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Modification of the Surface Elevation Probability Distribution in Ocean Swell by Nonlinear Spectral Broadening

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In this paper we investigate the effect of the Benjamin-Feir instability on the surface elevation probability distribution. A uniform wave train is unstable to modulational perturbations, giving rise to sidebands in the spectrum of the surface elevation (hence a broadening of the spectrum). Since these sidebands are phase-locked, one should therefore expect deviations from the Gaussian probability distribution. To calculate the surface elevation probability distribution, we assume that the end state of the instability is given by a stable cnoidal wave solution. The present effect could play a role in nature; especially, it may be relevant for the description of swell.

1. Introduction

It is well known that ocean waves can propagate over large distances. Early investigations [*Munk et al.*, 1959; *Snodgrass et al.*, 1966] demonstrated the possibility to relate the observed swell to remote source regions, sometimes more than half a global circumference away. The observed swell spectra turned out to be quite narrow (see also *Harris et al.* [1973]; *Thompson and Smith* [1974]). This narrowness, and the slow migration of the swell frequency with time, agreed nicely with expectations based on the dispersive character of deep water waves and the peakedness of wind sea spectra during wave growth under the influence of wind.

A striking feature of time records of swell is the occurrence of wave groups. The standard stochastic description of sea waves is based on the assumption that there is a large number of different Fourier modes with independent phases. This immediately leads to a Gaussian distribution for the surface elevation. Further, it can be shown [*Longuet-Higgins*, 1957] that the wave envelope function satisfies a Rayleigh distribution. This in turn determines, at least in case of a narrow spectrum, the wave height distribution and the statistics of the length of wave groups. (A wave group is defined by the requirement that successive maxima exceed a certain level). This idea was worked out further by *Ewing* [1973]. He showed how the average group length can be related to the spectral width. Naturally, narrow spectra imply long wave groups. Wave groups have been studied extensively because of their practical importance. A review was recently given by *Rye* [1980]. In general, there is a reasonable agreement between theory and observations, although studies have been reported [*Goda*, 1976] which seem to indicate a tendency for wave groups to be longer than expected. This is sometimes tentatively ascribed to phase relations between the individual waves, which would be the result of nonlinear interaction.

The effect of the nonlinear interactions on the probability distribution of sea waves has been studied by several people. The classical calculation was made by *Longuet-Higgins* [1963]. He showed how the upside-down asymmetry of deterministic sea waves had its counterpart in Gram-Charlier–like corrections to the Gaussian probability distribution.

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Later work along these lines was reported by *Tayfun* [1980] and *Longuet-Higgins* [1980]. The early experimental field verification (*Cartwright and Longuet-Higgins* [1956]; see also *Phillips* [1977]) was not geared toward a detection of nonlinear deviations. Later work [*Forristal*, 1978; *Longuet-Higgins*, 1980] is still somewhat controversial. In the laboratory interesting data were reported recently by *Huang and Long* [1980]. Their results could be described with a Gram-Charlier series, although the approximation was not uniform.

The theoretical work discussed so far was based on ordinary perturbation theory, and attention was confined to the case of uniform wave trains. However, surprisingly, it was discovered by *Lighthill* [1965] that a uniform wave train is unstable to the presence of sidebands [*Benjamin and Feir*, 1967]; when a sufficiently steep uniform wave train is produced, the presence of sidebands will result in a breaking up of the wave train and in generation of wave groups. Recently, this instability was studied experimentally by *Lake et al.* [1977]; the initial group formation was clearly seen. However, for long times an (approximate) recurrence to the initial state was observed: the so-called Fermi-Pasta-Ulam (FPU) recurrence [*Fermi et al.*, 1955].

A convenient tool for the description of the long-term behavior of a train of water waves is the nonlinear Schrödinger equation. This equation is used in many branches of physics. Its usefulness for water waves was demonstrated by *Davey and Stewartson* [1974]. The equation is valid for not too strong nonlinearities and not too strong dispersion (narrow spectra). It has several advantages over ordinary perturbation theory. Among these are its simplicity and the ease with which nonhomogeneous stochastic situations can be treated. For periodic boundary conditions, relevant for situations at open sea, both numerical [*Yuen and Lake*, 1980; *Yuen and Ferguson*, 1978] and analytical [*Hui and Hamilton*, 1979; *Janssen*, 1981] results from this equation have been reported. They are in agreement with the observations of *Lake et al.* [1977].

An interesting question that remains to be answered is the relevance of the Benjamin-Feir instability for the propagation of water waves in nature. In the case of steep waves such as those occurring during active wind generation, nonlinearities are certainly important [*Hasselmann*, 1962]. However, under these circumstances the influence of wind and wave breaking is so important that one can hardly expect to be able to observe a pure hydrodynamic instability. In the relatively simple case of swell, on the other hand, the steepness is rather small, so it has been argued that nonlinearities are less important. Nevertheless, we think it is interesting to explore possible nonlinear effects, since secular effects may be present and their importance depends not only on steepness but on duration as well.

By now several people have applied the nonlinear Schrödinger equation to the study of random sea waves. *Alber* [1978] performed a careful, two-dimensional analysis with the important result that the Benjamin-Feir instability does have a stochastic counterpart, provided that the normalized spectral bandwidth does not exceed a certain value. Later, *Alber and Saffman* [1978] elaborated further on this approach. In the first-mentioned paper the nonlinear Schrödinger equation is studied as a stochastic equation, in the other paper an equation for the Wigner distribution function is derived and studied. In both cases it was not possible to indicate the end state of the instability.

In this paper we speculate that the end state of the instability will be a stable steady solution of the nonlinear Schrödinger equation, periodic in space, and we explore the consequences of this Ansatz for the evolution of the wave height probability distribution. We could also investigate the case of a recurrent end state such as the FPU recurrence; however, such a solution seems less plausible to occur, since a small amount of dissipation is sufficient to induce limit cycle behavior. It will turn out that the stable steady end state can be written as a Jacobian elliptic function. The integration constants can be determined in terms of the parameters of the initial narrow wave train. To fix the stochastic meaning of this solution we make the additional assumption that initially the wave envelope is Rayleigh distributed, just as one would expect when initially many independent Fourier modes contribute. The main result then is that the probability distribution of the surface elevation in the end state is non-Gaussian, simply because side bands in the spectrum are present which are phase-locked.

