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Derivation of three-dimensional radiation stress based on Lagrangian solutions of progressive waves

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ABSTRACT

A new approach has been proposed to derive the expressions for three-dimensional radiation stress using solutions of the pressure and velocity distributions and the coordinate transformation function that are derived from a Lagrangian description wherein the pressure is zero (relative to the atmospheric pressure) at the sea surface. Using this approach, analytical expressions of horizontal and vertical depth-dependent radiation stress are derived at a uniform depth and for a sloping bottom, respectively. The results of the depth integration of the expressions agree well with the theory of Longuet-Higgins and Stewart. In the case involving a sloping bottom, the radiation stress expressions from this study provide a better balance of the net momentum compared to those from previous studies.

1 Introduction

Surface gravity waves play an important role in upper oceanic processes, which range from wave-induced upper-ocean mixing to wave setup and wave-induced currents (Bowen et al. 1968; Hasselmann 1971; Garrett 1976; Ting and Kirby 1994; Smith 2006; Liu et al., 2017). Two concepts have been utilized in the literature to represent the effects of waves on relatively long-term currents. The first is "radiation stress", which is included in the wave-averaged equations for the total momentum (i.e., the sum of the mean current and wave momenta) and can be regarded as the sum of the horizontal Reynolds stress term and the negative of the form stress term (Mellor 2003; Aiki and Greatbatch 2012, 2013; Ardhuin et al. 2017). The second is the so-called "vortex force", which is included in the wave-averaged equations only for the current momentum and is used to decompose wave-averaged effects into a Bernoulli head and a vortex force (Craik and Leibovich 1976; Andrews and McIntyre 1978; McWilliams et al. 2004; Lane et al. 2007; Ardhuin et al. 2008b; Aiki and Greatbatch 2014). Compared to the concept of radiation stress, the concept of the vortex force is relatively new. The advantage of the vortex force theory is that more of the mechanisms that govern wave-current interactions can be explained (Lane et al. 2007; Bennis et al. 2011; Ardhuin et al. 2017), while it is disadvantageously more complicated than the traditional radiation stress method (Mellor 2016). In this paper, we aim to improve the capability of the existing three-dimensional radiation stress (3DRS) formulations, e.g., those of Mellor (2003, 2015), rather than compare radiation stress and vortex force.

Since its concept was first introduced by Longuet-Higgins and Stewart (1962, 1964; hereinafter LHS), radiation stress has been widely used to render depth-integrated equations, which have clear physical significance, more suitable for 2D oceanic flows with surface waves (Phillips 1977). Dolata and Rosenthal (1984) extended the 2D radiation stress into three dimensions, including variations in the vertical direction, based on the LHS concept.

However, Dolata and Rosenthal (1984) neglected the impact of the wave pressure, and thus, their product was unable to return to an LHS form following vertical integration (Mellor 2003). Zheng and Yan (2001) reported a 3D solution of radiation stress, and their results in the vertical direction were described as functions of three vertical layers from the bottom to the surface: below the wave trough, the wave trough to the mean, and above the mean water level. Nevertheless, their method made it difficult to describe the vertical profile of radiation stress, which is defined as an average variable within wave periods (Xia et al. 2004). To improve the vertical structures of radiation stress, a number of studies have been conducted involving a transformation of the vertical coordinate through expression derivations. For instance, Xia et al. (2004) employed a vertical coordinate transformation that Sheng and Liu (2011) subsequently revealed as having the potential to induce obvious errors in the definition of radiation stress. Using an alternative coordinate transformation considering the vertical wave-induced motion of linear waves, Mellor (2003) derived a set of 3D expressions of radiation stress. However, those expressions were found to lack self-consistency for a simple case of waves that shoal over a slope without energy dissipation (Ardhuin et al. 2008a). Mellor (2008) reported an updated version of 3DRS using a hydrostatic pressure assumption for the surface layer and a pressure distribution derived from the vertical momentum equation for the subsurface layers. However, Bennis and Ardhuin (2011) disputed the revised technique since different averaging methods were used for both the pressure gradient term and for the other terms. The error in the hydrostatic pressure assumption for the top layer was subsequently investigated by Mellor (2013, 2015) by integrating the vertical momentum equation to derive the pressure distribution over the whole water column, and an expression for radiation stress was consequently obtained with a new pressure distribution. However, this pressure term could not precisely satisfy a zero pressure (relative to the atmospheric pressure) condition for the surface, and the derivation lacked a term that integrates over the vertical to a value of zero

(Ardhuin et al. 2017).

Various uncertainties remain within 3DRS studies, including (1) the expression of the vertical pressure distribution in the water column, especially within the top layer, where the pressure is zero at the sea surface and (2) an explicit description of the complex vertical flux of the wave momentum. The former is a minor concern since a mismatch of higher-order terms will not alter the dynamics of the system (Ardhuin et al. 2017), while the latter is regarded as a fundamental issue within 3DRS problems (Bennis et al. 2011; Ardhuin et al. 2008a, 2017). A relatively complete wave momentum flux requires an asymptotic expansion of the first order to account for the wave steepness, bottom slope, vertical current gradients and temporal scales of the amplitudes, and it is difficult to include such a variety of effects in the analytical 3DRS expressions at present. Ardhuin et al. (2008a, 2017) noted that finding a first-order analytical solution for the bottom slope is a fundamental step toward improving the 3DRS formulation. Therefore, in the present study, we attempt to derive an expression of the vertical wave momentum flux over a sloping bottom as the first step toward improving the 3DRS equations.

The linear wave theory, in which the treatment of pressure at the surface is ambiguous (Mellor 2003, 2008, 2015), has commonly been used in the derivation of 3DRS. In fact, a zero pressure at the free surface at any instantaneous time can be accomplished with a Lagrangian description (Chen 1994, Chen and Hsu 2009a, 2009b, Chen et al. 2010, Chen et al. 2012). Furthermore, first-order analytical solutions for wave motion in the bottom slope have been derived by Chen et al. (2012). Therefore, the wave solutions developed using a Lagrangian description are chosen to derive the 3DRS equations in this study. The velocity and pressure distributions and the coordinate transformation function, which are derived following their transformations into Eulerian descriptions, each utilize the Lagrangian solutions of progressive waves by Chen et al (2010, 2012).

Before addressing the vertically dependent radiation stress, we briefly introduce the Lagrangian solutions in Section 2. In Section 3, the derivation of 3DRS is presented. The formulations are evaluated using the test case proposed by Ardhuin et al. (2008a) in Section 4. The conclusions of this study and recommendations for future work are found in Section 5.

2 Lagrangian solutions of progressive waves

a. Lagrangian solutions at a uniform water depth

Chen (1994) and Chen et al. (2010) reported a set of Lagrangian solutions for vertical 2D progressive waves at a uniform water depth with the coordinate origin at the still water level. The formation of the general solutions can be described as follows:

$$x = a + \sum_{n=1}^{\infty} \varepsilon^n \left[f_n(a,b,\sigma_L t) + f_n'(a,b,\sigma_0 t) \right] \tag{1a}$$

$$z = b + \sum_{n=1}^{\infty} \varepsilon^n \left[g_n(a,b,\sigma_L t) + g_n'(a,b,\sigma_0 t) \right] \tag{1b}$$

$$P = -\rho g b + \sum_{n=1}^{\infty} \varepsilon^n P_n(a,b,\sigma_L t) \tag{1c}$$

$$\sigma_L = \sum_{n=0}^{\infty} \varepsilon^n \sigma_n(a,b) = \frac{2\pi}{T} \tag{1d}$$

where ε is an ordering parameter that can be set to 1 in the final solutions; x and z are the horizontal and vertical particle positions, respectively; a and b are two characteristic parameters in the Lagrangian system; P is the pressure (relative to the atmospheric pressure); σ_L is the angular frequency of the particle motion; and T is the period of the motion of a particle reappearing at the same level. The first-order Lagrangian solutions are described as

$$f_1 = -a_0 \frac{\cosh k(b+d)}{\sinh kd} \sin\psi \ , \quad g_1 = a_0 \frac{\sinh k(b+d)}{\sinh kd} \cos\psi \tag{2a,b}$$

