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Instabilities and oscillations of one- and two-dimensional Kadomtsev–Petviashvili waves and solitons II. Linear to nonlinear analysis

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We further investigate the dynamics of nonlinear structures that arise from the Kadomtsev–Petviashvili equation. When the analysis is linear, assuming perturbations grow exponentially in time, we find the growth rates of two important instabilities numerically. These are the only purely growing modes. The wavelength-doubling instability is seen to dominate its rival, that of Benjamin and Feir, at least when the amplitude of the wave is not too large. Approximate formulae, found to higher order than in part I (referenced in § 1), are checked against the numerically found values. The models are seen to be better than expected. For the dominant wavelength-doubling instability, our model extends beyond the assumed region of validity. It is surprisingly close, almost up to the soliton limit.

When we depart from linear stability analysis and include terms nonlinear in the perturbation, a simple analysis shows that the linear instability eventually drives a doubly space-periodic hyperbolic secans pulse in time. After a long time, initial conditions are reproduced. A proof that the maximum amplitude achieved by the perturbation is approximately proportional to the linear growth rate is given within the limitations of the calculation. This fact was suspected from numerics. A second class of possible dynamic behaviour, not arising from initially linear growth of a perturbation, is found. This class involves fully two-dimensional stationary solutions and their possible oscillations.

Keywords: Kadomtsev–Petviashvili equation; Benjamin–Feir instability; wavelength-doubling instability; cnoidal waves; solitons; Landau-type equation

1. Introduction

The equation formulated by Kadomtsev & Petviashvili (1970),

$$u_t + uu_x + u_{xxx} - \partial_x^{-1} u_{yy} = 0, \tag{1.1}$$

is generally designated KPI. It is integrable by inverse scattering (Zakharov & Shabat 1971). New mathematical results, embracing interpretations of the Kadomtsev–Petviashvili hierarchy, as well as new methods of finding solutions, can be found in

Iliev (1997), Bagrov *et al*. (1998) and Deoniuk & Segur (1998). Physical contexts, as well as methods of derivation in hydrodynamics, plasma theory and superfluidity, have been described by Infeld & Rowlands (2000), Jones & Roberts (1982), and in several other references listed by Infeld *et al*. (1999), henceforth referred to as part I. This last reference described a model for the growth rates of perturbations to nonlinear wave and soliton solutions of equation (1.1). In that reference, cnoidal wave solutions to this equation were derived. Next, charts of unstable regions in the space of the wavevector $\boldsymbol{K}$ of the linear perturbations were drawn. When the amplitude of the cnoidal wave was small (small m of the $\mathrm{sn}(x \mid m)$ function), approximate formulae for the growth rates, $\Gamma(\boldsymbol{K}, m)$, were given. Formulae for the soliton limit ($m = 1$) have been known for some time (Zakharov *et al*. 1980). Further large-amplitude information ($m < 1$, but not small) was taken from Infeld *et al*. (1978) and Infeld & Rowlands (1979a, b). All in all, a reasonably complete picture emerged. This information would be extremely difficult to extract from the formal solution of Kuznetsov *et al*. (1984).

In the present extension of part I, we find the growth rates Γ numerically and thus test our model equations (which we improve). Two instabilities, one like Benjamin & Feir (1967) (see also Lighthill 1965), and the wavelength-doubling mode, in which K_x is exactly one half of $2\pi/\lambda$ of the nonlinear structure, are treated. The latter instability is seen to be dominant. This is in contradistinction to the gravity wave case, where this only happens for some cases (see Saffman & Yuen 1985). Somewhat unexpectedly, an extension of the formula for Γ derived in part I is surprisingly good almost all the way up to the soliton limit ($m = 1$) for the dominant instability.

When terms nonlinear in the perturbation are included, we find that the initially exponential growth, characteristic of the linear regime, levels off. After a long time, the initial structure is recreated. We give a proof that the maximum amplitude of the perturbation is approximately proportional to Γ within the limitations of our calculation (for small m, the proportionality coefficient is even m independent). This was pointed out by Casali *et al*. (1998) and has been suspected to be true from numerics for some time. Thus, in a Landau-type model, given schematically by (see, for example, Infeld & Rowlands 2000, ch. 11)

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \Gamma a - \beta a^n, \quad n \geqslant 2, \tag{1.2}$$

an important subclass, in which β is not arbitrary, but can only depend on Γ, should be recognized. For our KPI calculation, β would then be a numerical coefficient divided by Γ, $n = 3$, and $a_{\max} \propto \Gamma$. We will come back to this theme in § 5. A second example will be quoted. Finally, different solutions for the dynamics of the perturbations, not having any relevance to the linear regime, are found. Presumably, an outside influence would be required to set one of them up. The common denominator of all these solutions is their recurrent character.

The plan of the paper is as follows. First, in § 2, we rederive the formulae for the nonlinear wave, so that part I need not be consulted. Next, in § 3, we derive the formula for Γ of the wavelength-doubling instability, central to further considerations. Derivation will be of higher order than in part I and geared to nonlinear extensions. Numerical values of Γ for a wide range of situations are then given in two figures in § 4. These are compared to what follows from the models. Next, in § 5, we perform a small-amplitude expansion including terms nonlinear in the perturbation and corresponding to the dominant instability. Removal of secular terms leads to a pulse in time, of a chessboard spatial structure. After a long time, the pulse disappears and the initial wave structure is recreated. The section ends with a discussion of the limitations of our treatment.

