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Ocean Engineering 33 (2006) 1552–1564

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On an improvement of a nonlinear iterative scheme for nonlinear wave profile prediction

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Received 18 May 2005; accepted 11 October 2005
Available online 24 January 2006

Abstract

The authors of the present paper present an iterative scheme to calculate the nonlinear wave profiles [Jang, T.S., Kwon, S.H., 2005. Application of nonlinear iteration scheme to the nonlinear water wave problem: Stokian wave. Ocean Engineering, in press]. The nonlinear operator was constructed from the dynamic boundary condition of the free surface. The initial input of the iterative process was linear potential. The linear dispersion relation was utilized. The authors of the present paper suggest an improved scheme in terms of accuracy and speed of convergence by utilizing the nonlinear dispersion relation. The existence and uniqueness of the improved scheme are illustrated in this paper. The calculation results together with Fast Fourier transform revealed that the improved scheme made it possible to predict higher-order nonlinear characteristics of the Stokes' wave.

Keywords: An improved scheme Stokes' wave; The nonlinear dispersion relation

1. Introduction

A considerable number of studies have been made on the nonlinear water wave profiles. Most of the nonlinear water wave profiles have been treated based on the

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0029-8018/$ - see front matter
doi:10.1016/j.oceaneng.2005.10.018

perturbation theory and computational fluid dynamics (CFD) methods ever since the appearance of Stokes' nonlinear wave theory (Stokes, 1847). For example, computational and theoretical studies of the nonlinear wave profiles could be found (Tsai and Yue, 1996; Dias and Kharif, 1999; Clamond, 2001; Nicholls, 2001). On the other hand, there has been a fixed-point approach to water wave problem. One example of the application of the fixed-point theorem to the wave problem is given in Bona and Bose (1974). They examined the question of the existence of solitary wave solutions to simple one-dimensional models for long waves in nonlinear dispersive systems.

With the help of the fixed-point theorem, Jang and Kwon (2005) proposed an iterative method to estimate nonlinear wave profiles, which was quite different from the traditional perturbation theory. The present paper is aimed at improving accuracy and speed of convergence of the previous iterative scheme suggested by the present authors. The improvement was accomplished by adopting nonlinear dispersion relation. The change in the dispersion relation changes a whole picture of the uniqueness, existence and stability. The uniqueness and existence result in the modification were illustrated in this paper. An equation which defines stable region in terms of wave number times amplitude of the wave was derived. The corresponding stability chart was drawn. The improvement achieved in the present scheme was demonstrated by analyzing the Fast Fourier transform (FFT) results. The FFT analysis revealed that the improved iterative scheme yielded high-order nonlinear Fourier components, which could be found in Stokes' high-order nonlinear wave theory.

In the study, we began with the derivation of the contraction operator from the Bernoulli equation. The detailed construction of the operator in the case of deep water is shown in Section 3. Section 4 discusses the condition for stability of the improved method. The existence and uniqueness of the solutions are examined in Section 5. The numerical results including FFT analysis for several wave slopes are shown in order to demonstrate the effectiveness of the improved scheme, in Section 6. To test the efficiency of the scheme, the results were compared with high-order Stokes' wave profiles. Numerical convergence tests were conducted based on sup-norm errors in order to examine the characteristics of the numerical convergences of the nonlinear iterative solutions. The comparison revealed that the results showed quite a good agreement with each other. The rate of convergence for the proposed operator was also very fast. As a result, all the computations achieved convergence in less than 10 iterations with respect to a specific tolerance.

2. Bernoulli's equation and Banach fixed-point theorem

Based on nonlinear contraction mapping (Zeidler, 1986), the Banach fixed-point theorem has been successfully applied to many fields, especially nonlinear problems. Through this study, we have found that Bernoulli's equation has a Banach fixed-point.

The fluid is assumed to be homogeneous, incompressible and inviscid. In addition, the fluid motion is irrotational, such that a velocity potential function exists. Suppose that we consider a free surface flow. A Cartesian coordinate system (x, y, z) is adopted, with $z = 0$ the plane of the undisturbed free surface and the z-axis positive upwards. The vertical elevation of any point on the free surface may be defined by a function $z = \eta(x, y, t)$. The surface tension being negligible, then, Bernoulli's equation applied on the free surface is

$$\varphi_t + \frac{1}{2}\nabla\varphi \cdot \nabla\varphi + \frac{P_\mathrm{a}}{\rho} + zg = f(t), \tag{1}$$

where φ, P_a and ρ stand for the velocity potential, the pressure of the atmosphere, and the constant fluid density, respectively. Taking Bernoulli's constant $f(t) = P_\mathrm{a}/\rho$, we have the expression for the free surface:

$$\eta = -\frac{1}{g}\left[\varphi_t + \frac{1}{2}\nabla\varphi \cdot \nabla\varphi\right]\Bigg|_{z=\eta}. \tag{2}$$