$$f_1' = g_1' = 0 \tag{2c}$$

$$\frac{P_1}{\rho} = -a_0 g \frac{\sinh kb}{\sinh kd \cosh kd} \cos\psi \tag{2d}$$

$$\sigma_0^2 = gk \tanh kd \tag{2e}$$

where $\psi = ka - \sigma_L t$ is the phase function in the Lagrangian system; $k = 2\pi/L$ is the wave number; a_0 represents the wave amplitude; and d is the water depth. The second-order solutions are

$$f_2 = -\frac{3}{8} a_0^2 k \frac{\cosh 2k(b+d)}{\sinh^4 kd} \sin 2\psi + \frac{1}{4} a_0^2 k \frac{1}{\sinh^2 kd} \sin 2\psi \tag{3a}$$

$$f_2' = \frac{1}{2} a_0^2 k \frac{\cosh 2k(b+d)}{\sinh^2 kd} \sigma_0 t \tag{3b}$$

$$g_2 = \frac{3}{8} a_0^2 k \frac{\sinh 2k(b+d)}{\sinh^4 kd} \cos 2\psi + \frac{1}{4} a_0^2 k \frac{\sinh 2k(b+d)}{\sinh^2 kd} \tag{3c}$$

$$g_2' = 0 \tag{3d}$$

$$\begin{aligned} \frac{P_2}{\rho} = {} & g a_0^2 k \left[\frac{3}{4} \frac{\cosh 2k(b+d)}{\sinh^3 kd \cosh kd} - \frac{3}{8} \frac{\sinh 2k(b+d)}{\sinh^4 kd} - \frac{3}{4} \frac{1}{\sinh kd \cosh kd} \right] \cos 2\psi \\ & + \frac{1}{4} g a_0^2 k \left[\frac{\cosh 2k(b+d)}{\sinh kd \cosh kd} - \frac{\sinh 2k(b+d)}{\sinh^2 kd} + \frac{1}{\sinh kd \cosh kd} \right] \end{aligned} \tag{3e}$$

$$\sigma_1 = 0 \tag{3f}$$

b. Lagrangian solutions for a sloping bottom

Chen et al. (2012) also derived asymptotic solutions in a Lagrangian description for nonlinear water waves propagating over a sloping bottom. The first-order asymptotic solutions can be obtained as follows (Chen et al. 2012; Li et al. 2013):

$$x = a + f_{1,0} + \varepsilon_2 f_{1,1} \tag{4a}$$

$$z = b + g_{1,0} + \varepsilon_2 g_{1,1} \tag{4b}$$

$$P = -\rho g b + P_{1,0} + \varepsilon_2 P_{1,1} \tag{4c}$$

where the ordering parameter ε_2 represents the bottom slope. The solutions have the following forms:

$$f_{1,0}=f_1\ ,\quad g_{1,0}=g_1\ ,\quad P_{1,0}=P_1 \tag{5a-c}$$

$$\begin{aligned} f_{1,1}=a_0\Bigg\{&\left[\frac{k^2\left(b+d\right)^2}{R\sinh 2kd}-k\left(b+d\right)+\frac{1}{R^2\tanh kd}\right]\frac{\cosh k\left(b+d\right)}{\sinh kd}\\ &+\left[\frac{k\left(b+d\right)}{R^2\tanh kd}+\frac{2k\left(b+d\right)}{R\sinh 2kd}-1\right]\frac{\sinh k\left(b+d\right)}{\sinh kd}\Bigg\}\cos\psi \end{aligned} \tag{5d}$$

$$\begin{aligned} g_{1,1}=a_0\Bigg\{&\left[\frac{k^2\left(b+d\right)^2}{R\sinh 2kd}-k\left(b+d\right)+\frac{1}{R^2\tanh kd}\right]\frac{\sinh k\left(b+d\right)}{\sinh kd}\\ &+\left[\frac{k\left(b+d\right)}{R^2\tanh kd}+\frac{2k\left(b+d\right)}{R\sinh 2kd}-1\right]\frac{\cosh k\left(b+d\right)}{\sinh kd}\Bigg\}\sin\psi \end{aligned} \tag{5e}$$

$$\begin{aligned} \frac{P_{1,1}}{\rho}=ga_0\Bigg\{&\left[\frac{k^2\left(b+d\right)^2}{R\sinh 2kd}-k\left(b+d\right)\right]\frac{\cosh k\left(b+d\right)}{\cosh kd}+\frac{k\left(b+d\right)}{R^2\tanh kd}\frac{\sinh k\left(b+d\right)}{\cosh kd}\\ &-\left[\frac{k^2\left(b+d\right)^2}{R\sinh 2kd}-k\left(b+d\right)+\frac{1}{R^2\tanh kd}\right]\frac{\sinh k\left(b+d\right)}{\sinh kd}\\ &-\left[\frac{k\left(b+d\right)}{R^2\tanh kd}+\frac{2k\left(b+d\right)}{R\sinh 2kd}-1\right]\frac{\cosh k\left(b+d\right)}{\sinh kd}\Bigg\}\sin\psi \end{aligned} \tag{5f}$$

$$R=1+\frac{2kd}{\sinh 2kd} \tag{5g}$$

c. Lagrange-Euler transformation

To successfully transform a set of given Eulerian solutions into completely unknown Lagrangian solutions, Chen and Hsu (2009a) used a successive Taylor series expansion for the water particle path and the period of particle motion. These transformed Lagrangian mean level and wave period expressions, which account for all of the water particles at different elevations, represent generic expressions compared with those in Longuet-Higgins (1979, 1986), which describe only particles at the free surface. Subsequently, Chen and Hsu (2009b) identified the corresponding reversible process (i.e., the transformation method from a

Lagrangian into an Eulerian description), and thus, the second-order transformation can be written as

$$F_m\left(a,b,\sigma_L t\right)=\left[\varepsilon' F_m-\varepsilon'^2\left(f_1\frac{\partial F_m}{\partial a}+g_1\frac{\partial F_m}{\partial b}\right)\right]_{\substack{a=x\\ b=y\\ \sigma_L=\sigma}} \tag{6}$$

where F_m represents the general expression; ε' is an ordering parameter that can be set equal to 1; and σ is the Eulerian angular frequency of the waves.

It should be noted that only the formulations used in the present study are reviewed. Details regarding the derivation of higher-order Lagrangian solutions can be found in the abovementioned corresponding studies.

3 Derivation of three-dimensional radiation stress (3DRS)

a. Momentum equations

Consider a classical problem involving 2D progressive waves in which the *x*-axis parallels the direction of wave propagation. Following Mellor (2003, 2015, 2016), the dependent variables ϕ are divided into both current variables $\hat{\phi}$ (whose temporal and spatial scales are large compared to those of the inverse wave frequency and wave number) and wave variables $\tilde{\phi}$ as follows:

$$\eta=\hat{\eta}+\tilde{\eta} \tag{7a}$$

$$u=\hat{u}+\tilde{u} \tag{7b}$$

$$w=\hat{w}+\tilde{w} \tag{7c}$$

$$p=\hat{p}+\tilde{p} \tag{7d}$$

where η is the free surface elevation; $\hat{\eta}$ is the phase-averaged surface elevation; u and w are the horizontal and vertical velocity, respectively; and p is the pressure. The vertical coordinate transformation proposed by Mellor is written as

$$z = s(x,\ \varsigma,\ t) = \hat{\eta} + \varsigma D + \tilde{s} \tag{8}$$

in which s is the coordinate transformation function, and $\tilde{s}$ is expressed as

$$\tilde{s} = a_0 \frac{\sinh kD(1+\varsigma)}{\sinh kD} \cos\varphi \tag{9}$$

where $\varphi = kx - \sigma t$ is the phase function; the coordinate ς ranges from -1 ($z = -h$) to 0 ($z = \eta = \hat{\eta} + \tilde{\eta}$); and $D = \hat{\eta} + h$ is the mean water column depth (h is the bottom depth).