Details of the numerics of § 4, as well as some further solutions to the nonlinear problem, not in general associated with the build-up of a linear instability, are relegated to Appendices A–C.

This paper can be read independently of part I.

2. Form of the background nonlinear wave

We now solve equation (1.1) for functions of $x-u_0t$, working in the coordinate system of the wave or soliton. Take $U = u(x) - u_0$, $x \to x - u_0t$, to obtain from (1.1)

$$U_{xxx} + UU_x = 0, \tag{2.1}$$

$$U_{xx} + \tfrac{1}{2}U^2 = 8[(m+1)^2 - 3m]. \tag{2.2}$$

The choice of constant will simplify calculations. We multiply equation (2.2) by U_x and integrate, once again choosing our constant with an eye on the final result,

$$U_x^2 + \tfrac{1}{3}(U - 4m - 4)(U - 4m + 8)(U + 8m - 4) = 0. \tag{2.3}$$

Equation (2.3) is solved by

$$U_0 = 4(m+1) - 12m\,\mathrm{sn}^2(x \mid m), \quad 0 \leqslant m \leqslant 1. \tag{2.4}$$

(When $m > 1$, we find $U_0 = 4(m+1) - 12\,\mathrm{sn}^2(\sqrt{m}x \mid m^{-1})$ by multiplying equation (2.3) through by m^{-3}, and so we will limit our considerations to $m \leqslant 1$.) Here, sn is the Jacobian elliptic function (Milne-Thomson 1950). Its period is $4K(m)$, where K is the complete elliptic integral. The square of sn, appearing above, has period $2K(m)$. For small m, we have

$$\mathrm{sn}(x \mid m) \simeq (1 + \tfrac{1}{16}m)\sin(\xi) + \tfrac{1}{16}m\sin(3\xi), \tag{2.5}$$

$$\xi \simeq x(1 + \tfrac{1}{4}m + \tfrac{9}{64}m^2)^{-1}. \tag{2.6}$$

For $m = 1$, the wavelength is infinite and $U_0 = 12\,\mathrm{sech}^2\,x - 4$. Our basic wave only depends on one parameter, though it is often described by two or three in the literature (two in Casali *et al*. 1998). However, as rescaling U_0 and x will reproduce these dependencies, we will work with this simplified version.

3. Small *m* models of two unstable modes

We now perturb the solution of § 2 and linearize (1.1),

$$\left.\begin{aligned} U &= U_0 + \delta U, \\ \delta U &= \tilde{u}(x)\mathrm{e}^{\Gamma t + \mathrm{i}K_y y}, \\ \tilde{u}(x) &= u(x)\mathrm{e}^{\mathrm{i}K_x x}. \end{aligned}\right\} \tag{3.1}$$

All these quantities will be needed. Now, from (1.1),

$$\Gamma\frac{\mathrm{d}\tilde{u}}{\mathrm{d}x} + \frac{\mathrm{d}^4\tilde{u}}{\mathrm{d}x^4} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\{[4(m+1) - 12m\,\mathrm{sn}^2(x \mid m)]\tilde{u}\} + K_y^2\tilde{u} = 0. \tag{3.2}$$

We next perform an expansion around $K_x = 1$, $K_y = \sqrt{3}$, keeping K_x fixed, but allowing K_y to vary. We know from part I that this region corresponds to the largest growth rates, hence the physical motivation. We also simplify calculations by so doing. We assume m small and expand in this quantity. This means that we consider small background waves. We also change the independent variable to ξ,

$$\left.\begin{aligned} \tilde{u} &= \tilde{u}_1 + \tilde{u}_2 + \cdots, \\ K_y^2 &= 3 + \delta K_{y1}^2 + \delta K_{y2}^2 + \cdots. \end{aligned}\right\} \tag{3.3}$$

Here, the subscript denotes a particular order in m. We find, defining $L = \partial_\xi^4 + 4\partial_\xi^2 + 3$,

$$L\tilde{u}_1 = 0, \tag{3.4}$$

$$\tilde{u}_1 = a\cos(\xi + \alpha). \tag{3.5}$$

We assume a to be a small amplitude in $\tilde{u}_1$. This assumption is not necessary in a linear calculation such as this, but will be in the extension to come in § 5. In the next order, we obtain

$$L\tilde{u}_2 = [3m\sin 2\alpha + \Gamma_1]a\sin(\xi+\alpha) + [3m(\cos 2\alpha - 1) - \delta K_y^2]a\cos(\xi+\alpha) + \text{non-secular}. \tag{3.6}$$

