J. Fluid Mech. (2005), *vol.* 545, *pp.* 29–40.
doi:10.1017/S0022112005006841 Printed in the United Kingdom

Conditional second-order short-crested water waves applied to extreme wave episodes

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(Received 1 December 2004 and in revised form 4 May 2005)

A derivation of the mean second-order short-crested wave pattern and associated wave kinematics, conditional on a given magnitude of the wave crest, is presented. The analysis is based on the second-order Sharma and Dean finite-water wave theory. A comparison with a measured extreme wave profile, the Draupner New Year Wave, shows a good agreement in the mean, indicating that this second-order wave can be a good identifier of the shape and occurrence of extreme wave events. A discussion on its use as an initial condition for a fully nonlinear three-dimensional surface wave analysis is given.

1. Introduction

In the past few years, a significant effort has been made to understand and model extreme ocean waves. The research has in part been initiated by the measurement on 1 January 1995 of an extreme wave in the location of the Draupner platform in the North Sea, Haver & Andersen (2000). Two main questions concern the physics and the statistical properties of such extreme waves. Obviously, the answers can have a significant influence on the safety of ships and offshore structures and may even explain some severe accidents and disasters encountered at sea, e.g. Faulkner (2000). Currently, nonlinear self-modulation is believed to be a governing phenomenon for the occurrence of such extreme waves, and models based on the nonlinear Schrödinger equation have been widely used, e.g. Osborne, Onorato & Serio (2000), Dysthe *et al.* (2003), Janssen (2003) and Krogstad *et al.* (2004). The governing equations are solved by numerical simulation and thus depend on the initial conditions applied. Encouraging results have been obtained for the understanding of the conditions under which these extreme waves may occur. However, to obtain statistical estimates for the extreme wave, the effect of the initial conditions as well as of the form of the wave spectral density needs to be studied in greater detail. If the initial conditions are chosen with a prescribed probability of occurrence then this probability can be assigned to the resulting extreme waves provided there is a uniqueness of the numerical simulations. This approach has been adopted by Bateman, Swan & Taylor (2001) using a conditional mean linear wave profile (Tromans, Anaturk & Hagemeijer 1991), and the associated kinematics as input. The linear solution is worked backwards in time to become the initial condition for the fully nonlinear computations. This approach has the clear advantage that the fully nonlinear computations can be limited to a small set of short-duration (100–200 s) simulations each with a predefined probability of occurrence. It was shown that directional spreading was very important to nonlinear wave–wave interaction and must be included in a model for extreme waves. It was also found that the linear input provided a good indicator for the magnitude and position in time and space of the fully nonlinear crest. They used a fully nonlinear spectral wave model, but other fully three-dimensional wave models, for instance based on the Boussinesq theory, Madsen, Bingham & Schaffer (2003), can equally well be applied to account for the possible strong energy transfer between wavenumbers in the vicinity of an extreme wave.

The aim of the present paper is to present a complete conditional-mean second-order short-crested wave theory for finite water. It is hoped that inclusion of second-order terms in the initial conditions might trigger nonlinear phenomena in the fully nonlinear simulations better than a linear input and thus give more consistent results. Some corrections for second-order effects have already been incorporated by Bateman *et al.* (2001), based on the Creamer transformation (Taylor 1992) for deep water waves. In the present paper conditional-mean wave profiles and associated wave kinematics as functions of space and time relative to the position of the given wave crest are obtained in closed form and reflect the spectral content of the waves. The accuracy of the second-order model is illustrated by a comparison with the extreme wave measured at the Draupner platform on 1 January 1995. The second-order conditional-mean wave taking into account wave spreading as well as finite water depth yields approximately the same crest to trough profile as the measured wave, supporting the assumption that the second order model can be a good identifier for a fully nonlinear extreme wave.

The paper starts with a derivation of nonlinear conditional processes to any order followed by the derivation of second-order short-crested wave models for finite water depth. Finally numerical results related to the Draupner New Year Wave are presented.

2. Nonlinear conditional processes

Consider two correlated normalized stationary stochastic processes $U(x)$, $V(x)$. Provided both processes are slightly non-Gaussian, their properties may conveniently be expressed as Gram–Charlier series by means of the cumulant generating function. This is so because the higher-order cumulants vanish in the limit of Gaussian processes.

