

# Evolution of ocean wave statistics in shallow water: Refraction and diffraction over seafloor topography

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[1] We present a stochastic model for the evolution of random ocean surface waves in coastal waters with complex seafloor topography. First, we derive a deterministic coupled-mode model based on a forward scattering approximation of the nonlinear mild slope equation; this model describes the evolution of random, directionally spread waves over fully two-dimensional topography, while accounting for wide angle refraction/diffraction, and quadratic nonlinear coupling. On the basis of the deterministic evolution equations, we derive transport equations for the wave statistical moments. This stochastic model evolves the complete wave cross-correlation matrix and thus resolves spatially coherent wave interference patterns induced by topographic scattering as well as nonlinear energy transfers to higher and lower frequencies. In this paper we focus on the linear aspects of the interaction with seafloor topography. Comparison to analytic solutions and laboratory observations confirms that (1) the forward scattering approximation is suitable for realistic two-dimensional topography, and (2) the combined effects of wide angle refraction and diffraction are accurately captured by the stochastic model.

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# 1. Introduction

[2] As ocean surface waves propagate across continental shelves into coastal areas, their evolution is strongly affected by the diminishing depth. Waves are refracted by the bottom topography, nonlinear effects result in the amplification of harmonics and radiation of long waves ("surfbeat"), and the breaking of the waves in the surf zone cascades the energy from the ordered wave motion to small-scale turbulence and heat. Apart from wave breaking in the surf zone, these linear and nonlinear processes are fairly well understood, and incorporated in numerous deterministic shallow water wave models [see, e.g., *Freilich and Guza*, 1984; *Kirby*, 1995; *Kaihatu and Kirby*, 1995; *Wei et al.*, 1995; *Dingemans*, 1997; *Madsen and Schäffer*, 1999; *Bredmose et al.*, 2004, 2005; *Janssen et al.*, 2006].

[3] However, for many science and engineering applications, such as the design of marine structures and the study of coastal sediment transport, the forcing by random ocean waves is represented by statistical averages of wave spectra and integral parameters (e.g., significant wave height, peak period). Such statistics can be estimated through Monte Carlo simulations with a deterministic model. However, this approach is numerically intensive and often prohibitive on larger scales of application due to the large number of

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realizations required, and the need to resolve intrawave spatial and temporal scales. Hence large-scale oceanic wave models are inherently stochastic [e.g., *Komen et al.*, 1994], generally based on the radiative transfer equation [e.g., *Hasselmann*, 1968; *Willebrand*, 1975], which transports the spectral distribution of wave action (or energy) through a slowly varying medium. Modern (third generation) implementations of such models [e.g., *The WAMDI Group*, 1988; *Tolman*, 1991; *Komen et al.*, 1994; *Booij et al.*, 1999; *Janssen*, 2004] include parameterized forcing (source) terms to account for the effects of, e.g., wind generation, wave breaking, and wave-wave interactions. This class of models is routinely applied to predict and hindcast wind-generated ocean wave fields on regional and global scales, with considerable success.

[4] The underlying premise of the radiative transfer equation is that the wave field's spectral constituents are slowly varying and mutually independent, implying a quasihomogeneous and Gaussian sea state. However, nature provides many examples where these assumptions are violated: for instance, the crescent-shaped (horseshoe) waves appearing when the wind starts to blow over the ocean's surface [e.g., Su, 1982; Fuhrman et al., 2004], the characteristic saw-tooth wave shapes seen at the onset of wave breaking just outside the surf zone [e.g., Elgar and *Guza*, 1985], the interference patterns observed in the focal region of a lens-like topographical feature [e.g., Berkhoff et al., 1982; O'Reilly and Guza, 1991], and wave diffraction patterns around breakwater tips and harbor mouths [e.g., Penney and Price, 1952]. Less conspicuous, inhomogeneity and nonnormality affect deep-water wave (in)stability processes [e.g., Alber, 1978; Crawford et al., 1980; Yuen and

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Lake, 1975], and the occurrence of extremely large ("rogue") waves on the ocean [e.g., Janssen, 2003]. The assumption of a quasi-homogeneous, Gaussian wave field is particularly restrictive in shallow water, where topographyinduced refraction and the shoaling amplification of nonlinearity often result in strongly inhomogeneous and non-Gaussian wave statistics, which in turn affects the wave-induced coastal circulation and (sediment) transport processes [e.g., Hoefel and Elgar, 2003]. Shallow-water stochastic models include extensions to higher-order statistics to account for non-Gaussianity [Herbers and Burton, 1997; Agnon and Sheremet, 1997; Eldeberky and Madsen, 1999; Herbers et al., 2003] but so far these models have been restricted to laterally homogeneous wave fields evolving across parallel depth contours. Here we derive a stochastic model that includes two-dimensional seafloor topography and accounts for both non-Gaussianity and spatial heterogeneity of the wave field.

[5] Our starting point is a deterministic model based on a forward scattering approximation of the nonlinear mildslope equation (section 2). We derive a set of transport equations for the statistical moments and verify the representation of wave-bottom interaction through comparison of model predictions to laboratory observations of waves traversing a topographical lens (section 3). In section 4 we relate the model to the concepts of geometrical optics and the radiative transfer equation, and we demonstrate its wide angle diffraction capability. Although our derivation includes quadratic nonlinearity, our discussion of the sto-chastic model focuses on the interaction with the topography; nonlinearity and associated closure issues will be addressed in a subsequent paper. Our main findings are summarized in section 5.

# 2. A Deterministic Forward Scattering Approximation

[6] We consider the propagation of waves on the surface of an inviscid and incompressible fluid and adopt a conventional Cartesian description with the origin of the reference frame at the undisturbed free surface of the fluid. We let z denote the vertical, positive pointing upward, and x = (x, y) the horizontal dimensions. Since we are interested in random (but stationary) wave fields, we write the associated velocity potential function  $\Phi(x, z, t)$  and surface elevation  $\eta(x, t)$  as a Fourier sum

$$\begin{bmatrix} \Phi(\mathbf{x}, z, t) \\ \eta(\mathbf{x}, t) \end{bmatrix} = \sum_{p_1 = -\infty}^{\infty} \begin{bmatrix} \phi_1(\mathbf{x}, z) \\ \zeta_1(\mathbf{x}) \end{bmatrix} \exp\left[-i\omega_1 t\right], \tag{1}$$

where  $\omega_1 = \omega_{p_1} = p_1 \Delta \omega$  with  $\Delta \omega$  the discrete angular frequency spacing. The numerical subscript on wave-related variables is introduced for convenience; for example,  $\phi_1$  is short for  $\phi_{\omega_1}$ .