So as to avoid secular terms, we demand that

$$\left.\begin{aligned} [3m\sin 2\alpha + \Gamma_1]a &= 0, \\ [3m(\cos 2\alpha - 1) - \delta K_y^2]a &= 0, \end{aligned}\right\} \tag{3.7}$$

which is satisfied if

$$\Gamma_1 = \sqrt{(3 - K_y^2)[K_y^2 - 3(1 - 2m)]}. \tag{3.8}$$

Instability thus appears between $K_y = \sqrt{3}$ and $K_y = \sqrt{3}\sqrt{1-2m}$ in this, small-m, approximation. The exact value of the lower limit is $\sqrt{3}(1-m)$ (see part I). We can therefore improve our model with impunity by writing

$$\Gamma_1 = \sqrt{(3 - K_y^2)[K_y^2 - 3(1 - m)^2]}, \tag{3.9}$$

equivalent in our approximation, but incorporating the exact lower limit. The upper limit is exact as it is (see part I). The perturbation in question is y dependent and has a wavelength twice that of the basic nonlinear structure (2.4). It thus breaks the y symmetry and alters the x symmetry. It is sometimes called the symmetry-breaking instability. Up to now, we have just been quoting part I, so as to render this paper readable independently of its predecessor. From now on, results for this mode are new. Calculation to next order vindicates (3.9). When secular terms are removed, we obtain $\Gamma_2 = \frac{1}{4}m\Gamma_1$. Thus

$$\Gamma = (1 + \tfrac{1}{4}m)\Gamma_1. \tag{3.10}$$

The maximum value of Γ following from (3.10) is $\Gamma_{\max} = 3m(1 + \frac{1}{4}m)(1 - \frac{1}{2}m)$.

The Benjamin–Feir (BF) instability is triggered by a perturbation having the same wavelength in x as the background wave,

$$\tilde{u}_1 = b\cos(2\xi + \alpha). \tag{3.11}$$

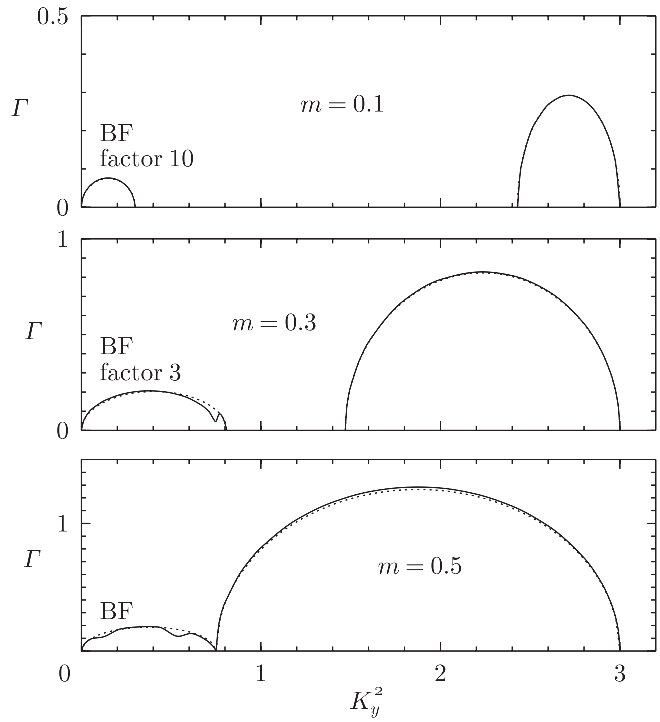

Figure 1. Growth rate Γ versus K_y^2 for both the BF and wavelength-doubling instabilities. Numerical results designated by solid lines, our model by broken lines. The graphs emerging from the origin for $m = 0.1$ are blown up by a factor of 10, and those for $m = 0.3$ by a factor of 3.

Here, the name Benjamin–Feir is used somewhat loosely. In that original instability, the perturbation was long wave in the x direction. Here, it is long wave in y. The long-wave character of the perturbation, albeit in a different direction, might justify the label. Derivation of Γ for small m is given in part I. If $K_x = 0$, the result is

$$\Gamma = \tfrac{1}{2} K_y \sqrt{3m^2 - K_y^2}, \qquad \Gamma_{\max} = \tfrac{3}{4} m^2, \tag{3.12}$$

and thus is weaker than the wavelength-doubling mode for all m, at least as follows from our model. We will see in § 4 that numerics basically vindicate this. Unstable K_y appear between 0 and $\sqrt{3}m$, shown in part I to be exact limits.

Thus we have models of both instabilities that, though derived for small m, give correct limits of unstable regions. In § 4 we will compare these formulae with values found from a simulation. The purpose of this exercise will be twofold. First, it will give a check on our models. Also, we will see how far our formulae can be pushed. Though derived for small m, there is always the possibility of extended validity.

A common feature of the two modes treated above is their purely growing character (Γ real). In this respect, they are unique, as shown in part I.

4. Numerical results

We calculated growth rates Γ for the two instabilities numerically. This seems to be simpler than trying to unravel the formulae of Kuznetsov *et al*. (1984). (See also

