left, as shown in Fig. 2. However, the prescribed acoustic envelope travels relatively fast to the right minimising the interaction time. Consequently, the resultant acoustic wave envelope might be significantly smaller. As the two acoustic beams concurrently move away from the gravity wave, with disparate group velocities, the resonant interaction gradually vanishes.

The analysis presented in this paper neglects the effect of gravity in the field equation, while it is retained in the combined surface boundary condition. This assumption might no longer be valid in the case of two low frequency hydro-acoustic modes, i.e. acoustic–gravity waves, interacting resonantly with a gravity mode.

Appendix A. Definition of coefficients

$$\begin{aligned}\Gamma_s^+ = {} & \sin[(\lambda_1+\lambda_2)h]\Big\{(q_1\lambda_2 + q_1\lambda_1)k^3 + (-q_1^2\lambda_1 - q_1^2\lambda_2 \\ & + \lambda_1\lambda_2^2 + \lambda_1^2\lambda_2)k^2 + (q_1\lambda_2^3 - q_1\lambda_2\lambda_1^2 - q_1\lambda_2^2\lambda_1 \\ & + q_1\lambda_1^3)k + (q_1^2\lambda_2\lambda_1^2 + \lambda_1^4\lambda_2 - q_1^2\lambda_1^3 + q_1^2\lambda_2^2\lambda_1 \\ & + \lambda_1\lambda_2^4 - q_1^2\lambda_2^3 - \lambda_1^3\lambda_2^2 - \lambda_1^2\lambda_2^3)\Big\}\end{aligned} \tag{A.1}$$

$$\begin{aligned}\Gamma_s^- = {} & \sin[(\lambda_1-\lambda_2)h]\Big\{(-q_1\lambda_2 + q_1\lambda_1)k^3 + (-q_1^2\lambda_1 + q_1^2\lambda_2 \\ & + \lambda_1\lambda_2^2 - \lambda_1^2\lambda_2)k^2 + (-q_1\lambda_2^3 + q_1\lambda_2\lambda_1^2 - q_1\lambda_2^2\lambda_1 \\ & + q_1\lambda_1^3)k + (-q_1^2\lambda_2\lambda_1^2 - \lambda_1^4\lambda_2 - q_1^2\lambda_1^3 + q_1^2\lambda_2^2\lambda_1 \\ & + \lambda_1\lambda_2^4 + q_1^2\lambda_2^3 - \lambda_1^3\lambda_2^2 + \lambda_1^2\lambda_2^3)\Big\}\end{aligned} \tag{A.2}$$

$$\Gamma_c^+ = \cos[(\lambda_1+\lambda_2)h]\Big\{k^4q_1 + (\lambda_1\lambda_2 - q_1^2)k^3 + (-2q_1\lambda_1\lambda_2 + q_1\lambda_2^2 + q_1\lambda_1^2)k^2 + (-q_1^2\lambda_2^2 + 2q_1^2\lambda_1\lambda_2 - q_1^2\lambda_1^2 + \lambda_1^3\lambda_2 - 2\lambda_1^2\lambda_2^2 + \lambda_1\lambda_2^3)k\Big\} \tag{A.3}$$

$$\Gamma_c^- = \cos[(\lambda_1-\lambda_2)h]\Big\{k^4q_1 + (-\lambda_1\lambda_2 - q_1^2)k^3 + (2q_1\lambda_1\lambda_2 + q_1\lambda_2^2 + q_1\lambda_1^2)k^2 + (-q_1^2\lambda_2^2 - 2q_1^2\lambda_1\lambda_2 - q_1^2\lambda_1^2 - \lambda_1^3\lambda_2 - 2\lambda_1^2\lambda_2^2 - \lambda_1\lambda_2^3)k\Big\} \tag{A.4}$$

$$\beta = q_1\,(k-q_1)\cos(\lambda_1 h)\cos(\lambda_2 h) - \lambda_1\lambda_2\sin(\lambda_1 h)\sin(\lambda_2 h) \tag{A.5}$$

$$\alpha_1 = \frac{1}{2g^2\cos(\lambda_1 h)}\Big\{\left(2g\sigma q_1 + \omega_1^3 + 2\sigma^2\omega_1 + 2\sigma\omega_1^2 - \sigma^3\right)\sigma\cos(\lambda_2 h) + 2g\sigma^2\lambda_2\sin(\lambda_2 h)\Big\} \tag{A.6}$$

$$\alpha_2 = \frac{1}{2g^2\cos(\lambda_2 h)}\Big\{\left(2g\sigma q_1 + \omega_1^3 + \sigma^2\omega_1 + \sigma\omega_1^2\right)\times\sigma\cos(\lambda_1 h) - 2g\sigma^2\lambda_1\sin(\lambda_1 h)\Big\}. \tag{A.7}$$

