The difference in the magnitude of the interference effects is sufficiently obvious from these curves. The variation in the form of the models shown in fig. 1 is considerable, and it would have been of interest to compare forms intermediate between those shown for the rear part of the model; but equations for such curves led to expressions for the wave resistance which were too complicated for numerical calculation. However, it may be inferred that for any case in which the lines of the model are smoothed out in this manner there will be a very considerable reduction in the magnitude of the interference effects.

On the Formation of Water Waves by Wind (Second Paper).

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In a previous paper* I investigated the problem of the formation of waves on deep water by wind, and found that the available data were consistent with the hypothesis that the growth of the waves is due principally to a systematic difference between the pressures of the air on the front and rear slopes. Lamb† had already discussed the maintenance of waves against viscosity by an approximate method, but without obtaining numerical results. Being under the incorrect impression that Lamb's approximation would not hold for the short waves I was chiefly considering, I proceeded on more elaborate lines. It now appears, however, that Lamb's method is not only applicable to the problem of waves on deep water, but is readily extended to cover the case when the water is comparatively shallow, and to allow for surface tension. The fundamental approximations are first, the usual one that squares of the displacements from the steady state can be neglected, and second, that viscosity modifies the motion of the water to only a small extent. The motion of the water can then, to a first approximation, be considered as irrotational.

With the previous notation, let ζ be the elevation of the free surface, x, y, z the position co-ordinates, t the time, U the undisturbed velocity of the water, h

* 'Roy. Soc. Proc.,' A, vol. 107, pp. 189–206 (1925).

† 'Hydrodynamics,' §§ 349, 350.

the depth, and ϕ the velocity potential. Also let σ, p, q, and ϑ denote respectively $\partial/\partial t$, $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, and write

$$p^2 + q^2 = - r^2. \tag{1}$$

Further, let the ratio of surface tension to density be T′. Then the velocity potential satisfies Laplace's equation

$$(\vartheta^2 - r^2)\,\phi = 0, \tag{2}$$

whence

$$\phi = Ux + e^{rz}A + e^{-rz}B \tag{3}$$

where A and B are independent of z. At the free surface we have

$$w = \vartheta\phi = r\,(A - B), \tag{4}$$

and also

$$= \frac{d\zeta}{dt} = (\sigma + Up)\,\zeta. \tag{5}$$

At the bottom $z = -h$, and $w = 0$. Hence

$$e^{-rh}A - e^{rh}B = 0, \tag{6}$$

and

$$\phi = Ux + \frac{\sigma + Up}{r}\,\frac{\cosh r\,(z + h)}{\sinh rh}\,\zeta. \tag{7}$$

The disturbance of pressure at the surface $z = 0$ is $g\,(\rho - \rho')\,\zeta + \rho T' r^2 \zeta$, allowing for the weight of the air displaced, but not for the disturbance of its velocity. It is also equal to $-\rho\,(\sigma + Up)\,(\phi - Ux)$.

Hence

$$\left[\frac{(\sigma + Up)^2}{r}\coth rh + g\,\frac{\rho - \rho'}{\rho} + T' r^2\right]\zeta = 0. \tag{8}$$

Suppose now that the water is initially at rest, so that $U = 0$, and assume that

$$\zeta = a\cos(\gamma t - \kappa x)\cos\kappa' y. \tag{9}$$

Then

$$\phi = -\frac{\gamma a}{r}\,\frac{\cosh r\,(z + h)}{\sinh rh}\sin(\gamma t - \kappa x)\cos\kappa' y, \tag{10}$$

where now

$$r^2 = \kappa^2 + \kappa'^2. \tag{11}$$

The kinetic energy of the water is given by

$$2T = \rho\iint \phi\,\frac{\partial\phi}{\partial n}\,dS \tag{12}$$

taken over the bounding surfaces, dn being an element of the outward normal.