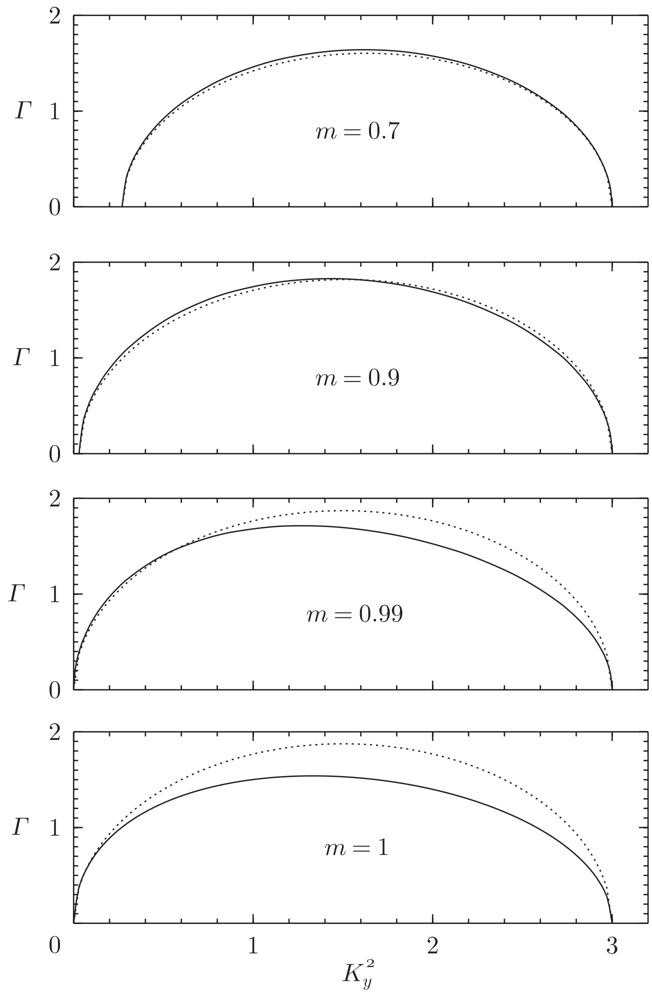

Figure 2. As in figure 1, but just the dominant wavelength-doubling instability, $0.7 \leqslant m \leqslant 1$. For the soliton, $m = 1$, the solid line represents the exact formula of Zakharov *et al*. (1980). Here, since the wavelength is infinite, the two instabilities merge.

part I for a discussion.) The results are presented in figures 1 and 2. The three frames of figure 1 show numerically found values of Γ for both instabilities up to $m = 0.5$, solid lines (details of the numerical simulation are given in Appendix A).

Clearly, as expected, wavelength doubling is dominant. Our model, indicated by broken lines, yields a good approximation for both instabilities.

The first three frames of figure 2 are limited to the dominant mode; the fourth, for the soliton, corresponding to the two merging. For wavelength doubling, agreement up to $m = 0.9$ is surprisingly good. It so happens that $\Gamma(K_y)$ for a soliton is known (Zakharov *et al*. 1980),

$$\Gamma_{\mathrm{sol}} = (4/3^{1/2})K_y\sqrt{1 - K_y/3^{1/2}}, \quad m = 1. \tag{4.1}$$

In conclusion, our model yields good approximations to Γ of the wavelength-doubling instability up to $m = 0.9$. Critical values $K_{y\text{crit}}$ are, on the other hand, all exact. All this is surprisingly close for a small-m model.

5. Nonlinear behaviour of the perturbation

We will now concentrate on the further fate of the wavelength-doubling instability. Exponential growth cannot continue past the stage when the perturbation becomes comparable to the basic wave. Thus we are looking at a class of dynamics of the perturbation such that initially equation (3.10) holds. However, the nonlinear term in δU then curbs the growth. Equation (3.2) must be generalized to

$$\frac{\partial^2 \delta U}{\partial t \partial x} + \frac{\partial^4}{\partial x^4}\delta U + \frac{\partial^2}{\partial x^2}\{[4(m+1) - 12m\,\mathrm{sn}^2(x \mid m)]\delta U + \tfrac{1}{2}\delta U^2\} - \frac{\partial^2}{\partial y^2}\delta U = 0, \quad (5.1)$$

yielding (3.2) when the δU^2 term is neglected and $\delta U \propto \mathrm{e}^{\mathrm{i}K_y y + \Gamma t}$ is assumed. Now removal of secular terms is no longer possible when δU_1 is merely proportional to the right-hand side of (3.5). We will need all physically meaningful eigenfunctions of

$$L = \partial_\xi^4 + 4\partial_\xi^2 - \partial_{y_0}^2, \quad (5.2)$$

where the last operator is of lowest order in m. Thus we start the calculation off with a hybrid combination of eigenfunctions, extending (3.5), including (3.11),

$$L\delta U_1 = 0, \qquad \delta U_1 = a\cos(\xi + \alpha_1)\cos(K_y y) + b\cos(2\xi + \alpha_2) + c. \quad (5.3)$$

The end result is not influenced by taking α_1 and α_2 to be different. In this exposition we will therefore take $\alpha_1 = \alpha_2 = \alpha$ for simplicity. Nonlinearity will generate the second and third component from the first, etc. Here, a, b, c and α are functions of time only. We demand

$$\lim_{a\to 0}\frac{a_t}{a} = \Gamma, \quad (5.4)$$

where Γ is given by (3.10). The calculation extends that leading to (3.9), but, of course, there will now be more terms. Removal of secularities leads to four conditions following from

$$L\delta U_2 = \text{secular} + \text{non-secular}. \quad (5.5)$$

Removal of secular terms yields

$$3am\sin 2\alpha + a_t = 0 \quad \text{removes} \quad \sin(\xi+\alpha)\cos K_y y, \quad (5.6\,a)$$

$$[3m(\cos 2\alpha - 1) - \delta K_y^2 + \alpha_t + c + \tfrac{1}{2}b]a = 0 \quad \text{removes} \quad \cos(\xi+\alpha)\cos K_y y, \quad (5.6\,b)$$

$$b_t + 12mc\sin 2\alpha = 0 \quad \text{removes} \quad \sin(2\xi+\alpha), \quad (5.6\,c)$$

$$b(\alpha_t + c) + \tfrac{1}{8}a^2 + 6mc\cos 2\alpha = 0 \quad \text{removes} \quad \cos(2\xi+\alpha). \quad (5.6\,d)$$

