On the Convolution Method of Describing Bottom Friction in Depth-Averaged Models

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11 August 1987 and 19 April 1988

1. Introduction

The convolution method for describing bottom friction in two-dimensional hydrodynamic models of homogeneous seas was introduced by Jelesnianski (1970) following an idea by Welander (1957). The conventional method of representing bottom friction in such models uses a term which is linear, or quadratic, in the depth-averaged velocity, a technique that is clearly invalid when the current has significant vertical shear. The convolution method was developed to provide a superior description of bottom friction in such cases, and was initially applied to storm surge models where bottom stress is of major importance.

The convolution method can provide an exact description of bottom stress, provided that a linear eddy viscosity model is appropriate. A two-dimensional model will then accurately reproduce the depth-averaged currents and elevations that would be obtained from a solution of the full three-dimensional equations. Furthermore, the vertical structure of the current can then be reconstructed at any chosen site by using the results of the two-dimensional model (Forristall 1974; Forristall et al. 1977). This latter technique is now widely used with the conventional formulation of bottom friction (e.g., Rothlisberg et al. 1983). Jordan and Baker (1980) have given full details of the convolution method for vertical eddy viscosities with various functional dependencies on vertical position in the water column.

We have been using the convolution method to investigate the steady-state flows produced by a constant wind stress. During this study it has been found that the convolution method gives rise to undamped inertial oscillations. Similar oscillations have recently and independently been noted by Davies (1987).

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If a constant wind stress is applied to an open, flat-bottomed basin at time $t = 0$, the analytical solution for the depth-averaged current contains the steady-state, wind-driven, long-time limit plus transient inertial oscillations of the form (for constant eddy viscosity, N),

$$\exp\left(-if - \frac{Nk_n^2}{h^2}\right)t \tag{1}$$

where k_n are the eigenvalues of the eddy viscosity operator ($n = 1, 2, 3 \cdots$); h is the water depth and f the Coriolis parameter. For two particular bottom boundary conditions:

$$\text{Zero slip:} \quad k_n = \left(n - \frac{1}{2}\right)\pi$$

$$\text{Zero stress:} \quad k_n = n\pi.$$

Therefore the lowest, and least damped, mode ($n = 1$) has a decay time

$$T_1 \approx 0.4\frac{h^2}{N} \quad \text{(zero slip, case a)}$$

$$\text{or} \quad \approx 0.1\frac{h^2}{N} \quad \text{(zero stress, case b)}.$$

For example, Forristall (1974) considered a constant wind stress applied to 100 m. of water with $N = 0.02$ m^2 s^{-1}, corresponding to $T_1 = 56$ h (a) or 14 h (b). In shallow water, T_1 is shorter so that, for example, if $h = 10$ m, and $N = 10^{-3}$ m^2 s^{-1} (corresponding to a moderate wind stress), then $T_1 = 11$ h (a) or 3 h (b). Thus, the decay time does not typically exceed one or two inertial periods and may be as small as a few hours. The convolution method, however, in contrast to the damped modes of the real system, induces undamped inertial oscillations such that the model does not converge to a steady state.

Numerical models of storm surges are typically run for a period of a few days duration (compatible with the time scale of a storm surge), which is usually too short to reveal the presence of undamped inertial oscillations. For example, in the case of steady winds cited above, with $T_1 = 56$ h (i.e., a zero-slip condition), Forristall (1974) ran the model for only 30 h. Furthermore, much of the structure in the evolution of real storm-generated currents arises from the time-dependence of the wind stress. Nevertheless, if T_1 is less than, or of similar magnitude to the time of duration of the storm, the undamped inertial oscillations will give considerable errors in the predicted surge; this effect is clearly greater in shallow water, due to the smaller values of T_1. Long-term inertial oscillations are often observed to follow storm winds, but these are usually associated with baroclinic effects (e.g., Gordon 1982).

2. Theory

In complex notation, the linear horizontal dynamic equation is

$$\frac{\partial w}{\partial t} = -ifw + \frac{\partial}{\partial z}\left(N(z)\frac{\partial w}{\partial z}\right) + \frac{G}{\rho h} \tag{2}$$

where z is the vertical coordinate ($-h$ at the seabed, η at the sea surface), $w = u + iv$, (u, v) being the components of the horizontal velocity, N the vertical eddy viscosity and G the complex slope-forcing variable:

$$G = -\rho g h\left(\frac{\partial \eta}{\partial x} + i\frac{\partial \eta}{\partial y}\right) \tag{3}$$

where g is gravity, η the surface elevation and (x, y) the horizontal coordinates.