[7] To obtain transport equations for the evolution of the wave variables in space, we simplify the boundary value problem [see, e.g., *Chu and Mei*, 1970; *Liu and Dingemans*, 1989; *Janssen et al.*, 2006] by assuming a weakly nonlinear wave field and (spatially) slowly varying sea floor topography. The nonlinearity is characterized by the (small) wave steepness  $\epsilon = a_0 k_0 \ll 1$  and the slow depth variability by the

nondimensional bottom slope  $\beta = |\nabla h_0|/(k_0h_0) \ll 1$ ; here  $k_0$ and  $a_0$  are a representative wave number and amplitude of the wave field, respectively, and  $h_0$  and  $|\nabla h_0|$  denote a characteristic water depth and bottom gradient. We assume  $O(\epsilon) \sim O(\beta)$ , omit terms of higher order than  $O(\epsilon^2)$ , and from the outset we assume that the lowest-order wave-wave (quadratic) interactions are near resonant (we will return to the implications of this assumption below). The evolution of the lowest-order wave field is governed by a solvability condition that takes the form of the mild slope equation (MSE) with a quadratic nonlinear coupling term [see also, e.g., *Kaihatu and Kirby*, 1995; *Janssen*, 2006]

$$\boldsymbol{\nabla}^2 \boldsymbol{\varphi}_1 + k_1^2 \boldsymbol{\varphi}_1 = i \sum_{\omega_1, \omega_2} \mathcal{W}_{23} \boldsymbol{\varphi}_2 \boldsymbol{\varphi}_3 \widehat{\delta}_{1;23}^{\omega}.$$
(2)

where  $\hat{\delta}_{1;23}^{\omega} = \hat{\delta}(\omega_2 + \omega_3 - \omega_1)$  and  $\hat{\delta}$  is a discrete Dirac delta or unit impulse function [see, e.g., *Oppenheim and Schafer*, 1989]. The wave number  $k_1$  is given by the linear dispersion relation for progressive gravity waves  $\omega_1^2 = gk_1 \tanh k_1h$ ; the variable  $\varphi_1 = \mathcal{P}_1 \phi_1|_{z=0}$  where  $\mathcal{P}_1 = \sqrt{C_1 C_{g,1}}$ , with  $C_1$  and  $C_{g,1}$  the phase and group speed corresponding to frequency  $\omega_1$  (in the linear approximation). In (2), the coupling coefficient in the nonlinear term is given by

$$\mathcal{W}_{23} = \frac{1}{2\mathcal{P}_{2+3}\mathcal{P}_2\mathcal{P}_3} \left[ \omega_2 k_3^2 \left( 1 - T_3^2 \right) + \omega_3 k_2^2 \left( 1 - T_2^2 \right) \right. \\ \left. + 2(\omega_2 + \omega_3) k_2 k_3 (\operatorname{sgn}(\omega_2 \omega_3) - T_2 T_3) \right]$$
(3)

where  $T_i = \tanh k_i h$  and sgn denotes the signum function. Consistent with the assumption of quadratic near-resonance the coupling coefficient (3) is based on a small-crossing angle approximation such that  $\nabla \varphi_2 \cdot \nabla \varphi_3 \approx -\text{sgn}(\omega_2 \omega_3) k_2 k_3 \varphi_2 \varphi_3$ .

[8] The nonlinear mild-slope equation (2) is an elliptic equation and represents an isotropic description of the wave evolution, i.e., waves are allowed to propagate in all directions of the horizontal plane. In the following we take into account that backscattering from seafloor topography and reflection from shore are usually weak at wind-wave and swell frequencies [see, e.g., Elgar et al., 1994; Ardhuin and Herbers, 2002]. We apply a forward scattering approximation to (2) and consider waves propagating in the halfplane of positive x; we will refer to x and y as the principal and lateral direction, respectively. Moreover, since we are interested primarily in waves propagating across the continental shelf toward the coast, we let x and y coincide with the cross- and alongshore directions, respectively. In this approximation the MSE (equation (2)) can be written as (see Appendix B)

$$\partial_{x}\varphi_{1}(\mathbf{x}) = \left(i\widetilde{\varkappa}_{1} - \frac{1}{2\widetilde{\varkappa}_{1}}\frac{\partial\widetilde{\varkappa}_{1}}{\partial x}\right)\varphi_{1}(\mathbf{x}) + \sum_{\omega_{1},\omega_{2}}\frac{\mathcal{W}_{23}}{2\widetilde{\varkappa}_{1}}\varphi_{2}(\mathbf{x})\varphi_{3}(\mathbf{x})\widehat{\delta}_{1,23}^{\omega}$$
(4)

where  $\tilde{\varkappa}_1 = \text{sgn}(\omega_1) \sqrt{k_1^2 + \partial_y^2}$ . For a plane wave over a laterally uniform bottom,  $\tilde{\varkappa}$  is the principal (x-component) wave number, and the linear part of (4) represents a WKB-type solution, which accounts for the slowly varying depth in the principal direction. For more general wave fields over two-dimensional topography we can solve the transport

equation (4) as a set of ordinary differential equations utilizing the Fourier transform pair with respect to the lateral space variable y

$$\varphi_1(\mathbf{x}) = \sum_{q_1 = -\infty}^{\infty} \varphi_1^1(x) \exp\left[i\lambda_1 y\right],\tag{5}$$

$$\varphi_1^1(x) = \frac{1}{L_y} \int_{-L_y/2}^{L_y/2} \varphi_1(x) \exp\left[-i\lambda_1 y\right] dy.$$
 (6)

Here  $L_y$  is the lateral dimension of the domain, and  $\lambda_1 = q_1 \Delta \lambda = q_1 2\pi/L_y$ . The numerical superscript is used to indicate the lateral wave number component (this convention will be used throughout). For instance,  $\varphi_1^1$  is shorthand notation for  $\varphi_{\omega_1}^{\lambda_1}$  and represents the spectral amplitude of component ( $\omega_1$ ,  $\lambda_1$ ). Applying the lateral transform (6) to the transport equation (4), while replacing the  $\partial_y$  by  $i\lambda_1$ , yields the amplitude evolution equation

$$\frac{d\varphi_1^l}{dx} = \mathcal{G}_{1;2}^{\lambda} \left\{ \mathcal{N}_1^l(\mathbf{x}) \varphi_1^2 \right\} + \mathcal{G}_{1;23}^{\lambda} \left\{ \sum_{\omega_2,\omega_3} \frac{\mathcal{W}_{23}}{2\varkappa_1^l} \varphi_2^2 \varphi_3^3 \widehat{\delta}_{1;23}^{\omega} \right\}, \quad (7)$$

with

$$\mathcal{N}_1^{\mathbf{l}}(\mathbf{x}) = i\varkappa_1^{\mathbf{l}}(\mathbf{x}) - \frac{1}{2\varkappa_1^{\mathbf{l}}(\mathbf{x})} \frac{d\varkappa_1^{\mathbf{l}}(\mathbf{x})}{dx}$$
(8)

where  $\varkappa_1^1 = \operatorname{sgn}(\omega_1)\sqrt{k_1^2 - \lambda_1^2}$ . The operator  $\mathcal{G}$  is defined in Appendix A and symbolically denotes the (discrete) back and forth transformation between the (lateral) wave number and the physical domain such that the interactions are evaluated as products in (lateral) space rather than convolutions in the wave number domain.