The mean value of $V(x)$ conditional on the value $U(0)$ and the slope $U'(0)$ of $U(x)$ at $x=0$ is given by

$$E[V(x)|U(0)=u,\, U'(0)=\dot{u}] = \frac{1}{p(u,\dot{u})}\int_{-\infty}^{\infty} v p(u,\dot{u},v)\mathrm{d}v \tag{2.1}$$

where p denotes the joint probability density of the arguments. From a Gram–Charlier series representation of p, it follows, see Jensen (1996), that p can be written as a series involving terms of the type

$$A_{abc} = \frac{1}{(2\pi)^{3/2}(1-\rho^2-\dot{\rho}^2)^{1/2}} H_{abc} J \tag{2.2}$$

where ρ, $\dot{\rho}$ are the correlation coefficients between $U(0)$, $V(x)$ and $U'(0)$, $V(x)$, respectively. The generalized Hermite polynomials H_{abc} are defined by

$$H_{abc} J \equiv (-1)^{a+b+c} \frac{\partial^a}{\partial u^a}\frac{\partial^b}{\partial \dot{u}^b}\frac{\partial^c}{\partial v^c} J \tag{2.3}$$

with

$$J = J(u,\dot{u},v) = \exp\left[-\frac{u^2(1-\dot{\rho}^2)+\dot{u}^2(1-\rho^2)+v^2-2\rho u v-2\dot{\rho}\dot{u}v+2\rho\dot{\rho}u\dot{u}}{2(1-\rho^2-\dot{\rho}^2)}\right]. \tag{2.4}$$

The integration in (2.1) is then easily performed as terms of the type, see Jensen (1996),

$$M_{abc}(u,\dot{u}) \equiv 2\pi \exp\left(\tfrac{1}{2}(u^2+\dot{u}^2)\right) \int_{-\infty}^{\infty} v\Lambda_{abc}\,\mathrm{d}v \tag{2.5}$$

become

$$\left.\begin{aligned} M_{ab0} &= \rho He_{a+1}(u)He_b(\dot{u}) + \dot{\rho} He_a(u)He_{b+1}(\dot{u}),\\ M_{ab1} &= He_a(u)He_b(\dot{u}),\\ M_{abc} &= 0 \quad \text{if } c \geqslant 2, \end{aligned}\right\} \tag{2.6}$$

where He_i are the usual Hermite polynomials ($He_1(u)=u$, $He_2(u)=u^2-1$, $He_3(u)=u^3-3u$, etc.). Thus, the result can be written

$$\left.\begin{aligned} &E[V(x)|U(0)=u,\,U'(0)=\dot{u}] = \frac{\rho u + \dot{\rho}\dot{u} + \sum\sum_{a+b\ \geqslant 3} [\Lambda_{ab0}H_0 + \Lambda_{ab1}H_1]}{1+\sum\sum_{a+b\ \geqslant 3} \Lambda_{ab0}H_1},\\ &H_0 \equiv \rho He_{a+1}(u)He_b(\dot{u}) + \dot{\rho}He_a(u)He_{b+1}(\dot{u}),\\ &H_1 \equiv He_a(u)He_b(\dot{u}). \end{aligned}\right\} \tag{2.7}$$

The coefficients Λ_{abc} are functions of the joint cumulants λ_{ijk} of u, $\dot{u}$, v:

$$\begin{aligned} \Lambda_{abc} = K_{abc} &+ \frac{1}{2!}\sum\sum\sum_{i+j+k\geqslant 3} K_{ijk}K_{a-i,b-j,c-k}\\ &+\frac{1}{3!}\sum\sum\sum_{i+j+k\geqslant 3}\sum\sum\sum_{l+m+n\geqslant 3} K_{ijk}K_{lmn}K_{a-i-l,b-j-m,c-k-n} + \ldots \end{aligned} \tag{2.8}$$

where

$$\left.\begin{aligned} K_{abc} &= \frac{\lambda_{abc}}{a!b!c!},\\ K_{abc} &= 0 \quad \text{if}\, a+b+c<3;\ a,b,c\geqslant 0. \end{aligned}\right\} \tag{2.9}$$

For slightly non-Gaussian processes, (2.7) can be approximated as

$$\begin{aligned} E[V(x)|U(0)=u,\,U'(0)=\dot{u}] \simeq \rho u + \dot{\rho}\dot{u} + &\sum\sum_{a+b\geqslant 3} [\Lambda_{ab1}He_a(u)He_b(\dot{u})\\ &- \Lambda_{ab0}(a\rho He_{a-1}(u)He_b(\dot{u}) + b\dot{\rho}He_a(u)He_{b-1}(\dot{u}))]. \end{aligned} \tag{2.10}$$

To lowest order ($a+b+c=3$):

$$\begin{aligned} E[V(x)|U(0)=u,\,U'(0)=\dot{u}] &= \rho u + \dot{\rho}\dot{u} + \tfrac{1}{2}(u^2-1)(\lambda_{201}-\rho\lambda_{300})\\ &+ \tfrac{1}{2}(\dot{u}^2-1)(\lambda_{021}-\rho\lambda_{120}-\dot{\rho}\lambda_{030}) + u\dot{u}(\lambda_{111}-\dot{\rho}\lambda_{120}) \end{aligned} \tag{2.11}$$

as $\lambda_{210}=0$. The second-order result, (2.11), is derived in Jensen (1996), together with applications to ocean wave kinematics. The linear result

$$E[V(x)|U(0)=u,\,U'(0)=\dot{u}] = \rho u + \dot{\rho}\dot{u} \tag{2.12}$$

is identical to the result in Tromans *et al.* (1991). The general result, (2.7), is believed to be new.