The surface wind stress is specified by

$$\tau_s = \rho N(\eta)\left.\frac{\partial w}{\partial z}\right|_{z=\eta} \tag{4}$$

where ρ is the water density; w is subject to some linear, homogeneous, boundary conditions at $z = -h$ (e.g., no slip or linear slip), the bottom friction stress being given by

$$\tau_B = \rho N(-h)\left.\frac{\partial w}{\partial z}\right|_{z=-h}. \tag{5}$$

If (4) is linearized ($\eta \to 0$), the system is linear so that, if w is zero at time $t = 0$, its subsequent evolution can be expressed as a sum of two convolutions involving the forcing functions τ_s and G. It follows from (5) that τ_B can also be written as the sum of two such convolutions.

The depth-average of (2) is, using (4) and (5),

$$\rho h\left(\frac{\partial \bar{w}}{\partial t} + if\bar{w}\right) = \tau_s - \tau_B + G \tag{6}$$

where $\bar{w}$ is the depth-averaged velocity.

Taking a Laplace transform of (6) and using a tilde for transformed functions, i.e.,

$$\tilde{\bar{w}} = \int_0^\infty \bar{w}(t)e^{-st}dt, \tag{7}$$

where

$$\left.\begin{aligned} \rho h(s + if)\tilde{\bar{w}} &= \tilde{H}(s) \\ \tilde{H}(s) &\equiv \tilde{\tau}_s(s) - \tilde{\tau}_B(s) + \tilde{G}(s) \end{aligned}\right\}. \tag{8}$$

The transform of the convolution for τ_B has the form

$$\tilde{\tau}_B(s) = \tilde{C}_s(s)\tilde{\tau}_s(s) + \tilde{C}_G(s)\tilde{G}(s) \tag{9}$$

where C_s and C_G are the kernels.

Inverting (8) for $\bar{w}(t)$ involves finding the singularities of the function $\tilde{H}(s)/(s + if)$. The function $\tilde{H}(s)$ has simple poles which lead to an infinite series of terms like (1), which are damped inertial oscillations and other singularities arising from the time dependence of τ_s and G.

If the convolution formulation (9) is employed, the function $\tilde{H}(s)/(s + if)$ does not have a singularity at $s = -if$. This was noted by Jelesnianski (1970) for the case of constant eddy viscosity. More generally, an investigation of the kernels in (9) shows that

$$\left.\begin{aligned} \tilde{C}_s(s) &= 1 - (s + if)\alpha_s(s) \\ \tilde{C}_G(s) &= 1 - (s + if)\alpha_G(s) \end{aligned}\right\} \tag{10}$$

where α_s, α_G are analytic and nonzero at $s = -if$ (Hearn and Hunter 1988). Hence, by Eqs. (8), (9) and (10), $\tilde{H}(s)$ has a first-order zero at $s = -if$ and so $\tilde{\bar{w}}$ is analytic. The absence of a pole at $s = -if$ is vital because it is associated with undamped inertial oscillations in $\bar{w}(t)$.

Numerical solution of (6) produces a solution for $\bar{w}$ that depends on the form adopted for τ_B. If the convolution formulation is employed, some error is introduced by the approximation used for the numerical integration. At first sight, it might appear that any reasonable approximation would produce a solution $\bar{w}(t)$ which would tend smoothly to the true solution as the approximation is refined. However, (8) shows that this is not so, because the behavior of $\tilde{\bar{w}}$ at $s = -if$ depends critically on the vanishing of $\tilde{H}(-if)$. Any finite difference approximation for (9) will, in general, produce a value of $\tilde{H}(-if)$ which is nonzero. Thus, assuming that $\tilde{H}$ is analytic at this point, $\tilde{\bar{w}}$ has a simple pole and hence $\bar{w}(t)$ will have a component of the form

$$\frac{\tilde{H}(-if)}{\rho h}e^{-ift} \tag{11}$$

corresponding to undamped inertial oscillations.

The term $\tilde{H}(-if)$ is a singular perturbation, in that it critically changes the behavior of the system by inducing undamped inertial oscillations.

For conventional bottom friction,

$$\tau_B = \rho r\bar{w} \tag{12}$$

where r is a "friction velocity," so (8) becomes

$$\rho h(s - s_0)\tilde{\bar{w}} = \tilde{\tau}_s(s) + \tilde{G}(s); \quad s_0 = -if - \frac{r}{h}. \quad (13)$$

The previously discussed pole in $\tilde{\bar{w}}$ now occurs at s_0, and τ_B is no longer a driving term on the right-hand side of the equation. Hence (11) is replaced by

$$\frac{\tilde{\tau}_s(s_0) + \tilde{G}(s_0)}{\rho h} e^{s_0 t} \quad (14)$$

which has a damping time of h/r.

