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A new approximation for ocean waves propagating over a beach with variable depth

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Abstract

In this paper, the phenomenon of ocean waves propagating over a beach with variable water depth is re-examined. Based on the assumption of shallow water, a linearised shallow water equation is solved with an arbitrary beach profile. These irregular beach profiles form a set of partial differential equation with variable coefficient as the governing equation, which is the main obstacle in obtaining analytical solutions. In this paper, two families of beach profile are used as examples. A parametric study is conducted to investigate the influence of the beach profiles on the water surface elevation (η) and velocities (u).

Keywords: Shallow water expansion; Variable water depth; Ocean waves; Wave propagation

1. Introduction

The evolution of ocean waves on beaches is the quintessential problem of coastal engineering. The phenomenon of ocean waves traveling from offshore (deep water) to nearshore (shallow water) is particularly important for the design and protection of coastline. This includes the topics of wave breaking, the stability of coastline, and beach nourishment. Also, the transmission of wave energy in the nearshore region is a dominant factor in the design of coastal structures.

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To understand the wave phenomenon in nearshore region is commonly required to solve the shallow water equation. The solution of shallow water wave equation is one of the classic problems of coastal hydrodynamics, and most great hydrodynamicists and mathematicians of the past two centuries have contributed ideas and solutions to specific bathymetric configuration. Numerous investigations of ocean waves propagating over a sloping seabed have been carried out. Carrier and his co-author (Carrier and Greenspan, 1958; Carrier, 1966) developed a series of analytical solutions for gravity waves propagating on the water of variable depths. Their solutions are limited to a beach with constant slope, although the solution for a beach with arbitrary bottom was suggested.

To date, a common-used step model is based on the uniform depth model, with an application of the conservation of energy flux to solve the wave fluctuation with step-varying depth (Chen and Hwung, 1982; Le Mehaute and Web, 1964). This type of approach cannot represent the effect of a variable slope seabed bottom in the solution. Few researchers have attempted to directly take the slope into account in the whole problem (Keller, 1958). However, their approach is only limited to cases with small slope and small relative water depth. Recently, with advances in numerical methods, a wave propagating over a sloping seabed, even to the point of breaking, can be described numerically. For example, the parabolic wave model proposed by Li (1994) has been widely used and extended to various situations (Liu, 1994; Li, 1997, Suh et al., 1997, Synolakis, 1999).

Besides analytical approximations for wave problems, significant advances have been made in developing mathematical models to describe fully non-linear and weakly dispersive waves propagating over an impermeable bottom (Gobbi and Kirby, 1999; Suh et al., 1997, Hsu and Wen, 2001, Ehrenmark and Williams, 2001). Based on the inviscid fluid assumption, these models reduce the three-dimensional Euler equations to a set of two-dimensional governing equations, and are usually expressed in terms of the free surface displacement and representative horizontal velocity components, which are either evaluated at a certain elevation, or depth averaged.

In this study, a new analytical solution is developed for the phenomenon of ocean wave propagating over a sloping beach. Unlike previous analytical approximations, we consider the beach with a structured shape, rather than a linear beach. Two types of beach shapes are used as examples, and their effects on the water surface elevation will be investigated.

2. Theoretical formulations

2.1. Boundary value problem

In this study, we consider ocean gravity waves propagating over a sloping beach, as depicted in Fig. 1. In the figure, h_a is the reference water depth far from the beach, $h(x,t)$ is the water surface elevation, which is defined by

$$h(x,t) = h_o(x) + \eta(x,t), \tag{1}$$

where $\eta(x,t)$ represents the fluctuation in the water height, and $h_o(x)$ is the water depth at the location (x).

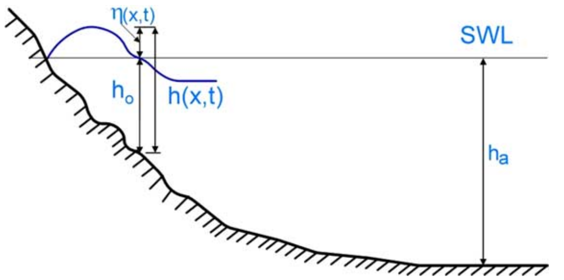

Fig. 1. Geometry of the general propagating problem.