The mean kinetic energy of a column of the water, of unit cross section, if κ' is not zero, is then evidently

$$\tfrac{1}{8}\rho \frac{\gamma^2 a^2}{r} \coth rh, \tag{13}$$

since $\partial\phi/\partial n$ is zero at the bottom. The gravitational potential energy is $\frac{1}{2}g(\rho - \rho')\,\zeta^2$ per unit area, and its mean value is $\frac{1}{8}g\,(\rho - \rho')\,a^2$. The potential energy due to the extension of the surface is $\frac{1}{2}\rho T'\,[(p\zeta)^2 + (q\zeta)^2]$, and its mean value is $\frac{1}{8}\rho T' r^2 a^2$. The mean potential energy per unit area is therefore $\frac{1}{8}a^2\,[g(\rho - \rho') + \rho T' r^2]$, which, by (8), is equal to the mean kinetic energy. The total energy per unit area is therefore

$$E = \tfrac{1}{4}\rho \frac{\gamma^2 a^2}{r} \coth rh. \tag{14}$$

The rate of dissipation of energy is

$$\mu \iint \frac{\partial Q^2}{\partial n}\, dS \tag{15}$$

where μ is the mechanical viscosity, otherwise denoted by $\nu\rho$, and Q is the resultant velocity. Now

$$Q^2 = (p\phi)^2 + (q\phi)^2 + (\vartheta\phi)^2, \tag{16}$$

and its mean value at a given depth is found, after some simplification, to be

$$Q^2 = \frac{\gamma^2 a^2}{4 \sinh^2 rh} \cosh 2r\,(z + h). \tag{17}$$

The mean value of $\partial Q^2/\partial n$ at the bottom is zero. At the surface

$$\left(\frac{\partial Q^2}{\partial n}\right)_{\text{mean}} = r\gamma^2 a^2 \coth rh. \tag{18}$$

The rate of dissipation in a column of unit cross section is therefore equal, on an average, to

$$\mu r\gamma^2 a^2 \coth rh. \tag{19}$$

Now suppose the wind to have velocity V, and call the wave velocity c. Then

$$c = \gamma/\kappa. \tag{20}$$

The velocity of the wind relative to the crests is $V - c$, and if s be the sheltering coefficient* the part of the pressure of the air that contributes to wave growth is $s\rho'\,(V - c)^2\,p\zeta$. The rate at which the pressure does work per unit

* Jeffreys, *loc. cit.*, p. 193.

area is equal to the product of the pressure and the downward normal velocity. Thus it is equal to

$$s\rho'[(\mathrm{V}-c)^2 p\zeta][-\sigma\zeta] = s\rho'(\mathrm{V}-c)^2\gamma\kappa a^2 \sin^2(\gamma t-\kappa x)\cos^2\kappa' y,$$

and its mean value is

$$\tfrac{1}{4}s\rho'(\mathrm{V}-c)^2\gamma\kappa a^2. \tag{21}$$

From (14) and (19), the energy, in the absence of wind, satisfies the equation

$$\frac{d}{dt}\left(\tfrac{1}{4}\rho\,\frac{\gamma^2 a^2}{r}\coth rh\right) = -\mu r\gamma^2 a^2 \coth rh, \tag{22}$$

and hence a varies like $\exp(-2\nu r^2 t)$. The only effect of the three-dimensional character of the wave is then to replace κ by r. If κ' were zero the mean value of $\cos^2\kappa' y$ would be 1 instead of $\frac{1}{2}$, but this affects all the quantities concerned and the growth or decay of waves involves only their ratios. Neither finiteness of depth nor surface tension affects the rate of decay.

