The Influence of the Directional Energy Distribution on the Nonlinear Dispersion Relation in a Random Gravity Wave Field

NORDEN E. HUANG

NASA Wallops Flight Center, Wallops Island, Va. 23337

CHI-CHAO TUNG

Department of Civil Engineering and Center for Marine and Coastal Studies, North Carolina State University, Raleigh 27607

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ABSTRACT

The influence of the directional distribution of wave energy on the dispersion relation is calculated numerically using various directional wave spectrum models. The results indicate that the dispersion relation varies both as a function of the directional energy distribution and the direction of propagation of the wave component under consideration. Furthermore, both the mean deviation and the random scatter from the linear approximation increase as the energy spreading decreases. Limited observational data are compared with the theoretical results. The agreement is favorable.

1. Introduction

In the study of surface gravity waves, there is a basic unique relationship between the wavenumber k and the frequency n known as the dispersion relation

$$n^2 = gk, \tag{1}$$

with g as the gravitational acceleration. Though this relationship is derived from linearized equations of motion, it has been used even for nonlinear waves. In the case of single trains of waves, both laboratory (Defant, 1960, pp. 41–42) and theoretical (De, 1955) studies indicate that the deviation of the dispersion relation from (1) only amounts to a few percent at most. As a result, it has been widely used in all the wave studies including even the random wave field which represents almost all the wave conditions observed in the open ocean. The justification of this liberal application of the simple dispersion relation is founded not so much on accuracy but on convenience. It is well known that when more than one wave train is present, nonlinear resonant interactions will occur (Phillips, 1960; McGoldrick *et al.*, 1966). Since there are infinitely many components of waves in the ocean, there will be infinitely many possible combinations for the interaction to occur. Furthermore, the bodily and oscillatory convection of the shorter waves by the longer waves can cause additional random scatter from the mean. Consequently, the very existence of the simple universal dispersion relation as given by (1) has been questioned by Phillips (1966, p. 77). Nevertheless, for lack of a better alternative, (1) has served a useful purpose.

Recently, a more general relationship has been derived by Huang and Tung (1976). It was shown that in a nonlinear random wave field, the dispersion relation is random. The mean and the mean square value of the scatter from the mean dispersion relation were derived in general form. Both the mean and the scatter were shown to be energy dependent and were expressed in terms of the directional energy spectrum of the waves, yet the influence of the directional energy distribution was not explored in detail. Due to the great potential and hence the increasing utilization of remote sensing techniques in ocean wave studies, explicit results are urgently needed to relate the spatial characteristics measured by the remote sensors to the temporal characteristics measured by the conventional methods. In this paper, various empirical ocean wave models are used to illustrate the importance of the directional energy distribution on the dispersion relation. Comparisons with a few recent observational data are also given as confirmations of the theoretical results.

2. The dispersion relation

The theoretical dispersion relation given by Huang and Tung (1976) is derived under the assumptions of an inviscid, incompressible fluid and irrotational motion using the full nonlinear kinematic and dynamic boundary conditions. It is shown that, in deep water, to the third order of approximation, the mean dispersion relation is

$$g - \frac{n^2}{|\mathbf{k}|} = -\int_{\mathbf{k}'}\int_{n'} f(\mathbf{k},n;\mathbf{k}',n')X(\mathbf{k}',n')d\mathbf{k}'dn', \tag{2}$$

in which

$$f=\frac{n^2}{|\mathbf{k}|}\left[\frac{1}{2}(\mathbf{k}+2\mathbf{k}')\cdot\mathbf{k}-\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')(\mathbf{k}-\mathbf{k}')\cdot\mathbf{k}}{|\mathbf{k}||\mathbf{k}-\mathbf{k}'|}\right]$$
$$+\frac{(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}}{|\mathbf{k}|}n(n+n')-\frac{1}{2}n^2|\mathbf{k}|-\frac{1}{2}\left(1-\frac{\mathbf{k}\cdot\mathbf{k}'}{|\mathbf{k}||\mathbf{k}'|}\right)$$
$$\times(|\mathbf{k}|+|\mathbf{k}'|)nn'+\frac{1}{2}\left(1-\frac{(\mathbf{k}-\mathbf{k}')\cdot\mathbf{k}'}{|\mathbf{k}-\mathbf{k}'||\mathbf{k}'|}\right)\frac{(\mathbf{k}-\mathbf{k}')\cdot\mathbf{k}}{|\mathbf{k}|}nn',$$

and $X(\mathbf{k},n)$ is the wavenumber frequency spectrum of the wave field. From (2), the phase speed $c=n/|\mathbf{k}|$ can be easily obtained.