[9] The set of transport equations (7) constitutes a deterministic model that evolves the angular spectrum components across the computational domain while accounting for topographical scattering and quadratic nonlinear effects. Although wide angle diffraction is accounted for, the forward scattering approximation limits the model to propagating modes traveling into the half plane of positive x, thus neglecting back-scattered wave components. Moreover, the model does not include evanescent modes ( $|\lambda_1| > k_1$ ) that, although potentially important locally on (very) steep slopes and near model boundaries [e.g., *Stamnes*, 1986; *Janssen et al.*, 2006], are generally of limited importance in the far field of topographical scatterers [e.g., *Magne et al.*, 2007].

[10] The assumption of forward propagating waves in equation (7) requires that variations in water depth over a typical wavelength are small ( $\beta \ll 1$ ). The two-dimensionality of the slowly varying depth appears in the slow lateral variation of the (principal) wave number (and its cross-shore gradient), and is incorporated through a convolution across the lateral wave number components. For weak lateral depth variability, the present model reduces to earlier angular-spectrum models [*Dalrymple et al.*, 1989; *Suh et al.*, 1990; *Janssen et al.*, 2006]; if instead the wave aperture is limited and a small-angle (parabolic) approximation is invoked, equation

(7) reduces to the model by *Kaihatu and Kirby* [1995] or wider-angle approximations thereof [*Kaihatu*, 2001].

[11] Equation (7) describes the evolution of the (transformed) velocity potential amplitude at z = 0. In terms of the (transformed) free-surface amplitudes  $A_1^1$  such that

$$\eta(x, y, t) = \sum_{q_1, p_1} \frac{A_1^1(x)}{\mathcal{P}_1(x, y)} \exp\left[i(\lambda_1 y - \omega_1 t)\right]$$
(9)

the evolution can be expressed as

$$\frac{dA_1^1}{dx} = \widetilde{\mathcal{G}}_{1,2}^{\lambda} \left\{ \mathcal{N}_1^1 A_1^2 \right\} - i \widetilde{\mathcal{G}}_{1,23}^{\lambda} \left\{ \sum_{\omega_2,\omega_3} \frac{g\omega_1 \mathcal{W}_{23}}{2\varkappa_1^1 \omega_2 \omega_3} A_2^2 A_3^3 \widetilde{\delta}_{1,23}^{\omega} \right\}.$$
(10)

where we used the dynamic free surface boundary condition in the linear approximation

$$A_1^1 = i \frac{\omega_1}{g} \varphi_1^1. \tag{11}$$

The use of a linear approximation in transforming the potential to the surface elevation function discards (quadratically forced) nonlinear contributions. These local boundwave corrections contribute to the surface elevation but are without dynamical consequences for the wave evolution [*Janssen et al.*, 2006] and their omission is consistent with the present approximation. After all, the nonlinear MSE (equation (2)), the starting point of our analysis, is a solvability condition, valid on the premise that the nonlinear forcing is near-secular: bound-wave corrections are thus neglected from the outset. If higher-order nonlinearity is pursued, accurate representation of nonsecular terms at second order is essential [*Janssen et al.*, 2006], but for the second-order nonlinear dynamics such terms are generally negligible [*Bredmose et al.*, 2005].

[12] We verify the two-dimensional capability of the deterministic forward-scattering model (10) with laboratory observations by *Chawla* [1995]. This particular topography is strongly two-dimensional, consisting of a circular shoal placed on a horizontal bottom (Figure 1), with depth varying from 45 cm away from the shoal to 8 cm at the shoal center (x = 5 m, y = 8.98 m). The incident wave field is monochromatic with 1.0 s period and 1.165 cm amplitude. We compare observed and predicted wave heights (Figure 2) along transects indicated in Figure 1. The spatial domain is discretized with  $\Delta y = 20$  cm and  $\Delta x = 20$  cm; the frequency array consists of the primary frequency (1 Hz) and its first harmonic.

[13] Although the topographic scattering is quite severe, and some wave energy is likely back-refracted [*Chawla*, 1995], the predictions are in excellent agreement with observations at all instrument locations (Figure 2). Inclusion of nonlinearity improves predictions inside the refractive focus region.

#### 3. A Coupled-Mode Stochastic Model

[14] To describe the evolution of wave statistics in nearshore regions, including the surf zone, we derive transport equations for the statistical moments of the sea surface elevation based on the results of section 2. First, we add a



**Figure 1.** Plan view experimental setup [*Chawla*, 1995]; solid lines indicate depth contours (unit meter); dashed lines indicate instrumented transects.

linear damping term to the transport equation (10) to parameterize energy loss due to depth-induced wave breaking

$$\frac{dA_1^1}{d\mathbf{x}} = \widetilde{\mathcal{G}}_{1;2}^{\lambda} \left\{ \left( \mathcal{N}_1^1 - \mathcal{D}_1(\mathbf{x}) \right) A_1^2 \right\} - i \widetilde{\mathcal{G}}_{1;23}^{\lambda} \left\{ \sum_{\omega_2,\omega_3} \frac{g\omega_1 \mathcal{W}_{23}}{2\varkappa_1^1 \omega_2 \omega_3} A_2^2 A_3^3 \widehat{\delta}_{1;23}^{\omega} \right\}.$$
(12)

Explicit expressions for the damping term  $\mathcal{D}_1$  [Janssen and Battjes, 2007] are given in Appendix C. We multiply (12) by  $(A_1^2)^*$  (with \* denoting the complex conjugate), add  $A_1^1 d_x (A_1^2)^*$ , and ensemble average the result. Upon letting  $\Delta \lambda$ ,  $\Delta \omega \to 0$ , we obtain the transport equation

$$\frac{d\mathcal{E}_{1}^{12}}{dx} = \widetilde{\mathcal{G}}_{1;3}^{\lambda} \left\{ \left( \mathcal{N}_{1}^{1} - \mathcal{D}_{1} \right) \mathcal{E}_{1}^{32} \right\} + \left( \widetilde{\mathcal{G}}_{2;3}^{\lambda} \left\{ \left( \mathcal{N}_{1}^{2} - \mathcal{D}_{1} \right) \mathcal{E}_{1}^{31} \right\} \right)^{*} \\
- i \left[ \widetilde{\mathcal{G}}_{1;34}^{\lambda} \left\{ \int \int \delta_{1;23}^{\omega} d\omega_{2} d\omega_{3} \frac{g\omega_{1} \mathcal{W}_{23}}{2\varkappa_{1}^{1} \omega_{2} \omega_{3}} \mathcal{C}_{23}^{342} \right\} \\
- \widetilde{\mathcal{G}}_{2;34}^{\lambda} \left\{ \int \int \delta_{1;23}^{\omega} d\omega_{2} d\omega_{3} \frac{g\omega_{1} \mathcal{W}_{23}}{2\varkappa_{1}^{2} \omega_{2} \omega_{3}} \left( \mathcal{C}_{23}^{341} \right)^{*} \right\} \right], \quad (13)$$