The right-hand side of (2) may be viewed as an operator for the free surface η, in such a way that we can define a new operator B, which shall be called a Bernoulli's operator in this study:

$$B(\eta) \equiv -\frac{1}{g}\left[\varphi_t + \frac{1}{2}\nabla\varphi \cdot \nabla\varphi\right]\Bigg|_{z=\eta}. \tag{3}$$

Then Bernoulli's operator B can be easily conformed to be nonlinear, and (2) can be simply written as

$$\eta = B(\eta). \tag{4}$$

As shown in (4), the free surface η is invariant under the nonlinear operator B: it is called a Banach fixed point (or function) η. Because (4) has a Banach fixed point, Bernoulli's equations (1) and (2), which are equivalent to (4), should have a Banach fixed point. If the operator B satisfies the following inequality:

$$||B(\eta_1) - B(\eta_2)||_\infty \leqslant \beta||\eta_1 - \eta_2||_\infty, \quad 0 < \beta < 1, \tag{5}$$

then B is considered a contraction and the fixed point is realized as the limit of the following sequence:

$$\eta_{k+1} = B(\eta_k) \tag{6}$$

with the initial condition for zero function $\eta_0 = 0$ (Roman, 1975).

3. Construction of Bernoulli operator $B(\eta)$

We shall restrict the discussion to plane harmonic waves that travel in the x-axis. To construct the nonlinear Bernoulli operator B, we begin with the Stokes third-order solution for periodic plane waves in deep water:

$$\varphi(\theta, z) = -aC_0 e^{kz} \sin\theta + \frac{a^3k^2C_0}{8} e^{kz} \sin\theta, \quad C_0 = \left(\frac{g}{k}\right)^{1/2}, \tag{7}$$

where k, a, and ω represent the wave number, the wave amplitude and frequency, respectively. The product ka, the wave slope, is assumed small. The symbol θ denotes the phase function $kx-\omega t$, where the nonlinear amplitude dispersion relation in deep water is assumed: $\omega^2 = gk(1 + a^2k^2)$.

Using the nonlinear amplitude dispersion relation, the combining of (7) and (3) and a simple calculation yield the explicit form of Bernoulli's operator:

$$\begin{aligned} B(\eta) = {} & a\sqrt{1 + a^2k^2}\left(1 + \frac{3}{8}a^2k^2 - \frac{1}{16}a^4k^4\right)e^{k\eta}\cos\theta \\ & - \frac{1}{2}a^2k\sqrt{1 + a^2k^2}\left(1 + \frac{a^4k^4}{64}\right)e^{2k\eta}. \end{aligned} \tag{8}$$

If we consider the second-order quantity for the contraction coefficient β in (11), then β would have an approximation as follows:

$$B(\eta) \approx a e^{k\eta}\cos\theta - \tfrac{1}{2}a^2k e^{2k\eta}. \tag{9}$$

Eq. (9) is identical with the result (Jang and Kwon, 2005), where the linear progressive wave potential and the linear dispersion relation were assumed.

4. Condition for contraction

Because η represents the free surface, it must be continuous. Therefore, it is convenient to introduce the sup-norm $||\cdot||_\infty$ into this study (Roman, 1975). Furthermore, it is sufficient to consider $0 \leqslant \theta \leqslant 2\pi$, since we are restricted to plane harmonic waves.

Suppose that two different wave elevations η_1, η_2, then their metric $||B(\eta_1) - B(\eta_2)||_\infty$, that is, the distance between them, is represented as

$$||B(\eta_1) - B(\eta_2)||_\infty = \sup_{\theta\in[0,2\pi]} |B(\eta_1) - B(\eta_2)|. \tag{10}$$

If we employ triangle inequality and smallness of $k\eta_1$ and $k\eta_2$ in (10), then we arrive at the inequality (11).

$$\begin{aligned} ||B(\eta_1) - B(\eta_2)||_\infty \leqslant {} & ak\sqrt{1 + a^2k^2}\left(1 + ak + \tfrac{3}{8}a^2k^2 - \tfrac{1}{16}a^4k^4 + \tfrac{1}{64}a^5k^5\right) \\ & \times ||\eta_1 - \eta_2||_\infty. \end{aligned} \tag{11}$$