The first two conditions generalize (3.7), yielding them when $\alpha_t = b = c = 0$. The middle two can be combined to yield

$$\frac{\mathrm{d}w}{\mathrm{d}t} = 6mw\sin 2\alpha, \qquad w = c + \alpha_t. \quad (5.7)$$

Solutions to equations (5.6) for $w \neq 0$ are given in Appendices B and C. Here, we are only concerned with solutions that describe a build-up from a linear perturbation, satisfying (5.4). These are found in Appendix C to correspond to $w = 0$. This restriction leads to one equation for a when (5.6 b) is differentiated by time and combined with (5.6 c) to remove b_t. Finally,

$$\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 = \Gamma^2 a^2 - \tfrac{1}{8}a^4; \tag{5.8}$$

a Landau-type equation when a is small. Importantly, there is only one parameter (Γ). The solution is

$$a = \pm\sqrt{8}\Gamma \operatorname{sech}[\Gamma(t - t_0)]. \tag{5.9}$$

Our solution uniquely describes a pulse driven by an initially linear instability. In other words, this is what we will observe if we start out with a tiny perturbation that falls within the instability range. Obviously, $a_{\max} \propto \Gamma$, and hence $(\delta U_1)_{\max}$ will be approximately proportional to Γ. This result was suggested in the literature as following from numerics (see, for example, Casali *et al.* 1998).† The proportionality is only approximate, as b can be seen from (5.6) and (5.9) to be non-zero but of higher order than a in this calculation. (All terms in each of (5.6) are of the same order, but not in (5.3) for our solution.) Ours is a simple confirmation of the above approximate relation within the limitations of our ansatz (small amplitude of both background wave and perturbation, periodicity in x with period double that of the background wave). More generally, it argues for the recognition of a subclass of Landau-type models in which there is essentially one parameter instead of two. A simple variant of this calculation shows that the equation of Zakharov & Kuznetsov (1974) also leads to equation (5.8). As mentioned in part I, and described in some detail for solitons by Murakani & Tajiri (1992) and Pelinovsky & Stepanyants (1993), and for waves by Infeld *et al.* (1995) and Infeld & Rowlands (2000, ch. 10), the appearance of a pulse that disappears after a while does not tell the whole story. A two-dimensional soliton, known as a lump, will detach itself from each crest of the wave (peaked at $\xi + \alpha = 2n\pi$ and $K_y y = 2m\pi$ in our model) and proceed forward at a greater speed than that of the background wave. This, however, is outside the scope of the present model. The lump was investigated numerically in part I and also by Senatorski & Infeld (1998) in three dimensions, and in two dimensions by Infeld *et al.* (1995), Minzouri (1996) and Feng *et al.* (1999).

In contradistinction to Casali *et al.* (1998), involving numerical work, the above calculation is simple and straightforward.

Appendix A.

To determine the eigenmodes numerically, we first admit arbitrary x and t dependence of the disturbance, i.e. we assume

$$U = U_0 + \tilde{u}(x, t)\cos(K_y y), \tag{A 1}$$

† Actually, when we check the statement of Casali *et al.* that $(\delta U_1)_{\max} \propto \Gamma$, by comparing their figure 5 with their figure 1, that is, $(\delta U_1)_{\max}$ against Γ, we find that the ratio of the two values is not strictly independent of K_y and varies between 5 and 6.4. This, however, may well be due to the fact that they invite us to compare curves for different background waves (this for the weaker instability of Benjamin & Feir)! Their theoretical argument substantiating $(\delta U_1)_{\max} \propto \Gamma$ is also unconvincing, as there is no way of knowing that their complicated term that corresponds to β in our (1.2) (their last term in equation (11)) has the required Γ dependence for this to be the case.

where U_0 is given by (2.4) and $\tilde{u}(x,t)$ is periodic in x (with period $2L$ equal to $2K(m)$ for the BF mode, and $4K(m)$ for the wavelength-doubling mode). Inserting this into (1.1) and linearizing, we obtain

$$[\tilde{u}_t + (U_0(x)\tilde{u})_x + \tilde{u}_{xxx}]_x + K_y^2\tilde{u} = 0. \tag{A 2}$$

This equation is not an evolution equation for $\tilde{u}$, but can be reduced to one by integrating over x, starting from some x_0. The result can be written as

$$\left.\begin{aligned} &\tilde{u}_t + \mathrm{F}[\tilde{u}] = 0,\\ &\mathrm{F}[\tilde{u}] \equiv -\tilde{u}_t(x=x_0,t) + [(U_0(x)\tilde{u})_x + \tilde{u}_{xxx}]_x|_{x_0}^{x} + K_y^2\int_{x_0}^{x}\tilde{u}(x',t)\,\mathrm{d}x'. \end{aligned}\right\} \tag{A 3}$$