where  $\delta_{1;23}^{\omega} = \delta(\omega_2 + \omega_3 - \omega_1)$  and  $\delta$  is a Dirac delta function. The operator  $\tilde{\mathcal{G}}$  in (13) denotes the back and forth transformation between the lateral physical and wave number domains; it is defined in Appendix A and is equivalent to  $\mathcal{G}$ , but operates on continuous spectral variables. The variables  $\mathcal{E}$  and  $\mathcal{C}$  in (13) are defined as

$$\mathcal{E}_{1}^{12}(x) = \mathcal{E}(\omega_{1}, \lambda_{1}, \lambda_{2}, x) = \lim_{\Delta\lambda, \Delta\omega \to 0} \frac{\left\langle A_{1}^{1} \left( A_{1}^{2} \right)^{*} \right\rangle}{\Delta\lambda^{2} \Delta\omega}, \quad (14)$$

$$\mathcal{C}_{12}^{123}(x) = \mathcal{C}(\omega_1, \omega_2, \lambda_1, \lambda_2, \lambda_3, x) = \lim_{\Delta \lambda, \Delta \omega \to 0} \frac{\left\langle A_1^1 A_2^2 \left( A_{(1+2)}^3 \right)^* \right\rangle}{\Delta \lambda^3 \Delta \omega^2}$$
(15)

where  $\langle \rangle$  denotes the ensemble average. We will refer to these quantities as the mutual spectrum and mutual bispectrum respectively to distinguish them from conven-



**Figure 2.** Comparison of predicted and observed [*Chawla*, 1995] normalized wave heights for normally incident monochromatic waves with height  $H_0 = 2.3$  cm and 1.0 s period. Thick solid lines represent nonlinear model predictions (equation (7)); dashed lines are linear predictions (same equation without nonlinear coupling).



**Figure 3.** (top) Plan view experimental setup [*Vincent and Briggs*, 1989]. Depth contours (unit meter) are indicated by solid lines; instrumented transects are indicated by dashed lines. (bottom) Predicted normalized wave height for monochromatic, unidirectional incident wave train ( $\omega = 1.45\pi$  rad) for the same area as shown in Figure 3, top.

tional spectra and bi-spectra. The sum of the wave number components  $\lambda_j$  in (14) and (15) does not generally add up to zero, allowing for the mutual coupling (in the statistical sense) between crossing waves, induced by the seafloor scattering.

[15] The transport equation (13) is the main result of this section. It governs the transformation of the mutual spectrum  $\mathcal{E}$  along the principal direction (*x*), whereas the statistical variability in the lateral direction is captured by the two-dimensionality of  $\mathcal{E}_1^{12}$  in wave number space. Evaluation of the nonlinear contribution to the spectral evolution requires an estimate of the bispectrum  $\mathcal{C}_{12}^{123}$ ; this may be obtained either directly from measurements [*Herbers et al.*, 2000], or by solving an additional evolution equation for the third-order statistics [*Herbers et al.*, 2003; *Janssen*, 2006]. The extension to higher-order statistics, and the associated closure issues, will be discussed in a subsequent paper.

[16] The local wave variance can be expressed as

$$\langle \eta^{2}(t,x,y) \rangle$$

$$= \int \int \mathcal{P}_{\omega}^{-2} \left( \int \mathcal{E}(\omega,\lambda+\lambda'/2,\lambda-\lambda'/2,x) \exp\left[i\lambda'y\right] d\lambda' \right) d\lambda d\omega$$
(16)

where  $\lambda = (\lambda_1 + \lambda_2)/2$ , and  $\lambda' = \lambda_1 - \lambda_2$ . The expression (16) includes potentially rapid spatial variations in the wave variance associated with correlations between wave components propagating at large mutual angles. Such cross-mode coupling, which occurs for example in the refractive

focus of topographical features, is neglected in conventional stochastic spectral models (see section 4).

[17] In the linear approximation, the stochastic model (equation (13)) is closed, i.e., no additional (closure) approximations were introduced. Consequently, the stochastic model inherits the complete linear refraction and diffraction characteristics embedded in the deterministic model (equation (10)). This implies that for sufficiently large ensembles, Monte Carlo simulations with the linear deterministic model converge to predictions by the (linear) stochastic model (equation (13) without the nonlinear and dissipation terms). In particular, for monochromatic, unidirectionally incident waves, the relative spatial distribution of wave variance is determined entirely by the interaction with the (deterministic) topography, and does not depend on the initial conditions. Therefore for such cases, the linear stochastic and deterministic model, differing only in the statistical averaging operation, predict the exact same normalized wave height.

[18] We have verified numerically both the convergence of Monte Carlo simulations, and the exact equivalence of the deterministic and stochastic model for monochromatic, unidirectional incident waves (not shown, see *Janssen* [2006]). In particular, we have verified, as a validation of the numerical implementation, that the linear stochastic model prediction for the laboratory case of Figures 1 and 2 is identical (not shown) to the (linear) deterministic model result shown in Figure 2.

[19] To further verify the representation of topographyinduced wave field inhomogeneity, we compare predictions of the linear stochastic model to another set of laboratory observations of wave evolution over a submerged shoal, including random, directionally spread incident waves [*Vincent and Briggs*, 1989]. The bottom consists of an elliptic shoal, with its crest 15.24 cm below still-water level, placed on an otherwise uniform depth of h = 45.72 cm (Figure 3). Further details are found in the work of *Vincent and Briggs* [1989].

[20] The computational domain measures  $L_x = 20$  m by  $L_y = 30$  m in principal and lateral direction, respectively. Consequently, the alongshore wave number spacing is  $\Delta \lambda = 2\pi/L_y = \pi/15$  rad/m. The spatial domain is discretized with  $\Delta x = 0.1$  m,  $\Delta y = 0.24$  m. Comparison is made to wave heights observed along the instrumented transects that are indicated in Figure 3.

[21] In our comparison we consider two cases: (1) a monochromatic, unidirectional incident wave field, with angular frequency  $\omega = 1.45\pi$  rad/s (Figure 4), and (2) a random wave field (TMA spectrum) with moderate directional spreading (case N1 of *Vincent and Briggs* [1989]) and peak frequency  $\omega_p = 1.54\pi$  rad/s (Figure 5). The monochromatic waves do not break, but for the random wave case intermittent breaking was observed in the vicinity of the mound [*Vincent and Briggs*, 1989]. We model the monochromatic case with a single frequency; the random wave frequency spectrum is discretized into 20 equidistant positive frequencies, with  $\Delta \omega = 0.2\pi$  rad/s.

[22] From geometrical optics we anticipate that this topography results in wave-ray convergence in the lee of the mound, reminiscent of focusing of light by a burning lens. For the unidirectional, monochromatic case (Figure 4), which is the archetype of a fully coherent incident wave



**Figure 4.** Comparison of observed (circles) [*Vincent and Briggs*, 1989] and predicted (solid line) wave heights; normally incident, monochromatic waves ( $\omega = 1.45\pi$  rad/s) (case M1 [*Vincent and Briggs*, 1989]).

field, the crossing waves in the caustic region originate from the same source, and their interference causes rapid modulations in the wave field statistics. The stochastic model accurately resolves the observed wave height variations across the refractive zone (Figure 4). For random, directionally spread incident waves (Figure 5), the lateral wave height variations are much more gradual. In effect, the decrease in coherency of the incident waves "smoothes out" the caustic in the convergence region (as also discussed in the work of *Vincent and Briggs* [1989]). Again, also for this case, predicted wave heights are in good agreement with observations although dissipation over the shoal is apparently underestimated in the model.

