Statistical Properties of the Kinematics and Dynamics of Nonlinear Waves

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(Manuscript received 16 June 1983, in final form 16 October 1983)

ABSTRACT

The probability density function and the first three statistical moments of velocity, acceleration and pressure of a gravity wave field, for points in the vicinity of still water level, are obtained taking into consideration the effects of free surface fluctuation and using the second-order Stokes wave model. These results reduce to those obtained previously by Tung using linear wave theory.

1. Introduction

Knowledge of wave field kinematics and dynamics is of importance to wave researchers and particularly ocean engineers interested in assessing wave force and pressure. To calculate fluid particle velocity, acceleration and pressure of points fixed in space in the vicinity of still water level in a wave field, one must take into account the fact that due to fluctuation of the free surface, the point under consideration is in the water only intermittently. Tung (1975), Tung and Pajouhi (1976) and Tung (1976) calculated the probability density function, the first three statistical moments and covariances and spectra of the kinematics and dynamics of deep-water linear, zero-mean Gaussian waves and showed that the free surface fluctuation phenomenon drastically affects these statistical properties. We now propose to extend the work to consider nonlinear, non-Gaussian waves.

In 1963, Longuet-Higgins studied the effects of nonlinearities on statistical distribution of wave elevation by using the Gram-Charlier series. For the problem at hand, if this line of investigation is followed, the mathematics would be rather involved. Recently, in consideration that sea waves are mostly narrow-band, Tayfun (1980) and Huang *et al.* (1983) used the Stokes wave to model nonlinear narrow-band random waves; the calculation becomes simpler especially when a perturbation scheme (Huang *et al.*, 1983) is introduced.

In essence, surface wave elevation of deep-water gravity wave is given by

$$\zeta = \frac{1}{2}a^2k + a\cos\chi + \frac{1}{2}a^2k\cos 2\chi + \frac{3}{8}a^3k^2\cos 3\chi + \cdots$$
$$= a\cos\chi + a^2k\cos^2\chi + \frac{3}{8}a^3k^2\cos 3\chi + \cdots, \qquad (1)$$

where the amplitude a and phase $\chi = kx - \omega t + \phi$ of the linear component $X = a\cos\chi$ are assumed to be slowly varying and are respectively Rayleigh and uniformly distributed so that X is zero-mean Gaussian. The determination of the probability density function of the nonlinear non-Gaussian ζ as a function of X then becomes a matter of transformation of random variables. In doing so, the inverse of ζ must be found. This is achieved by perturbation (Huang *et al.*, 1983) in view of the smallness of wave slope.

In this study, we calculate the probability density function and the first three statistical moments of velocity, acceleration and pressure using the second-order Stokes wave model and taking into account fluctuations of the free surface for points in the vicinity of still water level.

2. Velocity

To the second order, wave elevation $z = \zeta$, measured from still water level upwards, is

$$\zeta = a\cos\chi + a^2k\cos^2\chi, \qquad (2)$$

where a is Rayleigh distributed, $\chi = kx - \omega t + \phi$, ϕ (and hence χ) being uniformly distributed and k and ω are the deterministic wavenumber and frequency of the zero-mean Gaussian linear component $X = a\cos\chi$ so that in terms of X,

$$\zeta = X + X^2k. \qquad (3)$$

The horizontal and vertical velocity components are, to that order, (Bowden, 1948),

$$U_1 = -(gk)^{1/2}e^{kz}a\cos\chi, \qquad (4)$$

$$U_2 = -(gk)^{1/2}e^{kz}a\sin\chi, \qquad (5)$$

respectively where g is gravitation acceleration. Noting

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that U_1 and U_2 are zero-mean and letting $Y = a \sin\chi$, we may write

$$U_1 = -\left(\frac{\sigma_U}{\sigma}\right)X, \quad (6)$$

$$U_2 = -\left(\frac{\sigma_U}{\sigma}\right)Y, \quad (7)$$

where σ is the standard deviation of X or Y. That is,

$$\sigma^2 = E[a^2 \cos^2\chi] = E[a^2 \sin^2\chi] = \frac{E[a^2]}{2}, \quad (8)$$

$E[\ ]$ being the expected value of the quantity enclosed in the brackets, and

$$\sigma_U = (gk)^{1/2}e^{kz}\sigma \quad (9)$$

is the standard deviation of U_1 or U_2 as seen from (6) and (7).