Based on the shallow water theory (Carrier and Greenspan, 1958), the governing equations for the gravity waves on an incompressible and invisid fluid can be expressed in Eulerian system as

$$u_t + uu_x + g\eta_x = 0, \tag{2}$$

$$h_t + (uh)_x = 0, \tag{3}$$

where u is the velocity in the horizontal direction, t is the time, g is the gravitational acceleration, and the subscripts 'x' and 't' denote the partial differentiation respective to x and t, respectively.

Now, we consider the problem in a Lagrangian system, and choose h_a as the reference height, the relationship between two co-ordinates is

$$\frac{\partial x}{\partial X} = \frac{h_a}{h(x,t) + \eta(x,t)}. \tag{4}$$

To simplify the problem, we linearise (4) as,

$$\frac{\partial x}{\partial X} = \frac{h_a}{h(x,t)}. \tag{5}$$

Then the linearised governing equations in a Lagrangian system can be expressed as

$$\frac{\partial u}{\partial t} + \frac{h_o}{h_a} g \frac{\partial \eta}{\partial X} = 0, \tag{6}$$

$$\frac{\partial \eta}{\partial t} + \frac{h_0^2}{h_a} \frac{\partial u}{\partial X} = 0. \tag{7}$$

To simplify the mathematical expression, we non-dimensionalise the variables as follows

$$(X^*, \eta^*, h^*) = (X, \eta, h)/h_a, \qquad t^* = t/\left(h_a/\sqrt{gh_a}\right), \tag{8}$$

where the superscript '*' denotes a non-dimensional variable. To avoid complicated expressions, the '*' will be ignored, and all physical variables are non-dimensional parameters in the following section, unless specified.

Introducing (8) into (6) and (7), the governing equations can be re-written as

$$\frac{\partial u}{\partial t} + h_0 \frac{\partial \eta}{\partial X} = 0, \tag{9}$$

$$\frac{\partial \eta}{\partial t} + h_0^2 \frac{\partial u}{\partial X} = 0. \tag{10}$$

2.2. Analytical solution

In this paper, we solve the above governing Eqs. (9) and (10) analytically. Herein, we define a new variable by,

$$\frac{\mathrm{d}R}{\mathrm{d}X} = \frac{1}{h_0^{3/2}}, \tag{11}$$

so that Eqs. (9) and (10) become

$$\frac{\partial u}{\partial t} + \frac{1}{\sqrt{h_0}} \frac{\partial \eta}{\partial R} = 0, \tag{12}$$

$$\frac{\partial \eta}{\partial t} + \sqrt{h_0} \frac{\partial u}{\partial R} = 0, \tag{13}$$

or, eliminating $u(x, t)$,

$$\frac{\partial^2 \eta}{\partial t^2} - \sqrt{h_0} \frac{\partial}{\partial R} \left(\frac{1}{\sqrt{h_0}} \frac{\partial \eta}{\partial R} \right) = 0. \tag{14}$$

Let $C(R) = \sqrt{h_0(X)}$, we have

$$\frac{\partial^2 \eta}{\partial t^2} - C(R) \frac{\partial}{\partial R} \left(\frac{1}{C(R)} \frac{\partial \eta}{\partial R} \right) = 0. \tag{15}$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{C(R)} \frac{\partial}{\partial R} \left(C(R) \frac{\partial u}{\partial R} \right) = 0 \tag{16}$$

in which

$$R = \int_0^X \frac{\mathrm{d}s}{C^3(s)} \text{ and } X = \int_0^R C^3(s)\, \mathrm{d}s. \tag{17}$$

Note that Eqs. (15) and (16) contains the *beach shape function*, $C(R) = \sqrt{h_0(X)}$, which describes the variation in the beach profile with depth. In general, it is difficult to obtain analytical solutions for equations of the form (15) and (16). However, an exact solution is possible using the approach of Varley and Seymour (1988).