Throughout this paper, we denote phase average variables with an overbar, that is, $\overline{(\ )} = (2\pi n)^{-1} \int_0^{2\pi n} (\ ) d\varphi$, wherein n is an integer equal or greater than 1. Thus, the phase average of z can be expressed as $\bar{z} = \hat{\eta} + \varsigma D$.

Neglecting buoyancy, Coriolis forces, and mixing and viscous effects, the horizontal mean momentum in Mellor (2003) [his Eq. (34a)] in which motion is restricted to the vertical plane is given as

$$\frac{\partial DU}{\partial t} + \frac{\partial DU^2}{\partial x} + \frac{\partial \Omega U}{\partial \varsigma} + gD\frac{\partial \hat{\eta}}{\partial x} = F_{xx} + F_{x3} \tag{10}$$

where U is the mean drift velocity (i.e., the Lagrangian mean current), which contains both the quasi-Eulerian current (Jenkins, 1989) and the Stokes drift, and Ω is the vertical mean velocity. On the right-hand side, the first term denotes the horizontal divergence of the horizontal flux of the wave momentum and is expressed as

$$F_{xx} = -\frac{\partial S_{xx}}{\partial x} = -\frac{\partial}{\partial x}\left(\overline{D\tilde{u}^2 + \tilde{p}\frac{\partial \tilde{s}}{\partial \varsigma}}\right) \tag{11}$$

Meanwhile, the second term denotes the vertical divergence of the vertical flux of the wave momentum:

$$F_{x3} = -\frac{\partial S_{x3}}{\partial \varsigma} = \frac{\partial}{\partial \varsigma}\left(\overline{\tilde{p}\frac{\partial \tilde{s}}{\partial x}}\right) \tag{12}$$

b. Derivation of the horizontal radiation stress S_{xx}

Throughout this paper, we assume that the timescales of the variations in the quantities are small compared to the timescales of the waves (i.e., $\left|\left(\partial a_0/\partial t\right)/\left(\sigma a_0\right)\right|$, $\left|\left(\partial \hat{u}/\partial t\right)k/\sigma^2\right|$ and $\left|\left(\partial D/\partial t\right)k/\sigma\right| \ll 1$). We also neglect the effects of vertical current shear ($\left|\left(\partial \hat{u}/\partial z\right)/\sigma\right| \ll 1$). These hypotheses allow us to employ the Lagrangian solutions of Chen et al. that were discussed in Section 2. Furthermore, the wave slope ka_0 and bottom slope $\left|\left(\partial D/\partial x\right)\right|$ are stipulated to be small, and the respective fourth-order $\left(ka_0\right)^4$ and second-order $\left(\partial D/\partial x\right)^2$ terms are consequently neglected during the derivation.

The derivation consists of three steps. We first develop a modified transformation from a Lagrangian to an Eulerian description based on the method of Chen and Hsu (2009b). Then, we transform the wave solutions from a Lagrangian to an Eulerian description. In the third step, we derive the radiation stress expression in Eulerian form based on the transformed velocity and pressure distributions and the coordinate transformation function.

Step 1:

For the first-order Lagrangian approximation, Eq. (1b) can be written as[1]

$$z = b + a_0 \frac{\sinh k(b+h)}{\sinh kD} \cos\left(ka - \sigma_L t\right) \tag{13}$$

where the Lagrangian variable b ranges from $-h$ ($z=-h$) to $\hat{\eta}$ ($z=\eta$). Chen and Hsu (2009b) developed a transformation from a Lagrangian to an Eulerian system, which is demonstrated in Eq. (6). The first-order approximate transformation for the phase part $\cos(ka - \sigma_L t)$ is reproduced here:

[1] The mean surface elevation $\hat{\eta} \equiv 0$ in the investigations of Chen et al. (2010, 2012); however, $\hat{\eta}$ is considered in the present paper for completeness and for consistency with the investigations of Mellor (2003, 2015).

$$F_m(a,b,\sigma_L t) = (F_m)_{\substack{a=x \\ \sigma_L=\sigma}} \tag{14}$$

The parameters a and σ_L in Eq. (13) are transformed using Eq. (14), and the parameter b remains unchanged here:

$$z = b + a_0 \frac{\sinh k(b+h)}{\sinh kD}\cos\left(kx - \sigma t\right) \tag{15}$$

Combining Eqs. (8), (9) and (15), the following algebraic relationship can be given:

$$b = \hat{\eta} + \varsigma D \tag{16}$$

Following Chen et al. (2010) and Chen and Chen (2014), the Lagrangian variable b is equal to the wavelength-averaged (or phase-averaged) $\overline{z}$ of the vertical displacement z. Based upon this physical definition, Eq. (16) is evidently true and exact. Using Eqs. (14) and (16), we develop a modified transformation from a Lagrangian to an Eulerian system (Fig. 1a):

$$F_m(a,b,\sigma_L t) = (F_m)_{\substack{a=x \\ b=\hat{\eta}+\varsigma D \\ \sigma_L=\sigma}} \tag{17}$$

Step 2:

The expressions of the wave velocity and pressure terms in a Lagrangian form are reproduced as follows:

$$\tilde{u} = \frac{\partial x}{\partial t} = a_0 \sigma_L \frac{\cosh k(b+h)}{\sinh kD}\cos\psi \tag{18}$$

$$\tilde{p} = g a_0 \left[\frac{\cosh k\left(b+h\right)}{\cosh kD} - \frac{\sinh k\left(b+h\right)}{\sinh kD}\right]\cos\psi \tag{19}$$

Using Eq. (17), both Eqs. (18) and (19) can be converted into an Eulerian description as follows:

$$\tilde{u} = a_0 \sigma \frac{\cosh kD(1+\varsigma)}{\sinh kD}\cos\varphi \tag{20}$$

$$\tilde{p} = g a_0 \left[\frac{\cosh kD\left(1+\varsigma\right)}{\cosh kD} - \frac{\sinh kD\left(1+\varsigma\right)}{\sinh kD}\right]\cos\varphi \tag{21}$$

where the pressure $\tilde{p}=0$ at the free surface ($\varsigma=0$).

Step 3:

Inserting Eqs. (9), (20) and (21) into Eq. (11) yields

$$S_{xx}=\overline{D\tilde{u}^2+\tilde{p}\frac{\partial\tilde{s}}{\partial\varsigma}}=kDE\left[F_{cs}F_{cc}+\left(F_{cs}F_{cc}-F_{ss}F_{cs}\right)\right] \tag{22}$$

where a set of convenient definitions are as follows:

$$\left.\begin{aligned} F_{ss}&=\frac{\sinh kD(1+\varsigma)}{\sinh kD}\\ F_{cs}&=\frac{\cosh kD(1+\varsigma)}{\sinh kD}\\ F_{sc}&=\frac{\sinh kD(1+\varsigma)}{\cosh kD}\\ F_{cc}&=\frac{\cosh kD(1+\varsigma)}{\cosh kD}\end{aligned}\right\} \tag{23a-d}$$

Eq. (22) is easily extended into 3DRS with the following transformation:

$$S_{ij}=kDE\left[\frac{k_ik_j}{k^2}F_{cs}F_{cc}+\delta_{ij}\left(F_{cs}F_{cc}-F_{ss}F_{cs}\right)\right] \tag{24}$$

where the wave number $k=|\mathbf{k}|$,[2] $E=ga_0^2/2$ is the wave energy, and the subscripts i and j denote the horizontal coordinates. Eq. (24) is identical to the expression of the horizontal radiation stress of Mellor (2003, 2015, 2016), but the pressure term at the surface is guaranteed to be equal to zero. When Eq. (24) is integrated from the bottom ($\varsigma=-1$) to the free surface ($\varsigma=0$), we can obtain the classical 2D radiation stress of LHS.

In the derivation based on the first-order Lagrangian solutions, the wave solutions are given in terms of the order $\varepsilon_1=ka_0$, and the expression of the pressure term is equivalent to that of Mellor (2003), which is incomplete (Mellor 2008, 2015). Following Steps 1–3, the subsequent derivation is based on a second-order Lagrangian approximation, and it includes

[2] The equation $k=|\mathbf{k}|$ is available only for Eqs. (24), (31) and (39).

terms of the order ε_1^2 in the wave solutions.