Now, due to free surface fluctuation, the velocity components actually observed are

$$\bar{U}_i = U_i H(\zeta - z), \quad i = 1, 2, \quad (10)$$

where $H(\ )$ is the Heaviside unit function. The determination of the probability distribution of $\bar{U}_i$ is facilitated by the use of the theorem of total probability. Thus,

$$F_{\bar{U}_i}(\bar{u}_i) = P(\bar{U}_i \leqslant \bar{u}_i) = P(\bar{U}_i \leqslant \bar{u}_i | \zeta > z)P(\zeta > z) + P(\bar{U}_i \leqslant \bar{u}_i | \zeta \leqslant z)P(\zeta \leqslant z), \quad (11)$$

where $P(\ )$ and $P(|)$ denote respectively probability and conditional probability. The event $\zeta \leqslant z$, to the second order, is equivalent to $X + X^2k \leqslant z$ by virtue of (3). If we restrict the points under consideration to those in the vicinity of still water level so that kz is small, then $X + X^2k = z$ implies

$$X = \frac{1}{2k}[-1 \pm (1 + kz)^{1/2}]$$

$$\approx \frac{1}{2k}[-1 \pm (1 + 2kz - 2k^2z^2)] = z - z^2k$$

to the second order of accuracy. Equivalently, by perturbation, write

$$X = z - X^2k.$$

To the first order,

$$X = z,$$

and, to the second order,

$$X = z - z^2k.$$

Thus, the event $(\zeta \leqslant z) = (X \leqslant z - z^2k) = (X \leqslant \sigma\bar{\beta})$ where

$$\bar{\beta} = \frac{z - z^2k}{\sigma} = \frac{z}{\sigma} - \frac{z^2}{\sigma^2}\sigma k. \quad (12)$$

That is, $\bar{\beta}$ depends on the parameter z/σ, the location of the point under consideration, and σk, a measure of wave slope.

We thus have

$$P(\zeta \leqslant z) = P(X \leqslant \sigma\bar{\beta}) = \int_{-\infty}^{\sigma\bar{\beta}} f_X(x)dx = 1 - Q(\bar{\beta}), \quad (13)$$

where

$$f_X(x) = \frac{1}{\sigma} Z\left(\frac{x}{\sigma}\right)$$

is the probability density function of X,

$$Z(x) = \frac{1}{(2\Pi)^{1/2}} e^{-x^2/2},$$

$$Q(x) = \int_x^{\infty} Z(\xi)d\xi.$$

The event $(\bar{U}_i \leqslant \bar{u}_i | \zeta \leqslant z)$ corresponds to the situation in which the point under consideration is not in the water and is therefore a certain event for all $\bar{u}_i > 0$. That is,

$$P(\bar{U}_i \leqslant \bar{u}_i | \zeta \leqslant z) = H(\bar{u}_i). \quad (14)$$

To calculate the second term in (11), we must treat $\bar{U}_1$ and $\bar{U}_2$ separately. For $\bar{U}_1$, since $\zeta > z$ implies $\bar{U}_1 = U_1$,

$$P(\bar{U}_1 \leqslant \bar{u}_1 | \zeta > z)P(\zeta > z) = P(U_1 \leqslant \bar{u}_1, \zeta > z)$$

$$= P\left[-\left(\frac{\sigma_U}{\sigma}\right)X \leqslant \bar{u}_1, X > \sigma\bar{\beta}\right]$$

$$= P\left[X > -\left(\frac{\sigma}{\sigma_U}\right)\bar{u}_1, X > \sigma\bar{\beta}\right]$$

$$= \int_{\sigma\bar{\beta}}^{\infty} f_X(x)dx \quad \text{if} \quad \sigma\bar{\beta} > -\left(\frac{\sigma}{\sigma_U}\right)\bar{u}_1$$

$$= \int_{-(\sigma/\sigma_U)\bar{u}_1}^{\infty} f_X(x)dx \quad \text{if} \quad \sigma\bar{\beta} \leqslant -\left(\frac{\sigma}{\sigma_U}\right)\bar{u}_1. \quad (15)$$

Substituting (13), (14) and (15) into (11) and taking derivative with respect to $\bar{u}_1$, we get the probability density function of $\bar{U}_1$:

$$f_{\bar{U}_1}(\bar{u}_1) = [1 - Q(\bar{\beta})]\delta(\bar{u}_1) + \frac{1}{\sigma_U} Z\left(\frac{\bar{u}_1}{\sigma_U}\right) H\left(-\frac{\bar{u}_1}{\sigma_U} - \bar{\beta}\right), \quad (16)$$

where $\delta(\ )$ is the Dirac delta function.

The second term in (11), for U_2, is

$$P(\bar{U}_2 \leqslant \bar{u}_2 | \zeta > z)P(\zeta > z) = P(U_2 \leqslant \bar{u}_2, \zeta > z) = P\left[Y > -\left(\frac{\sigma}{\sigma_U}\right)\bar{u}_2\right]P(X > \sigma\bar{\beta}), \quad (17)$$

in view of (3), (7) and the statistical independence of X and Y. Noting that

$$P\left[Y > -\left(\frac{\sigma}{\sigma_U}\right)\bar{u}_2\right] = \int_{-(\sigma/\sigma_U)\bar{u}_2}^{\infty} f_Y(y)dy. \quad (18)$$

and $P(X > \sigma\bar{\beta})$ is the complement of $P(X \leqslant \sigma\bar{\beta})$ given by (13), the probability density function of $\bar{U}_2$ is, after substituting (13), (14), (17) and (18) into (11) and taking derivative with respect to $\bar{u}_2$,

$$f_{\bar{U}_2}(\bar{u}_2) = [1 - Q(\bar{\beta})]\delta(\bar{u}_2) + \frac{1}{\sigma_U} Z\left(\frac{\bar{u}_2}{\sigma_U}\right)Q(\bar{\beta}). \quad (19)$$