The plan of this paper is as follows. In section 2 we illustrate our method of calculation with a simple example, namely, a uniform wave train with Stokes frequency correction. The main calculation of the probability distribution for a nonuniform wave train is given in section 3, whereas our conclusions are given in section 4.

2. The Nonlinear Schrödinger Equation

To investigate the effect of nonlinearities such as the Stokes frequency correction and the effect of the Benjamin-Feir instability on the formation of wave groups we propose to study the nonlinear Schrödinger equation. This equation may be applied to the case of water waves with a narrow band spectrum and small wave steepness, so that in a good approximation the surface elevation is given by

$$\eta \simeq \mathrm{Re}\,(\rho(x, t) \exp i(-\omega_0 t + k_0 x)) \tag{1}$$

Here ω_0 and k_0 are the angular frequency and wave number of the carrier wave obeying the deep-water dispersion relation $\omega_0 = (gk_0)^{1/2}$, whereas $\rho(x, t)$ is the slowly varying, complex envelope of the wave.

Application of the multiple scale technique to the exact deep-water equations then gives the following nonlinear Schrödinger equation for $\rho(x, t)$,

$$i\left(\frac{\partial}{\partial t} + \omega_0' \frac{\partial}{\partial x}\right)\rho + \frac{\omega_0''}{2}\frac{\partial^2}{\partial x^2}\rho - \frac{1}{2}\omega_0 k_0^2|\rho|^2\rho = 0 \tag{2}$$

Here, the prime denotes differentiation with respect to k_0, and we assume that the complex envelope ρ is a function of x (the propagation direction of the carrier wave) and t only.

We emphasize, however, that the nonlinear Schrödinger equation has only a restricted validity. As shown by, e.g., *Longuet-Higgins* [1980], *Lake et al.* [1977], and *Dysthe* [1979], the steepness $k_0\rho$ of the carrier wave must be smaller than, say, 10–15%, since for greater wave steepness the effect of higher order dispersion and the effect of wave-induced current becomes important. Consequently, we have a restriction on the width of the spectrum in wave number space. Noting that in the derivation of the nonlinear Schrödinger equation the dispersion is assumed to be of the same order as the effect of nonlinearity, we obtain for the width δk of the spectrum in wave number space the upper bound

$$\frac{\delta k}{k_0} = 0(2k_0\rho_0) \leqq 20\text{–}30\% \tag{3}$$

The width of the frequency spectrum is, therefore, at most 10%, since $\delta\omega/\omega_0 = \delta k/2k_0$.

Because of (3) and the restriction on the wave steepness, the nonlinear Schrödinger equation seems to be an appropriate model for the description of swell.

To investigate some special solutions of the nonlinear Schrödinger equation (2), we transform (2) to a frame moving with the group velocity ω_0' to obtain

$$i\frac{\partial}{\partial t}\rho - \frac{1}{2}\frac{\partial^2}{\partial \xi^2}\rho - \kappa|\rho|^2\rho = 0 \tag{4}$$

where $\xi = (x - \omega_0' t)/(-\omega_0'')^{1/2}$ and $\kappa = \omega_0 k_0^2/2$.

The rest of this section is devoted to the case of uniform wave trains, while in the next section the modulated wave train is considered.

For a uniform wave train (i.e., the envelope ρ is independent of ξ), equation (4) may be solved at once with the result

$$\rho = a_0 \exp i\sigma \qquad \sigma = -\kappa a_0^2 t + \sigma_0 \tag{5}$$

for the initial condition $\rho(0) = a_0 \exp i\sigma_0$. It should be noted that $\partial\sigma/\partial t$ is just the Stokes correction to the frequency. We next assume that a_0 and σ_0 are random variables with joint probability distribution

$$p_{a_0,\sigma_0}(a_0, \sigma_0) = \frac{a_0}{\pi\langle a_0^2\rangle} \exp -a_0^2/\langle a_0^2\rangle \tag{6}$$

i.e., we have random phases, whereas a_0 obeys the Rayleigh distribution law. Hence at $t = 0$ we deal with a Gaussian process. Let us determine the statistics for $t \neq 0$ by means of a method described by, e.g., *Soong* [1973]. To that end, it is most convenient to write (4) in matrix form. For the uniform case we then obtain

$$\frac{\partial}{\partial t}\mathbf{x} = \begin{pmatrix} 0 & 0 \\ -\kappa a & 0 \end{pmatrix}\mathbf{x} \qquad \mathbf{x}(0) = (a_0, \sigma_0) \tag{7}$$

where $\mathbf{x} = (a, \sigma)$.

The solution of (7) is given by

$$\mathbf{x} = \begin{pmatrix} 1 & 0 \\ -\kappa a_0 t & 1 \end{pmatrix} \mathbf{x}(0) \tag{8}$$

while its inverse reads

$$\mathbf{x}(0) = \begin{pmatrix} 1 & 0 \\ +\kappa a t & 1 \end{pmatrix} \mathbf{x} \tag{9}$$

Equation (7) is a first-order differential equation. As shown by Soong in that case, a Liouville equation for the probability distribution function may be derived. The result is that one can calculate $p_{\mathbf{x}}(\mathbf{x})$ as follows:

$$p_{\mathbf{x}} = p_{\mathbf{x}_0}(\mathbf{x}_0 = \mathbf{x}_0(\mathbf{x}, t))|J| \tag{10}$$

where the Jacobian J is given by

$$J = \frac{\partial \mathbf{x}_0{}^T}{\partial \mathbf{x}}$$

In our case $|J| = 1$, hence

$$p(a, \sigma) = \frac{a}{\pi \langle a_0{}^2 \rangle} \exp -a^2/\langle a_0{}^2 \rangle \tag{11}$$

We therefore have shown that a uniform wave train with Stokes frequency corrections included obeys the Gaussian statistics for all times.