The general solution for (15) and (16) can be expressed as

$$\eta = \sum_{n=0}^{N} f_n(R)\frac{\partial^{N-n}F}{\partial R^{N-n}}, \tag{18}$$

$$u = \sum_{n=0}^{N} e_n(R)\frac{\partial^{N-n}E}{\partial R^{N-n}}, \tag{19}$$

where $f_0 = 1/e_0 = \sqrt{C(R)} = h_0^{1/4}(X)$, and E and F satisfy

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial R} = 0 \text{ and } \frac{\partial F}{\partial t} + \frac{\partial E}{\partial R} = 0. \tag{20}$$

The function E and F can be expressed as

$$E = A(t+R) + B(t-R), \tag{21}$$

$$F = -A(t+R) + B(t-R). \tag{22}$$

where $A(t+R)$ is given as the incident wave components, and $B(t-R)$ is an unknown function, which needs to be determined later.

To find B, the following boundary conditions are required:

$$\eta \to 0 \text{ as } C(R) \to 0 \tag{23}$$

$$u \text{ is bounded as } C(R) \to 0 \tag{24}$$

Using $N=1$ as the first approximation, we have

$$\eta = \sqrt{C}\frac{\partial F}{\partial R} + \lambda_1 F, \tag{25}$$

$$u = \frac{1}{\sqrt{C}}\frac{\partial E}{\partial R} + k_1 E. \tag{26}$$

With the boundary conditions, we have $A(t)=B(t)$, which gives us

$$E = A(t+R) + A(t-R), \tag{27}$$

$$F = -A(t+R) + A(t-R). \tag{28}$$

If we consider the incident wave $A(t+R)$ to be periodic of the form for an incoming wave traveling to the left

$$A(x,t) = A_0 \cos\left(\frac{2\pi}{L}(ct-x)\right) = A_0 \cos\left(\frac{2\pi h_a}{L}(X^* - t^*)\right). \tag{29}$$

where L is the wavelength of incident wave in deep water, A_0 is the amplitude of waves.

3. Results and discussions

3.1. Beach profile

In this paper, two types of beach profiles are used:—need to make these more general—include a parameter

- Case I: $h_0(R) = [a_1 \tanh(b_1 R)]^4$
- Case II: $h_0(R) = (c_1 R)^4$

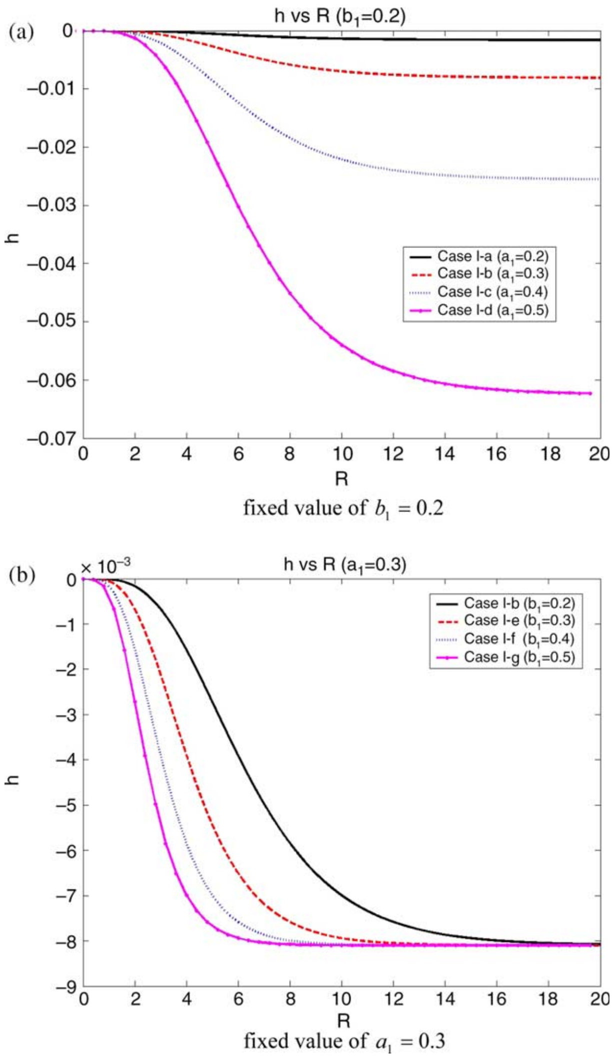

Fig. 2. Beach profile—Case I.