The probability density functions for $\bar{U}_1$ and $\bar{U}_2$ obtained previously (Tung, 1975) are

$$f_{\bar{U}_i}(\bar{u}_i) = \left[1 - Q\left(\frac{z}{\sigma}\right)\right]\delta(\bar{u}_i) + \frac{1}{\sigma_{U_i}} Z\left(\frac{\bar{u}_i}{\sigma_{U_i}}\right)Q\left[\left(\frac{z}{\sigma} - \frac{r_i\bar{u}_i}{\sigma_{U_i}}\right)\Big/(1 - r_i^2)^{1/2}\right], \quad i = 1, 2. \quad (20)$$

To the first order, our present model reduces to the narrow-band linear processes $\zeta = X$, $U_1 = -(\sigma_U/\sigma)X$, and $U_2 = -(\sigma_U/\sigma)Y$ and $\bar{\beta} = z/\sigma$. Thus, $\sigma_{U_1} = \sigma_{U_2} = \sigma_U$ as given by (9),

$$r_1 = E[\zeta U_1]/\sigma_U\sigma = E\left[X\left(-\frac{\sigma_U}{\sigma}\right)X\right]\Big/\sigma_U\sigma = -E[X^2]/\sigma^2 = -1,$$

$$r_2 = E[\zeta U_2]/\sigma_U\sigma = E\left[X\left(-\frac{\sigma_U}{\sigma}\right)Y\right]\Big/\sigma_U\sigma = -E[XY]/\sigma^2 = 0.$$

The concurrence between (20) for $i = 2$ and (19) is obvious. For $i = 1$, (20) also agrees with (16) if it is recognized that

$$Q\left[\left(\frac{z}{\sigma} - \frac{\bar{u}_1}{\sigma_U}r_1\right)\Big/(1 - r_1^2)^{1/2}\right]$$

for $r_1 = -1$ is in fact $H\left(-\frac{\bar{u}_1}{\sigma_U} - \frac{z}{\sigma}\right)$.

The jth statistical moment of $\bar{U}_i$, $i = 1, 2$, may be obtained from

$$E[\bar{U}_i^j] = \int_{-\infty}^{\infty} \xi^j f_{\bar{U}_i}(\xi)d\xi, \quad i = 1, 2. \quad (21)$$

For $\bar{U}_1$, the first three moments are

$$E[\bar{U}_1] = -\sigma_U Z(\bar{\beta}), \quad (22)$$

$$E[\bar{U}_1^2] = \sigma_U^2[\bar{\beta}Z(\bar{\beta}) + Q(\bar{\beta})], \quad (23)$$

$$E[\bar{U}_1^3] = -\sigma_U^3 Z(\bar{\beta})(2 + \bar{\beta}^2). \quad (24)$$

For $\bar{U}_2$, all odd moments vanish and the second moment is

$$E[\bar{U}_2^2] = \sigma_U^2 Q(\bar{\beta}). \quad (25)$$

These moments may be obtained without resort to explicit expressions of $f_{\bar{U}_i}(\bar{u}_i)$. Thus, for $\bar{U}_1$, from (3), (6) and (10),

$$E[\bar{U}_1^j] = \left(-\frac{\sigma_U}{\sigma}\right)^j E[X^j H(X - \sigma\bar{\beta})] = \left(-\frac{\sigma_U}{\sigma}\right)^j \int_{\sigma\bar{\beta}}^{\infty} x^j f_X(x)dx, \quad (26)$$

which gives, for $j = 1, 2, 3$, (22), (23) and (24) as can be easily verified. For $\bar{U}_2$, from (3), (7) and (10),

$$E[\bar{U}_2^j] = \left(-\frac{\sigma_U}{\sigma}\right)^j E[Y^j]E[H(X - \sigma\bar{\beta})], \quad (27)$$

giving $E[\bar{U}_2^j] = 0$ for $j = 1, 3$ and (25) for $j = 2$. The first three statistical moments obtained previously (Tung, 1975) using linear wave theory are

$$E[\bar{U}_i] = r_j\sigma_{U_j}Z\left(\frac{z}{\sigma}\right), \quad (28)$$

$$E[\bar{U}_i^2] = \sigma_{U_i}^2\left[Q\left(\frac{z}{\sigma}\right) + r_i^2\frac{z}{\sigma}Z\left(\frac{z}{\sigma}\right)\right], \quad (29)$$

$$E[\bar{U}_i^3] = \sigma_{U_i}^3 Z\left(\frac{z}{\sigma}\right)\left[3r_i - r_i^3\left(1 - \frac{z^2}{\sigma^2}\right)\right]. \quad (30)$$

After dropping all higher order terms in our present model and noting that $\sigma_{U_1} = \sigma_{U_2} = \sigma_U$ and $r_1 = -1$, $r_2 = 0$, the agreement between (22), (23), (24) and (25) and (28), (29) and (30) for $i = 1, 2$ is readily seen.