Since we are interested in wave groups, we compute the average length of the wave groups by means of an expression derived by *Ewing* [1973]. For an envelope ρ, which has a Rayleigh distribution, Ewing has shown that the average length $\langle l_1 \rangle$ of the wave groups is given by

$$\langle l_1 \rangle = \frac{1}{k}\left(\frac{m_2}{2\pi\mu_2}\right)^{1/2} \tag{12}$$

where $k = \rho/m_0{}^{1/2}$ ($k = 2$ for a level corresponding to the significant wave height) and

$$\mu_{2n} = \int_0^\infty (\omega - \bar{\omega}_0)^{2n} S(\omega) d\omega \qquad m_{2n} = \int_0^\infty \omega^{2n} S(\omega) d\omega$$

Here, $S(\omega)$ is the spectrum of the waves and $\bar{\omega}_0$ its central frequency. Note that for linear waves $m_0 = \langle a_0{}^2 \rangle/2$. Combining (1) and (5) we obtain for the elevation

$$\eta = a_0 \cos(k_0 x - \omega_0 t - \kappa a_0{}^2 t + \sigma_0) \tag{13}$$

The spectrum of the signal η is determined through the correlation function $R(\tau)$ defined by

$$R(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \eta(t)\eta(t + \tau) dt \tag{14a}$$

Next, defining the spectrum according to

$$S(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\tau \, e^{i\omega t} R(\tau) \qquad \omega > 0 \tag{14b}$$

we obtain for the process (13) the spectrum

$$S(\omega) = \frac{a_0{}^2}{2} [\delta(\omega - \omega_0 - \kappa a_0{}^2)] \tag{15}$$

where δ denotes the Dirac function. This is the usual expression for the spectrum of a harmonic function in the deterministic case. However, a_0 is a random variable with a Rayleigh distribution, and therefore the average spectrum of the stochastic process (13) reads

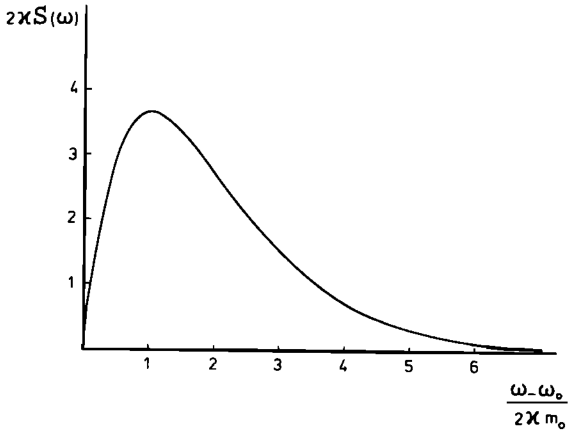

Fig. 1. Spectral shape resulting from the nonlinear Stokes correction to a monochromatic spectrum.

$$\langle S(\omega) \rangle = \int da_0 \, d\sigma_0 \, p_{a_0,\sigma_0}(a_0, \sigma_0) \, S(\omega) \tag{16}$$

where $p_{a_0,\sigma_0}(a_0, \sigma_0)$ is given by (6). The result is

$$\begin{aligned} \langle S(\omega) \rangle &= \frac{\omega - \omega_0}{4\kappa^2 m_0} \exp - \frac{(\omega - \omega_0)}{2\kappa m_0} \qquad \omega > \omega_0 \\ \langle S(\omega) \rangle &= 0 \qquad 0 < \omega < \omega_0 \end{aligned} \tag{17}$$

The form of $S(\omega)$ is given in Figure 1, which clearly shows that for stochastic processes the Stokes correction gives rise to a nonlinear broadening of the spectrum.

The maximum of the spectrum is given by

$$\langle S(\omega_{\max}) \rangle > = \frac{1}{2\kappa e} = \frac{g^2}{\omega_0{}^5 e} \tag{18}$$

where $\omega_{\max} = 2\kappa m_0 + \omega_0 = \omega_0 + \kappa \langle a^2 \rangle = \omega_0 + \Omega_{st}$.

It is now a simple matter to compute the moments of the spectrum, and the end result for $\langle l_1 \rangle$ reads

$$\langle l_1 \rangle = \frac{1}{k}\left\{\frac{1 + (\bar{\omega}_0/\sigma)^2}{2\pi}\right\}^{1/2} \tag{19}$$

where σ is the width of the spectrum defined by

$$\sigma^2 = \frac{\mu_2}{m_0}$$

which in this case in given by

$$\sigma^2 = 8\kappa^2 m_0{}^2 \tag{20}$$

In terms of the average wave steepness $\varepsilon = k_0 \langle a^2 \rangle^{1/2} = 2^{1/2} k_0 m_0{}^{1/2}$, the expression for $\langle l_1 \rangle$ may also be written as

$$\langle l_1 \rangle = \frac{1}{k\varepsilon^2}\left(\frac{\varepsilon^4 + 2(1 + \varepsilon^2)^2}{2\pi}\right)^{1/2} \simeq \frac{1}{k\varepsilon^2}\left(\frac{1}{\pi}\right)^{1/2} \qquad \varepsilon \ll 1$$

In practice, $\varepsilon \simeq 10\%$, giving a wave group length of the order $\langle l_1 \rangle \simeq 28$ for $k = 2$. This corresponds to a small width of the spectrum: $\sigma \simeq 10^{-2}\bar{\omega}_0$. We therefore conclude that for deep-water waves the broadening of the frequency spectrum due to the Stokes correction could not explain the observed width of the spectrum, since usually $\langle l_1 \rangle \simeq 2$. Apparently, there must be additional effects responsible for the observed width of the spectrum. Parameters such as the distance from the source region and the extent of the source region are relevant for determining the width of the spectrum [*Walden,* 1956]. It is well known, however, that for deep-water waves a uniform wave train is unstable to perturbations in its envelope [*Benjamin and Feir,* 1967]. As a result of this instability, sidebands will occur in the spectrum which give rise to an additional broadening of the spectrum. Especially, far away from the source region the latter effect may be important, and the next section is therefore devoted to the study of the effect of the Benjamin-Feir instability on the statistical properties of the wave train. We incidentally note that for shallow water waves the effect of the nonlinear frequency correction may be really important. Starting from the one-dimensional Davey-Stewartson equations [*Davey and Stewartson,* 1974], we obtain in the limit $k_0h \ll 1$ (where h is the water depth) a nonlinear Schrödinger equation for the envelope of the wave train with

$$\kappa = -\frac{9}{16}\frac{\omega_0 k_0^2}{(k_0h)^4} \tag{21}$$

Owing to the wave-induced current, nonlinearity now gives rise to a decrease of the frequency of the wave train, whereas the wave train is modulationally stable (at least in one-dimensional theory). The expression for the average length of the wave group is the same as given in (19). In terms of the wave steepness we obtain

$$\langle l_1 \rangle = k^{-1}\pi^{-1/2}\delta^{-1}(1-\delta) \qquad \delta = \frac{9}{8}\varepsilon^2/(k_0h)^4$$

For $k_0h = 0.5$, an average wave steepness of 10%, and $k = 2$, we obtain $\langle l_1 \rangle \simeq 1.3$ and $\sigma \simeq 0.13\,\omega_0$, showing that in shallow water the broadening of the spectrum due to the nonlinear frequency correction may be considerable.