The derivation process is similar, and thus, we directly derive the 3DRS equations here. The detailed second-order solutions can be found in Eqs. (3a)–(3f). Eq. (1b), which is a general Lagrangian solution, can be re-written as

$$
\begin{aligned}
z &= b + g_1 + g_2 \\
&= b + a_0 \frac{\sinh k(b+h)}{\sinh kD}\cos\psi + \frac{3}{8}a_0^2 k \frac{\sinh 2k(b+h)}{\sinh^4 kD}\cos 2\psi + \frac{1}{4}a_0^2 k \frac{\sinh 2k(b+h)}{\sinh^2 kD}
\end{aligned}
\tag{25}
$$

The algebraic relationship $b = \hat{\eta} + \varsigma D$ remains unchanged, and thus, the second-order transformation method of Chen and Hsu (2009b) is written as (Fig. 1b)

$$
F_m(a,b,\sigma_L t) = \left(F_m\right)_{\substack{a=x-f_1 \\ b=\hat{\eta}+\varsigma D \\ \sigma_L=\sigma}} \approx \left(F_m - f_1 \frac{\partial F_m}{\partial a}\right)_{\substack{a=x \\ b=\hat{\eta}+\varsigma D \\ \sigma_L=\sigma}}
\tag{26}
$$

By using Eq. (26), Eq. (25) can be converted into a coordinate transformation expression within an Eulerian system:

$$
\begin{aligned}
z &= \hat{\eta} + \varsigma D + a_0 \frac{\sinh kD(1+\varsigma)}{\sinh kD}\cos\varphi + \frac{3}{8}a_0^2 k \frac{\sinh 2kD(1+\varsigma)}{\sinh^4 kD}\cos 2\varphi \\
&\quad + \frac{1}{4}a_0^2 k \frac{\sinh 2kD(1+\varsigma)}{\sinh^2 kD}\cos 2\varphi - \frac{3}{4}a_0^3 k^2 \frac{\cosh kD(1+\varsigma)\sinh 2kD(1+\varsigma)}{\sinh^5 kD}\sin 2\varphi \sin\varphi
\end{aligned}
\tag{27}
$$

and

$$
\begin{aligned}
\tilde{s} &= a_0 \frac{\sinh kD(1+\varsigma)}{\sinh kD}\cos\varphi + \frac{3}{8}a_0^2 k \frac{\sinh 2kD(1+\varsigma)}{\sinh^4 kD}\cos 2\varphi + \frac{1}{4}a_0^2 k \frac{\sinh 2kD(1+\varsigma)}{\sinh^2 kD} \\
&\quad \times \cos 2\varphi - \frac{3}{4}a_0^3 k^2 \frac{\cosh kD(1+\varsigma)\sinh 2kD(1+\varsigma)}{\sinh^5 kD}\sin 2\varphi \sin\varphi
\end{aligned}
\tag{28}
$$

The wave velocity and pressure terms in Eulerian form are taken to be

$$
\begin{aligned}
\tilde{u}_i = \frac{k_i}{k}\Bigg\{ & a_0\sigma \frac{\cosh kD(1+\varsigma)}{\sinh kD}\cos\varphi + \frac{3}{4}a_0^2 \sigma k \frac{\cosh 2kD(1+\varsigma)}{\sinh^4 kD}\cos 2\varphi \\
& - \frac{1}{2}a_0^2 \sigma k \frac{1}{\sinh^2 kD}\cos 2\varphi + \frac{1}{2}a_0^2 \sigma k \frac{\cosh 2kD(1+\varsigma)}{\sinh^2 kD} \\
& - a_0^2 \sigma k \frac{\cosh^2 kD(1+\varsigma)}{\sinh^2 kD}\sin^2\varphi \Bigg\} + O\left(\varepsilon_1\right)^3
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
\tilde{p} = {} & g a_0 \left[\frac{\cosh kD\left(1+\varsigma\right)}{\cosh kD} - \frac{\sinh kD\left(1+\varsigma\right)}{\sinh kD} \right] \cos\varphi - \frac{3}{2} g k a_0^2 \frac{1}{\sinh 2kD} \cos 2\varphi \\
& + g k a_0^2 \left[\frac{\sinh kD\left(1+\varsigma\right)}{\sinh kD} - \frac{\cosh kD\left(1+\varsigma\right)}{\cosh kD} \right] \frac{\cosh kD(1+\varsigma)}{\sinh kD} \sin^2\varphi \\
& + \frac{3}{4} g k a_0^2 \frac{\cosh 2kD(1+\varsigma)}{\sinh^3 kD \cosh kD} \cos 2\varphi - \frac{3}{8} g k a_0^2 \frac{\sinh 2kD(1+\varsigma)}{\sinh^4 kD} \cos 2\varphi \\
& + \frac{1}{2} g k a_0^2 \frac{1}{\sinh 2kD} + \frac{1}{2} g k a_0^2 \frac{\cosh 2kD(1+\varsigma)}{\sinh 2kD} - \frac{1}{4} g k a_0^2 \frac{\sinh 2kD(1+\varsigma)}{\sinh^2 kD} + O\left(\varepsilon_1\right)^3
\end{aligned}
\qquad (30)
$$

where the pressure term also satisfies $\tilde{p} = 0$ at the free surface ($\varsigma = 0$). Inserting Eqs. (28)–(30) into Eq. (11) and neglecting the terms of the order and higher than ε_1^4, we obtain

$$
S_{ij} = kDE\left[\frac{k_i k_j}{k^2} F_{cs} F_{cc} + \delta_{ij}\left(F_{cs} F_{cc} - F_{ss} F_{cs} \right) \right] \qquad (31)
$$

We can determine that Eq. (31) is identical to Eq. (24).

c. Derivation of the vertical radiation stress S_{x3}

The vertical radiation stress S_{x3} in Eq. (12) represents the pressure-induced flux through the sloping iso-surfaces in the vertical plane, and it is considered representative of the fundamental problem concerning the wave-averaged equations for the total momentum. Either utilizing an inappropriate approximation of the vertical radiation stress (Mellor 2003) or omitting the term entirely (Xia et al. 2004; Mellor 2008, 2015) will produce incorrect wave-induced forcing profiles for steady shoaling waves over a sloping bottom. This problem was first noted by Ardhuin et al. (2008a), who suggested that $\tilde{p}$ and $\tilde{s}$ must be estimated to the first order in the bottom slope ε_2, for which the linear wave theory is insufficient. Furthermore, Ardhuin et al. (2008a, 2017) estimated S_{x3} using a numerical method with a series of modes, but it required an enormous amount of computational power (Magne et al. 2007). In this subsection, we focus on this particular issue.

Since the vertical flux of the wave momentum must be estimated to the first order in the bottom slope, we derive the expression using the Lagrangian solutions for waves propagating along a sloping bottom.[3] Inserting Eqs. (5b) and (5e) into Eq. (4b), we obtain

$$\begin{aligned} z &= b + g_{1,0} + \varepsilon_2 g_{1,1} \\ &= b + a_0 \frac{\sinh k(b+h)}{\sinh kD}\cos\psi + \varepsilon_2 a_0 \left\{ \left[\frac{k^2 (b+h)^2}{R\sinh 2kD} - k(b+h) + \frac{1}{R^2 \tanh kD} \right] \right. \\ &\quad \left. \times \frac{\sinh k(b+h)}{\sinh kD} + \left[\frac{k(b+h)}{R^2 \tanh kD} + \frac{2k(b+h)}{R\sinh 2kD} - 1 \right] \frac{\cosh k(b+h)}{\sinh kD} \right\} \sin\psi \end{aligned} \tag{32}$$