3. Short-crested second-order waves

In the following the theory is applied to short-crested finite water waves in water depth h. The second-order wave theory derived by Sharma & Dean (1979) & Dalzell (1999) is applied. The mean wave profile is considered conditional on a peak (crest) at $x \equiv s = 0$, $t = 0$, i.e. $\dot{u} = 0$. The derivative can be taken with respect to either space or time. Here the wave profile as a function of space for a given instant of time is of most concern and thus $()' = \partial/\partial s$. The analogous formulas using $()' = \partial/\partial t$ become very similar and should be applied when the variation in time is needed at different space locations, e.g. for buoy measurements.

With M wave components in each of N wave directions, the linear wave profile $H^{(1)}(s, \psi, t)$ in the direction ψ relative to the wind direction can be expressed as

$$H^{(1)}(s, \psi, t) = \sum_{i}^{M} \sum_{j}^{N} a_{ij} \cos(\psi_{ij}) \tag{3.1}$$

in a polar coordinate system (s, ψ). Here

$$\psi_{ij} = k_i s \cos(\psi - \varphi_j) - \omega_i t + \theta_{ij} = \phi_{ij} + \theta_{ij}. \tag{3.2}$$

Furthermore, a_{ij} is the wave amplitude for the long-crested wave component ij with wavenumber k_i and wave frequency ω_i travelling in the direction ϕ_j relative to the wind direction and θ_{ij} is the random phase lag for this wave. The time t included in (3.2) is retained in order to be able to work the solution backwards in time for application as initial condition in a fully nonlinear three-dimensional analysis.

Within the second-order wave theory by Sharma & Dean (1979), the second order part of the wave elevation can be written as follows (Madsen 1987):

$$H^{(2)}(s, \psi, t) = \sum_{i}^{M} \sum_{j}^{N} a_{ij} \sum_{m}^{M} \sum_{n}^{N} a_{mn} [h^{+}_{ijmn} \cos(\psi_{ij} + \psi_{mn}) + h^{-}_{ijmn} \cos(\psi_{ij} - \psi_{mn})] \tag{3.3}$$

where

$$\left.\begin{aligned} h^{\pm}_{ijmn} &= \frac{1}{4}\left[\frac{D^{\pm}_{ijmn} - \boldsymbol{k}_{ij} \cdot \boldsymbol{k}_{mn} \pm R_i R_m}{\sqrt{R_i R_m}} + R_i + R_m\right], \\ D^{\pm}_{ijmn} &= \frac{(\sqrt{R_i} \pm \sqrt{R_m})\left\{\sqrt{R_m}\left(k_i^2 - R_i^2\right) \pm \sqrt{R_i}\left(k_m^2 - R_m^2\right)\right\}}{(\sqrt{R_i} \pm \sqrt{R_m})^2 - k^{\pm}_{ijmn} \tanh(k^{\pm}_{ijmn} h)} \\ &\quad + \frac{2(\sqrt{R_i} \pm \sqrt{R_m})^2 (\boldsymbol{k}_{ij} \cdot \boldsymbol{k}_{mn} \mp R_i R_m)}{(\sqrt{R_i} \pm \sqrt{R_m})^2 - k^{\pm}_{ijmn} \tanh(k^{\pm}_{ijmn} h)}, \\ \boldsymbol{k}_{ij} \cdot \boldsymbol{k}_{mn} &= k_i k_m \cos(\varphi_j - \varphi_n), \\ k^{\pm}_{ijmn} &= |\boldsymbol{k}^{\pm}_{ijmn}| = |\boldsymbol{k}_{ij} \pm \boldsymbol{k}_{mn}| = \sqrt{k_i^2 + k_m^2 \pm 2 k_i k_m \cos(\varphi_j - \varphi_n)}, \\ R_i &= \frac{\omega_i^2}{g} = k_i \tanh(k_i h). \end{aligned}\right\} \tag{3.4}$$

In a stationary stochastic seaway the first-order wave phase angles θ_{ij} can be taken to be uniformly distributed. Thus, the stochastic variables ξ_{ij}

$$\xi_{ij} = a_{ij} \cos\theta_{ij}, \quad \xi_{i+M,j} = a_{ij} \sin\theta_{ij} \tag{3.5}$$

become pairwise jointly normally distributed with zero mean values and variances V_{ij} related to the directional wavenumber spectral density $S(k,\varphi)$ through

$$V_{ij} = S(k_i, \varphi_j)\Delta k_i \Delta\varphi_j. \tag{3.6}$$

The variance of the linear part of the waves is

$$\sigma_h^2 = \sum_i^M \sum_j^N S(k_i, \varphi_j)\Delta k_i \Delta\varphi_j. \tag{3.7}$$