3. Acceleration

To the second order, the horizontal and vertical components of acceleration are respectively

$$A_1 = -gke^{kz}Y, \quad (31)$$

$$A_2 = gke^{kz}X + gk^2e^{2kz}(X^2 + Y^2). \quad (32)$$

Noting that $\sigma_{\dot{U}_1} = \sigma_{\dot{U}_2} = gke^{kz}\sigma = \sigma_{\dot{U}}$ where the overdot denotes local time derivative,

$$A_1 = -\left(\frac{\sigma_{\dot{U}}}{\sigma}\right)Y, \quad (33)$$

$$A_2 = \left(\frac{\sigma_{\dot{U}}}{\sigma}\right)X + \left(\frac{\sigma_{\dot{U}}^2}{\sigma^2 g}\right)(X^2 + Y^2). \quad (34)$$

Considering free surface fluctuation,

$$\bar{A}_i = A_i H(\zeta - z) i = 1, 2. \quad (35)$$

Because of the resemblance of A_1 in (31) and U_2 in (7), it is seen that the probability density function of $\bar{A}_1$ may be written as follows:

$$f_{\bar{A}_1}(\bar{a}_1) = [1 - Q(\bar{\beta})]\delta(\bar{a}_1) + \frac{1}{\sigma_{\dot{U}}} Z\left(\frac{\bar{a}_1}{\sigma_{\dot{U}}}\right)Q(\bar{\beta}). \quad (36)$$

The derivation of the probability density function of

$\bar{A}_2$, however, is much more involved because of the nonlinear terms in X and Y in (34). Thus,

$$P(\bar{A}_2 \leqslant \bar{a}_2) = P(\bar{A}_2 \leqslant \bar{a}_2|\zeta \leqslant z)P(\zeta \leqslant z) + P(\bar{A}_2 \leqslant \bar{a}_2, \zeta > z). \quad (37)$$

The first term in (37) is simply

$$P(\bar{A}_2 \leqslant \bar{a}_2|\zeta \leqslant z)P(\zeta \leqslant z) = H(\bar{a}_2)[1 - Q(\beta)]. \quad (38)$$

To compute the second term, for convenience, let

$$S_1 = \left(\frac{\sigma_{\dot{U}}^2}{\sigma^2 g}\right)Y^2, \quad (39)$$

$$S_2 = \left(\frac{\sigma_{\dot{U}}}{\sigma}\right)X + \left(\frac{\sigma_{\dot{U}}^2}{\sigma^2 g}\right)X^2, \quad (40)$$

so that

$$A_2 = S_1 + S_2. \quad (41)$$

In this case ζ, as given by (3), may be expressed in terms of S_2 from (40) by perturbation, to the second order as

$$\zeta = \left(\frac{\sigma}{\sigma_{\dot{U}}}\right)S_2 - \left(\frac{\sigma}{\sigma_{\dot{U}}}\right)^2\left(\frac{\sigma_{\dot{U}}}{\sigma g} - k\right)S_2^2. \quad (42)$$

The event $\zeta > z$ in the second term in (37) is therefore

$$S_2 > \left(\frac{\sigma_{\dot{U}}}{\sigma}\right)z + \left(\frac{\sigma_{\dot{U}}}{\sigma}\right)^2\left(\frac{1}{g} - \frac{\sigma k}{\sigma_{\dot{U}}}\right)z^2 \equiv \alpha. \quad (43)$$

Thus,

$$P(\bar{A}_2 \leqslant \bar{a}_2, \zeta > z) = P(S_1 + S_2 \leqslant \bar{a}_2, S_2 > \alpha) = \int_{s_2=\alpha}^{\bar{a}_2}\int_{s_1=0}^{\bar{a}_2 - s_2} f_{S_1}(s_1)f_{S_2}(s_2)ds_1 ds_2, \quad (44)$$

(see Fig. 1) where, from (39) and (40) and by transformation of random variables,

$$f_{S_1}(s_1) = \left(\frac{g}{2\Pi s_1}\right)^{1/2}\frac{1}{\sigma_{\dot{U}}}\exp\left(-\frac{s_1 g}{2\sigma_{\dot{U}}^2}\right), \; s_1 > 0, \quad (45)$$