To summarize, we have investigated in this section uniform wave trains. Whereas in the deterministic case a single peaked spectrum is obtained, the combination of nonlinear frequency correction and randomness gives rise to a broadening of that spectrum. This nonlinear line broadening is quite small for deep-water waves, although it may be considerable for shallow-water waves. Furthermore, we have shown that a uniform wave train is a Gaussian process if it is Gaussian initially.

For deep water, a uniform wave train is, however, unstable to modulational perturbations, and it is the purpose of the next section to investigate the effect of the modulations on the statistics of the wave train.

3. Effect of the Benjamin-Feir Instability

This section is devoted to the study of nonuniform wave trains, and we are especially interested in the effect of modulations in the envelope of the wave train on the statistics.

Let us suppose that we have initially a uniform wave train with a small perturbation in the envelope; e.g., the energy in the modulation is small compared with the energy in the carrier wave. To apply the method given by, e.g., *Soong* [1973], we need to solve the one-dimensional nonlinear Schrödinger equation subject to the above-mentioned initial conditions and periodic boundary conditions in x space. Then, inverting the solution, one can in principle obtain the probability distribution for the wave train. Unfortunately, the solution to the initial value problem with periodic boundary conditions is not known to us. We therefore attack this problem in a different fashion. Thereby, we assume that the physical system (in this case the narrow-band wave train) evolves from the unstable initial state (which is the uniform wave train) to a more complicated, stable equilibrium. Against the assumption of limit cycle behavior the objection may be raised that according to the experiments of *Lake et al.* [1977] the time evolution of the narrow-band wave train shows the Fermi-Pasta-Ulam recurrence [*Fermi et al.,* 1955]. These authors remark, however, that the recurrence is not perfect because of the presence of dissipation.

Although the nonlinear Schrödinger equation (2) does not include the effect of dissipation, we assume that at least the end state is approximately well described by this equation.

For an end state of the form

$$\rho = a\,e^{i(\sigma_0 - \Omega t)} \tag{22}$$

where (for simplicity) Ω is a constant and a is a function of ξ only, we obtain from (4) the following equation for a:

$$\frac{1}{2}\frac{d^2}{d\xi^2}a + \kappa a(a^2 - a_0^2) = 0 \qquad a_0^2 = +\Omega/\kappa \tag{23}$$

The solution of (23) is given by

$$a = a_0\left(\frac{2}{2-m}\right)^{1/2} dn\left(\xi a_0\left(\frac{2\kappa}{2-m}\right)^{1/2}, m\right) \tag{24}$$

$$0 \le m \le 1$$

where m is the modulus of the dn function. We note that (24) represents a whole class of solutions parameterized by the constant m. In the limit $m \to 0$ we rediscover the uniform solution (5), which is, as already mentioned, modulationally unstable. This follows from a linear stability analysis of (4) with the result

$$\omega^2 = \tfrac{1}{4}k^2(k^2 - 4\kappa a_0^2) \tag{25}$$

where ω and k are frequency and wave number of the modulation ($\sim \exp(ik\xi - \omega\tau)$), respectively, and a_0 is the amplitude of the uniform wave train. Hence ω^2 is negative (i.e., ω is complex) if $k^2 < 4\kappa a_0^2$, and maximum growth is found for

$$k = k_{\max} = (2\kappa a_0^2)^{1/2} \tag{26}$$

As was conjectured by *Lake and Yuen* [1978] and *Longuet-Higgins* [1980], we assume that the mode with the wave number $k_{\max}$ dominates the behavior of the envelope for large times.

In other words, the wave number of the end state (24) is given by $k = k_{\max}$. From this requirement we obtain the following condition

$$K = \frac{\pi}{k_{\max}}a_0\left(\frac{2\kappa}{2-m}\right)^{1/2} = \frac{\pi}{(2-m)^{1/2}} \tag{27}$$

where K is the complete elliptic integral of the first kind. We have solved (27) for m numerically. It turns out that $m \simeq 1$ so that $K \simeq \frac{1}{2} \ln [16/(1 - m)]$, and we have

$$m \simeq 1 - 16e^{-2\pi} = 0.96 \tag{28}$$

Hence the envelope a is now completely fixed. In the appendix it is shown that the solution (24) with m given by (28) is stable to one-dimensional perturbations, so that, in agreement with a conjecture made by *Whitham* [1974], the cnoidal wave is a possible end state of the Benjamin-Feir instability.

Combination of (22) and (24), and substitution of the result in equation (1), gives

$$\eta = a_0 \left(\frac{2}{2 - m}\right)^{1/2} dn \left\{\xi a_0 \left(\frac{2\kappa}{2 - m}\right)^{1/2}, m\right\} \cdot \cos \{k_0 x - (\omega_0 + \Omega)t + \sigma_0\} \tag{29}$$

where $\Omega = \kappa a_0^2$, $\xi = (x - \omega_0' t)/(-\omega_0'')^{1/2}$, and m is given by (28). We next determine the spectrum of η by using the Fourier series expansion of the dn function, as given by *Abramowitz and Stegun* [1964],

$$dn(u, m) = \frac{\pi}{2K}\left(1 + 4 \sum_{n=1}^{\infty} \frac{q^n}{1 + q^{2n}} \cos 2\, n\nu\right) \tag{30}$$

$$\nu = \pi u/(2K)$$

where q is the nome,

$$q = \exp(-\pi K'/K) \qquad K' = K(1 - m)$$

In an analogous fashion, as was done in the previous section, one can determine the spectrum $S(\omega)$ through the correlation function $R(\tau)$ with the result

$$S(\omega) = \tfrac{1}{4} a_0^2 \left\{\delta(\omega - \omega_0) + 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{(1 + q^{2n})^2} (\delta(\omega - \Omega_+) + \delta(\omega - \Omega_-))\right\} \tag{31}$$

where $\Omega_\pm = \omega_0 (1 \pm n k_0 a_0)$. Here, we have neglected the Stokes frequency correction, because it only gives a small contribution (see the previous section).