Combining Eqs. (4c), (5c) and (5f), the expression of the wave pressure term can be written as

$$\begin{aligned} \tilde{p} &= g a_0 \left[\frac{\cosh k(b+h)}{\cosh kD} - \frac{\sinh k(b+h)}{\sinh kD} \right] \cos\psi + \varepsilon_2 g a_0 \left\{ \left[\frac{k^2 (b+h)^2}{R\sinh 2kD} - k(b+h) \right] \right. \\ &\quad \times \frac{\cosh k(b+h)}{\cosh kD} + \frac{k(b+h)}{R^2 \tanh kD} \frac{\sinh k(b+h)}{\cosh kD} - \left[\frac{k^2 (b+h)^2}{R\sinh 2kD} - k(b+h) \right. \\ &\quad \left. \left. + \frac{1}{R^2 \tanh kD} \right] \frac{\sinh k(b+h)}{\sinh kD} - \left[\frac{k(b+h)}{R^2 \tanh kD} + \frac{2k(b+h)}{R\sinh 2kD} - 1 \right] \frac{\cosh k(b+h)}{\sinh kD} \right\} \sin\psi \end{aligned} \tag{33}$$

Eqs. (32) and (33) can be converted into an Eulerian description using Eq. (26):

$$\begin{aligned} z &= \hat{\eta} + \varsigma D + a_0 \frac{\sinh kD(1+\varsigma)}{\sinh kD}\cos\varphi + \varepsilon_2 a_0 \left\{ \left[\frac{k^2 D^2 (1+\varsigma)^2}{R\sinh 2kD} - kD(1+\varsigma) + \frac{1}{R^2 \tanh kD} \right] \right. \\ &\quad \times \frac{\sinh kD(1+\varsigma)}{\sinh kD} + \left[\frac{kD(1+\varsigma)}{R^2 \tanh kD} + \frac{2kD(1+\varsigma)}{R\sinh 2kD} - 1 \right] \frac{\cosh kD(1+\varsigma)}{\sinh kD} \\ &\quad \left. - \varsigma \frac{\cosh kD(1+\varsigma)}{\sinh kD} \right\} \sin\varphi + a_0 \left[(S_1 + S_2)\sin\varphi + (C_1 + C_2)\cos\varphi \right] \sin\varphi \end{aligned} \tag{34}$$

---

[3] Upon using the Lagrangian solutions for waves propagating along a sloping bottom to derive the horizontal radiation stress S_{xx} in Eq. (11), we find that the final expression is identical to Eq. (22) because the bottom slope ε_2 is included in the $O(\varepsilon_2)^2$ terms, which are neglected. Thus, the wave solutions at a uniform depth are sufficient to estimate the horizontal radiation stress. This conclusion is consistent with Ardhuin et al. (2008a, 2017).

$$
\begin{aligned}
\tilde{p} = {} & g a_0 \left[\frac{\cosh kD\left(1+\varsigma\right)}{\cosh kD} - \frac{\sinh kD\left(1+\varsigma\right)}{\sinh kD} \right] \cos\varphi + \varepsilon_2 g a_0 \left\{ \left[\frac{k^2 D^2 \left(1+\varsigma\right)^2}{R \sinh 2kD} - kD\left(1+\varsigma\right) \right] \right. \\
& \times \frac{\cosh kD\left(1+\varsigma\right)}{\cosh kD} + \frac{kD\left(1+\varsigma\right)}{R^2 \tanh kD} \frac{\sinh kD\left(1+\varsigma\right)}{\cosh kD} - \left[\frac{k^2 D^2 \left(1+\varsigma\right)^2}{R \sinh 2kD} - kD\left(1+\varsigma\right) \right. \\
& \left. + \frac{1}{R^2 \tanh kD} \right] \frac{\sinh kD\left(1+\varsigma\right)}{\sinh kD} - \left[\frac{kD\left(1+\varsigma\right)}{R^2 \tanh kD} + \frac{2kD\left(1+\varsigma\right)}{R \sinh 2kD} - 1 \right] \frac{\cosh kD\left(1+\varsigma\right)}{\sinh kD} \\
& \left. + \varsigma \frac{\cosh kD\left(1+\varsigma\right)}{\sinh kD} \right\} \sin\varphi + g a_0 \left[\left(S_3 + S_4\right) \sin\varphi + \left(C_3 + C_4\right) \cos\varphi \right] \sin\varphi
\end{aligned}
\tag{35}
$$

where S_1, S_3, C_1, and C_3 are terms of the order ε_1; S_2, S_4, C_2, and C_4 are terms of the order $\varepsilon_1 \varepsilon_2$; and the bottom slope $\varepsilon_2 = -\partial D / \partial x$. Combining Eqs. (8) and (34) yields the following:

$$
\begin{aligned}
\tilde{s} = {} & a_0 \frac{\sinh kD(1+\varsigma)}{\sinh kD} \cos\varphi + \varepsilon_2 a_0 \left\{ \left[\frac{k^2 D^2 \left(1+\varsigma\right)^2}{R \sinh 2kD} - kD\left(1+\varsigma\right) + \frac{1}{R^2 \tanh kD} \right] \right. \\
& \left. \times \frac{\sinh kD\left(1+\varsigma\right)}{\sinh kD} + \left[\frac{kD\left(1+\varsigma\right)}{R^2 \tanh kD} + \frac{2kD\left(1+\varsigma\right)}{R \sinh 2kD} - 1 \right] \frac{\cosh kD\left(1+\varsigma\right)}{\sinh kD} \right\} \sin\varphi \\
& - \varepsilon_2 a_0 \varsigma \frac{\cosh kD\left(1+\varsigma\right)}{\sinh kD} \sin\varphi + a_0 \left[\left(S_1 + S_2\right) \sin\varphi + \left(C_1 + C_2\right) \cos\varphi \right] \sin\varphi
\end{aligned}
\tag{36}
$$

Eq. (36) corresponds to the definition of $\tilde{s}$ in Ardhuin et al. (2008a) [their Eq. (11)]. The first and second lines correspond to ξ_3 in their same equation. The first term in the third line corresponds to $-\xi_1 \partial \overline{s} / \partial x$, while the other terms in the third line are of a higher order. Inserting Eqs. (35) and (36) into Eq. (12), we obtain

$$
S_{x3} = \overline{-\tilde{p} \frac{\partial \tilde{s}}{\partial x}} = -\left[\frac{1}{2} g a_0 M_2 \frac{\partial a_0 M_1}{\partial x} + \frac{1}{2} \varepsilon_2 k g a_0{}^2 \left(M_2 N_1 - M_1 N_2 \right) \right] \tag{37}
$$

where the convenient definitions are as follows:

$$
M_1 = \frac{\sinh kD\left(1+\varsigma\right)}{\sinh kD} \tag{38a}
$$

$$N_1 = \left[\frac{k^2 D^2 (1+\varsigma)^2}{R \sinh 2kD} - kD(1+\varsigma) + \frac{1}{R^2 \tanh kD} \right] \frac{\sinh kD(1+\varsigma)}{\sinh kD} + \left[\frac{kD(1+\varsigma)}{R^2 \tanh kD} + \frac{2kD(1+\varsigma)}{R \sinh 2kD} - 1 \right] \frac{\cosh kD(1+\varsigma)}{\sinh kD} - \varsigma \frac{\cosh kD(1+\varsigma)}{\sinh kD} \tag{38b}$$

$$M_2 = \frac{\cosh kD(1+\varsigma)}{\cosh kD} - \frac{\sinh kD(1+\varsigma)}{\sinh kD} \tag{38c}$$

$$\begin{aligned} N_2 = & \left[\frac{k^2 D^2 (1+\varsigma)^2}{R \sinh 2kD} - kD(1+\varsigma) \right] \frac{\cosh kD(1+\varsigma)}{\cosh kD} \\ & + \frac{kD(1+\varsigma)}{R^2 \tanh kD} \frac{\sinh kD(1+\varsigma)}{\cosh kD} - \left[\frac{k^2 D^2 (1+\varsigma)^2}{R \sinh 2kD} - kD(1+\varsigma) + \frac{1}{R^2 \tanh kD} \right] \\ & \times \frac{\sinh kD(1+\varsigma)}{\sinh kD} - \left[\frac{kD(1+\varsigma)}{R^2 \tanh kD} + \frac{2kD(1+\varsigma)}{R \sinh 2kD} - 1 \right] \frac{\cosh kD(1+\varsigma)}{\sinh kD} \\ & + \varsigma \frac{\cosh kD(1+\varsigma)}{\sinh kD} \end{aligned} \tag{38d}$$