The "friction velocity," r, is roughly $C_D U$, where C_D is a bottom drag coefficient and U is a typical value of the total water velocity (which may consist of other motions, such as tides, not explicitly included in w, Hunter 1975). Taking $C_D = 0.0025$ and $U = 0.2$ m s^{-1}, $r = 0.0005$ m s^{-1} and the damping times become 56 and 6 h, for $h = 100$ m and $h = 10$ m, respectively, roughly in agreement with the estimates given in the Introduction.

It should be noted that, with the convolution method, at the inertial frequency, (8) contains no terms in $\tilde{\bar{w}}$, and hence $\bar{w}$ is effectively decoupled from the wind stress and surface slope; however, with conventional friction, (13) and (14) show that the coupling is retained.

3. Discussion

Model runs using the convolution method for bottom friction have shown undamped inertial oscillations. These start immediately the wind stress is applied and remain constant, or grow, with time. The theory of section 2 shows that the oscillations will always be present, no matter what numerical scheme is adopted. They reflect a basic sensitivity of the exact two-dimensional equations to numerical error at the inertial frequency.

The numerical scheme used to approximate the Coriolis term in the dynamic equations can cause further problems. Many schemes require the interpolation of one component of velocity in the evaluation of this term, which leads to an apparent reduction in the inertial frequency (Simons 1980). Common techniques also utilize a forward (first-order accurate) difference for the time integration of the Coriolis term, yielding a scheme which, in the absence of friction, is *unconditionally unstable.* The simple pole in $\tilde{\bar{w}}$ will hence not occur at exactly $s = -if$, possibly giving rise to growth of the inertial oscillation if the pole is displaced appropriately from the imaginary axis. Varying the numerical methodology alters the amplitude, and possible growth rate, of the oscillations.

Although any finite-difference method will, in general, produce a nonzero value for $\tilde{H}(-if)$, it is important to note that, for a linear system, the value of $\tilde{H}(-if)$ is proportional to $\tilde{\tau}_s(-if)$; this assumes that $\tilde{G}(-if)$ arises only through a linear coupling of the continuity and momentum equations. Consequently, the unwanted inertial oscillations can be avoided by filtering the wind stress (or other appropriate boundary forcing) so that the system is not excited at the model inertial frequency. If the required forcing variable is denoted by $F(t)$, then it is common to force the model with the variable, $P(t)$, given by

$$P = \begin{cases} 0, & t < 0 \\ F, & 0 \leq t, \end{cases}$$

where the model simulation is effectively started at $t = 0$. Such "step" forcing will in general introduce significant unwanted inertial oscillations. These oscillations may be removed by convolving P with a suitable low-pass or band-pass filter that effectively removes all components in the vicinity of the inertial period (bearing in mind that this period may be modified by the numerical scheme). Experiments involving a steady-state wind-driven simulation have indicated that the use of such a filter can reduce the unwanted inertial oscillations to a negligible level.

Acknowledgments. This work was funded by the Australian Research Grants Scheme.

REFERENCES

Davies, A. M., 1987: On extracting current profiles from vertically integrated numerical models. *Coastal Eng.*, **11**, 445–477.

Forristall, G. Z., 1974: Three-dimensional structure of storm-generated currents. *J. Geophys. Res.*, **79**, 2721–2729.

——, R. C. Hamilton and V. J. Cardone, 1977: Continental shelf currents in tropical storm Delia: Observations and theory. *J. Phys. Oceanogr.*, **7**, 532–546.

Gordon, R. L., 1982: Coastal ocean current response to storm winds. *J. Geophys. Res.*, **87**, 1939–1951.

Hearn, C. J., and J. R. Hunter, 1988: A new method of describing bottom stress in two-dimensional hydrodynamical models of shallow homogeneous seas, estuaries and lakes. *Appl. Math. Modelling,* in press.

Hunter, J. R., 1975: A note on quadratic friction in the presence of tides. *Estuar. Coastal Mar. Sci.*, **3**, 473–475.

Jelesniansky, C. P., 1970: Bottom stress time history in linearised equations of motion for storm surges. *Mon. Wea. Rev.*, **98**, 462–478.

Jordan, T. F., and J. R. Baker, 1980: Vertical structure of time-dependent flow dominated by friction in a well-mixed fluid. *J. Phys. Oceanogr.*, **10**, 1091–1103.

Rothlisberg, P. C., J. A. Church and A. M. G. Forbes, 1983: Modelling the advection of vertically migrating shrimp larvae. *J. Mar. Res.*, **41**, 511–538.

Simons, T. J., 1980: Circulation models of lakes and inland seas. *Can. Bull. Fish. Aquat. Sci.*, **203**.

Welander, P., 1957: Wind action on a shallow sea: Some generalizations of Ekman's theory. *Tellus,* **9**(1), 45–52.