# 4. Discussion

### 4.1. A Coupled-Mode Spectrum

[23] The mutual spectrum  $\mathcal{E}(\omega, \lambda_1, \lambda_2, x)$  is a complete representation of the lowest-order statistics of the wave field, including the spatial heterogeneity associated with



**Figure 5.** Comparison of observed (circles) [*Vincent and Briggs*, 1989] and predicted (solid line) wave heights; directionally spread, random incident waves (TMA spectrum), peak frequency  $\omega_p = 1.54\pi$  rad/s (case N1 [*Vincent and Briggs*, 1989]).

caustics. However, since this representation is not 'local' in the lateral sense, the geometrical interpretation is somewhat obscured. A local spectrum can be obtained by taking the inverse Fourier transform of the mutual spectrum  $\mathcal{E}(\omega, \lambda_1, \lambda_2, x)$  with respect to the difference wave number  $\lambda_1 - \lambda_2$ , written here as

$$\widehat{\mathcal{E}}(\omega,\lambda,x,y) = \int \mathcal{E}(\omega,\lambda+\lambda'/2,\lambda-\lambda'/2,x) \exp\left[i\lambda'y\right] d\lambda'$$
(17)

where  $\lambda = (\lambda_1 + \lambda_2)/2$  and  $\lambda' = \lambda_1 - \lambda_2$ . The spectrum  $\hat{\mathcal{E}}(\omega, \lambda, x, y)$  is a function both of the lateral wave number and the lateral physical coordinate (in contrast to the mutual spectrum  $\mathcal{E}$ , which is three-dimensional in spectral space and one-dimensional in physical space). This form of the spectrum we will refer to as a coupled-mode (CM) spectrum. The surface wave variance can be written as

$$\langle \eta^2(t,x,y) \rangle = \int \int \mathcal{S}(\omega,\lambda,x,y) d\lambda d\omega,$$
 (18)

where

$$S(\omega, \lambda, x, y) = \frac{\widehat{\mathcal{E}}(\omega, \lambda, x, y)}{\mathcal{P}^2(\omega)}$$
(19)

The expression (18) suggests that S is a variance density spectrum [e.g., *Tolman*, 1991; *Komen et al.*, 1994; *Booij et al.*, 1999]. However, the resemblance is misleading. Whereas a variance density must be positive, the CM spectrum S can attain negative values without violating causality. To substantiate this, consider for instance the surface elevation associated with a bidirectional, monochromatic wave field, written in discrete form as

$$\eta(t, x, y) = \left[\zeta^{\lambda_1}(x) \exp\left[i\lambda_1 y\right] + \zeta^{\lambda_2}(x) \exp\left[i\lambda_2 y\right]\right] \exp\left[-i\omega t\right] + *.$$
(20)

with  $|\lambda_1|$ ,  $|\lambda_2| < k(\omega)$  and  $\lambda_1 \neq \lambda_2$ . The two-point correlation of laterally separated surface observations can be written as

$$\langle \eta(t, x, y + y'/2)\eta(t, x, y - y'/2) \rangle$$

$$= \left\langle |\zeta^{\lambda_1}|^2 \right\rangle \exp\left[i\lambda_1 y'\right] + \left\langle |\zeta^{\lambda_2}|^2 \right\rangle \exp\left[i\lambda_2 y'\right]$$

$$+ \left[ \left\langle \zeta^{\lambda_1} \left(\zeta^{\lambda_2}\right)^* \right\rangle \exp\left[i\lambda' y\right] \exp\left[i\lambda y'\right] + * \right]$$

$$(21)$$

where again  $\lambda = (\lambda_1 + \lambda_2)/2$  and  $\lambda' = \lambda_1 - \lambda_2$ . Thus the variance  $(y' \rightarrow 0)$  consists of spatially invariant contributions (first two terms in the right-hand side of equation (21)) and modulations (last term in brackets of equation (21)), associated with the coherence between  $\zeta^{\lambda 1}$  and  $(\zeta^{\lambda 2})^*$ ; the latter contributions capture potentially fast modulations associated with wave interference patterns. The CM spectrum S thus contains both variance contributions of

the individual spectral components and variance modulations associated with their mutual coupling; only when the cross-mode coupling is negligible does it represent a spectral variance distribution.

[24] A coupled-mode stochastic approach that includes spatial inhomogeneity through cross-mode correlations appears not to have been used before in the context of shallow water gravity waves. However, it has been used to study instability processes occurring in narrow-band random wave fields in deep water [e.g., Alber, 1978; Crawford et al., 1980; Yuen and Lake, 1975], and the occurrence of rogue waves [Janssen, 2003]. In fact, across various branches of science similar concepts have surfaced independently, and under different names: for instance, the Wigner distribution in quantum mechanics and optics [e.g., Wigner, 1932; Mori et al., 1962; Wigner, 1971; Bremmer, 1932; Bastiaans, 1979], the Wigner-Ville distribution in signal analysis [e.g., Ville, 1948; Mallat, 1998], and the concept of generalized radiance in radiometry [e.g., Walther, 1968, 1973; Marchand and Wolf, 1974; Wolf, 1978]. Historically, the CM spectrum can be regarded as a manifestation of the Wigner distribution [Wigner, 1932]. However, since in the present context the CM spectrum is in many ways a generalization of the widely used variance density spectrum, we refer to it as a coupled-mode spectrum to emphasize its physical significance rather than its eponymy.

# 4.2. Radiative Transfer Equation

[25] Our stochastic model (equation (13)) transports the mutual spectrum. Alternatively, it can be expressed in terms of the CM spectrum directly. Thereto we apply an inverse Fourier transform with respect to the difference wave number  $\lambda_1 - \lambda_2$  on the linear (conservative) part of (13), which can be written as [e.g., *Bremmer*, 1932]

$$\partial_{x}\widehat{\mathcal{E}}(\omega,\lambda,x,y) = i\left[\left(\widehat{\varkappa}^{+} + i\frac{\partial_{x}\widehat{\varkappa}^{+}}{2\widehat{\varkappa}^{+}}\right) - \left(\widehat{\varkappa}^{-} - i\frac{\partial_{x}\widehat{\varkappa}^{-}}{2\widehat{\varkappa}^{-}}\right)\right]\widehat{\mathcal{E}}(\omega,\lambda,x,y)$$
(22)

where

$$\widehat{\varkappa}^{s} = \sqrt{k^{2} \left(\omega, x, y + s\frac{i}{2}\frac{\partial}{\partial\lambda}\right) + \left(\frac{1}{2}\frac{\partial}{\partial y} + si\lambda\right)^{2}}$$
(23)

and the sign index  $s = \pm$ . If the surface wave field is spatially incoherent (broad directional spreading), wave components scattered at large angles generally originate from independent sources, and are thus uncorrelated. For such cases only near-collinear wave components remain correlated, such that the CM spectrum is a slowly varying quantity. If we then assume that  $k^{-1}\partial_y \sim O(\epsilon) \ll 1$ , Taylor expand (23) around  $(y, \lambda)$ , we find

$$\left[\frac{\partial}{\partial x} - \frac{\partial \varkappa_1^1}{\partial \lambda_1}\frac{\partial}{\partial y} + \frac{\partial \varkappa_1^1}{\partial y}\frac{\partial}{\partial \lambda_1}\right] \left(V_1^1 \mathcal{S}(\omega_1, \lambda_1, x, y)\right) = 0.$$
(24)