From (19) and (21) we see that the wave will grow if

$$\tfrac{1}{4}s\rho'(\mathrm{V}-c)^2\gamma\kappa > \nu\rho r\gamma^2\coth rh, \tag{23}$$

which may be written

$$\frac{(\mathrm{V}-c)^2}{c} > \frac{4\nu\rho}{s\rho'}\,\frac{r}{\tanh rh}. \tag{24}$$

We recall that, by (8),

$$c^2 = \left[g\,\frac{\rho-\rho'}{\rho} + \mathrm{T}'r^2\right]\frac{r}{\kappa^2}\tanh rh. \tag{25}$$

Now if V is given the left side of (24) increases steadily as c diminishes, and therefore has its greatest value when c is a minimum. The right side increases steadily with r, and therefore, subject to κ being the same, is an increasing function of κ'. By (25), c is also an increasing function of κ' if κ is kept constant. Hence if κ' is not zero, we can increase the left side of (24) and decrease the right side by decreasing κ'. The inequality (24) therefore has the best chance of being satisfied if $\kappa'=0$. Thus the easiest waves to excite are always two-dimensional.

Taking now $\kappa'=0$, (25) becomes

$$c^2 = \left[\frac{g}{\kappa}\,\frac{\rho-\rho'}{\rho} + \mathrm{T}'\kappa\right]\tanh\kappa h, \tag{26}$$

whence

$$2c\frac{dc}{d\kappa} = \left[-\frac{g}{\kappa^2}\,\frac{\rho-\rho'}{\rho} + \mathrm{T}'\right]\tanh\kappa h + \left[\frac{g}{\kappa}\,\frac{\rho-\rho'}{\rho} + \mathrm{T}'\kappa\right]h\,\mathrm{sech}^2\kappa h. \tag{27}$$

When κ is zero, this is negative ; when κ is infinite, it is positive ; and, indeed, when κ is such that

$$\frac{g}{\kappa^2}\frac{\rho - \rho'}{\rho} = \mathrm{T}', \tag{28}$$

$2c\,(dc/d\kappa)$ is positive. Thus there is always a minimum velocity, and the corresponding wave-length is greater than that which gives the minimum velocity on deep water. The minimum velocity is evidently less than the minimum velocity on deep water.

If now κ is such that the wave-length is less than that which corresponds to the minimum wave velocity, we see that, as κ decreases, c decreases, and therefore $(\mathrm{V} - c)^2/c$ increases. At the same time $\kappa/\tanh \kappa h$ decreases. Thus the inequality (24), if satisfied by any value of κ in this range, will be satisfied by the value giving the minimum velocity. The easiest wave to excite is therefore never a ripple, in the Kelvin sense.

We are therefore limited to gravity waves in two dimensions. Now the depth enters (24) and (25) only through $\tanh \kappa h$, which is practically unity if $\kappa h > 1\cdot 5$, and practically κh if $\kappa h < 0\cdot 5$. The former case is that of deep water, which was examined in the previous paper. If indeed we ignore T′ and put $\tanh \kappa h = 1$,

$$c^2 = \frac{g}{\kappa}\frac{\rho - \rho'}{\rho}$$

and

$$\kappa c = \frac{g}{c}\frac{\rho - \rho'}{\rho}.$$

Then (24) becomes

$$(\mathrm{V} - c)^2 c > \frac{4\nu g\,(\rho - \rho')}{s\rho'}, \tag{29}$$

which is identical with equation 3 (14) of the previous paper. This was there found to imply that the easiest wave-length to excite was about 8 cm., based on the observational fact that waves first appear when V is about 110 cm./sec., and itself agreeing with observation so far as yet tested. Then $\kappa = 0\cdot 7/1$ cm. nearly, and then if $\kappa h = 1\cdot 5$ we must have $h = 2$ cm. nearly. The former theory should, then, hold for water of all depths greater than about 2 cm. The critical wave-length is great enough for the neglect of surface tension to be justifiable.