Equation (A 3) defines $\tilde{u}(x,t)$ for any x and t if we specify

$$\tilde{u}(x,t=0) = \tilde{u}_0(x), \quad \text{initial condition at } t=0, \tag{A 4a}$$

$$\tilde{u}_t(x=x_0,t) = \tilde{u}_{t0}(t), \quad \text{boundary condition at } x=x_0. \tag{A 4b}$$

As both $U_0(x)$ and $\tilde{u}(x,t)$ are periodic in x, it is quite natural to represent $\tilde{u}(x,t)$ on the mesh by the inverse discrete Fourier transform (DFT). With this in mind, we first linearly transform the x coordinate, so that the interval $(-L,L)$ is mapped into $(0,2\pi)$, i.e. replace x by $\bar{x}$,

$$\bar{x} = \alpha_x(x+L), \qquad \alpha_x = \pi/L. \tag{A 5}$$

Also transforming t into $\bar{t} = \alpha_t t$, $\alpha_t = \alpha_x^3$, and leaving out the tilde, we end up with

$$\left.\begin{aligned} &u_{\bar{t}} + \bar{\mathrm{F}}[u] = 0,\\ &\bar{\mathrm{F}}[u] \equiv -u_{\bar{t}}(\bar{x}=\bar{x}_0,\bar{t}\,) + [c_1(\bar{U}_0(\bar{x})u)_{\bar{x}} + u_{\bar{x}\bar{x}\bar{x}}]_{\bar{x}}|_{\bar{x}_0}^{\bar{x}} + c_2\int_{\bar{x}_0}^{\bar{x}} u(\bar{x}',\bar{t}\,)\,\mathrm{d}\bar{x}',\\ &\bar{U}_0(\bar{x}) = \mathrm{cn}^2(x \mid m) + c_3,\\ &c_1 = \frac{12m}{\alpha_x^2}, \qquad c_2 = \frac{K_y^2}{\alpha_x^4}, \qquad c_3 = \frac{1-2m}{3m},\\ &0 \leqslant \bar{x} \leqslant 2\pi, \qquad \bar{t} \geqslant 0. \end{aligned}\right\} \tag{A 6}$$

We introduce the mesh $\bar{x}_j = j\Delta\bar{x}$, $j = 0,1,\dots,N-1$, $\Delta\bar{x} = 2\pi/N$, where N is an even integer. To determine $\bar{\mathrm{F}}[u]$ on the mesh, we first calculate the DFT,

$$\hat{u}_k = N^{-1/2}\sum_{j=0}^{N-1} u(\bar{x}_j)\mathrm{e}^{-\mathrm{i}k\bar{x}_j}, \quad k = 0,\pm1,\pm2,\dots,\pm\tfrac{1}{2}N. \tag{A 7}$$

The inverse DFT defines $u(\bar{x}_j)$ in terms of $\hat{u}_k$,

$$u(\bar{x}_j) = N^{-1/2}\sum_{k=-N/2}^{N/2}\hat{u}_k\mathrm{e}^{\mathrm{i}k\bar{x}}\bigg|_{\bar{x}=\bar{x}_j}, \tag{A 8}$$

where only one half of the contributions at $k = \pm\frac{1}{2}N$ should be included in the sum over k. Equation (A 8) is, in fact, an interpolation formula for $u(\bar{x}_j)$, which can be used to determine the $\bar{x}$ derivatives and the integral in $\bar{\mathrm{F}}[u]$. Thus, for example,

$$u_{\bar{x}\bar{x}\bar{x}}(\bar{x}_j) = N^{-1/2}\sum_k(\mathrm{i}k)^3\hat{u}_k\mathrm{e}^{\mathrm{i}k\bar{x}_j},$$

and similarly for $(\bar{U}_0 u)_{\bar{x}}(\bar{x}_j)$. The integral in $\bar{\mathrm{F}}[u]$ must be periodic in $\bar{x}$, which requires $\hat{u}_0(\bar{t}\,) \equiv 0$. In that case,

$$\int_{\bar{x}_0}^{\bar{x}_j} u(\bar{x}', \bar{t}\,)\,\mathrm{d}\bar{x}' = N^{-1/2} \sum_{k \neq 0} \frac{\hat{u}_k}{\mathrm{i}k} \mathrm{e}^{\mathrm{i}k\bar{x}} \bigg|_{\bar{x}_0}^{\bar{x}_j}.$$

Equation (A 6) was integrated using the leapfrog time-step (second-order accuracy in $\delta\bar{t}\,$),

$$u(\bar{t} + \delta\bar{t}\,) - u(\bar{t} - \delta\bar{t}\,) + 2\delta\bar{t}\bar{\mathrm{F}}[u(\bar{t}\,)] = 0. \tag{A 9}$$

This explicitly defines $u(\bar{t} + \delta\bar{t}\,)$ in terms of $u(\bar{t} - \delta\bar{t}\,)$ and $u(\bar{t}\,)$. Starting from

$$u(\bar{x}, \bar{t} = 0) = u_0(\bar{x}) \quad \text{and} \quad u(\bar{x}, \bar{t} = \delta\bar{t}\,) \simeq u_0(\bar{x}) - \delta\bar{t}\bar{\mathrm{F}}[u_0(\bar{x}) - (\tfrac{1}{2}\delta\bar{t}\,)\bar{\mathrm{F}}[u_0(\bar{x})]]$$

(Euler–leapfrog algorithm, second order in $\delta\bar{t}\,$), we can determine $u(\bar{x}, \bar{t} = 2\delta\bar{t}\,)$, then $u(\bar{x}, \bar{t} = 3\delta\bar{t}\,)$ (from $u(\bar{x}, \bar{t} = \delta\bar{t}\,)$ and $u(\bar{x}, \bar{t} = 2\delta\bar{t}\,)$), etc.