**Figure 6.** Contours of wave height (normalized by the offshore values) behind (top) semi-infinite breakwater and (bottom) breakwater gap. Comparison between stochastic angular spectrum model (equation (28), left) and analytic expression [*Penney and Price*, 1952] (right). The (x', y') are the horizontal coordinates normalized with the wavelength. Normally incident monochromatic waves with  $\omega = \pi$  rad/s and kh = 1.2. Contours are drawn between 0 and 1.5 at equidistant intervals of 0.15.

where  $V_1^1 = (\varkappa_1^1/k_1)C_{g,1}$ . Equation (24) is a Liouville-type equation and represents a forward-scattering approximation of the radiative transfer equation, implying the conservation of  $S_1^1$  in a slowly varying medium in the absence of time-varying currents. In this approximation, the spectrum  $S_1^1$  regains its physical significance of a variance density function, transported along the rays of geometrical optics.

#### 4.3. Wide-Angle Diffraction

[26] Ocean waves generated by local storms are often fairly broad-banded. This results in a smoothing out of caustics such that the variations in the wave statistics are gradual, and generally well described by the approximation of geometrical optics [see, e.g., *Magne et al.*, 2007]. However, as a result of dispersion, remotely generated swell can be narrow-banded and its interaction with seafloor topography can result in caustic regions where a diffraction theory (equation (13)) is needed [e.g., *O'Reilly and Guza*, 1991].

[27] The diffraction capability of our stochastic model is implicit in the comparison between observations and model predictions in section 3. However, to illustrate the wideangle diffraction capability by means of a classical example, we consider waves propagating through a gap in a thin, rigid but absorbing barrier along the line x = 0. The gap through which the waves can penetrate extends over  $-G_1 < y < G_2$ . Waves originate from sources in the half plane x < 0, in which region the spectrum is assumed homogeneous and known. To determine the spectrum in the half plane x > 0 from the matching condition at x = 0 we use the physical optics or Kirchhoff approximation

$$\frac{d\Phi}{dx}\Big|_{0^+} = \begin{cases} \frac{d\Phi_i}{dx}\Big|_{0^-}, & -G_1 < y < G_2\\ 0, & \text{elsewhere,} \end{cases}$$
(25)

where  $x = 0^{\pm}$  denotes locations just inside/outside the domain x > 0; the subscript *i* on  $\Phi$  in (25) designates the incident wave field, which is assumed the same as when the barrier would have been absent. We write the incident potential function at  $x = 0^{-}$  as

$$\Phi_{i}(0^{-}, y, z, t) = \sum_{\lambda_{1}, \omega_{1}} -i \frac{g}{\omega_{1}} \frac{A_{1,i}^{1}(x)}{\mathcal{P}_{1}} \frac{\cosh k_{1}(z+h)}{\cosh k_{1}h}$$
$$\cdot \exp\left[i(\lambda_{1}y - \omega_{1}t)\right]; \quad \frac{dA_{1,i}^{1}}{dx} = i\varkappa_{1}^{1}A_{1,i}^{1}, \tag{26}$$

From the matching condition at the gap (namely, (25)) we obtain  $A_1^1$ , from which we, by forming the statistical moment  $\langle A_1^1(A_1^2)^* \rangle$ , obtain the mutual spectrum  $\mathcal{E}_1^{12}$  at  $x = 0^+$ . Since we assume a uniform depth, and omit dissipation and nonlinear effects, the transport equation for  $\mathcal{E}$  (equation (13)) for x > 0 simplifies to

$$\frac{d}{dx}\mathcal{E}_{1}^{12} = i(\varkappa_{1}^{1} - \varkappa_{1}^{2})\mathcal{E}_{1}^{12}.$$
(27)



**Figure 7.** Comparison of normalized wave height along lateral transects (left) x' = 4 and (right) x' = 15 behind the breakwater; solid lines represent analytic solution [*Penney and Price*, 1952], circles are predictions stochastic model (equation (28)). The (x', y') are the horizontal coordinates normalized with the wavelength. Normally incident monochromatic waves with  $\omega = \pi$  rad/s and kh = 1.2.

so that the spectrum  $\mathcal{E}_1^{12}$  in the region x > 0 can be expressed analytically as

$$\mathcal{E}(\omega_1, \lambda_1, \lambda_2, x) = \mathcal{E}(\omega_1, \lambda_1, \lambda_2, x = 0^+) \exp\left[i(\varkappa_1^1 - \varkappa_1^2)x\right]$$
$$= \frac{1}{\pi^2} \int d\lambda_3 \frac{(\varkappa_1^3)^2}{\varkappa_1^1 \varkappa_1^2} \widehat{\mathcal{E}}(\omega_1, \lambda_3, x = 0^-)$$
$$\cdot \frac{\sin\left((\lambda_3 - \lambda_1)G_m\right)}{(\lambda_3 - \lambda_1)} \frac{\sin\left((\lambda_3 - \lambda_2)G_m\right)}{(\lambda_3 - \lambda_2)}$$
$$\cdot \exp\left[i\left((\varkappa_1^1 - \varkappa_1^2)x + (\lambda_1 - \lambda_2)G_\Delta\right)\right]. \quad (28)$$

Here  $G_m = (G_1 + G_2)/2$  and  $G_{\Delta} = (G_1 - G_2)/2$ . Thus for linear waves across a barrier gap in a region of uniform depth, the convolution in (28) relates the mutual spectrum in the half plane of positive x to the incident wave spectrum at 0<sup>-</sup>. This solution is not exact. Its approximate nature, apart from the simplifications implied by the use of an inviscid theory, originates from the use of geometrical optics for the matching condition (25), and the neglect of nonlinearity and evanescent modes [*Stamnes*, 1986].