Introducing the variables ξ_{ij} into (3.1)–(3.3) the second order stochastic wave profile can be expressed as

$$H(s,\psi,t) = \sum_i^{2M}\sum_j^N \beta_{ij}(s,\psi,t)\xi_{ij} + \sum_i^{2M}\sum_j^N\sum_m^{2M}\sum_n^N \beta_{ijmn}(s,\psi,t)\xi_{ij}\xi_{mn} \tag{3.8}$$

where, with $\phi_{i+M,j} = \phi_{ij}$; $i \leqslant M$ and $\beta_{ij} = \beta_{ij}\ (s,\psi,t)$,

$$\beta_{ij} = \begin{cases} \cos(\phi_{ij}), & i \leqslant M \\ -\sin(\phi_{i-Mj}), & i > M \end{cases} \tag{3.9}$$

and, with $\beta_{ijmn} = \beta_{ijmn}\ (s,\psi,t)$,

$$\beta_{ijmn} = \begin{cases} h^+_{ijmn}G^{c+}_{ijmn} + h^-_{ijmn}G^{c-}_{ijmn}, & i,m \leqslant M \\ -h^+_{i-Mjmn}G^{s+}_{i-Mjmn} - h^-_{i-Mjmn}G^{s-}_{i-Mjmn}, & i > M, m \leqslant M \\ -h^+_{ijm-Mn}G^{s+}_{ijm-Mn} + h^-_{ijmn}G^{s-}_{ijm-Mn}, & i \leqslant M, m > M \\ -h^+_{i-Mjm-Mn}G^{c+}_{i-Mjm-Mn} + h^-_{i-Mjm-Mn}G^{c-}_{i-Mjm-Mn}, & i,m > M \end{cases} \tag{3.10}$$

with

$$G^{c\pm}_{ijmn} = \cos(\phi_{ij} \pm \phi_{mn}), \quad G^{s\pm}_{ijmn} = \sin(\phi_{ij} \pm \phi_{mn}). \tag{3.11}$$

Substitution of (3.9) into the correlation ρ yields

$$\rho(s,\psi,t) = \frac{E\,[H(0,0,0)H(s,\psi,t)]}{\sigma_h^2} \simeq \sum_i^{2M}\sum_j^N \nu_{ij}\alpha_{ij}\beta_{ij} = \sum_i^M\sum_j^N \nu_{ij}\cos(\phi_{ij}) \tag{3.12}$$

where the non-dimensional variances $\nu_{ij} = V_{ij}/\sigma_h^2$ have been introduced and terms of the type $\alpha_{ijmn}\beta_{ijmn}$ have been omitted since they are small compared to the leading terms in the normalized cumulants λ_{ijk}. Here

$$\begin{aligned} \alpha_{ij} &= \beta_{ij}|_{s=0,t=0}, \quad & \alpha_{ijmn} &= \beta_{ijmn}|_{s=0,t=0}, \\ \alpha'_{ij} &= \left.\frac{\partial\beta_{ij}}{\partial s}\right|_{s=0,t=0}, \quad & \alpha'_{ijmn} &= \left.\frac{\partial\beta_{ijmn}}{\partial s}\right|_{s=0,t=0}. \end{aligned} \tag{3.13}$$

At $s = 0$ the results do not depend on the angle ψ so $H(0,\psi,0) = H(0,0,0)$. The normalized cumulants λ_{ijk} needed in (2.11) become (see Longuet-Higgins (1963), Jensen (1996)), by using irreducible products and keeping only the leading terms:

$$\begin{aligned} \lambda_{201}(s,\psi,t) &= \frac{E[H(0,0,0)^2 H(s,\psi,t)]}{\sigma_h^3} \\ &\simeq 2\sigma_h \sum_i^{2M}\sum_j^N \nu_{ij} \sum_m^{2M}\sum_n^N \nu_{mn}[\alpha_{ij}\alpha_{mn}\beta_{ijmn} + 2\alpha_{ij}\beta_{mn}\alpha_{ijmn}] \end{aligned}$$

$$= 2\sigma_h \sum_i^M \sum_j^N \nu_{ij} \sum_m^M \sum_n^N \nu_{mn} [h^+_{ijmn} \{\cos(\phi_{ij} + \phi_{mn}) + \cos(\phi_{ij}) + \cos(\phi_{mn})\} + h^-_{ijmn} \{\cos(\phi_{ij} - \phi_{mn}) + \cos(\phi_{ij}) + \cos(\phi_{mn})\}] \quad (3.14)$$