Comparing (11) with (5), the contraction coefficient β becomes

$$\beta = ak\sqrt{1 + a^2k^2}\left(1 + ak + \tfrac{3}{8}a^2k^2 - \tfrac{1}{16}a^4k^4 + \tfrac{1}{64}a^5k^5\right). \tag{12}$$

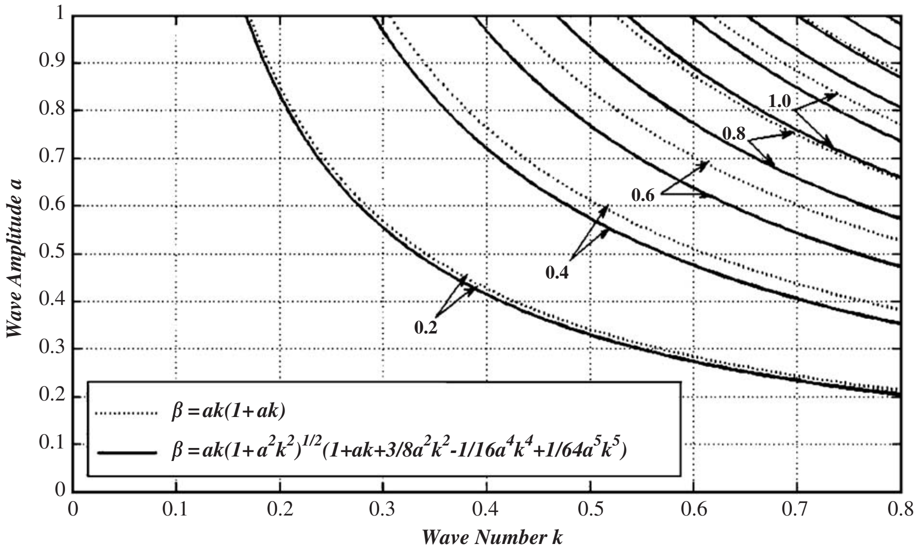

Fig. 1. Contour line for the contraction coefficient.

If we consider the second-order quantity for the contraction coefficient β in (12), then β would be $\beta = ak(1 + ak)$, which is identical with the result of Jang and Kwon (2004), where the linear dispersion relation was assumed.

From Contraction mapping theorem (Zeidler, 1986), it is well known that the iteration (6) converges when $0<\beta<1$: that is, the condition for the iteration to converge requires the following inequality:

$$0<ak\sqrt{1+a^2k^2}\left(1+ak+\tfrac{3}{8}a^2k^2-\tfrac{1}{16}a^4k^4+\tfrac{1}{64}a^5k^5\right)<1. \tag{13}$$

Fig. 1 shows the contour line for the contraction coefficient β, where the solid line is corresponding to the case of (12) and the dotted line to the case of $\beta = ak(1 + ak)$. It is easily seen that the convergence region $0<\beta<1$ described by solid line is reduced compared to that by dotted line.

5. Uniqueness and existence of the wave profiles

Now the necessary time to discuss a unique solution of (4) for Stokes wave profile. Suppose that we have two solutions of wave profiles η_α, η_β. Then we have

$$\eta_\alpha = B(\eta_\alpha) \quad \text{and} \quad \eta_\beta = B(\eta_\beta) \tag{14}$$

because η_α, η_β are invariant under nonlinear operator B for Stokes wave. From (14), we have

$$||\eta_\alpha - \eta_\beta||_\infty = ||B(\eta_\alpha) - B(\eta_\beta)||_\infty \leqslant \beta||\eta_\alpha - \eta_\beta||_\infty \tag{15}$$

which, for $\eta_\alpha \neq \eta_\beta$, can be divided by the norm $||\eta_\alpha - \eta_\beta||_\infty$ to yield the contradiction $\beta \geqslant 1$ because contraction mapping (13) is assumed in the study. Thus, $\eta_\alpha = \eta_\beta$ and (6) has at most one solution of wave profile.

Using the null initial condition, we consider the following inequality:

$$\begin{aligned} ||\eta_{n+1} - \eta_n||_\infty &= ||E\eta_n - E\eta_{n-1}||_\infty \leqslant \beta||\eta_n - \eta_{n-1}||_\infty \leqslant \cdots \leqslant \beta^n||\eta_1 - \eta_0||_\infty \\ &= \beta^n||\eta_1||_\infty, \end{aligned}$$

and hence for $p > m$,

$$\begin{aligned} ||\eta_p - \eta_m||_\infty &= ||(\eta_p - \eta_{p-1}) + (\eta_{p-1} - \eta_{p-2}) + \cdots + (\eta_{m+1} - \eta_m)||_\infty \\ &\leqslant ||\eta_p - \eta_{p-1}||_\infty + ||\eta_{p-1} - \eta_{p-2}||_\infty + \cdots + ||\eta_{m+1} - \eta_m||_\infty \\ &\leqslant [\beta^{p-1} + \cdots + \beta^m]||\eta_1 - \eta_0||_\infty \\ &\leqslant \beta^m[1 + \beta + \beta^2 + \cdots]||\eta_1 - \eta_0||_\infty \\ &= \frac{\beta^m}{1-\beta}||\eta_1||_\infty. \end{aligned}$$