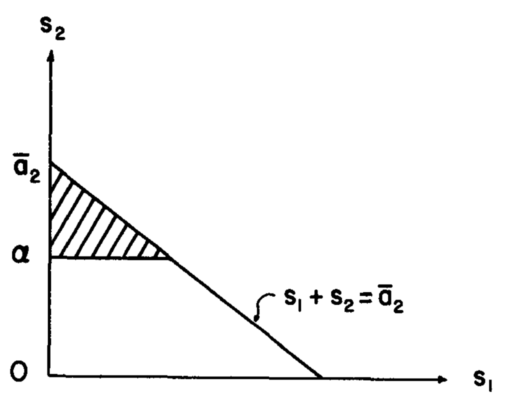

FIG. 1. Venn diagram in $(s_1 - s_2)$ space for computation of $f_{\bar{A}_2}(\bar{a}_2)$.

$$f_{S_2}(s_2) = \left(1 - \frac{2}{g}s_2 + \frac{2}{g^2}s_2^2\right)[(2\Pi)^{1/2}\sigma_{\dot{U}}]^{-1} \times \exp\left[-\frac{1}{2\sigma_{\dot{U}}^2}\left(s_2 - \frac{s_2^2}{g}\right)^2\right], \quad (46)$$

$$-\infty < s_2 < \infty.$$

To obtain the probability density function of $\bar{A}_2$, we need to take the derivative of (37) with respect to $\bar{a}_2$. While this is easily achieved for the first term from (38), the second term results in the following integral:

$$I = \int_{s_1=0}^{\bar{a}_2-\alpha}\left[1 - \frac{2}{g}(\bar{a}_2 - s_1) + \frac{2}{g}(\bar{a}_2 - s_1)^2\right] \times \left[2\Pi\sigma_{\dot{U}}^2\left(\frac{s_1}{g}\right)^{1/2}\right]^{-1}\exp\left(-\frac{s_1 g}{2\sigma_{\dot{U}}^2}\right) \times \exp\left\{-\frac{1}{2\sigma_{\dot{U}}^2}\left[(\bar{a}_2 - s_1) - \frac{(\bar{a}_2 - s_1)^2}{g}\right]^2\right\}ds_1. \quad (47)$$

Letting $x_2^2 = gs_1/\sigma_{\dot{U}}^2$, we have

$$I = 2\int_{x_2=0}^{\Delta}\left[1 - \frac{2\sigma_{\dot{U}}}{g}\left(x_1 - \frac{\sigma_{\dot{U}}}{g}x_2^2\right) + \frac{2\sigma_{\dot{U}}^2}{g^2}\left(x_1 - \frac{\sigma_{\dot{U}}}{g}x_2^2\right)^2\right](2\Pi\sigma_{\dot{U}})^{-1} \times \exp\left(-\frac{x_2^2}{2}\right)\exp\left\{-\frac{1}{2}\left[\left(x_1 - \frac{\sigma_{\dot{U}}}{g}x_2^2\right) - \frac{\sigma_{\dot{U}}}{g}\left(x_1 - \frac{\sigma_{\dot{U}}}{g}x_2^2\right)^2\right]^2\right\}dx_2, \quad (48)$$

where

$$x_1 = \frac{\bar{a}_2}{\sigma_{\dot{U}}}, \quad (49)$$

$$\Delta = \left[\frac{(x_1 - \beta)g}{\sigma_{\dot{U}}}\right]^{1/2}, \quad (50)$$

$$\beta = \frac{\alpha}{\sigma_{\dot{U}}}. \quad (51)$$

Expanding the second exponential function in (48) and keeping only terms to the second order, the integration may be carried out. The resulting probability density function of $\bar{A}_2$ is

$$f_{\bar{A}_2}(\bar{a}_2) = [1 - Q(\beta)]\delta(\bar{a}_2) + H(x_1 - \beta) \times \frac{2Z(x_1)}{\sigma_{\dot{U}}}\left\{\left[\frac{1}{2} - Q(\Delta)\right] \times \left[1 + \frac{\sigma_{\dot{U}}}{g}(-x_1 + x_1^3)\right] - \frac{\sigma_{\dot{U}}}{g}x_1\Delta Z(\Delta)\right\}. \quad (52)$$

To first order, (52) reduces to

$$f_{\bar{A}_2}(\bar{a}_2) = \left[1 - Q\left(\frac{z}{\sigma}\right)\right]\delta(\bar{a}_2) + H\left(x_1 - \frac{z}{\sigma}\right)\frac{Z(x_1)}{\sigma_{\dot{U}}}, \quad (53)$$

which agrees with the previous result (Tung, 1975) if we let $\zeta = X$, $A_2 = \frac{\sigma_{\dot{U}}}{\sigma} X$, $r_2 = E[A_2\zeta]/\sigma_{\dot{U}}\sigma = 1$ and recognize that for $r_2 = 1$, $Q\left[\left(\frac{z}{\sigma} - \frac{\bar{a}_2}{\sigma_{\dot{U}}} r_2\right)\Big/ (1 - r_2^2)^{1/2}\right]$ is the same as $H\left(x_1 - \frac{z}{\sigma}\right)$.