So far, we have investigated the deterministic aspects of the problem. We next assume that we deal with random initial conditions such that a_0 and σ_0 are distributed according to the distribution law (6). The purpose of the remaining part of this section is then to determine the spectrum of this stochastic system and its statistical properties for large times. The spectrum of the stochastic process with modulations included is obtained by averaging the spectrum of the deterministic process (31) by means of the Rayleigh distribution law (6). As a result we find

$$\langle S(\omega)\rangle = \tfrac{1}{4} \langle a_0^2\rangle \left[\delta(\omega - \omega_0) + 8 \sum_{n=1}^{\infty} \alpha_n^2 \beta_n |\omega - \omega_0|^3 \exp(-\alpha_n(\omega - \omega_0)^2)\right] \tag{32}$$

where $\alpha_n = (n\omega_0\varepsilon_0)^{-2}$, $\beta_n = q^{2n}/(1 + q^{2n})^2$, and ε_0 is the wave steepness at $t = 0$ ($\varepsilon_0 = k_0 \langle a_0^2\rangle^{1/2}$). In Figure 2 we have plotted the normalized spectrum $\hat{S} = 8\kappa \langle S(\omega)\rangle/\varepsilon_0$ as a function of normalized angular frequency $x = (\omega - \omega_0)/\varepsilon_0\omega_0$.

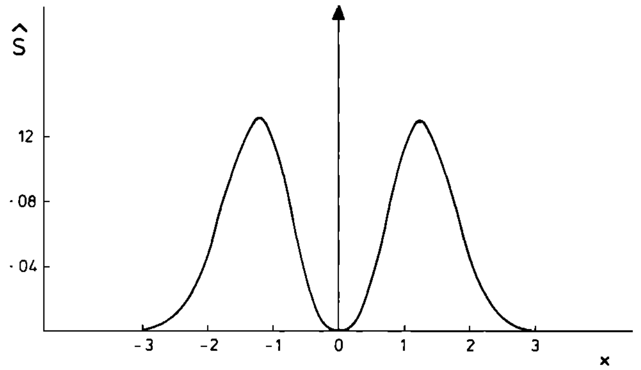

Fig. 2. Spectral shape resulting from our estimate of the effect of the Benjamin-Feir instability on a monochromatic spectrum. $\hat{S} = 8\kappa\langle S(\omega)\rangle/\varepsilon_0$, $x = (\omega - \omega_0)/(\varepsilon_0\omega_0)$.

Only the sidebands with $n = 1$ are presented, because even for $m = 0.96$ ($q = 0.2$), the sidebands with $n \geq 2$ are quite small and are therefore not presented in Figure 2.

The effect of randomness is clear from the broadening of the sidebands in the spectrum. The sidebands are symmetrical with respect to the main peak at $\omega = \omega_0$. Note that the effect of the Stokes frequency correction is to enhance the peak to the left of ω_0 and to reduce the peak to the right of ω_0, although this feature is quite small.

To summarize our results of the spectrum we offer the following picture. Initially, the width of the spectrum is infinitesimally small. Due to the Benjamin-Feir instability sidebands will appear in the spectrum, giving rise to a broadening of the spectrum. The spectrum will broaden until the instability is quenched. We note that the conjecture that a broad spectrum may be stable to modulational perturbations is in qualitative agreement with the theoretical results of *Alber* [1978]. He found, on the basis of a linear stability analysis, that a random uniform wave train is stable if its spectrum is sufficiently broad:

$$\frac{\sigma}{\omega_0} \geq \varepsilon \tag{33}$$

where σ is the spectral width and ε is the wave steepness, $\varepsilon = k_0 \langle a^2\rangle^{1/2}$. From (32) we obtain for the relative width σ/ω_0 of the spectrum

$$\frac{\sigma}{\omega_0} \simeq \frac{4(2\beta_1)^{1/2}}{1 + 8\beta_1} \varepsilon \tag{34}$$

Here, we have taken into account the width due to the first sideband only. Remarkably, for $m \simeq 0.96$ ($q \simeq 0.20$), the value of the relative width is just below the threshold value for stability of a random uniform wave train (see equation (33)).

From (34) we observe that the broadening of the spectrum is of the order of the wave steepness ε, rather than ε^2, as in the case of the uniform wave train treated in the previous section. One may therefore expect much shorter wave groups with an average length of the order of ε^{-1}, rather than ε^{-2} as found for the uniform wave train with Stokes frequency correction included.

We next investigate the statistical properties of the modulated wave train. We confine ourselves to a discussion of the probability distribution of the envelope a. Note that the envelope will no longer obey a Rayleigh distribution because the sidebands are phase locked instead of independent in the statistical sense. To see this we compute the probability distribution for the envelope of the stochastic process (29), assuming the distribution of a_0 to be given by the Rayleigh distribution. For convenience, we only take the first term of the series of the dn function, but our results can easily be extended to the full expression for the dn function. Thus

$$a = \tfrac{1}{2}(2)^{1/2}a_0 \left[1 + \frac{4q}{1+q^2}\cos(\gamma t)\right] \qquad (35)$$

where $\gamma = k_0 a_0 \omega_0$, and for convenience we have chosen $x = 0$. The process (35) is not stationary. This is evident, e.g., from the expression of the first moment $\langle a \rangle$,

$$\langle a \rangle = \int da_0 p_{a_0}(a_0) a$$

where p_{a_0} is the Rayleigh distribution for a_0. Clearly, the first moment depends on time. For large t, however, the integral involving the cosine function averages out due to phase mixing so that $\lim_{t\to\infty} \langle a \rangle = \langle a_0 \rangle (2)^{1/2}/2$. For this reason, we only calculate the probability distribution of a for large times.