and

$$\lambda_{021}(s, \psi, t) = \frac{E[H'(0,0,0)^2 H(s,\psi,t)]}{\sigma_{\dot{h}}^2 \sigma_h}$$
$$\simeq 2\frac{\sigma_h^3}{\sigma_{\dot{h}}^2} \sum_i^{2M} \sum_j^N \nu_{ij} \sum_m^{2M} \sum_n^N \nu_{mn} [\alpha'_{ij}\alpha'_{mn}\beta_{ijmn} + 2\alpha'_{ij}\beta_{mn}\alpha'_{ijmn}]$$
$$= 2\frac{\sigma_h^3}{\sigma_{\dot{u}}^2} \sum_i^M \sum_j^N \nu_{ij} \sum_m^M \sum_n^N \nu_{mn} \big[h^+_{ijmn}\big\{ - k^e_{ij} k^e_{mn} \cos(\phi_{ij} + \phi_{mn}) + \big(k^e_{ij} + k^e_{mn}\big)\big(k^e_{mn}\cos(\phi_{ij}) + k^e_{ij}\cos(\phi_{mn})\big)\big\} + h^-_{ijmn}\big\{k^e_{ij}k^e_{mn}\cos(\phi_{ij} - \phi_{mn}) - \big(k^e_{ij} - k^e_{mn}\big)\big(k^e_{mn}\cos(\phi_{ij}) - k^e_{ij}\cos(\phi_{mn})\big)\big\}\big] \quad (3.15)$$

with $k^e_{ij} = k_i \cos(\psi - \varphi_j)$ and

$$\sigma_{\dot{h}}^2 = \sum_i^M \sum_j^N k^2 S(k_i, \varphi_j)\Delta k_i \Delta\varphi_j. \quad (3.16)$$

If the derivative is with respect to time, then k^e_{ij} should be replaced by the wave frequency ω_i in (3.15). The skewness λ_{030} of the wave slope is zero due to the vertical symmetry of the Stokes wave profile and, furthermore, $\lambda_{300} = \lambda_{201}|_{s=0,t=0}$ and $\lambda_{120} = \lambda_{021}|_{s=0,t=0}$ as $V(x) = U(x) = H(s, \psi, t)$ in (2.11) for the wave profile.

Similar results can be obtained for the associated wave kinematics. For instance, the conditional mean horizontal wave particle velocity profile $V(s, \psi, z, t)$ to second order in direction ψ relative to the wind direction is obtained from, see Madsen (1987),

$$V^{(1)}(s, \psi, z, t) = \sum_i^M \sum_j^N a_{ij} f_{ij} \cos(\psi_{ij}),$$
$$V^{(2)}(s, \psi, z, t) = \sum_i^M \sum_j^N a_{ij} \sum_m^M \sum_n^N a_{mn} [f^+_{ijmn}\cos(\psi_{ij} + \psi_{mn}) + f^-_{ijmn}\cos(\psi_{ij} - \psi_{mn})], \quad (3.17)$$

with

$$f_{ij} = \frac{gk_i \cosh(k_i(z+h))}{\omega_i \cosh(k_i h)} \cos(\psi - \varphi_j),$$
$$f^\pm_{ijmn} = \frac{1}{4}\frac{g^2}{\omega_i\omega_m} \frac{\cosh(k^\pm_{ijmn}(z+h))}{\cosh(k^\pm_{ijmn}h)} \frac{D^\pm_{ijmn}}{\omega_i \pm \omega_m} \{k_i\cos(\psi - \varphi_j) \pm k_m \cos(\psi - \varphi_n)\}, \quad (3.18)$$

and where z is the vertical distance from the still water surface. Thus, λ_{201} and λ_{021} are obtained by substituting f_{ij}, $f^\pm_{ijmn}$ for β_{ij}, $\beta^\pm_{ijmn}$ in (3.14)–(3.15). The values of λ_{300} and λ_{120} are unchanged as they depend only on $H(0, 0, 0)$ and $H'(0, 0, 0)$.

4. Modelling an extreme wave episode

Numerical results for conditional mean wave elevations and associated horizontal wave particle velocity for a fixed point in time have been presented in Jensen (2004). The results show that finite water depth, directional spreading and superimposed current can have a significant effect on the conditional wave.

To investigate whether the conditional mean second-order wave is a good indicator of an extreme wave, a comparison is made with the Draupner New Year Wave, Haver and Andersen (2000). Averaged over 20 minutes, the sea state associated with this extreme wave event can be characterized by a significant wave height H_s of about 12 m and a zero-crossing period T_z around 12 s. The water depth is 70 m. The measurements were made by a downward-looking laser device and no information on directional spreading is available. Satellite measurements close to this location on the same day show, however, that the very large waves observed are fairly short-crested, Nieto-Borge *et al.* (2004). With only this information available, inevitably a number of assumptions must be made. Hence, no current is included and a standard Pierson–Moskowitz spectrum with an adiabatic transformation for finite water depth and a cosine square spreading function is considered here. Possible errors in the measurements are unknown and ignored. It has previously been shown by Haver & Andersen (2000) that a second-order representation of the wave elevation yields peak value statistics in good agreement with these measurements, except for the single Draupner extreme peak crest. The intention of the present comparison between the second-order conditional wave and the Draupner wave is therefore only to show that even if the magnitude of the wave crest of this extreme event cannot be predicted by standard statistics applied to a second-order wave theory the choice of phase relations inherent in the conditional second-order wave still yields a rather good agreement with the measured extreme crest profile. The comparison is thus a mathematical 'fitting' using the measured crest height as the prescribed second-order crest height and is not a physical second-order reproduction of the measured wave. Such a mathematical wave could be a candidate as a design wave similar to the standard Stokes' fifth-order wave, Tromans *et al.* (1991).