From (38), we see that all odd moments of $\bar{A}_1$ vanish and the second moment is the same as $E[\bar{U}_2^2]$ in (25) with σ_U replaced by $\sigma_{\dot{U}}$. The calculation of statistical moments of $\bar{A}_2$ from (52) entails integrations which can be achieved by integration by parts, appropriate substitutions of variables and keeping terms only to the second order.

After much tedious work, we have

$$E[\bar{A}_2] = \sigma_{\dot{U}} Z(\beta) + \frac{\sigma_{\dot{U}}^2}{g}[(1 + \beta^2)\beta Z(\beta) + 2Q(\beta)], \quad (54)$$

$$E[\bar{A}_2^2] = \sigma_{\dot{U}}^2[\beta Z(\beta) + Q(\beta)] + \frac{\sigma_{\dot{U}}^3}{g} Z(\beta)(6 + 2\beta^2 + \beta^4), \quad (55)$$

$$E[\bar{A}_2^3] = \sigma_{\dot{U}}^3 Z(\beta)(2 + \beta^2) + \frac{\sigma_{\dot{U}}^4}{g}[Z(\beta)(12\beta + 3\beta^3 + \beta^5) + 12Q(\beta)]. \quad (56)$$

These moments, however, may be calculated much more easily from (34) and (35) directly. For example,

$$E[\bar{A}_2] = \frac{\sigma_{\dot{U}}}{\sigma} E[XH(x - \sigma\bar{\beta})] + \frac{\sigma_{\dot{U}}^2}{\sigma^2 g}\{E[X^2H(X - \sigma\bar{\beta})] + E[Y^2]E[H(X - \sigma\bar{\beta})]\}$$

$$= \sigma_{\dot{U}} Z(\bar{\beta}) + \frac{\sigma_{\dot{U}}^2}{g}[\bar{\beta}Z(\bar{\beta}) + 2Q(\bar{\beta})]. \quad (57)$$

Similarly,

$$E[\bar{A}_2^2] = \sigma_{\dot{U}}^2[\bar{\beta}Z(\bar{\beta}) + Q(\bar{\beta})] + 2\frac{\sigma_{\dot{U}}^3}{g} Z(\bar{\beta})(3 + \bar{\beta}^2), \quad (58)$$

$$E[\bar{A}_2^3] = \sigma_{\dot{U}}^3 Z(\bar{\beta})(2 + \bar{\beta}^2) + \frac{\sigma_{\dot{U}}^4}{g}[Z(\bar{\beta})(12\bar{\beta} + 3\bar{\beta}^3) + 12Q(\bar{\beta})]. \quad (59)$$

Since

$$\beta = \bar{\beta} + \frac{\sigma_{\dot{U}}}{g}\frac{z^2}{\sigma^2}, \quad (60)$$

upon substituting (60) into (54), (55) and (56), expanding $Z(\beta)$ and $Q(\beta)$ about $\bar{\beta}$ and retaining appropriate orders, the results in (54), (55) and (56) are seen to agree with those in (57), (58) and (59). If higher order terms are dropped, these moments reduce to the results obtained earlier (Tung, 1975).

4. Pressure

To the second order, dynamic pressure is given by (Bowden, 1948)

$$P_1 \equiv \frac{P}{\rho g} = e^{kz} a \cos\chi - \frac{1}{2} a^2 k e^{2kx}$$

$$= e^{kz} a \cos\chi - \frac{1}{2} k e^{2kz}(a^2 \cos^2\chi + a^2 \sin^2\chi)$$

$$= e^{kz} X - \frac{1}{2} k e^{2kz}(X^2 + Y^2). \quad (61)$$

Letting σ_P be the standard deviation of $e^{kz} a \cos\chi$, the above may be written as

$$P_1 = \left(\frac{\sigma_P}{\sigma}\right) X - \frac{1}{2} k \left(\frac{\sigma_P}{\sigma}\right)^2 (X^2 + Y^2), \quad (62)$$

and the actually observed pressure is

$$\bar{P}_1 = P_1 H(\zeta - z). \quad (63)$$

The similarity between (62) and (34) of vertical acceleration shows that the derivation of the probability density function of $\bar{P}_1$ is much the same as that of $\bar{A}_2$. The resulting probability density function of $\bar{P}_1$ is

$$f_{\bar{P}_1}(\bar{p}_1) = [1 - Q(\beta)]\delta(\bar{p}_1) + H(x_1 - \beta)\frac{Z(x_1)}{\sigma_P} \times \left[1 + \frac{\sigma_P k}{2}(x_1 - x_1^3)\right] + H(\beta - x_1)\frac{2Z(x_1)}{\sigma_P} \times \left\{\left[1 + \frac{\sigma_P k}{2}(x_1 - x_1^3)\right]Q(\bar{\Delta}) - \frac{\sigma_P k}{2}\bar{\Delta} Z(\bar{\Delta})\right\}, \quad (64)$$

where

$$x_1 = \frac{\bar{P}_1}{\sigma_P}, \quad (65)$$

$$\beta = \bar{\beta} - \frac{\sigma_P k}{2}\frac{z^2}{\sigma^2}\left(= \frac{\alpha}{\sigma_P}\right), \quad (66)$$

$$\bar{\Delta} = \left[\frac{2}{\sigma_P k}(\beta - x_1)\right]^{1/2}. \quad (67)$$