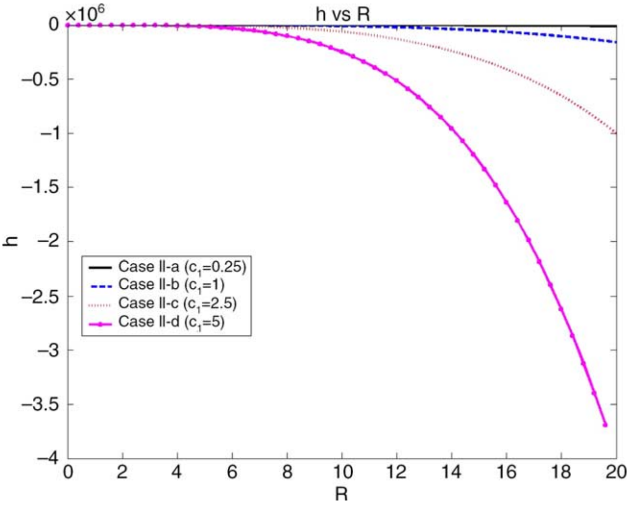

Fig. 3. Beach profile—Case II.

Two different beach shapes are plotted in Figs. 2 and 3. As shown in the figure, Case I represents a case with gentle slope, which is a function of '$[a_1 \tanh(b_1 R)]^4$', while Case II represents a case of rapidly slope, which is a function of '$(\sqrt{c_1}R)^4 R^4$'. The coefficients, a_1, b_1 and c_1 are listed in Table 1. In this study, we use $h_a = 0.05L$ as the reference water depth, because we are only concern with the case of shallow water. In the following discussion, the non-dimensional variables will be represented with '*' superscripts.

3.2. Case I—'$h(R) = [a_1 \tanh(b_1 R)]^4$' type beach profile

For the first family of beach profile, Case I, the values of coefficients a_1 and b_1 are varied to test the influences of these coefficients on the beach profiles, water surface

Table 1
Beach profiles used in the numerical examples

Case no	Coefficients
Case I-a	$a_1 = 0.2$ and $b_1 = 0.2$
Case I-b	$a_1 = 0.3$ and $b_1 = 0.2$
Case I-c	$a_1 = 0.4$ and $b_1 = 0.2$
Case I-d	$a_1 = 0.5$ and $b_1 = 0.2$
Case I-e	$a_1 = 0.3$ and $b_1 = 0.3$
Case I-f	$a_1 = 0.3$ and $b_1 = 0.4$
Case I-g	$a_1 = 0.3$ and $b_1 = 0.5$
Case II-a	$c_1 = 0.25$
Case II-b	$c_1 = 1$
Case II-c	$c_1 = 2.5$
Case II-d	$c_1 = 5$