The wavenumber range is taken from 0 to $5k_m$, where the mean wavenumber

$$k_m = \frac{1}{g}\left(\frac{2\pi}{T_z}\right)^2. \tag{4.1}$$

The upper limit on the wavenumber is chosen such that the bandwidth $\sqrt{1 - m_2^2/m_4 m_0}$ in terms of the usual spectral moments m_i of the Pierson–Moskowitz part of the wave spectrum becomes approximately 0.6, a number often quoted for deep-water ocean waves.

The conditional mean wave crest a at time $t = 0$ is taken to be equal to the Draupner New Year Wave crest of 18.5 m. The results are shown in figure 1 and compared with the measured elevation as a function of time. The linear conditional mean waves (both the deep-water unidirectional wave and the finite-water short-crested wave) cannot predict the rather small troughs measured just before and after the extreme crest. Neither can the conditional-mean second-order unidirectional deep-water wave. The conditional-mean finite-water second-order wave including wave spreading is seen to agree better with the measured adjacent troughs and peaks. However, the importance of wave spreading and water depth is seen to be not as important as the effect of the second-order contribution.

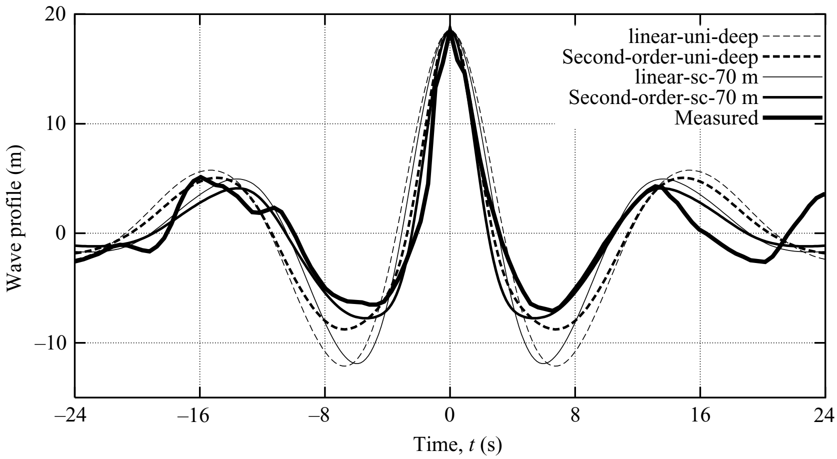

FIGURE 1. Draupner New Year Wave elevation: measurements from Haver & Andersen (2000). Conditional-mean wave elevations based on $H_s = 12\,\mathrm{m}$, $T_z = 12\,\mathrm{s}$, no current. Uni: unidirectional, sc: short-crested waves. Deep water and finite water depth $h = 70\,\mathrm{m}$.

For a second-order Stokes wave theory the results are symmetric with respect to $t = 0$. It should be recognized that the conditional-wave elevation is a statistical quantity except at $t = 0$ and that its standard deviation is already approaching the unconditional value at the first trough. This time-varying standard deviation is readily obtained from the conditional-mean wave analysis, see Lindgren (1972) and Tromans *et al.* (1991) for Gaussian waves and Jensen (1996) for second-order waves. Any strict agreement between the measurements and the conditional mean profile can of course not be expected. It is noted that the measured steepness very close to the crest is larger than obtained from the second-order model. Simulations using third-order reconstructions in the modified nonlinear Schrödinger model show a similar behaviour, Krogstad *et al.* (2004). This indicates that the second-order model is not adequate in the close vicinity of large crests as also found by Haver & Anderson (2000). For the Draupner wave no wave kinematics is recorded. However, to illustrate the wave kinematics by the present theory the horizontal wave-particle velocity at $s = 0$ at the passage of the crest ($t = 0$) and at $t = 2\,\mathrm{s}$ before the passage is shown in figure 2. Both the linear predictions and the present second-order results are included. The very large reductions in wave-particle velocity above the still water level due to the second-order terms are clearly seen. Generally, a much better agreement with measurements is thus obtained, e.g. Anastasiou, Tickell & Chaplin (1983), Stansberg & Gudmestad (1996), although the velocities close to the crest still are somewhat too large. The reason is that the Stokes theory applied does not account for the shorter waves riding on the longer waves. The results are thus very sensitive to the cut-off value (here $5k_m$) in the wavenumber range. For the wave profile no such strong dependence is seen.