The numerical stability of the leapfrog algorithm (A 9) was determined, as it is usually done (Richtmyer & Morton 1967; Fornberg & Whitham 1978; Infeld *et al*. 1995), by examining the (exponential) time behaviour of a single Fourier harmonic. The latter is described by a linear differential equation with *constant* coefficients, following from the algorithm. In our case, this equation is obtained by first replacing the variable coefficient $\bar{U}_0(\bar{x})$ in $\bar{\mathrm{F}}[u]$ by a constant, $\bar{U}_0(\bar{x}) \to a = \text{const.}$, $\operatorname{abs} a \leqslant 1 + \operatorname{abs} c_3$, and then differentiating equation (A 9) with respect to $\bar{x}$. Assuming

$$u = \kappa^{\bar{t}/\delta\bar{t}} \mathrm{e}^{\mathrm{i}k\bar{x}}, \quad k = \pm 1, \dots, \pm \tfrac{1}{2}N,$$

and using it in the differential equation just described, we end up with a quadratic in κ,

$$\kappa^2 - \mathrm{i}2f\kappa - 1 = 0, \qquad f = \delta\bar{t}[k^3 - c_1 ak + c_2/k],$$

i.e. $\kappa = \mathrm{i}f \pm \sqrt{1 - f^2}$. Numerical stability ($\operatorname{abs}\kappa \leqslant 1$) is obtained if and only if $\operatorname{abs} f \leqslant 1$, in which case $\operatorname{abs}\kappa = 1$ (marginal stability). As $k^3 + c_1 \operatorname{abs} ak + c_2/k$ has a single minimum for $k > 0$, the maximum of this function for $k = 1, \dots, \frac{1}{2}N$ is reached at k either 1 or $\frac{1}{2}N$. This leads to a stability condition,

$$\delta\bar{t} < \{\max[1 + g + c_2, \tfrac{1}{2}N(\tfrac{1}{4}N^2 + g) + 2c_2/N]\}^{-1}, \qquad g = c_1(1 + \operatorname{abs} c_3). \tag{A 10}$$

When this condition is fulfilled, the leapfrog algorithm (A 9) is both simple (i.e. fast) and accurate.

For the eigenfunction with $\varGamma > 0$, neither the initial nor the boundary condition is known (see (A 4)). We only know that $\tilde{u} \propto \exp(\varGamma t) = \exp[(\varGamma/\alpha_t)\bar{t}]$, i.e.

$$u_{\bar{t}}(\bar{x} = \bar{x}_0, \bar{t}\,) = \frac{\varGamma}{\alpha_t} u(\bar{x} = \bar{x}_0, \bar{t}\,). \tag{A 11}$$

To determine both the eigenfunction and $\varGamma$ by an iteration process, we start by choosing K_y close to one of the stability limits, where $\varGamma = 0$, and the eigenfunction is known (see equations (5.1) in part I). Putting $\varGamma = \varGamma_0$ in (A 11) (e.g. $\varGamma_0 = 0$, or $\varGamma_0$ given by (3.12)), and choosing the known eigenfunction as the initial condition, we perform a number of evolution steps N_{stp}. We then calculate

$$\varGamma_{0\mathrm{calc}} = \frac{\Delta \ln \operatorname{abs} u(\bar{x}_1, \bar{t}\,)}{\Delta t}, \qquad \Delta t = N_{\mathrm{stp}}\delta t, \qquad \delta t = \frac{\delta\bar{t}}{\alpha_t}, \tag{A 12}$$

where $\bar{x}_1 \neq \bar{x}_0$ must be chosen so that $u(\bar{x}_1, \bar{t}\,) \neq 0$. The next approximation to $\mathit{\Gamma}$ is defined as

$$\mathit{\Gamma}_1 = (1-\tau)\mathit{\Gamma}_0 + \tau \mathit{\Gamma}_{0\mathrm{calc}}, \tag{A 13}$$

where τ is the interpolation parameter ($0 \leqslant \tau \leqslant 1$). The procedure is then repeated (with the same $\bar{x}_1$), starting from $\mathit{\Gamma}_1$ in equation (A 11) and the already determined $\tilde{u}(x, t = \Delta t)$ as the initial condition. This leads to $\mathit{\Gamma}_2 = (1-\tau)\mathit{\Gamma}_1 + \tau\mathit{\Gamma}_{1\mathrm{calc}}$, etc. We conclude the calculation when successive approximations to $\mathit{\Gamma}$ converge. The eigenfunction thus determined is then used as the initial condition for the neighbouring value of K_y, for which again we assume $\mathit{\Gamma} = \mathit{\Gamma}_0$ and calculate $\mathit{\Gamma}_1, \mathit{\Gamma}_2, \dots$, and so on.