We first observe now that for large t the processes a_0 and $\cos(\gamma t)$ are independent in the statistical sense because of phase mixing. Hence the probability that a is smaller than some value x can easily be determined if we know the distribution functions for a_0 and $\cos(\gamma t)$. Since a_0 is Rayleigh distributed, we only need to calculate the probability distribution for $\cos(\gamma t)$.

The probability distribution function for

$$y = \cos z \qquad z = \gamma t = a_0 k_0 \omega_0 t \qquad (36)$$

is easily found. Suppose the value $y = y_0$ is realized for $z = z_0$. Then, because of the periodicity of the cosine function, y has the same value y_0 if

$$z = \pm z_0 + 2n\pi$$

where n is an integer. The probability density for the occurrence of y_0 is then given by

$$p_y(y_0) = \sum_n \{p_z(z_0 + 2n\pi) + p_z(-z_0 + 2n\pi)\} dz_0/dy_0 \qquad (37)$$

where the Jacobian $dz_0/dy_0 = 1/(1 - y_0^2)^{1/2}$. The variable z, being a multiple of a_0, is Rayleigh distributed,

$$p_z(z_0) = \frac{2z_0}{\langle z_0^2 \rangle} \exp(-z_0^2/\langle z_0^2 \rangle)$$

For large t, the summation (37) can be replaced by an integral, since the spacing goes to zero. The result is

$$p_y(y_0) = \frac{1}{\pi(1 - y_0^2)^{1/2}} \qquad t \to \infty \qquad (38)$$

We remark that this type of consideration may be applied to any periodic function, e.g., the dn function of (39); the only difference is the Jacobian dz_0/dy_0.

Since the stochastic variables y and a_0 are independent for large times, we obtain the joint probability distribution

$$p_{a_0,y}(a_0, y_0) = \frac{1}{\pi(1 - y_0)^{1/2}} \frac{2a_0}{\langle a_0^2 \rangle} \exp(-a_0^2/\langle a_0^2 \rangle) \qquad (39)$$

Then, the probability that $a < x$, with a given by (35), equals

$$P(a < x) = \int_D da_0\, dy_0\, p_{a_0,y}(a_0, y_0) \qquad (40)$$

where D is defined by

$$D\colon \tfrac{1}{2}(2)^{1/2} a_0 \left[1 + \frac{4q}{1+q^2} y_0\right] \le x$$

Differentiating (40) with respect to x, we obtain for the probability density of the envelope a

$$p_a(x) = \frac{x}{\pi \langle a_0^2 \rangle} \int_0^{\pi} d\theta \frac{\exp - [\alpha^2 x^2/(1 + 4\beta_1^{1/2}\cos\theta)^2]}{(1 + 4\beta_1^{1/2}\cos\theta)^2} \qquad (41)$$

where $\alpha^2 = 2/\langle a_0^2 \rangle$ and $\beta_1^{1/2} = q/(1 + q^2)$.

We remark that in the limit $\beta_1 \to 0$, corresponding to a uniform wave train, we rediscover from (41) the Rayleigh distribution. For small $\beta_1^{1/2}$ we obtain an approximate expression to (41) by means of a multiple scale analysis,

$$p_a(x) \simeq \frac{(2\pi)^{1/2}}{8(\langle a_0^2 \rangle \beta_1)^{1/2}} [\mathrm{erf}(z_1) - \mathrm{erf}(z_2)] \qquad \beta_1 \ll 1 \qquad (42)$$

where $z_1 = \alpha x/(1 - 4\beta_1^{1/2})$, $z_2 = \alpha x/(1 + 4\beta_1^{1/2})$, and erf is the error function. Equations (41) and (42) show that the stochastic process is non-Gaussian, because the envelope a does not obey the Rayleigh distribution. Unfortunately, in the case of interest, $\beta_1^{1/2} \simeq 0.2$ so that the approximation (42) fails. In addition, for such large values of $\beta_1^{1/2}$, (35) is only a poor approximation to the exact expression for the envelope. We therefore took more terms of the series for the dn function into account. By inspection, we found out that three terms of the series were sufficient for our purposes. The results are presented in Figure 3 and are compared with a Rayleigh distribution with the actual rms amplitude. We observe from Figure 3 that the modulated wave train has a probability distribution for the envelope that gives a larger probability for larger waves and also for small waves. Also shown is the result of a numerical simulation of the envelope of the random process (29) for large times. The agreement is good.

We finally note that the present method can also be applied to the random process (29). Following the same procedure, one can easily obtain the probability distribution for the surface elevation, but we shall not give the details.

4. Conclusions

In one dimension a narrow spectrum of water waves is unstable, which gives a broadening of the spectrum.

If initially the probability distribution of the surface elevation is Gaussian, we find in the end state deviations that result from phase relations between the different modes in the broadened spectrum.

The present effect could play a role in nature, although observational evidence is hard to obtain. This is because surface wave spectra have a natural line width that obscures the nonlinear broadening. *Walden* [1956], e.g., has shown how in the case of swell the finite extension of the source region and the finite generation time of the waves by the wind determine linearly the width of the wave spectrum.

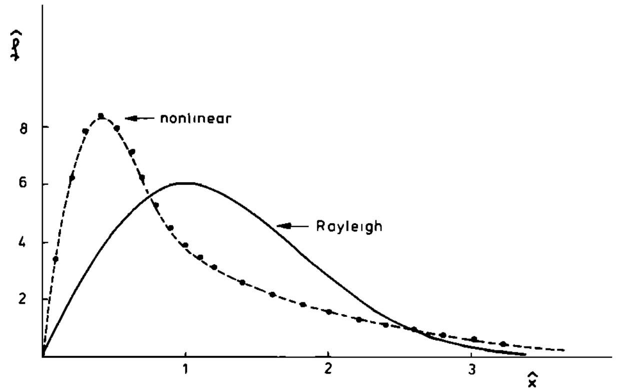

Fig. 3. The probability density function for our stable nonlinear solution (41) for the cnoidal wave envelope (24). The dashed line is the result of a Monte Carlo simulation, while the solid circles indicate the result of a numerical integration. For comparison, the Rayleigh distribution (solid line) is also given. We normalized such that $\hat{f} = m_0^{1/2}\, p_a(a)$ and $\hat{x} = a/m_0^{1/2}$, with $m_0 = \int d\omega\, S(\omega)$.