The first inequality in the chain stems from the polygonal inequality. Thus for all p, m, we have a formulation for mathematical convergence rate as follows (Kolmogorv and Fomin, 1970):

$$||\eta_p - \eta_m||_\infty \leqslant \frac{||\eta_1||_\infty}{1-\beta}\,\beta^{\min(m,p)} \tag{16}$$

and since the right-hand side can be arbitrarily small for sufficiently large p, m, $\{\eta_k\}$ is a Cauchy sequence (Roman, 1975). Because $\{\eta_k\}$ is Cauchy sequence and $\eta_0 = 0$ is continuous, $\{\eta_k\}$ converges to a continuous function η (Stakgold, 1998). Let us take limit on both sides of (6), then

$$\lim_{n\to\infty} \eta_{n+1} = \lim_{n\to\infty} B(\eta_n) \quad \text{or} \quad \eta = \lim_{n\to\infty} B(\eta_n).$$

Because contraction B is continuous, the lim notation and the operator are interchangeable (Stakgold, 1998):

$$\eta = E\left(\lim_{n\to\infty} \eta_n\right).$$

We obtain $\eta = B(\eta)$ as required. Therefore, it is concluded Stokes wave profile (6) is expressed as $\eta = \lim_{k\to\infty}\eta_k$; furthermore, its existence of continuous wave profile is always guaranteed under the condition for contraction (13).

6. Numerical results

In this section, we will present the numerical results of the nonlinear Stokes wave profile. For the solution of the unknown free surface of the nonlinear equation (4), (6) is iterated with an initial condition for zero function (for $n = 0$), that is, $\eta_0 = 0$. In this paper, the three different wave slopes ka are taken as 0.01, 0.1, 0.2, and 0.3, respectively. For the numerical convergence test, we calculate the following coefficient of norm error $\mu_\infty = ||\eta_{\mathrm{ST}} - \eta_n||_\infty / ||\eta_{\mathrm{ST}}||_\infty$, as a function of the number of iterations n. Here η_{ST} stands for Stokes wave profile of third order:

$$\eta_{\mathrm{ST}} = a \cos\left[kx - \omega\left(1 + \frac{a^2k^2}{2}\right)t\right] + \frac{1}{2} a^2 k \cos\left\{2\left[kx - \omega\left(1 + \frac{a^2k^2}{2}\right)t\right]\right\} + \frac{3}{8} a^3 k^2 \cos\left\{3\left[kx - \omega\left(1 + \frac{a^2k^2}{2}\right)t\right]\right\}.$$

The coefficient value for μ_∞ is plotted in Fig. 2 for the four different wave slopes, where the symbol n represents the number of iterations. It shows that our solution strategy η_n is converging to the Stokes nonlinear solution η_{ST} regardless of the wave slopes. From this analysis, it is clear that the norm error of μ_∞ is reduced as n increases. High convergence rates are found for all cases of wave slopes.

The convergence behavior of η_n for $ka = 0.2$ is illustrated in Fig. 3. From the figure, we can observe that the zero line (corresponding to the zero-initial wave form) approaches the nonlinear wave solution. It was only a few iterations which gave a dramatically converged wave profile.

The obtained converged solutions are compared with the corresponding Stokes and linear waves, as shown in Figs. 4–6. When $ka = 0.01$, the comparison indicates that it is difficult to distinguish the converged solution from the Stokes profile. As the wave slope becomes larger, we can examine the nonlinear wave characteristics of the actual shapes of waves, that is, the crests are steeper and the troughs are flatter. There are little differences of wave profiles between the converged solutions