To the first order, the probability density function in (64) reduces to

$$f_{\bar{P}_1}(\bar{p}_1) = \left[1 - Q\left(\frac{z}{\sigma}\right)\right]\delta(\bar{p}_1) + H\left(x_1 - \frac{z}{\sigma}\right)\frac{Z(x_1)}{\sigma_P}, \quad (68)$$

and our earlier result (Tung, 1975) is recovered.

The statistical moments of $\bar{P}_1$, as obtained from (64), are very much the same as those of $\bar{A}_2$ in (54), (55) and (56) with $\sigma_{\dot{U}}$ replaced by σ_P, and g by $-2/k$ except that for $\bar{A}_2$, α and β are given by (43) and (51), but for $\bar{P}_1$ these quantities are defined in (66). Similarly, when these moments of $\bar{P}_1$ are obtained directly from (62) and (63), they are of the same form as those of $\bar{A}_2$ in (57), (58) and (59).

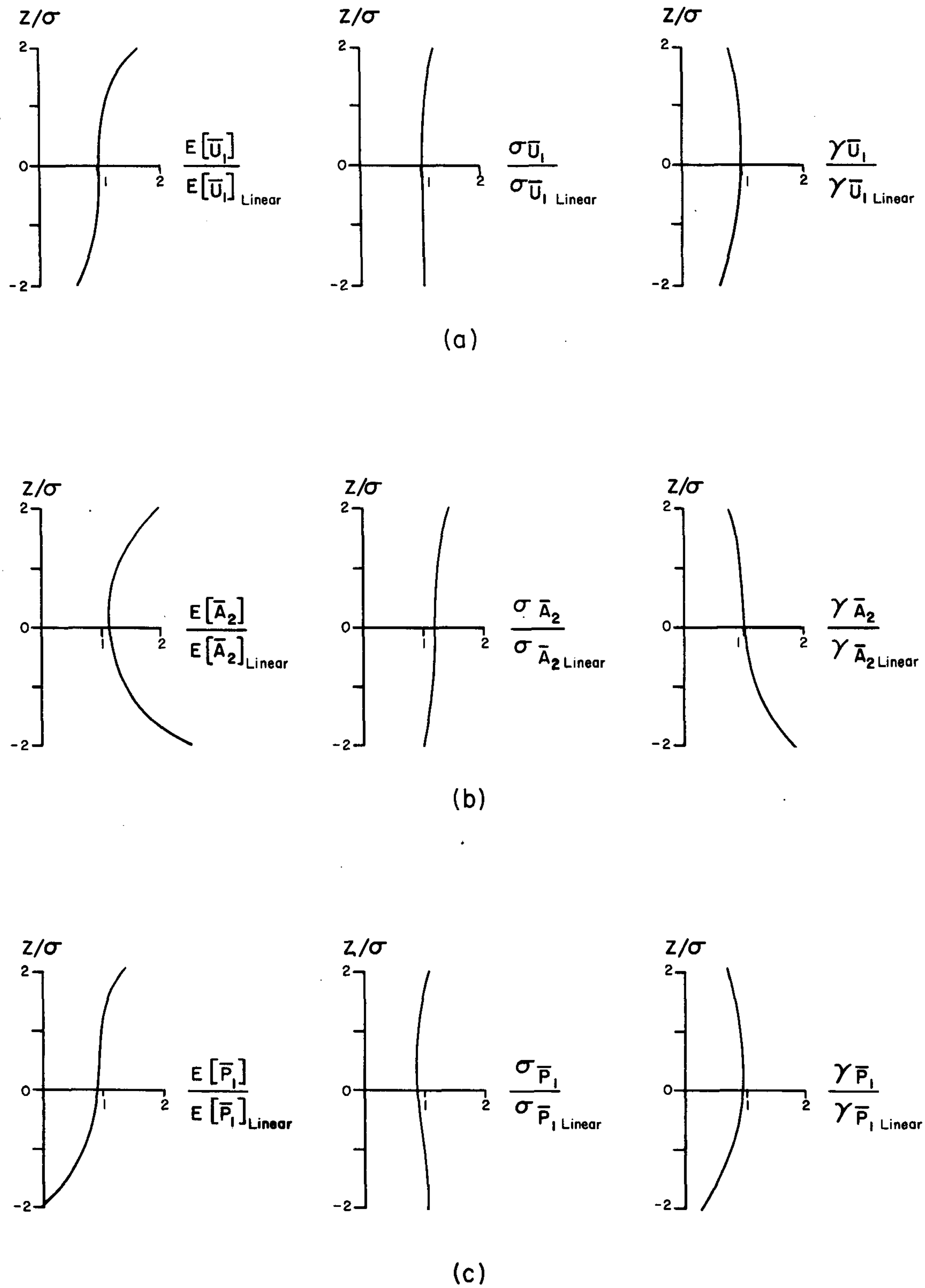

FIG. 2. Ratio of mean, standard deviation and skewness as computed by nonlinear and linear models for (a) horizontal velocity, (b) vertical acceleration and (c) dynamic pressure.

5. Numerical results

In order to determine the severity of nonlinear effects, we compute the ratios of nonlinear and linear results for the mean, standard deviation and skewness (γ) (skewness being defined as the ratio of third-order central moment and the cube of standard deviation) of $\bar{U}_1$, $\bar{A}_2$ and $\bar{P}_1$. That is, i.e., the mean of $\bar{U}_1$ using the nonlinear model is $E[\bar{U}_1] = -\sigma_U Z(\bar{\beta})$ given by (22) where $\bar{\beta} = (z/\sigma) - (z^2/\sigma^2)\sigma k$ is given by (12). The mean of $\bar{U}_1$ using linear wave theory is given by (28) and is $E[\bar{U}_1]_L = -\sigma_U Z(z/\sigma)$ so that the ratio of these mean values is

$$\frac{E[\bar{U}_1]}{E[\bar{U}_1]_L} = \frac{Z(\bar{\beta})}{Z(z/\sigma)}. \tag{69}$$

This ratio is seen to be a function of the location z/σ of the point under consideration and wave slope σk. It is noted that σk is related to the significant slope ξ

$$\xi = \frac{(E[\zeta^2] - E^2[\zeta])^{1/2}}{\lambda_0}, \tag{70}$$