[28] To validate this approximation we numerically integrate the differential equation (27), using the initial condition  $\mathcal{E}(\omega_1, \lambda_1, \lambda_2, x = 0^+)$  obtained from the matching condition (25), and compare to the analytical expressions in the work of *Penney and Price* [1952], for a rigid, absorbing barrier as assumed here. The numerical integration is performed for a discrete lateral wave number array  $[-79...79]\Delta\lambda$  for  $\lambda_1$  and  $\lambda_2$  with  $\Delta\lambda = k_1/80$ , and a spatial resolution  $\Delta x = \Delta y = 1$  m. The comparisons (Figures 6 and 7) are for monochromatic waves, normally incident on a semi-infinite screen  $(G_1 = L_y/2, G_2 = 0, \text{ with } L_y$  the lateral extent of the domain), and a finite barrier gap  $(G_1 = G_2 = 2.65 \text{ wavelengths} \ll L_y)$ . The agreement between the numerical and analytical solution is excellent, even at locations just a few wavelengths from the boundary. This shows that, despite the approximations implied by the use of the matching condition (25) and the transport equation (27), diffraction effects on directional wave spectra transmitted through a barrier gap are accurately described in this manner.

[29] For the purpose of illustration we considered here a classical pure diffraction problem with a well-known analytical solution [*Sommerfeld*, 1896]. However, the stochastic modeling approach is of course suitable for typically broadband incident wave spectra as commonly observed in coastal areas. Moreover, by using the matching condition (25) to initialize the more general transport equation (13), we can evolve the transmitted spectrum over variable depth, incorporating nonlinear effects and (parameterized) wave breaking.

# 5. Conclusions

[30] On the basis of a deterministic, forward scattering approximation of a nonlinear mild slope equation, we derived

a stochastic model for the evolution of random, directionally spread ocean surface waves over two-dimensional sea floor topography. In the present work we discuss the stochastic representation of spatial coherence and intermode coupling resulting from the interaction with the topography. We have related the general representation of the lowest-order wave statistics to the concepts of geometrical optics and physical optics. Comparison to analytic expressions for diffraction, and laboratory observations of waves over a topographical lens, confirms that the effects of wide angle refraction and diffraction for arbitrary coherency of the incident wave field can be accurately captured by a coupled-mode representation of the wave field statistics.

# Appendix A: Transform Operators

[31] For convenience we make use of a shorthand notation to describe repeated back-and-forth Fourier transform operations, which are used to evaluate spectral convolutions as products in the physical domain.

[32] The discrete Fourier transformation and its inverse are denoted by  $\mathcal{F}$ {} and  $\mathcal{F}^{-1}$ {}, respectively, and defined as

$$\mathcal{F}_{\lambda_1}\{f(y)\} \equiv \frac{1}{L_y} \int_{-L_y/2}^{L_y/2} f(y) \exp\left[-i\lambda_1 y\right] dy = F_{\lambda_1}, \quad (A1)$$

$$\mathcal{F}_{\lambda_{1}}^{-1}\{F_{\lambda_{1}}\} \equiv \sum_{q_{1}=-\infty}^{\infty} F_{\lambda_{1}} \exp\left[i\lambda_{1}y\right] = f(y).$$
(A2)

Here the *f* and *F* are dummy variables, *y* is the (continuous) physical variable, and the discrete lateral wave number  $\lambda_1 = q_1 \Delta \lambda = 2q_1 \pi / L_y$ . The function *f*(*y*) is periodic with *L<sub>y</sub>*, the extent of the domain in physical (*y*) space.

[33] For continuous spectral variables  $(\Delta \omega, \Delta \lambda \to 0)$  we denote the integral transformation and its inverse by  $\widetilde{\mathcal{F}}$ {} and  $\widetilde{\mathcal{F}}^{-1}$ {}, respectively, which are defined as

$$\widetilde{\mathcal{F}}_{\lambda_{1}}\{f(y)\} \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \exp\left[-i\lambda_{1}y\right] dy = F(\lambda_{1}), \quad (A3)$$

$$\widetilde{\mathcal{F}}_{\lambda_{1}}^{-1}\{F\lambda_{1}\} \equiv \int_{-\infty}^{\infty} F(\lambda_{1}) \exp\left[i\lambda_{1}y\right] d\lambda_{1} = f(y), \qquad (A4)$$

and are to be understood in the limit sense of generalized Fourier transforms [see, e.g., *Lighthill*, 1958; *Kinsman*, 1965].

[34] The repeated back-and-forth transformation operating on discrete spectral variables is denoted by the operator  $\mathcal{G}_{1;2...N}^{\lambda}$  and defined as

$$\begin{aligned} \mathcal{G}_{1;2...N}^{\lambda} \Big\{ f(y) \, G_{\lambda_2}^1 \dots G_{\lambda_N}^{(N-1)} \Big\} \\ &= \mathcal{F}_{\lambda_1} \Big\{ f(y) \mathcal{F}_{\lambda_2}^{-1} \Big\{ G_{\lambda_2}^1 \Big\} \dots \mathcal{F}_{\lambda_N}^{-1} \Big\{ G_{\lambda_N}^{(N-1)} \Big\} \Big\} \\ &= \mathcal{F}_{\lambda_1} \Big\{ f(y) g^1(y) \dots g^{(N-1)}(y) \Big\}, \end{aligned}$$
(A5)

where the  $g^i$ , i = 1...N are dummy variables and the transformed (dummy) variables are denoted by capitals subscripted by  $\lambda_i$  (as before).

[35] For continuous spectral variables this operation is denoted by  $\widetilde{\mathcal{G}}_{1;2,..,N}^{\lambda}$  and defined as

$$\begin{aligned} \widetilde{\mathcal{G}}_{1;2...N}^{\lambda} \Big\{ f(y) \, G^1(\lambda_2), \dots, G^{(N-1)}(\lambda_N) \Big\} \\ &= \widetilde{\mathcal{F}}_{\lambda_1} \Big\{ f(y) \widetilde{\mathcal{F}}_{\lambda_2}^{-1} \Big\{ G_{\lambda_2}^1 \Big\} \dots \widetilde{\mathcal{F}}_{\lambda_N}^{-1} \Big\{ G_{\lambda_N}^{(N-1)} \Big\} \Big\} \\ &= \widetilde{\mathcal{F}}_{\lambda_1} \Big\{ f(y) g^1(y) \dots g^{(N-1)}(y) \Big\}. \end{aligned}$$
(A6)

# Appendix B: A Forward Scattering Approximation

[36] To reduce the model (2) to a forward scattering approximation, we introduce slow spatial variables

$$\boldsymbol{X} = \boldsymbol{\epsilon} \boldsymbol{x}, \qquad \nabla = \nabla_{\boldsymbol{x}} + \boldsymbol{\epsilon} \nabla_{\boldsymbol{X}}. \tag{B1}$$

and assume that

$$\varphi_1 = \epsilon \widetilde{\varphi}_1(\mathbf{x}, \mathbf{X}) \tag{B2}$$

$$k_1 = k_1(\boldsymbol{X}). \tag{B3}$$

Insertion of (B1), (B2), and (B3) into the solvability condition (2) yields (to  $O(\epsilon^2)$ )

$$\nabla_{x}^{2} \widetilde{\varphi}_{1} + k_{1}^{2} \widetilde{\varphi}_{1} + \epsilon [\nabla_{x} \cdot \nabla_{x} + \nabla_{x} \cdot \nabla_{x}] \widetilde{\varphi}_{1}$$
  