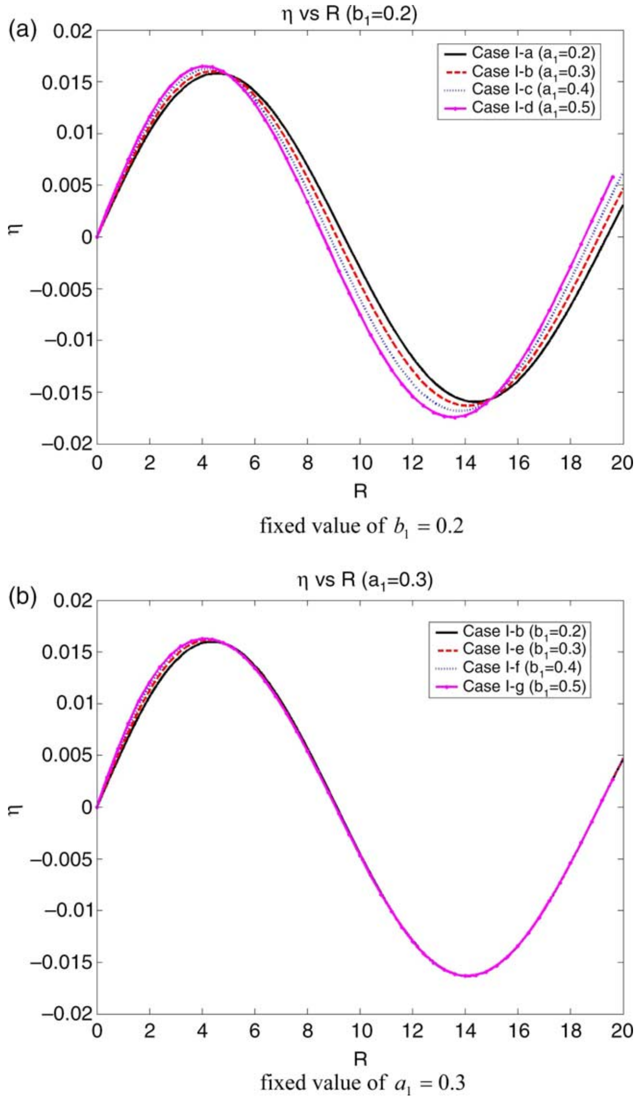

Fig. 4. Distribution of the water surface elevation (η) versus R for different cases (Case I).

elevation and velocities. As shown in Fig. 2, the beach profile is more sensitive to the coefficient a_1 (with fixed value of $b_1 = 0.2$, compared with the coefficient b_1.

The distribution of water surface elevation (η) versus R for the beaches with various values of a_1 is plotted in Fig. 4(a). The water surface elevation slightly increases as a_1 increases before it reach the crest. Then, an opposite trend is observed between the crest and troughs. Compared with the influence of a_1, the water surface elevation is in-sensitive to the coefficient b_1, as shown in Fig. 4(b).

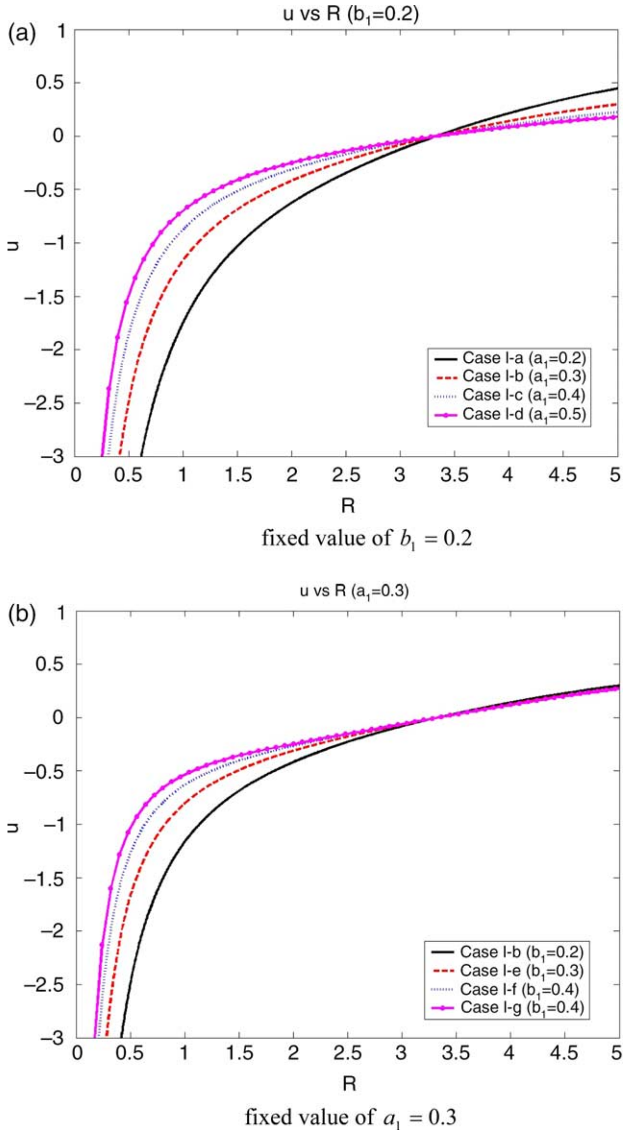

Fig. 5. Distribution of the Eulerian velocity (u) versus R for different cases (Case I).