$$R = 1 + \frac{2kD}{\sinh 2kD} \tag{38e}$$

The terms of the orders $\varepsilon_1^4 \varepsilon_2^q$ and $\varepsilon_1^r \varepsilon_2^2$, in which q and r are arbitrary natural numbers, are neglected. The first term of Eq. (37) is from the expression of Mellor (2003). The vertical radiation stress term S_{x3} has no flux across the surface or the bottom, and thus, it vertically integrates to zero; furthermore, the vertically integrated equations including this term conform to the conventional integrated equations of LHS and Smith (2006). Eq. (37) is easily extended into the 3DRS formulation:

$$S_{x_i 3} = -\left[\frac{1}{2} g a_0 M_2 \frac{\partial a_0 M_1}{\partial x_i} - \frac{1}{2} k g {a_0}^2 \frac{\partial D}{\partial x_i} \left(M_2 N_1 - M_1 N_2 \right) \right] \tag{39}$$

d. Doppler effect

The derivation above does not consider the Doppler effect of the current on a wave. The Doppler effect can be included for a given Doppler velocity $\hat{u}_A$, which is a weighted average of the vertical current according to Kirby and Chen (1989) and Mellor (2003):

$$\hat{u}_A = 2\int_{-1}^{0} \hat{u}\,\frac{kD\cosh 2kD(1+\varsigma)}{\sinh 2kD}\,d\varsigma \tag{40}$$

The Lagrangian solutions for waves propagating within a uniform current provided by Chen et al. (2013) and Chen and Chen (2014) can be written as Eqs. (1b)–(1d) with

$$x = a + \hat{u}_A t + \sum_{n=1}^{\infty} \varepsilon^n \left[f_n(a,b,\sigma_L t) + f_n'(a,b,\sigma_0 t) \right] \tag{41}$$

The modified first-order approximate transformation from a Lagrangian to an Eulerian system [Eq. (17)] becomes

$$F_m(a,b,\sigma_L t) = (F_m)_{\substack{a=x-\hat{u}_A t \\ b=\hat{\eta}+\varsigma D \\ \sigma_L=\sigma}} \tag{42}$$

and the phase function $\psi = ka - \sigma_L t$ in a Lagrangian system can be converted into an Eulerian description with the following formulation that is consistent with Mellor (2003, 2015):

$$\varphi = kx - k\hat{u}_A t - \sigma t = kx - \omega t \tag{43}$$

The wave frequency can be expressed as $\omega = \sigma + k\hat{u}_A$, where σ is the intrinsic frequency. It is easily proven that Eq. (43) is unchanged for higher-order Lagrange-Euler transformations.

Therefore, the phase function included in $\tilde{s}$ and in the wave variables (e.g., the velocity and pressure terms) in an Eulerian form can be expressed by Eq. (43) when considering the Doppler effect. However, the final forms of the 3DRS terms based on Eqs. (42) and (43) are identical to those derived within the present study because they are phase-averaged variables.

4 Test case proposed by Ardhuin et al. (2008a)

To evaluate the performances of the expressions of the wave-induced 3DRS developed by Mellor (2003), Ardhuin et al. (2008a) proposed a test case involving steady monochromatic waves shoaling over a sloping bottom without energy dissipation. In such a test case, the lowest-order momentum balance equation can be expressed as

$$\overbrace{-gD\frac{\partial\hat{\eta}}{\partial x}}^{F_\eta}-\overbrace{\frac{\partial S_{xx}}{\partial x}}^{F_{xx}}-\overbrace{\frac{\partial S_{x3}}{\partial\varsigma}}^{F_{x3}}=0 \tag{44}$$

In the equation above, the hydrostatic pressure gradient F_η is balanced with the sum of F_{xx} and F_{x3}. The present paper follows Ardhuin et al. (2008a) and uses this test case to evaluate the performances of the analytical expressions derived herein.

The bottom profile given by Roseau (1976) is used here (Fig. 2):

$$Z(x')=\frac{h_1(x'-i\alpha)+(h_2-h_1)\ln\left(1+e^{x'-i\alpha}\right)}{\alpha} \tag{45}$$

Following Ardhuin et al. (2008a), we use the following: $h_1=6$ m, $h_2=4$ m, offshore wave amplitude $a_{0,0}=0.12$ m, wave frequency $f=0.2$ Hz, and $\alpha=15\pi/180$. The change in the wave amplitude a_0 is calculated using the conservation of wave energy, and the phase function is regarded as the integral over x of the local wave number ($\partial\varphi/\partial x=k$). The maximum surface wave slope $\varepsilon_1=ka_{0,\max}=2.6\times10^{-2}$, which is equal to the maximum bottom slope ε_2 in this case. The length of the wave channel is 250 m, which varies by 4–6 m based on Eq. (45). The three terms in Eq. (44) are estimated using a second-order finite difference.

The modeled results from this paper (Fig. 2a) show that the instantaneous wave-induced pressure field varies between negative values below the wave troughs and positive values below the crests. The magnitude of the wave pressure increases from zero at the surface to a maximum at the bottom. The results of Mellor (2015) show spatial features that are similar to those of our model except at the surface, wherein the pressure varies around zero with negligible magnitudes (Fig. 2b). To illustrate the terms in Eq. (44) more clearly, we normalize the terms by a factor of $gD\varepsilon_1^2\varepsilon_2$, the results of which are shown in Fig. 3. As we can see, the hydrostatic pressure gradient F_η and the horizontal divergence of the horizontal flux of the wave momentum F_{xx} from this paper are identical to those of Mellor (2003). However, the value of F_{x3} significantly differs from that estimated using the theory from Mellor (2003). The analytical expression of F_{x3} in the present paper reveals negative values in the surface layer and positive values in the bottom layer (Fig. 3c), which play an important role in the momentum balance. The sum of the three terms from the present model is zero overall (Fig. 3d), which is plainly different from the results of Mellor (2003), wherein an imbalance can be observed both in the surface layer and in the bottom layer (Fig. 3h).

The vertical profiles of the net wave-induced forces from Eq. (44) that are based on the modeled results of our expressions and those from Mellor (2003) and Mellor (2015) are plotted in Fig. 4. The formulations in this study provide a net momentum balance close to zero from the surface to the bottom. In contrast, a significant imbalance can be observed in the results of Mellor (2003) and Mellor (2015) that comprise negative values in the bottom layer and positive values in the surface layer. The mean absolute error (MAE) of our result is 1.2×10^{-4}, which is only 0.025% and 0.021% of the MAEs of Mellor (2003) and Mellor (2015), respectively. This suggests that our formulations demonstrate better performances than those from the investigations of Mellor. We can easily determine that our results are also better than numerical results using the model of Athanassoulis and Belibassakis (1999), which is based on a local-mode series expansion (not shown, see Fig. 4 in Ardhuin et al. 2008a).

5 Conclusions and recommendations

A new approach has been proposed to derive the expressions for the wave-induced 3DRS using wave solutions transformed from a Lagrangian description where the pressure is zero (relative to the atmospheric pressure) at the sea surface. Using this approach, the horizontal depth-dependent radiation stress is first derived based on both first- and second-order Lagrangian approximate solutions at a uniform water depth. The analytical expression of the vertical radiation stress is then derived using the Lagrangian solutions of waves propagating along a sloping bottom. The wave-induced change in the vertical coordinate $\tilde{s}$ is obtained using a modified second-order Lagrange-Euler transformation method, and it corresponds well to the definition of Ardhuin et al. (2008a). The vertical integration of the derived results agrees with the earlier 2D expression proposed by LHS. During a basic test of shoaling waves over a sloping bottom without energy dissipation, the present formulations generate a net momentum balance close to zero from the surface to the bottom. Specifically, the MAE of the net wave-induced forces in this study is 1.2×10^{-4}, which is only 0.025% and 0.021% of the MAEs of Mellor (2003) and Mellor (2015), respectively.