=  $i\epsilon \sum_{\omega_{1},\omega_{2}} W_{23} \widetilde{\varphi}_{2} \widetilde{\varphi}_{3} \widehat{\delta}_{1;23}^{\omega} + O(\epsilon^{2}).$  (B4)

We introduce an angular-spectrum decomposition with amplitudes that vary slowly in the lateral direction, written as

$$\widetilde{\varphi}_1(x, y, X, Y) = \sum_{q_1 = -\infty}^{\infty} \widetilde{\varphi}_1^{\nu_1}(x, X, Y) \exp\left[i\nu_1 y\right], \qquad (B5)$$

where  $\nu_1 = q_1 \Delta \nu$ , with  $\Delta \nu$  the lateral wave number interval of the fast scale. Physically, this decomposition applies to a region large enough such that the  $\Delta \nu$  resolves the directional wave field and small enough such that the medium can be considered laterally homogeneous. Inserting (B5) in the lowest-order part of (B4) yields

$$\frac{\partial^2 \widetilde{\varphi}_1^{\nu_1}}{\partial x^2} = -\left(k_1^2 - \nu_1^2\right) \widetilde{\varphi}_1^{\nu_1}.$$
(B6)

so that for the forward propagating wave components we have

$$\frac{\partial \widetilde{\varphi}_1^{\nu_1}}{\partial x} = i\varkappa_1^{\nu_1} \widetilde{\varphi}_1^{\nu_1} \tag{B7}$$

where  $\varkappa_1^{\nu_1} = \text{sgn}(\omega_1)\sqrt{k_1^2 - \nu_1^2}$ . The  $\varkappa_1^{\nu_1}$  is a local, principal (or cross-shore) wave number, which varies on the slow space scales. Applying the same local decomposition (B5) to the second-order part of (B4), using (B7), and combining the first- and second-order results, yields

$$\frac{\partial \widetilde{\varphi}_{1}^{\nu_{1}}}{\partial x} + \epsilon \frac{\partial \widetilde{\varphi}_{1}^{\nu_{1}}}{\partial X} = \left( i \varkappa_{1}^{\nu_{1}} + \epsilon \frac{\partial \varkappa_{1}^{\nu_{1}}}{\partial \nu_{1}} \frac{\partial}{\partial Y} - \epsilon \frac{1}{2\varkappa_{1}^{\nu_{1}}} \frac{\partial \varkappa_{1}^{\nu_{1}}}{\partial X} \right) \widetilde{\varphi}_{1}^{\nu_{1}} + \epsilon \sum_{\substack{\omega_{1}, \omega_{2} \\ \nu_{2}, \nu_{3}}} \frac{\mathcal{W}_{23}}{2\varkappa_{1}^{\nu_{1}}} \widetilde{\varphi}_{2}^{\nu_{2}} \widetilde{\varphi}_{3}^{\nu_{3}} \widehat{\delta}_{1;23}^{\omega} \widehat{\delta}_{1;23}^{\nu} \right)$$
(B8)

We combine the first two terms on the right-hand side as

$$i\varkappa_{1}^{\nu_{1}} + \epsilon \frac{\partial \varkappa_{1}^{\nu_{1}}}{\partial \nu_{1}} \frac{\partial}{\partial Y} = i \operatorname{sgn}(\omega_{1}) \sqrt{k_{1}^{2} + \left(i\nu_{1} + \epsilon \frac{\partial}{\partial Y}\right)^{2} + O(\epsilon^{2})}.$$
(B9)

Then, to return to physical variables and coordinates, we apply the inverse transform with respect to  $\nu_1$ , effectively replacing  $i\nu_1$  with  $\partial_3$ , and absorb the small parameters, so that the end result can be written as equation (4).

### Appendix C: Wave Breaking Parameterization

[37] To parameterize the loss of wave energy in the breaking process, we introduced a frequency-dependent damping term  $\mathcal{D}_1$  in equation (12), which results in a sink term in our stochastic model (13). Here we derive an expression for  $\mathcal{D}_1$ , based on the bulk energy dissipation rate in a random wave field, D, for which we utilize the expressions derived by *Janssen and Battjes* [2007].

[38] Since it is unknown how breaking affects the crosscorrelations between noncollinear wave components, we assume, for our present purpose, slowly varying (e.g., quasi-homogeneous) wave statistics. In that approximation,  $D_1$  is a rate of variance loss with a frequency dependent weighting to accommodate the empirical observations that energy is lost more strongly at higher frequencies [see, e.g., *Chen et al.*, 1997; *Herbers et al.*, 2000], which can be expressed as

$$\mathcal{D}_1(\mathbf{x}) = \left(\sum_n \frac{r_n}{\widetilde{m}_n} |\omega_1|^n\right) \mathcal{D}(\mathbf{x}).$$
(C1)

Here the  $r_n$  are weighting coefficients ( $0 \le r_n \le 1$  and  $\sum_n r_n = 1$ ), which can be varied to allow different frequency weightings of the dissipation across the spectrum. After *Janssen and Battjes* [2007], we write the bulk variance dissipation rate  $\mathcal{D}$  in (C1) as

$$\mathcal{D} = \frac{3}{2\sqrt{\pi}} B \frac{m_1 \sqrt{2m_0}}{h} \left[ 1 + \frac{4}{3\sqrt{\pi}} \left( R^3 + \frac{3}{2} R \right) \exp\left[ -R^2 \right] - \operatorname{erf}(R) \right]$$
(C2)

where  $R = \gamma h/H_{\text{rms}}$ . The  $\tilde{m}_n$  and  $m_n$  in (C1) and (C2) are spectral moments defined as

$$\widetilde{m}_{n} = \int \int |\omega_{1}|^{n} V_{1}^{1} \mathcal{S}_{1}^{1} d\lambda_{1} d\omega_{1},$$

$$m_{n} = \int \int |\omega_{1}|^{n} \mathcal{S}_{1}^{1} d\lambda_{1} d\omega_{1}.$$
(C3)

Finally, to complete the parametric representation of wave breaking, we choose, based on a few trial runs [*Janssen*, 2006],  $r_0 = 0.1$ ,  $r_2 = 0.9$ , and B = 1. After *Baldock et al.* [1998] we set

$$\gamma = 0.39 + 0.56 \tanh 33S_0 \tag{C4}$$

where  $S_0$  denotes the deep-water wave steepness as defined by *Battjes and Stive* [1985]. [39] Acknowledgments. This research was supported by the Technology Foundation STW, applied science division of NWO, Netherlands. Part of this research was performed while TTJ held a National Research Council Research Associateship, funded by the National Science Foundation (Physical Oceanography Program) and the Naval Postgraduate School in the USA. THCH is supported by the U.S. Office of Naval Research (Coastal Geosciences Program) and the National Science Foundation (Physical Oceanography Program). We thank Ad Reniers for sharing his code for evaluating the analytic diffraction solutions. Arun Chawla, Michael Briggs, and Linwood Vincent are gratefully acknowledged for making their laboratory observations available.

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