Taking now the other extreme case, where $\tanh \kappa h$ is replaceable by κh, we find that (24) and (25) reduce to

$$\frac{(\mathrm{V} - c)^2}{c} > \frac{4\nu\rho}{s\rho' h}, \tag{30}$$

$$c^2 = (g + \mathrm{T}'\kappa^2)\,h, \tag{31}$$

where the unimportant factor $(\rho-\rho')/\rho$ has now been dropped. The right side of (30) is now a constant, and the easiest wave-length to excite, therefore, corresponds to the smallest wave-velocity. This is evidently $\sqrt{(gh)}$, provided the wave-length is several times $2\pi\sqrt{(\mathrm{T}'/g)}$ or $1\cdot8$ cm. Then

$$\{\mathrm{V}-(gh)^{\frac{1}{2}}\}^2 > \frac{4\nu\rho}{s\rho'}\left(\frac{g}{h}\right)^{\frac{1}{2}}. \tag{32}$$

Thus the velocity needed to raise waves is determinate. But this analysis is only valid if $\kappa h < 0\cdot5$, which for a wave-length of $1\cdot8$ cm. would make $h < 0\cdot14$ cm.; actually, with longer waves, it may be several times greater. But a further restriction is operative. The minimum velocity is less, and the corresponding wave-length greater, than are applicable in deep water. On both grounds the period is greater; the period corresponding to the least velocity on deep water is about $0\cdot08$ sec. Now viscosity is dominant through a region whose thickness is about the square root of the product of the kinematic viscosity and the period, or, with these values, $0\cdot038$ cm. The conditions of shallow water will increase this result several times, and further viscosity will be operative both at the top and the bottom, thus again doubling the thickness affected. Thus in long waves on shallow water the motion will be dominated by viscosity, and no wave-formation will be possible. The only waves capable of being formed, therefore, are those whose length is not great enough to make the shallow-water approximation valid; the easiest waves to form have, therefore, lengths not more than about four times the depth. This agrees with what is observable in shallow roadside puddles. When the wind is strong enough to raise considerable waves on deep water, it often produces no noticeable disturbance in these puddles; a strong wind will raise waves, but their length does not exceed about 2 cm. It is easily found from (24) that waves of this length would be formed on deep water by a wind of 160 cm./sec., and this would hold for depths down to about 1 cm.

Summary.

On the hypothesis that, in a first approximation, water waves may be considered irrotational, viscosity and other factors tending to change the amplitude being small, it has been found possible to investigate the conditions of growth of waves under the action of wind, even when the depth is finite and surface tension is allowed for. It is found that the rate of decay of waves, in the absence of wind, is independent of the depth and the surface-tension. The easiest waves for a wind to raise are always two-dimensional, and are

gravity waves, not ripples. The wind velocity needed to produce waves is greater in shallow water than in deep water. The longer waves on shallow water (less than about 1 cm. deep) are, however, so much affected by viscosity that the hypothesis that viscosity exerts only a modifying effect on the motion is invalid for them, and it seems that they can in no case be formed by wind. The waves produced on shallow water must therefore be very short, about 2–3 cm. say, which agrees with observation.

The Excitation of Soft X-Rays.

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§ 1. In a paper (1) by one of us (O. W. R.) and Prof. Bazzoni it was shown that it was possible to detect the excitation of characteristic soft X-rays, such as the K X-rays of carbon, by a photoelectric method. The substance under investigation was bombarded with electrons in a highly-evacuated bulb of quartz glass, and the radiation generated, after passing between two parallel plates of a vacuum condenser to filter out ions and electrons, was received on a metal plate from which the photoelectric emission could be measured. The photoelectric current increases with the thermionic electron current, and with the potential difference driving the latter. If the photoelectric current divided by the thermionic current is plotted against this potential difference the excitation of characteristic X-rays setting in at certain voltages could be detected by the existence of discontinuous changes of slope in the curves so obtained. Similar observations were made almost simultaneously by A. Ll. Hughes (2), E. H. Kurth (3), P. Holweck (4), and Mohler and Foote (5), and since that time a very considerable number of papers on this subject has appeared (6)—(16).

This paper gives an account of measurements made since the autumn of 1923 on the elements carbon, tungsten, nickel and iron, starting with carbon and proceeding in the order named. It is believed that they represent a progressive improvement in this branch of experimenting so that the accuracy of the results probably increases from carbon to iron. It should also be made clear that the present communication does not represent an immediate continuation