For the BF instability, better convergence was obtained when starting iterations from an analytical approximation to the eigenfunction (small-m expansion up to terms of order m^2),

$$\left.\begin{aligned} \tilde{u}_0(x) &\simeq \cos(2\xi + \alpha) + (\tfrac{1}{4}m + \tfrac{1}{8}m^2)\cos(4\xi+\alpha) + \tfrac{9}{256}m^2\cos(6\xi+\alpha), \\ \alpha &= -\arccos\sqrt{\frac{K_y^2}{3m^2}}, \qquad \xi = \tfrac{1}{2}\pi\frac{x}{K(m)}. \end{aligned}\right\} \tag{A 14}$$

Appendix B.

An equilibrium solution of equations (5.6) is given by one of $\alpha = 0$ or $\alpha = \frac{1}{2}\pi$. Starting with the first case, we can find α_0 and b_0 in terms of c_0,

$$a_0^2 = 16c_0(c_0 + 3m\epsilon) > 0, \tag{B 1}$$

$$b_0 = -2c_0 - 6m(1+\epsilon), \tag{B 2}$$

$$\epsilon = [3(1-m) - K_y^2]/(3m), \quad -1 \leqslant \epsilon \leqslant 1. \tag{B 3}$$

If we now assume

$$a = a_0 + \delta a \mathrm{e}^{\mathrm{i}\omega t}, \qquad b = b_0 + \delta b \mathrm{e}^{\mathrm{i}\omega t}, \qquad c = c_0 + \delta c \mathrm{e}^{\mathrm{i}\omega t}, \qquad \alpha = \delta\alpha \mathrm{e}^{\mathrm{i}\omega t}, \tag{B 4}$$

and linearize (5.6) around the equilibrium solution, we find that consistency demands

$$\omega^2 = 12c_0(c_0 + 2m\epsilon) > 0 \quad \text{from (B 1)}. \tag{B 5}$$

Similarly, for $\alpha = \frac{1}{2}\pi$, b_0 becomes $-2c_0 + 6m(1-\epsilon)$, but a_0^2 and ω^2 are still given by the above expressions.

We have found both stationary chessboard-like solutions and also oscillations of a perturbation with finite frequency around them. For the latter, curves in phase space (a, a_t) would be ellipses around the centre $(a_0, 0)$. These solutions are far from the linear regime and cannot be built up from arbitrarily small a. Phase curves never emerge from the origin. An outside influence other than a very small perturbation would be required to set them up.

Appendix C.

Equations (5.6 b) and (5.6 c) yield

$$\frac{\mathrm{d}w}{\mathrm{d}t} = 6wm\sin 2\alpha, \tag{C 1}$$

$$w = c + \alpha_t. \tag{C 2}$$

The trivial solution, $w = 0$, generates the Landau-type equation for a (5.8). Now, from (C 1) and (5.6 a), assuming $w \neq 0$, $\ln(wa^2) = \text{const.} = \ln k_1$, i.e.

$$a^2 = k_1/w. \tag{C 3}$$

Next we use this in (5.6 b) minus (5.6 d) divided by 2ω. This removes b. Further straightforward manipulation, once more using (C 1), finally gives just one equation for w,

$$\left(\frac{\mathrm{d}w}{\mathrm{d}t}\right)^2 = -4w^4 + 8(3m - 3 + K_y^2)w^3 + 6k_2w^2 - \tfrac{1}{2}k_1w = f(w), \tag{C 4}$$

where k_2 is a second arbitrary constant. The solution outlined in Appendix B is recovered by looking for values of k_i that yield

$$f(w) = 0, \tag{C 5}$$

$$\frac{\mathrm{d}f}{\mathrm{d}w} = 0, \tag{C 6}$$

where $w = c_0$. These values are

$$k_1 = c_0a_0^2, \tag{C 7}$$

$$k_2 = 2c_0(c_0 + 4m\epsilon). \tag{C 8}$$

When we take

$$w = c_0 + \delta w \mathrm{e}^{\mathrm{i}\omega t}, \tag{C 9}$$

using (C 7) and (C 8) in (C 4), we find $(\dot{\delta w})^2 = -\omega^2(\delta w)^2$ and

$$\omega^2 = 12c_0(c_0 + 2m\epsilon), \tag{C 10}$$

in agreement with the result of Appendix B. Phase curves in (a, a_t)-space are easily drawn from (C 3) and (C 4),

$$\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 = -k_1^2a^{-2} + 2(3m - 3 + K_y^2)k_1 + \tfrac{3}{2}k_2a^2 - \tfrac{1}{8}a^4. \tag{C 11}$$

For large amplitudes, these curves depart from ellipses, but they never emerge from the origin, due to the a^{-2} term and $k_1 \neq 0$. They generalize the solutions of Appendix B to larger amplitudes. (In general, we can express $a(t)$ in terms of Jacobian elliptic functions by first multiplying by a^2, and next using this squared function as our new variable. However, the general behaviour can be seen from the phase curve (C 11).) If we formally take $k_1 = 0$, $k_2 = \frac{2}{3}\Gamma^2$, we recover (5.8).

The only solution to equations (5.6) not covered by § 5 and this appendix is that for $a \equiv 0$. This, however, limits considerations to one space dimension, the Korteweg–de Vries (KdV) equation, the nonlinear wave solutions to which are known to be stable (Benjamin 1972; Whitham 1965). Consistency of equations (5.6 c) and (5.6 d), $a = 0$, with known properties of KdV solutions has been checked.

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