The mean square value of the random scatter is given by

$$E[\epsilon^2]=\int_{\mathbf{k}'}\int_{n'}f_1^2(\mathbf{k},n;\mathbf{k}',n')X(\mathbf{k}',n')d\mathbf{k}'dn', \quad (3)$$

in which

$$f_1=n^2-\frac{\mathbf{k}\cdot(\mathbf{k}+\mathbf{k}')}{|\mathbf{k}||\mathbf{k}+\mathbf{k}'|}n(n+n')+\frac{1}{2}\left(1-\frac{\mathbf{k}\cdot\mathbf{k}'}{|\mathbf{k}||\mathbf{k}'|}\right)nn'.$$

3. Numerical results based on specific spectral function

To compute the mean dispersive relationship (2), the phase speed and the mean square value of random scatter (3), knowledge of the wavenumber-frequency spectrum $X(\mathbf{k},n)$ is required. Due to difficulties in measuring, information on $X(\mathbf{k},n)$ is not directly available. $X(\mathbf{k},n)$, nevertheless, may be related to either the directional wavenumber spectrum $\psi_0(|\mathbf{k}|,\theta)$ or the directional frequency spectrum $\phi_0(|n|,\theta)$ in which θ is the angle between $\mathbf{k}$ and an arbitrary reference axis, usually taken in the direction of the wind when considering wind-generated waves (Phillips, 1966).

To obtain the above relations it is noted that $X(\mathbf{k},n)$ and the wavenumber vector spectrum $\psi(\mathbf{k})$ are related as

$$X(\mathbf{k},n)=\psi(\mathbf{k})\delta(n-\sigma), \quad (4)$$

where $\sigma^2=g|\mathbf{k}|$ and $\delta(\ )$ is the Dirac delta function. The wavenumber vector spectrum may in turn be expressed in terms of the directional wavenumber spectrum as

$$\psi(\mathbf{k})d\mathbf{k}=\psi_0(|\mathbf{k}|,\theta)|\mathbf{k}|d|\mathbf{k}|d\theta, \quad (5)$$

and the relationship between $\psi_0(|\mathbf{k}|,\theta)$ and $\phi_0(n,\theta)$ is given by

$$\psi_0(|\mathbf{k}|,\theta)|\mathbf{k}|d|\mathbf{k}|=\phi_0(n,\theta)ndn. \quad (6)$$

An overwhelming majority of available experimental data on the characteristics of surface waves are collected at single points from which the frequency spectrum $\phi(n)$ may be generated. The directional frequency spectrum $\phi_0(n,\theta)$ is related to $\phi(n)$ based on meager experimental evidence and physical grounds. One of the models that has been suggested (Kitaigorodskii, 1973) and adopted in this study is to assume that

$$\phi_0(n,\theta)=\phi(n)K(\theta)/n, \quad (7)$$

in which

$$K(\theta)=\begin{Bmatrix}\frac{8}{3\pi}\cos^4\theta, & |\theta|<\frac{\pi}{2}\\ 0, & |\theta|>\frac{\pi}{2}\end{Bmatrix}. \quad (8)$$

The directional wavenumber spectrum may likewise be expressed as

$$\psi_0(|\mathbf{k}|,\theta)=s(|\mathbf{k}|)K(\theta) \quad (9)$$

in which $s(|\mathbf{k}|)$ may be obtained from (6) and (9). Thus,

$$\psi_0(|\mathbf{k}|,\theta)=\phi_0(n,\theta)\frac{n}{|\mathbf{k}|}\frac{dn}{d|\mathbf{k}|} \quad (10)$$

in which $dn/d|\mathbf{k}|=g/2n$ and $n=\sigma$. Substitution of (7) and (9) into (10) yields

$$s(|\mathbf{k}|)=\frac{g}{2|\mathbf{k}|n}\phi(n)|_{n=\sigma}. \quad (11)$$

In carrying out the computation for the mean dispersive relationship, the quantity $X(\mathbf{k}',n')d\mathbf{k}'dn'$ in (2) is replaced by

$$X(\mathbf{k}',n')d\mathbf{k}'dn'=s(|\mathbf{k}'|)K(\theta')|\mathbf{k}'|\delta(n'-g|\mathbf{k}'|)d|\mathbf{k}'|d\theta'$$