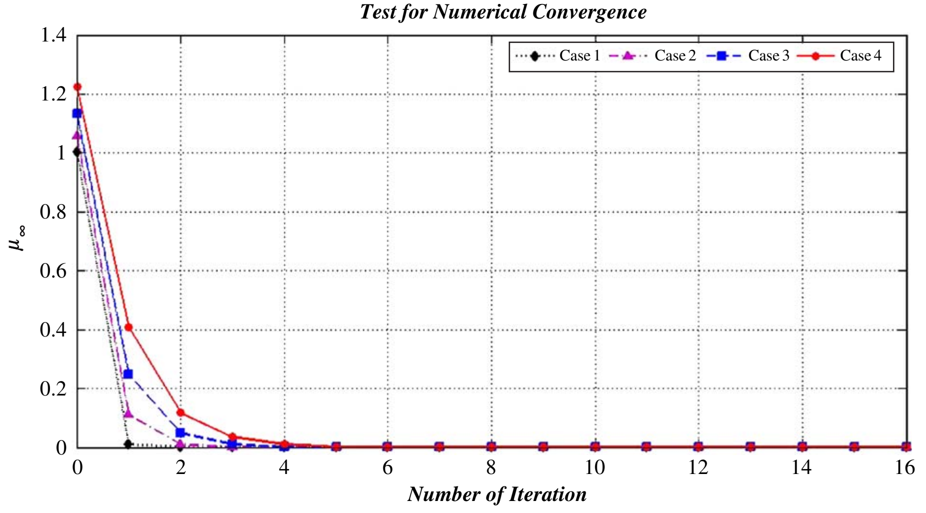

Fig. 2. Test for numerical convergence using $\mu_\infty = ||\eta_{\mathrm{ST}} - \eta_n||_\infty / ||\eta_{\mathrm{ST}}||_\infty$.

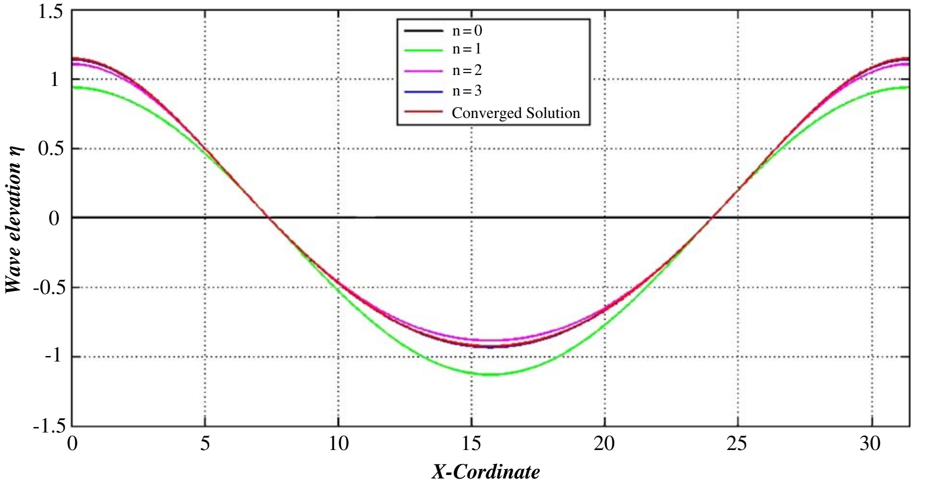

Fig. 3. Convergence behavior of η_n for $ka = 0.2$ (Case 3).

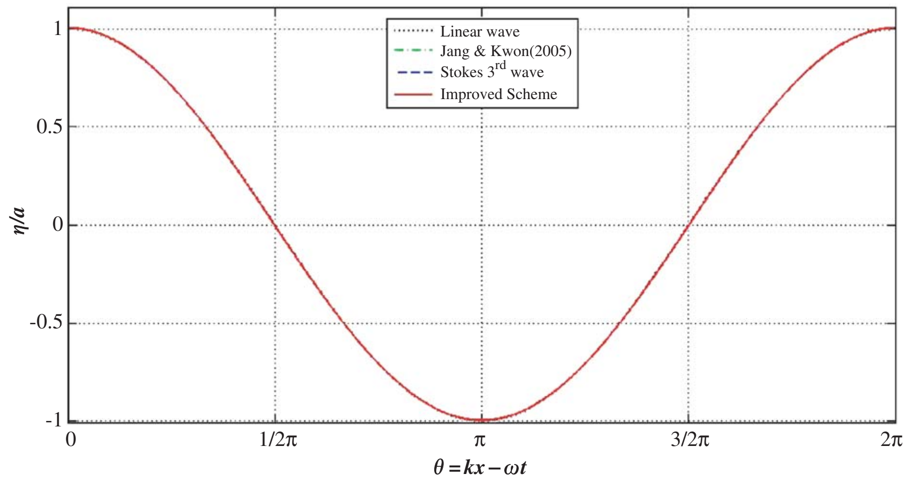

Fig. 4. Comparison of wave profiles for $ka = 0.01$ (case 1).