Recently, it was suggested [*Goda*, 1976; *Mollo-Christensen and Ramamonjiarisoa*, 1978; *Rye*, 1980] that nonlinear effects increase the average group length. This conjecture seems at variance with our result that nonlinearities may actually shorten the wave groups. However, we merely state that an initial line spectrum (with infinite wave group length) evolves into a broader spectrum with finite wave group length. To support the above-mentioned conjecture, one should calculate the wave group length according to linear theory and compare it with the expression for the wave group length, including nonlinear effects, for a spectrum with the same width. This is, however, beyond the scope of the present paper.

Our calculation was made under a few restrictive assumptions. In the first place, we considered one-dimensional wave propagation only. Extension to two dimensions is desirable but very hard at present, since no stable cnoidal solutions in two dimensions are known. Also, we restricted our attention to deep-water waves. In shallow water different results will certainly follow for the modulated wave train and the corresponding probability distribution. In the case of our simple example of the Stokes correction to a uniform wave train, we were able to discuss the shallow water limit. It was found that nonlinear effects are more important than in deep water. Finally, we made a conjecture about the stable end state of the Benjamin-Feir instability. It is important that we were able to find a stable end state at all, and it was interesting to investigate its consequences. However, more general solutions may be relevant (with nonuniform phases, for example), and it would be worthwhile to extend the present approach.

APPENDIX

In this appendix we investigate the linear stability of the cnoidal wave solution (24) in one-space dimension. Our starting point is the nonlinear Schrödinger equation (4). Substitution of

$$\rho = a \exp -i\kappa \hat{a}^2 t \tag{A1}$$

where $\hat{a}$ is an arbitrary reference amplitude, gives

$$i\frac{\partial}{\partial \tau}A - \frac{\partial^2}{\partial z^2}A - A(|A|^2 - 1) = 0 \tag{A2}$$

Here, we introduced the dimensionless quantities $A = a/\hat{a}$, $\tau = \kappa\hat{a}^2 t$ and $z = \hat{a}\xi(2\kappa)^{1/2}$. The cnoidal wave (24) reads in the present units

$$A_0 = \beta(2)^{1/2}\, dn(\beta z, m) \tag{A3}$$

where $\beta = 1/(2 - m)^{1/2}$. To investigate its stability, we write

$$A = A_0 + A_1 \qquad A_1 \ll A_0 \tag{A4}$$

and we linearize (A2) to obtain for solutions of the form $A_1 = (u + iv)\exp(\gamma t)$ the following differential equation for v,

$$L_1 L_2 v = -\gamma^2 v \tag{A5}$$

where $L_1 = \partial^2/\partial z^2 + 3A_0^2 - 1$; $L_2 = \partial^2/\partial z^2 + A_0^2 - 1$.

Since A_0 is periodic with period $2K/\beta$, we observe at once that (A5) is invariant under translation over $2K/\beta$. This suggests that, in complete analogy with the Ansatz of Bloch for periodic potentials, v may be written in the form

$$v = e^{ikz} g(z) \tag{A6}$$

where $g(z)$ has period $2K/\beta$, and k is an arbitrary wave number. The normal modes are thus modulated with period $2K/\beta$. Inserting (A6) into (A5), we obtain the following eigenvalue problem

$$L_1 L_2 g = -\gamma^2 g \tag{A7}$$

where g is periodic with period $2K/\beta$, $L_1 = (\partial/\partial z + ik)^2 + 3A_0^2 - 1$, and $L_2 = (\partial/\partial z + ik)^2 + A_0^2 - 1$. Unfortunately, however, no exact solutions to the eigenvalue problem (A7) are known to us. We therefore seek approximate solutions of (A7) for small k and small γ, following the method of *Zakharov and Rubenchik* [1974]. To that end, for wave numbers k which are small with respect to $\beta/2K$, we expand $g = g_0 + g_1 + \cdots$, $\gamma^2 = (\gamma^2)_1 + \cdots$. In lowest order we then obtain

$$L_1^0 L_2^0 g_0 = 0 \tag{A8}$$

where $L_1^0 = \partial^2/\partial z^2 + 3A_0^2 - 1$, $L_2^0 = \partial^2/\partial z^2 + A_0^2 - 1$. Equation (A8) gives two periodic solutions, namely, one which is even with respect to the crest of the steady state (A3) and one which is odd. In the following we consider the even mode in more detail. It is given by

$$g_0^+ = A_0 \tag{A9}$$

For this choice of g_0 we obtain in next order,

$$L_1^0 L_2^0 g_1 = -\gamma^2 g_0^+ + 2k^2[(g_0^+)^2 + 2(g_0^+)']' \equiv \mathfrak{B} \tag{A10}$$

where the prime denotes differentiation with respect to z, and we only retained terms up to order k^2. The growth rate γ^2 is now obtained from the solvability condition of (A10),

$$(\chi, \mathfrak{B}) \equiv \int_{-k/\beta}^{k/\beta} dz\, \chi^* \mathfrak{B} = 0 \tag{A11}$$

where χ is the solution of the adjoint problem $L_2^0 L_1^0 \chi = 0$. Obviously, (A11) is automatically satisfied for uneven χ because $\mathfrak{B}$ is even. The even solution of the adjoint problem is given by *Martin et al.* [1980]. Without presenting the details, the result in the neighborhood of $m = 1$ reads

$$\gamma^2 = 2k^2(1 - m)K(K - 2E)/\{(1 - m)K^2 - E^2\} \tag{A12}$$

where K and E are the complete elliptic integrals of the first and second kind, respectively. Clearly, γ^2 vanishes for $m = 1$ and is negative in the neighborhood of $m = 1$. Hence near $m = 1$ the cnoidal wave is stable to even perturbations. In addition, it can be shown by means of a similar analysis that the cnoidal wave near $m = 1$ is stable to odd perturbations.

REFERENCES

Abramowitz, M., and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1964.