With the abovementioned advantages, the present study provides improved 3DRS expressions. Nevertheless, several limitations and approximations remain (e.g., $\left|(\partial a_0/\partial t)/(\sigma a_0)\right|$ and $\left|(\partial \hat{u}/\partial z)/\sigma\right| \ll 1$). This paper uses wave solutions in a Lagrangian description provided by Chen et al. (2010, 2012, 2013); however, these solutions do not completely include the influences of wave-current interactions (e.g., the effects of vertical current shear). Further work must consider the additional effects of currents on waves.

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Figure Captions

FIG. 1. Illustration of the Lagrange-Euler transformation method. The first- and second-order Lagrangian solutions (for a wave amplitude of 3 m and a wavelength of 100 m at a water depth of 30 m) are converted into an Eulerian description using (a) the modified first-order transformation and (b) the modified second-order transformation method. The red bars are the connections between the fixed particles $\left(a,b\right)$ in the still water and the displaced particles $\left(x,z\right)$, which are situated along the blue lines that connect the positions of the particles with the same b values at a fixed point in time. The black arrows are several transformation examples from $\left(a,b\right)$ to $\left(x,z\right)$.

FIG. 2. Instantaneous wave-induced pressure fields in (a) the present study and (b) Mellor (2015) for the shoaling waves over the sloping bottom profile (thick black lines) given by Eq. (45).

FIG. 3. (a)–(c) The forces in Eq. (44) normalized by $gD\varepsilon_1^2\varepsilon_2$, (d) their sum in the present study, and (e)–(h) the respective terms estimated from the theory of Mellor (2003).

FIG. 4. Vertical profiles of the net forces for the shoaling waves over a sloping bottom. The net wave-induced forces are integrated over x and normalized by a similar integration of F_η.

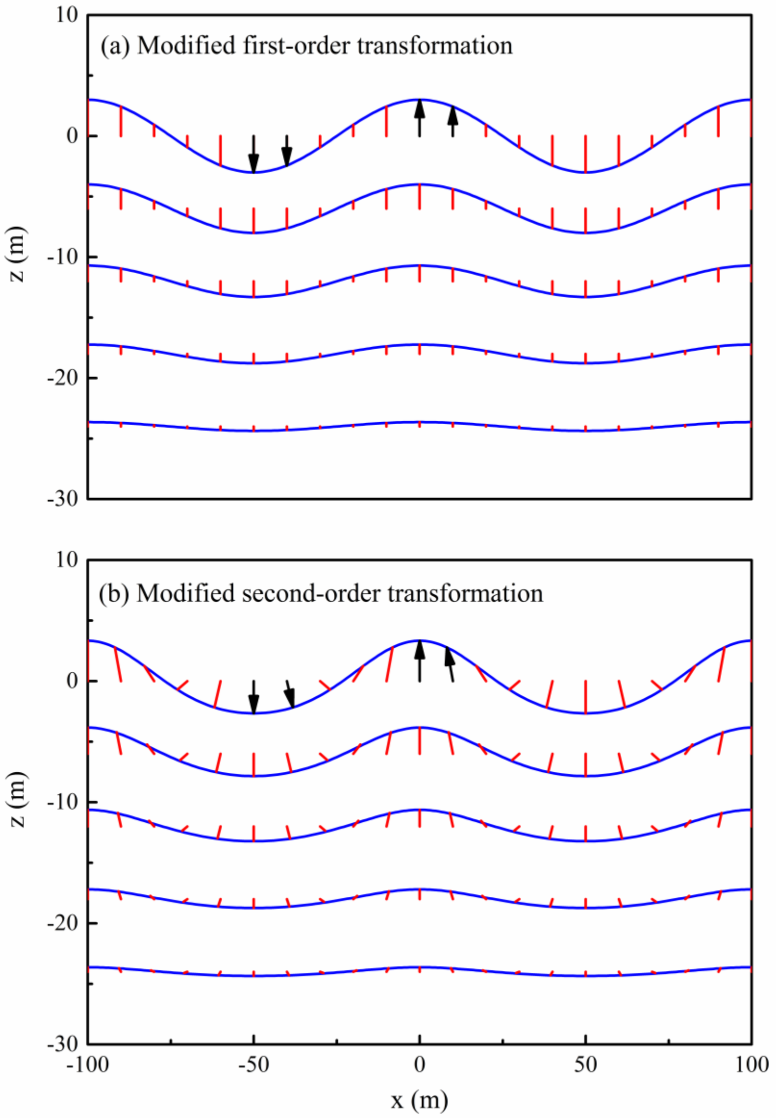

FIG. 1. Illustration of the Lagrange-Euler transformation method. The first- and second-order Lagrangian solutions (for a wave amplitude of 3 m and a wavelength of 100 m at a water depth of 30 m) are converted into an Eulerian description using (a) the modified first-order transformation and (b) the modified second-order transformation method. The red bars are the connections between the fixed particles (a,b) in the still water and the displaced particles (x,z), which are situated along the blue lines that connect the positions of the particles with the same b values at a fixed point in time. The black arrows are several transformation examples from (a,b) to (x,z).

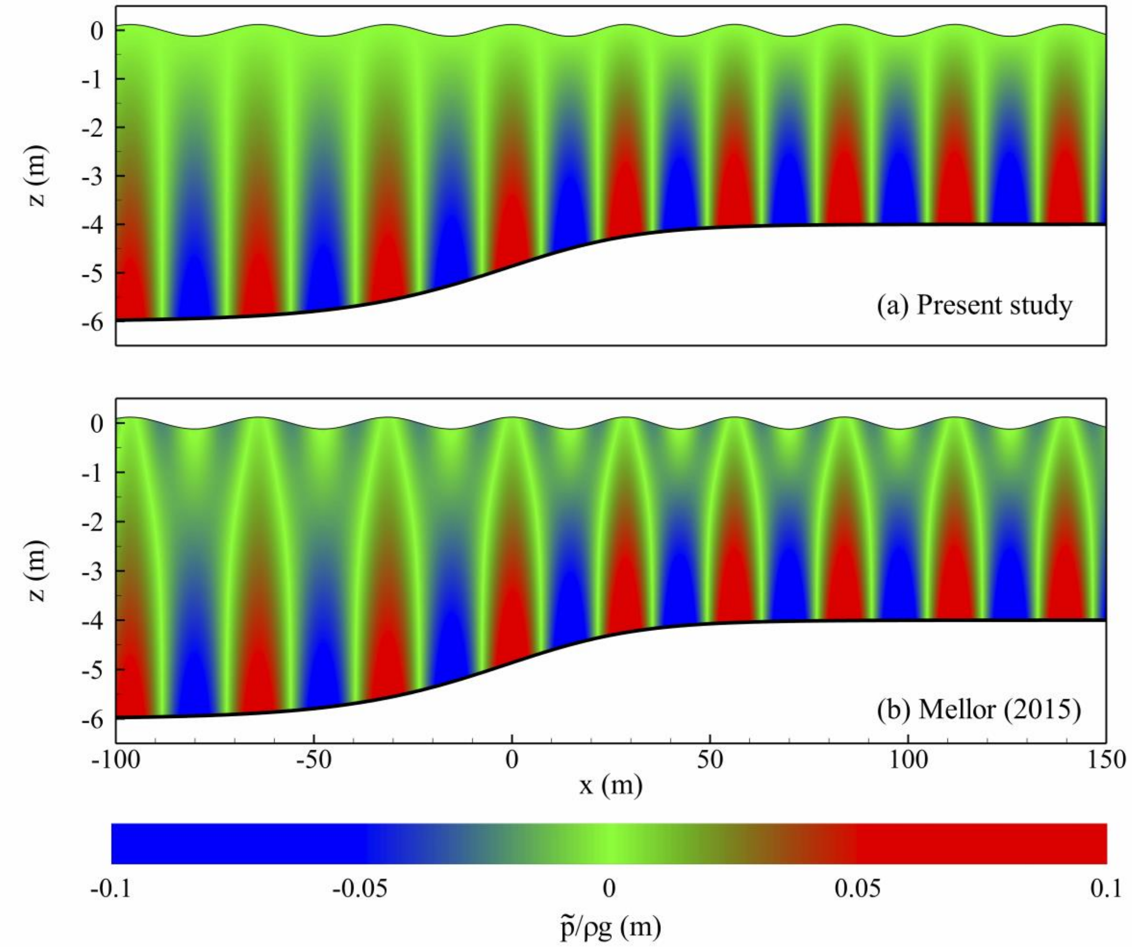

FIG. 2. Instantaneous wave-induced pressure fields in (a) the present study and (b) Mellor (2015) for the shoaling waves over the sloping bottom profile (thick black lines) given by Eq. (45).