(or improved solutions) and the Stokes profiles in the all cases of $ka = 0.1$, 0.2 and 0.3. Compared with the Stokes profiles, the result of Jang and Kwon (2005) has some errors of differences between wave profiles, especially in parts of the crests, when the larger wave slope is considered as shown in Fig. 7.

The Fourier transform analysis was done on the wave profiles which were generated by the improved iterative scheme to investigate the frequency contents (Figs. 8–11). Here wave amplitude a is normalized to be unit (i.e., $a = 1$).

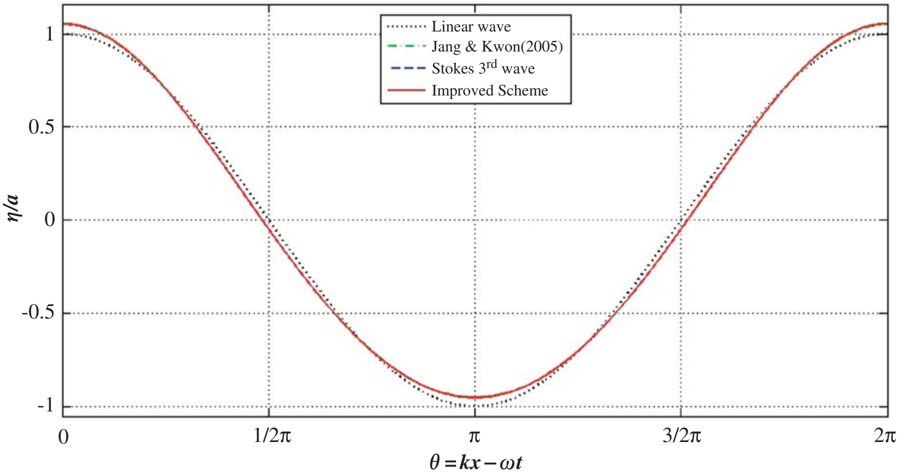

Fig. 5. Comparison of wave profiles for $ka = 0.1$ (case 2).

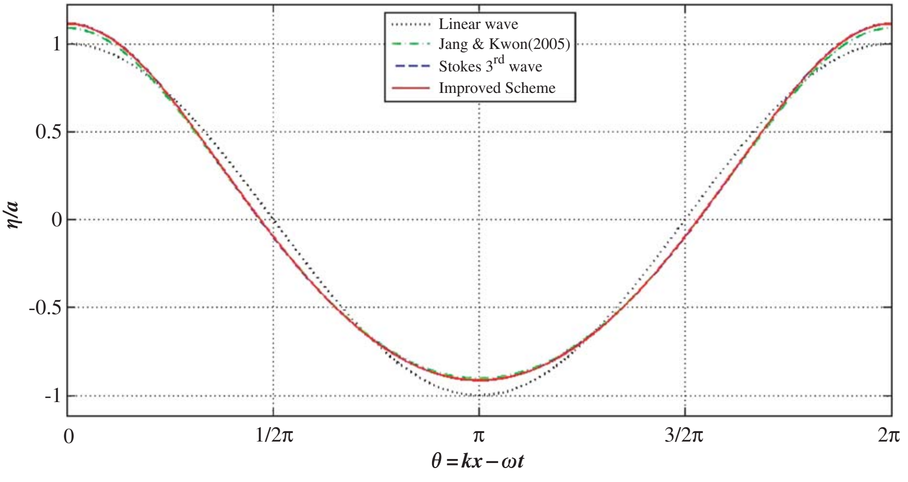

Fig. 6. Comparison of wave profiles for $ka = 0.2$ (case 3).

When wave slope is very small as in Fig. 8, the significant peak is observed at $k = 0.01$. It is the main wave number component. We can see the peak due to the double wave number component at 0.02 even though its magnitude is very small compared to that of the main wave number component.

As the wave slope becomes larger, we can investigate nonlinear wave characteristics of Stokes' wave, that is, nonlinear higher frequencies of wave number: three peaks of wave number can be examined in Fig. 9 (wave slope $ka = 0.1$), four peaks in Fig. 10