Alber, I. E., The effects of randomness on the stability of two-dimensional surface wavetrains, *Proc. R. Soc. London Ser. A*, *363*, 525–546, 1978.

Alber, I. E., and P. G. Saffman, Stability of random nonlinear deep water waves with finite bandwidth spectra, *TRW Rep. 31326-6035-RU-OO*, TRW, Inc., Redondo Beach, Calif., 1978.

Benjamin, T. B., and J. E. Feir, The disintegration of wave trains on deep water, I, Theory, *J. Fluid Mech.*, *27*, 417–430, 1967.

Cartwright, D. E., and M. S. Longuet-Higgins, The statistical distribution of the maxima of a random function, *Proc. R. Soc. London Ser. A*, *237*, 212–232, 1956.

Davey, A., and K. Stewartson, On three-dimensional packets of surface waves, *Proc. R. Soc. London Ser. A*, *338*, 101–110, 1974.

Dysthe, K. B., Note on a modification to the nonlinear Schrödinger equation for application to deep water waves, *Proc. R. Soc. London Ser. A*, *369*, 105–114, 1979.

Ewing, J. A., Mean length of runs of high waves, *J. Geophys. Res.*, *78*, 1933–1936, 1973.

Fermi, E., J. Pasta, and H. C. Ulam, Studies of nonlinear problems, in *Collected Papers of Enrico Fermi*, vol. 2, edited by E. Segre, pp. 978–988, University of Chicago Press, Chicago, Ill., 1955.

Forristall, G. Z., On the statistical distribution of wave heights in a storm, *J. Geophys. Res.*, *83*, 2353–2358, 1978.

Goda, Y., On wave groups, in *An International Conference on the Behavior of Off-Shore Structures*, vol. 1, pp. 115–128, The Norwegian Institute of Technology, Trondheim, Norway, 1976.

Harris, T. F. W., J. N. Marshall, and F. A. Shillington, Quasi-monochromatic 0.05 Hz waves in Cape waters, *Nature*, *244*, 154, 1973.

Hasselmann, K., On the nonlinear energy transfer in a gravity-wave spectrum, 1, General theory, *J. Fluid Mech.*, *12*, 481–500, 1962.

Huang, N. E., and S. R. Long, An experimental study of the surface elevation probability distribution and statistics of wind-generated waves, *J. Fluid Mech.*, *101*, 179–200, 1980.

Hui, W. H., and J. Hamilton, Exact solutions of a three-dimensional nonlinear Schrödinger equation applied to gravity waves, *J. Fluid Mech.*, *93*, 117–133, 1979.

Janssen, P. A. E. M., Modulational instability and the Fermi-Pasta-Ulam recurrence, *Phys. Fluids*, *24*, 23–26, 1981.

Lake, B. M., and H. C. Yuen, A new model for nonlinear wind waves, 1, Physical model and experimental evidence, *J. Fluid Mech.*, *88*, 33–62, 1978.

Lake, B. M., H. C. Yuen, H. Rungaldier, and W. E. Ferguson, Jr., Nonlinear deep-water waves: Theory and Experiment, 2, Evolution of a continuous wave train, *J. Fluid Mech*, *83*, 49–74, 1977.

Lighthill, M. J., Contributions to the theory of waves in nonlinear dispersive systems, *J. Inst. Math. Appl.*, *1*, 269–306, 1965.

Longuet-Higgins, M. S., The statistical analysis of a random, moving surface, *Philos. Trans. R. Soc. London Ser. A*, *249*, 321–387, 1957.

Longuet-Higgins, M. S., The effect of non-linearities on the statistical distributions in the theory of sea waves, *J. Fluid Mech.*, *17*, 459–480, 1963.

Longuet-Higgins, M. S., Modulation of the amplitude of steep wind waves, *J. Fluid Mech.*, *99*, 705–713, 1980.

Martin, D. U., H. C. Yuen, and P. G. Saffman, Stability of plane wave solutions of the two-space-dimensional nonlinear Schrödinger equation, *Wave Motion*, *2*, 215–229, 1980.

Mollo-Christensen, E., and A. Ramamonjiarisoa, Modeling the presence of wave groups in a random wave field, *J. Geophys. Res.*, *83*, 4117–4122, 1978.

Munk, W. H., F. E. Snodgrass, and M. J. Tucker, Spectra of low-frequency ocean waves, *Bull. Scripps Inst. Oceanogr.*, *7*, 283–362, 1959.

Phillips, O. M., *The Dynamics of the Upper Ocean*, 2nd ed., Cambridge University Press, New York, 1977.

Rye, H., Wave parameters studies and wave groups, Climatologie de la Mer, *Collect. Colloq. Semin. Inst. Fr. Pet.*, *34*, 89–123, 1980.

Snodgrass, F. E., G. W. Groves, K. F. Hasselmann, G. R. Miller, W. H. Munk, and W. H. Powers, Propagation of ocean swell across the Pacific, *Phil. Trans. R. Soc. London Ser. A*, *259*, 431–497, 1966.

Soong, T. T., *Random Differential Equations in Science and Engineering*, Academic, New York, 1973.

Tayfun, M. A., Narrow-band nonlinear sea waves, *J. Geophys. Res.*, *85*, 1548–1552, 1980.

Thompson, W. C., and R. C. Smith, Jr., Wave groups in ocean swell, in *Proceedings of the International Symposium on Ocean Wave Measurement and Analysis*, pp. 338–351, American Society of Civil Engineers, New York, 1974.

Walden, H., Zur Bandbreite des Periodenspektrums in der Dünung, *Dtsch. Hydrogr. Z.*, *9*, 132–136, 1956.

Whitham, G. B., *Linear and Nonlinear Waves*, John Wiley, New York, 1974.

Yuen, H. C., and W. E. Ferguson, Jr., Relationship between Benjamin-Feir instability and recurrence in the nonlinear Schrödinger equation, *Phys. Fluids*, *21*, 1275–1278, 1978.

Yuen, H. C., and B. M. Lake, Instabilities of waves on deep water, *Ann. Rev. Fluid Mech.*, *12*, 303–334, 1980.

Zakharov, V. E., and A. M. Rubenchik, Instability of waveguides and solitons in nonlinear media, *Sov. Phys. JETP*, *38*, 494–500, 1974.

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