Fig. 5 illustrates the distribution of velocity (u) versus R with various beach profiles. The amplitude of the fluctuations of the velocity increases as a_1 increases near wave crests, while it decreases near wave troughs. A similar trend is observed in Fig. 5(b). Since the influence of b_1 is less significant, we only plot the range between $R=0$ and 5 to demonstrate the influence of b_1 on the velocity.

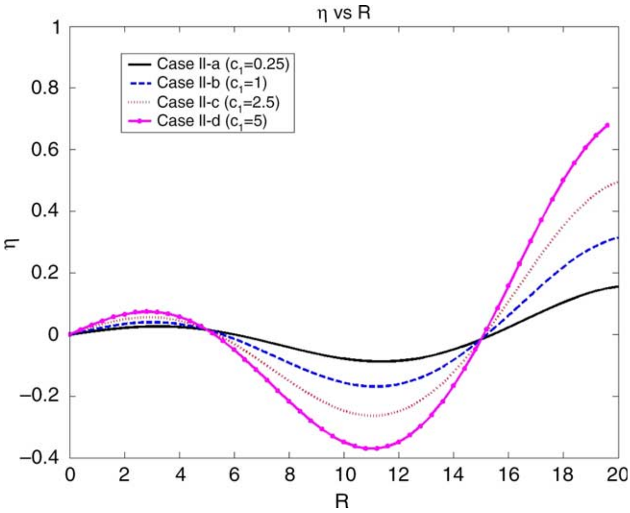

Fig. 6. Distribution of the water surface elevation (η) versus R for different cases (Case II).

3.3. Case II—‘$h(R) = c_1^2 R^4$’ type beach profile

Another family of beach profile, Case II, represents a steeper beach profile. The beach profile is very sensitive to the coefficient c_1, as shown in Fig. 3. Compared with Case I, the influence of this family of beach profile significantly affects the water surface elevation and velocity, as shown in Figs. 6 and 7. It is noted that the amplitude of η increases as R increases. This is the limitation of the present solution. That is, the present solution is only valid in shallow water. For deep water or intermediate water, the present solution will be invalid.

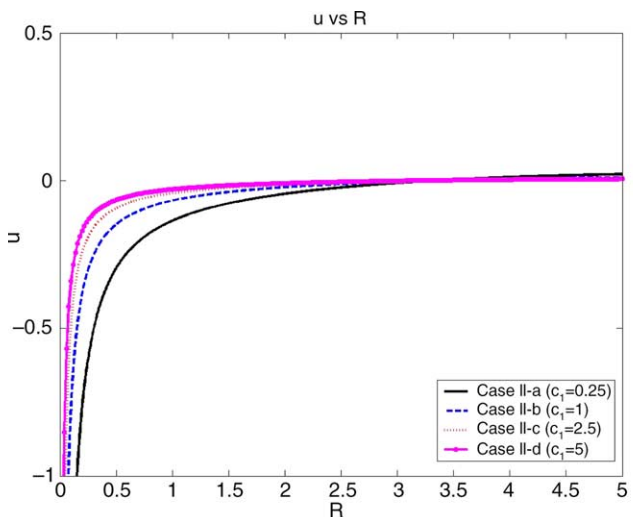

Fig. 7. Distribution of the Eulerian velocities (u) versus R for different cases (Case II).

4. Conclusions

In this paper, an exact solution for linearised shallow water equation with arbitrary beach profile is derived. Two families of beach profiles, Case I ($h(R)=[a_1 \tanh(b_1R)]^4$) and Case II ($h(R)=c_1^2R^4$) are considered. A parametric study indicates that coefficients a_1 and c_1 significantly affect the beach profile, water surface elevation and velocity.

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