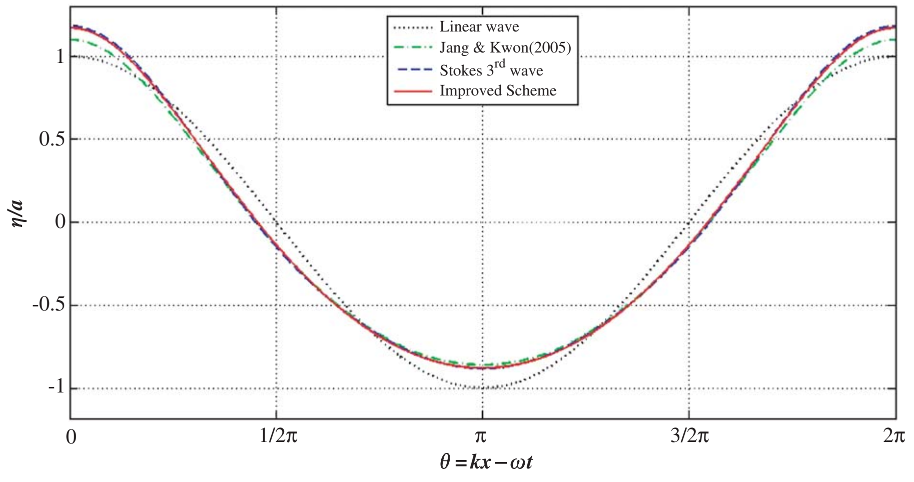

Fig. 7. Comparison of wave profiles for $ka = 0.3$ (case 4).

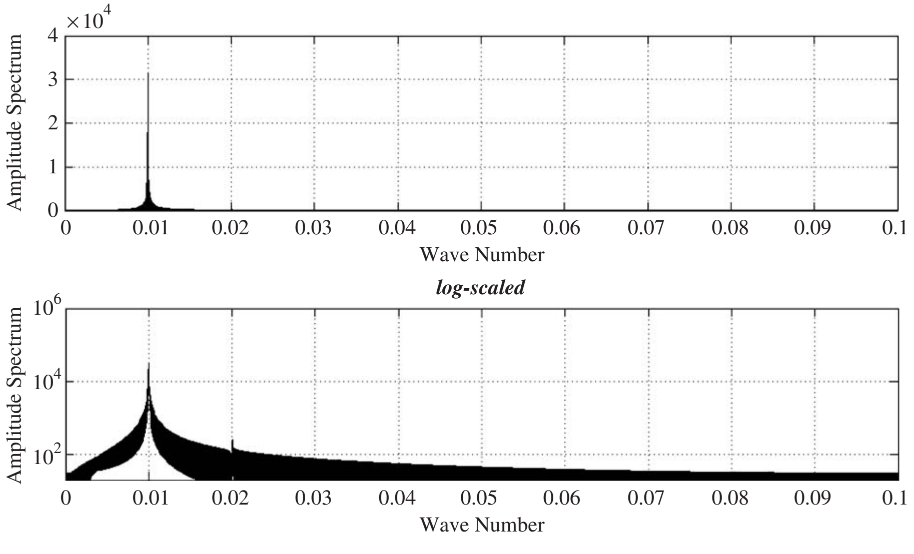

Fig. 8. Amplitude spectra for wave slope $ka = 0.01$.

(wave slope $ka = 0.2$), and five peaks in Fig. 11 (wave slope $ka = 0.3$). Eq. (8) for Stokes' third-order perturbation theory cannot predict the results for Figs. 10 and 11: that is, it was possible to predict higher-order nonlinear Stokes' wave number components by using the improved iterative scheme. It is not hard to see that these peak points shown in Figs. 10 and 11 are exactly same as Stokes' nonlinear higher frequencies of wave number (Debnath, 1994).

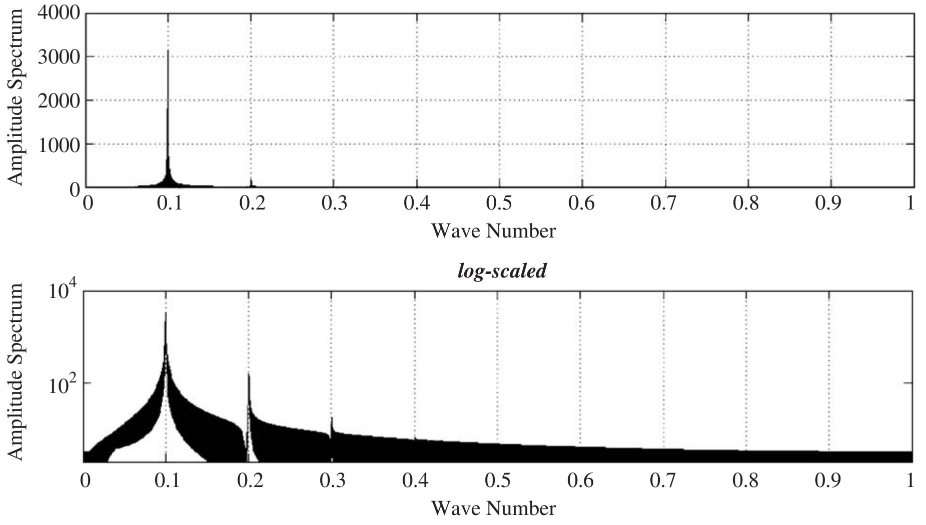

Fig. 9. Amplitude spectra for wave slope $ka = 0.1$.