as defined by Huang *et al.* (1981) where $\lambda_0 = 2\Pi/k$ is the length of the primary component of the waves. From (3),

$$E[\zeta] = E[X^2]k = \sigma^2 k, \tag{71}$$

$$E[\zeta^2] = E[X^2 + 2X^2k + X^4k^2]$$
$$= E[X^2] + 2kE[X^3] + k^2E[X^4] = \sigma^2 + 3\sigma^4k^2. \tag{72}$$

Thus,

$$(E[\zeta^2] - E^2[\zeta])^{1/2} = (\sigma^2 + 2\sigma^4k^2)^{1/2}$$
$$= \sigma(1 + 2\sigma^2k^2)^{1/2} \approx \sigma(1 + \sigma^2k^2) \approx \sigma \tag{73}$$

to the second order of accuracy. That is,

$$\xi = \frac{\sigma}{2\Pi/k} = \frac{\sigma k}{2\Pi}. \tag{74}$$

Similar to the ratio of the mean values for $\bar{U}_1$, we demonstrate that the ratio of the mean values of $\bar{A}_2$ using nonlinear and linear wave theories is also dependent only on the two parameters z/σ and $\sigma k = 2\Pi\xi$. Thus, the mean value of $\bar{A}_2$ is given by (57) and using linear wave theory, it is

$$E[\bar{A}_2]_L = \sigma_{\dot{U}} Z(z/\sigma). \tag{75}$$

The ratio is therefore

$$\frac{E[\bar{A}_2]}{E[\bar{A}_2]_L} = \frac{Z(\bar{\beta}) + (\sigma_{\dot{U}}/g)[\bar{\beta}Z(\bar{\beta}) + 2Q(\bar{\beta})]}{Z(z/\sigma)}. \tag{76}$$

Since

$$\frac{\sigma_{\dot{U}}}{g} = \sigma k e^{kz} = \sigma k e^{\sigma k(z/\sigma)},$$

the ratio obviously depends only on z/σ and σk.

For $\xi = 0.01$, the ratios are computed for values of $|z/\sigma| \leq 2$ and are plotted in Fig. 2. It is seen that nonlinearity has little effect on values of standard deviation of $\bar{U}_1$, $\bar{A}_2$ and $\bar{P}_1$ and has only moderate influence on the mean and skewness values of $\bar{U}_1$. However, for the mean and skewness of $\bar{A}_2$ and $\bar{P}_1$, nonlinear effects can be quite important. It is also mentioned that since the perturbation scheme employed here requires that $kz = \sigma k(z/\sigma)$ be small, the radius (z/σ) of convergence depends on the value of $\sigma k = 2\Pi\xi$ selected for consideration.

Acknowledgments. This work is supported in part by the National Aeronautics and Space Administration through a grant to North Carolina State University.

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