A theory based on Stokes' second-order wave theory, like the present one, might be of value in providing significant improvements compared to a linear analysis. However, it can also be applied as the initial condition to a fully nonlinear three-dimensional analysis. In that case inclusion of directional spreading of the waves is important to capture relevant wave–wave interactions. Bateman *et al.* (2001) investigated this possibility using a linear and a partly second-order solution (based on the Creamer transformation) for their fully non-linear three-dimensional spectral procedure. Thus,

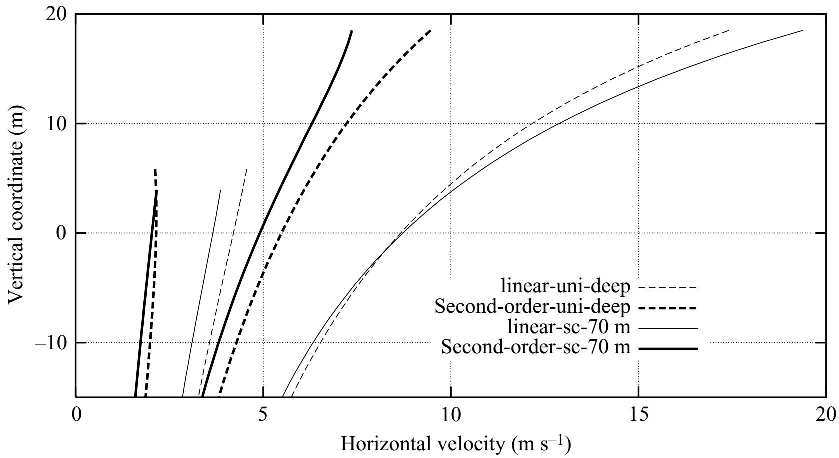

FIGURE 2. Conditional-mean horizontal wave particle velocities at $s = 0$ at the passage of the crest and 2 s before. Shown up to the second-order elevation. Legends and sea state as in figure 1. Both the linear and second-order results use exponential extrapolation above the mean water level.

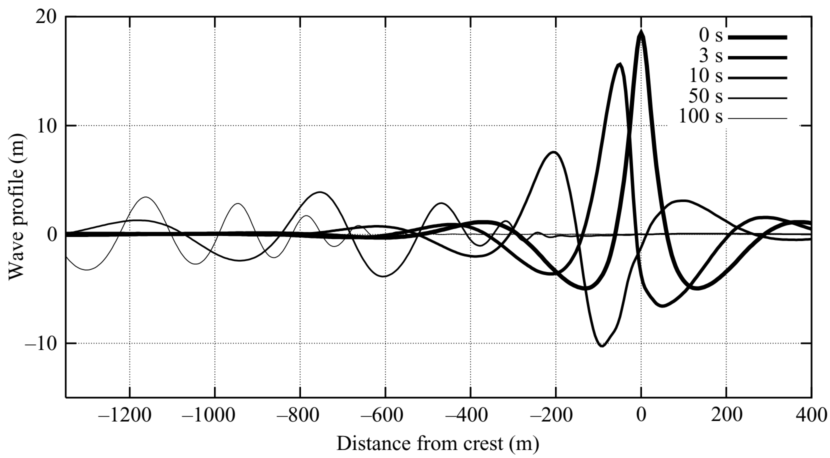

FIGURE 3. Calculated spatial variation of the second-order short-crested conditional-mean wave profiles in the direction of the main wave direction at the time instants 0, 3, 10, 50 and 100 s before the extreme wave crest appears. Water depth $h = 70$ m. Sea state as in figure 1.

the computational effort can be reduced significantly as a time range of only 100 to 200 s needs to be considered. Still, a fairly large space region has to be covered when three-dimensional waves are dealt with. As an example of possible initial conditions the second-order conditional mean wave elevations at different time instants before the appearance of the second order wave crest are shown in figure 3. The associated second-order conditional-mean wave potentials are readily determined analogously to the horizontal wave-particle velocity and, as for the velocity, a much larger difference between the linear and the second-order solution is found than for the wave elevation. Hence, the use of the second-order solution (elevation and potential) rather than the linear solution as initial condition is expected to result in more reliable fully nonlinear results. This is in line with the discussion given by Bateman *et al.* (2001). Provided the second-order extreme conditional-mean wave is a good identifier of an extreme fully nonlinear wave, such a procedure might give answers about the importance of wave–wave interactions involving four or more wave numbers for the magnitude of the extreme crest heights.