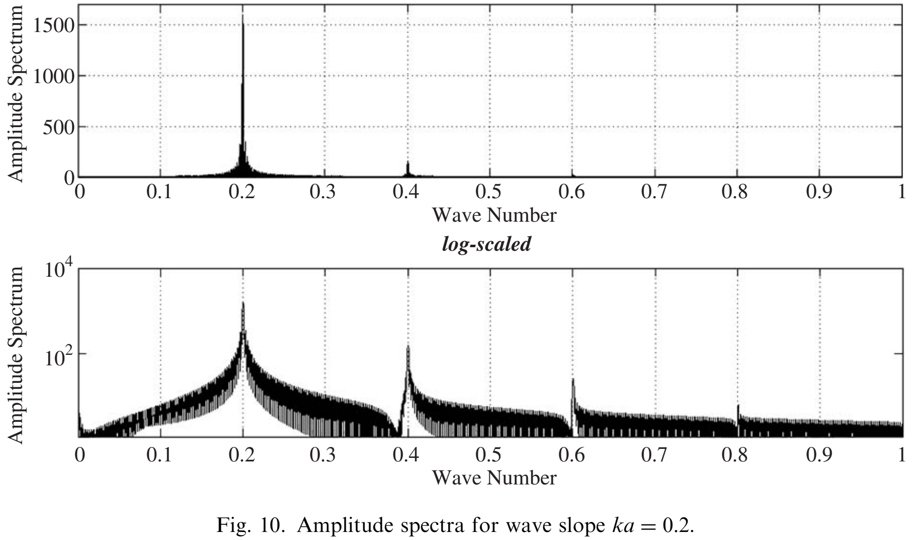

Fig. 10. Amplitude spectra for wave slope $ka = 0.2$.

7. Conclusions

Combining the contraction mapping theorem with the nonlinear dispersion relation, a nonlinear iterative scheme was proposed to achieve higher-order Stokes' wave profile in deep water. The stability of the proposed scheme was analyzed. The existence and uniqueness of the solution were also discussed. Although the proposed

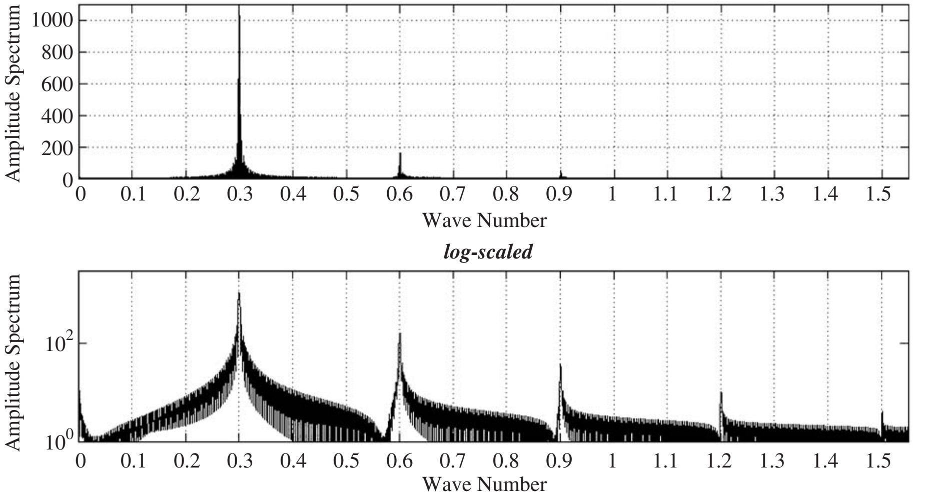

Fig. 11. Amplitude spectra for wave slope $ka = 0.3$.

iterative scheme was based on Stokes' third-order perturbation theory, the solution of the scheme enabled us to investigate higher-order nonlinear characteristic of Stokes' wave profile. In addition to that, it provided more accurate numerical results of Stokes' profiles when compared to the previous iterative scheme based on the linear dispersion relation (Jang and Kwon, 2005). Furthermore, the convergence rate was shown to be very fast in the present scheme.

Acknowledgment

This work was supported by the Korea Science and Engineering Foundation through the Advanced Ship Engineering Research Center at the Pusan National University.

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