Appendix B. Solution of the evolution equations

Eq. (24) can be written as

$$S^*S_\tau + SS_\tau^* = \beta\left(SA_1A_2^* + S^*A_1^*A_2\right) \tag{B.1}$$

$$A_1^*A_{1,\tau} + A_1A_{1,\tau}^* = \alpha_1\left(SA_1A_2^* + S^*A_1^*A_2\right) \tag{B.2}$$

$$A_2^*A_{2,\tau} + A_2A_{2,\tau}^* = -\alpha_2\left(SA_1A_2^* + S^*A_1^*A_2\right). \tag{B.3}$$

More compactly Eqs. (B.3), (B.1), and (B.2) can be written as

$$|S_\tau|^2 = 2\beta\,\mathrm{Re}\left\{SA_1A_2^*\right\}, \qquad |A_{1,\tau}|^2 = 2\alpha_1\,\mathrm{Re}\left\{SA_1A_2^*\right\}, \qquad |A_{2,\tau}|^2 = -2\alpha_2\,\mathrm{Re}\left\{SA_1A_2^*\right\}. \tag{B.4}$$

Now define $M_\tau = \mathrm{Re}\left\{SA_1A_2^*\right\}$ gives

$$|S|^2 = 2\beta M + |S_0|^2, \qquad |A_1|^2 = 2\alpha_1 M + |A_{10}|^2, \qquad |A_2|^2 = -2\alpha_2 M + |A_{20}|^2 \tag{B.5}$$

where $S_0 = S(\tau = 0)$, $A_{10} = A_1(\tau = 0)$, and without loss of generality we take $A_{20} = A_2(\tau = 0) = 0$. Now M is governed by

$$\frac{\mathrm{d}^2M}{\mathrm{d}\tau^2} = -8\alpha_2\beta\alpha_1 M\left(M + \frac{|S_0|^2}{2\beta}\right)\left(M + \frac{|A_{10}|^2}{2\alpha_1}\right). \tag{B.6}$$

The solution of Eq. (B.6) is given by [14], equation 236.00 on page 79:

$$M = -\frac{|A_{10}|^2}{2\alpha_1}\mathrm{sn}^2(u,\theta). \tag{B.7}$$

Substituting (B.7) in (B.5) we obtain the solution given in (25)–(26).

References

[1] M.S. Longuet-Higgins, Resonant interactions between two trains of gravity waves, J. Fluid Mech. 12 (03) (1962) 321–332.
[2] J. Hammack, D. Henderson, Resonant interactions among surface water waves, Annu. Rev. Fluid Mech. 25 (1) (1993) 55–97.
[3] K.M. Watson, B.J. West, B.I. Cohen, Coupling of surface and internal gravity waves: a mode coupling model, J. Fluid Mech. 77 (01) (1976) 185–208.
[4] K. Dysthe, K. Das, Coupling between a surface-wave spectrum and an internal wave: modulational interaction, J. Fluid Mech. 104 (1981) 483–503.
[5] W. Craig, P. Guyenne, C. Sulem, Coupling between internal and surface waves, Nat. Hazards 57 (3) (2011) 617–642.
[6] M.S. Longuet-Higgins, A theory of the origin of microseisms, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Sci. 243 (857) (1950) 1–35.
[7] U. Kadri, M. Stiassnie, Generation of an acoustic-gravity wave by two gravity waves, and their subsequent mutual interaction, J. Fluid Mech. 735 (2013) R6.
[8] U. Kadri, Wave motion in a heavy compressible fluid: Revisited, Eur. J. Mech. B Fluids 49 (0) (2015) 50–57. http://www.sciencedirect.com/science/article/pii/S0997754614001253.
[9] F. Ardhuin, T.H. Herbers, Noise generation in the solid earth, oceans and atmosphere, from nonlinear interacting surface gravity waves in finite depth, J. Fluid Mech. 716 (2013) 316–348.
[10] E. Eyov, A. Klar, U. Kadri, M. Stiassnie, Progressive waves in a compressible-ocean with an elastic bottom, Wave Motion 50 (5) (2013) 929–939.
[11] U. Kadri, M. Stiassnie, Acoustic-gravity waves interacting with the shelf break, J. Geophys. Res.: Oceans (1978–2012) 117 (C3) (2012).
[12] A. Dyachenko, V. Zakharov, Is free-surface hydrodynamics an integrable system? Phys. Lett. A 190 (2) (1994) 144–148.
[13] U. Kadri, M. Stiassnie, A note on the shoaling of acoustic-gravity waves, WSEAS Trans. Fluid Mech. 8 (2) (2013).
[14] P.F. Byrd, M.D. Friedman, P. Byrd, Handbook of Elliptic Integrals for Engineers and Scientists. Vol. 67, Springer, Berlin, 1971.
[15] A.D. Craik, Wave Interactions and Fluid Flows, Cambridge University Press, 1988.
[16] J. Armstrong, N. Bloembergen, J. Ducuing, P. Pershan, Interactions between light waves in a nonlinear dielectric, Phys. Rev. 127 (6) (1962) 1918.