The crest statistics of the fully nonlinear crest can then be derived from the statistics for the second-order wave crest. As a first estimate the standard deviation σ_h and the skewness λ_{300} of the waves can be used to estimate the probability distribution of the wave crest. On the assumption that the nonlinearities are of second order the wave elevation H can be modelled by a quadratic transformation of a standard normal distributed process W, e.g. Winterstein (1988):

$$H = \sigma_h\big(W + \tfrac{1}{6}\lambda_{300}(W^2 - 1)\big). \tag{4.2}$$

Provided (4.2) is monotonic and that the peak in W follows a Rayleigh distribution the most probable largest crest a among N crests becomes

$$a = \sigma_h\big(\sqrt{2\ln N} + \tfrac{1}{6}\lambda_{300}(2\ln N - 1)\big). \tag{4.3}$$

By comparison with measurements made in the North Sea, Vinje & Haver (1994) show that (4.3) gives good estimates of the measured crest values. For unidirectional deep water waves with a Pierson-Moskowitz spectrum, the skewness λ_{300} can be derived analytically, Vinje & Haver (1994):

$$\lambda_{300} = 6\pi^2\left(1 - \frac{1}{\sqrt{2}}\right)\frac{H_s}{gT_z^2} \tag{4.4}$$

whereas (3.14) with $s = 0, t = 0$ can be used in general cases. The skewness is around 0.2 for very steep waves. Hence, the difference between a linear and a second-order estimation of the most probable wave crest will be about 20 %, depending slightly on N. For the Draupner wave the skewness becomes 0.155 by using the present spectral density with a cut-off limit and including wave spreading and finite water depth. The second-order conditional-mean wave is, for large values of the wave crest, nearly independent of the significant wave height, but of course the wave crest itself depends approximately linearly on H_s. Using a standard deviation of $\sigma_h = 3$ m yields $N = 10^6$ which is a rare event given that the sea state itself is rare. Thus, nonlinear effects not included in a second-order wave model seem to be important for modelling of the Draupner New Year Wave. This is supported by the observation mentioned above, that the steepness very close to the crest is not accurately modelled by second-order waves. However, as noted by Krogstad *et al.* (2004), the question is also whether freak waves are real outliers from standard (second) order statistics, or merely at the wrong place at the wrong time.

The second-order crest values a will according to (4.2) have a probability density function $p(a)$:

$$p(a) = \frac{u\,\mathrm{e}^{-0.5u^2}}{\sigma_h\big(1 + \frac{1}{3}\lambda_{300}u\big)} \quad \text{with } u = \frac{-3 + \sqrt{9 + \lambda_{300}(\lambda_{300} + 6a/\sigma_h)}}{\lambda_{300}} \tag{4.5}$$

and this probability might be associated with the fully nonlinear wave crest a_{nl} through

$$p(a_{nl}) = p(a)\left(\frac{\mathrm{d}a_{nl}}{\mathrm{d}a}\right)^{-1} \tag{4.6}$$

using the numerically derived monotonic relation between a_{nl} and a.

In the application of the conditional second-order wave as an initial condition for fully nonlinear three-dimensional computations, the present conditional mean wave can be superimposed with a time and space varying Gaussian component to investigate the importance of slight changes in the initial conditions for the development of the fully nonlinear extreme wave. This Slepian concept is exact for linear waves and expected to be a good approximation for the second-order waves as the main second-order contribution is due to the conditional mean wave when considering crest values much larger than the linear standard deviation. The statistics of the fully nonlinear wave are then obtained by unconditioning the conditional probability $p(a_{nl}|a)$ with respect to the second-order wave crests considered

$$p(a_{nl}) = \int_0^\infty p(a_{nl}|a)p(a)\,\mathrm{d}a. \tag{4.7}$$

Such an approach has been used by Taylor, Johathan & Harland (1995) in the analysis of jack-up platforms. Linear random conditional waves are used as input and the statistics of the jack-up response is obtained by unconditioning with respect to the linear wave amplitude. A somewhat similar approach is used by Dietz, Friis-Hansen & Jensen (2004) in a recent study on the wave-induced bending moment in ships. Whether or not such a 'simple' input model will be sufficient as an input to fully nonlinear simulations of extreme wave events should of course be clarified further before such models are accepted as a replacement of full-length simulations.

5. Conclusion

A derivation of nonlinear conditional processes to any order has been given followed by the derivation of a second-order short-crested wave model for finite water depth. Numerical results related to the Draupner New Year Wave have been presented. Fairly good average agreement with the measured wave profile is obtained.

The second-order wave model might be used directly as a design wave, but for the analysis of extreme wave events the second-order model could also be applied as an initial condition for a fully nonlinear three-dimensional procedure to obtain very accurate results with a reasonable computational effort. It could be interesting to investigate this approach, possibly by including also a random wave superimposed on the conditional mean wave to measure the sensitivity of slight changes in the initial conditions to the extreme wave event.

A third-order conditional-mean wave model can be derived using the present results for nonlinear conditional processes together with an appropriate third-order irregular wave theory, e.g. Zhang & Chen (1999). Additional hypotheses might, however, be needed to account for four-wave interactions. The advantage of a third-order wave as initial conditions to the nonlinear simulations compared to the second-order wave might, however, be limited.

The author would like to thank Dr Sverre Haver for providing the measured time trace of the Draupner New Year Wave.

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