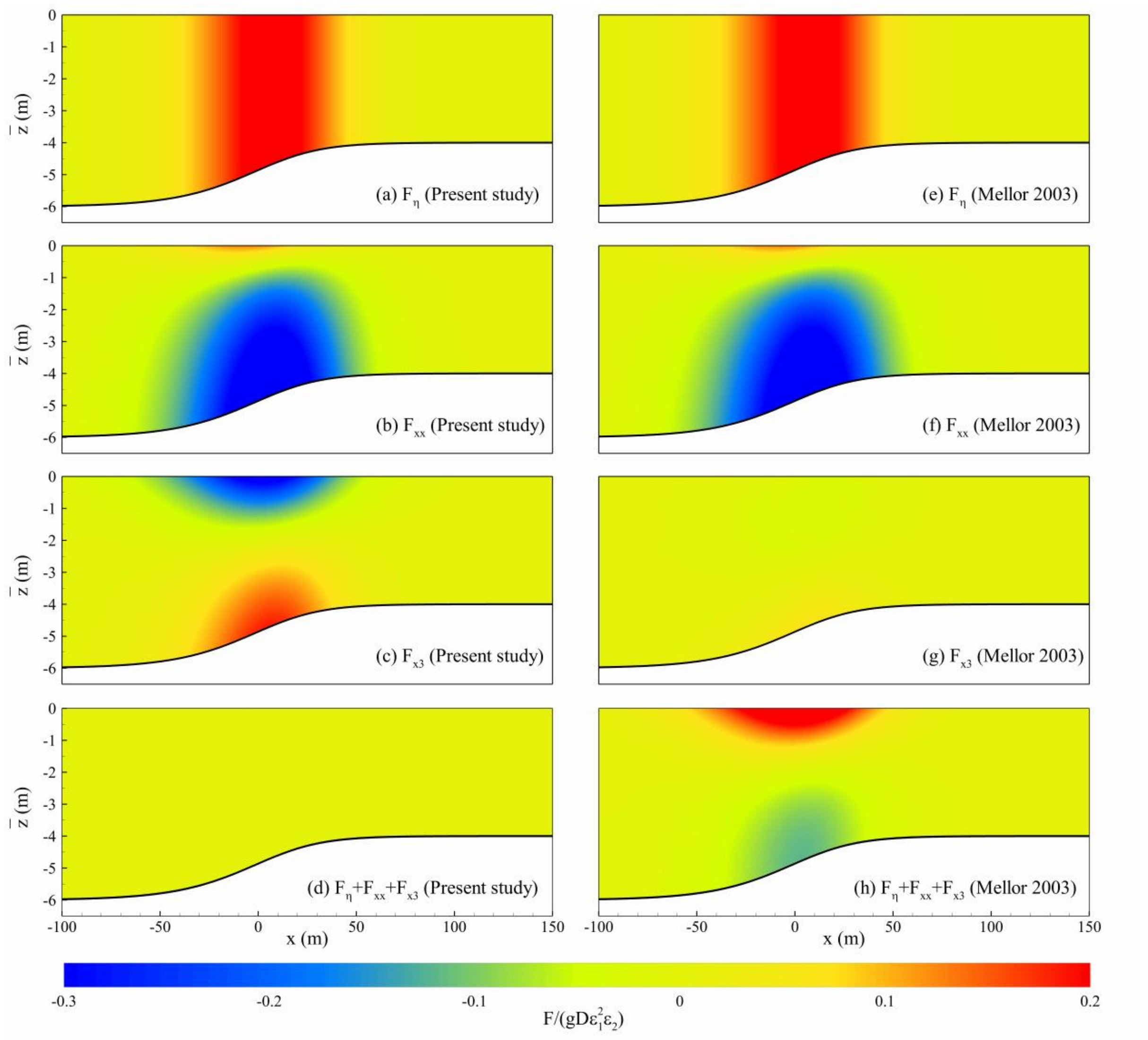

FIG. 3. (a)–(c) The forces in Eq. (44) normalized by $gD\varepsilon_1^2\varepsilon_2$, (d) their sum in the present study, and (e)–(h) the respective terms estimated from the theory of Mellor (2003).

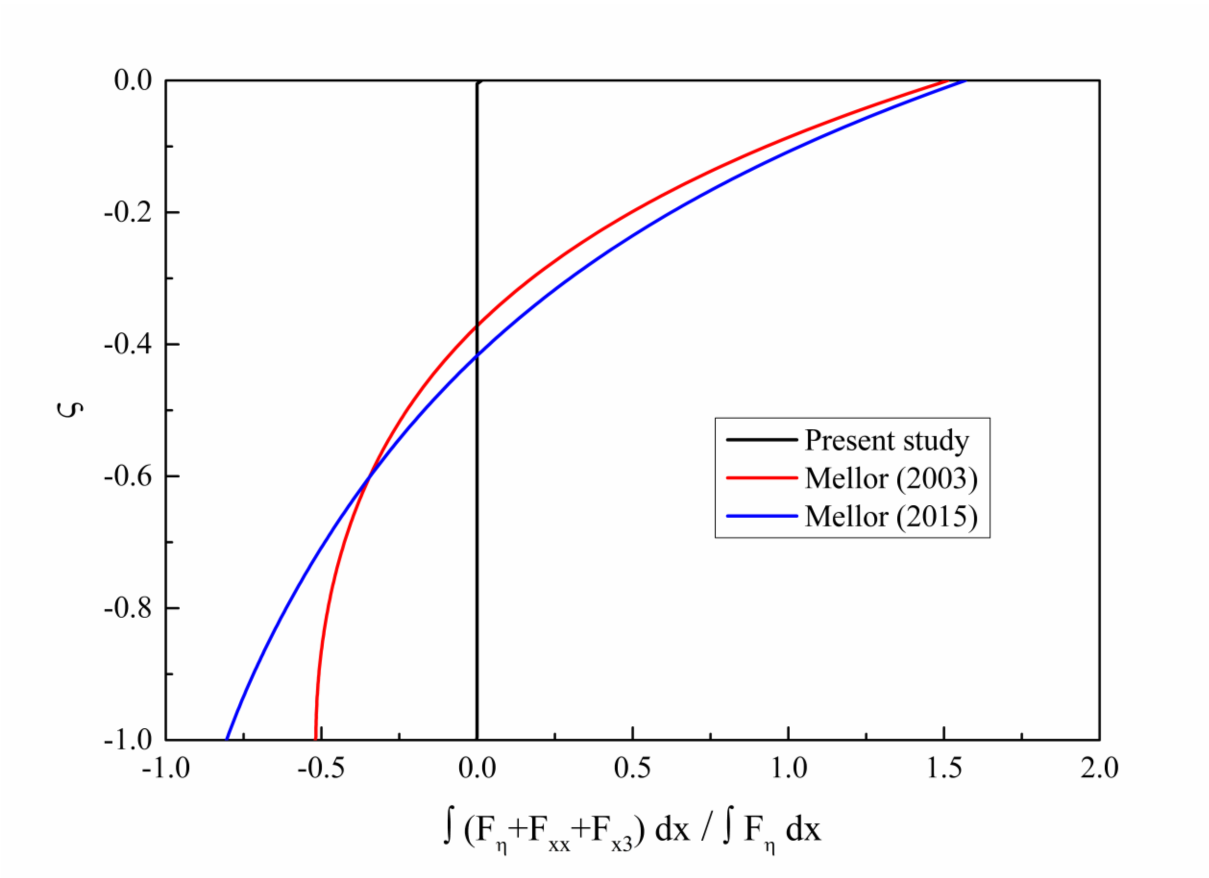

FIG. 4. Vertical profiles of the net forces for the shoaling waves over a sloping bottom. The net wave-induced forces are integrated over x and normalized by a similar integration of F_η.