and f is changed to

$$f(|\mathbf{k}|,n,\theta;|\mathbf{k}'|,n',\theta')=n^2\left[\tfrac{1}{2}|\mathbf{k}|+|\mathbf{k}'|\cos(\theta'-\theta)-\frac{[|\mathbf{k}|-|\mathbf{k}'|\cos(\theta'-\theta)]^2}{[|\mathbf{k}|^2+|\mathbf{k}'|^2-2|\mathbf{k}||\mathbf{k}'|\cos(\theta'-\theta)]^{\frac{1}{2}}}\right]$$
$$+[|\mathbf{k}|+|\mathbf{k}'|\cos(\theta'-\theta)]n(n+n')-\tfrac{1}{2}n^2|\mathbf{k}|-\tfrac{1}{2}[1-\cos(\theta'-\theta)](|\mathbf{k}|+|\mathbf{k}'|)nn'$$
$$+\frac{1}{2}\left[1-\frac{|\mathbf{k}|\cos(\theta'-\theta)-|\mathbf{k}'|}{[|\mathbf{k}|^2+|\mathbf{k}'|^2-2|\mathbf{k}||\mathbf{k}'|\cos(\theta'-\theta)]^{\frac{1}{2}}}\right][|\mathbf{k}|-|\mathbf{k}'|\cos(\theta'-\theta)]nn',$$

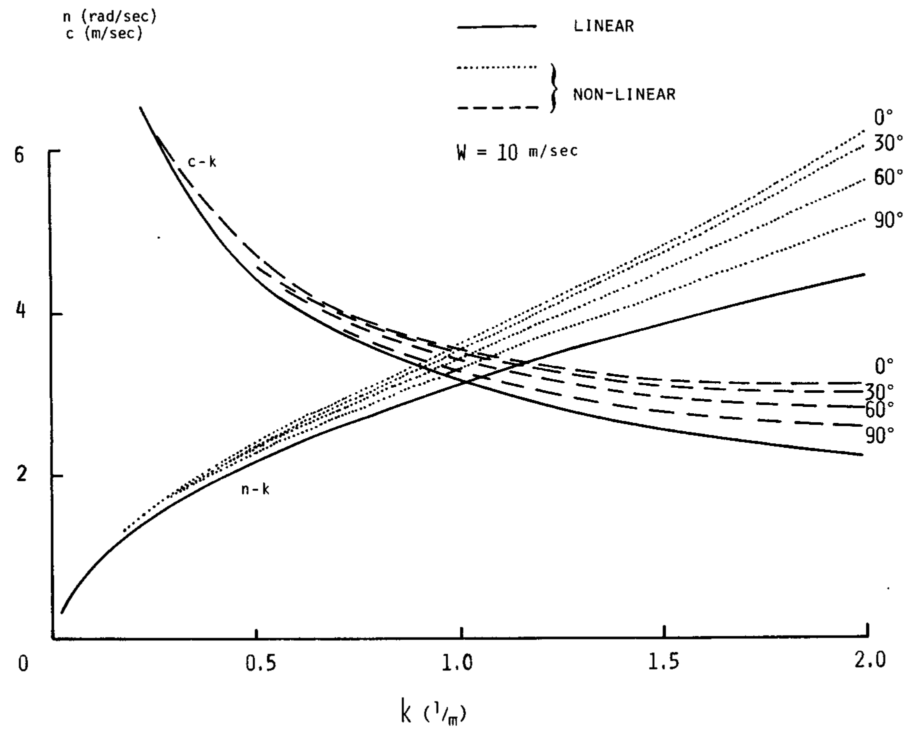

FIG. 1. Comparisons of the nonlinear and linear dispersion relations and phase velocity using Kitaigorodskii's spectral function with $K(\theta)=\cos^4\theta$ for $W=10$ m s^{-1}. The wave components under consideration are propagating at 0°, 30°, 60° and 90° with respect to the wind direction.

where θ' is the angle between $\mathbf{k}'$ and the arbitrary reference axis. Upon substitution of the above equalities into (2), the mean dispersion relation may be obtained numerically. The computation of the mean squared scatter $E[\epsilon^2]$ by (3) follows exactly the same line as that of (2).

Numerical results are obtained first for $s(|\mathbf{k}|)$ derived from the frequency spectrum proposed by Pierson and Moskowitz (1964) for fully developed wind generated waves given by

$$\phi(n)=\frac{\alpha g^2}{n^5}\exp\left[-\beta\left(\frac{g}{Wn}\right)^4\right],\quad n>0,$$

in which $\alpha=0.81\times10^{-2}$ and $\beta=0.74$ are nondimensional constants and W is the mean wind speed. The corresponding $s(|k|)$ is obtained from (11) and is

$$s(|k|)=\frac{\alpha}{2|\mathbf{k}|^4}\exp\left[-\beta\left(\frac{g}{W^2|\mathbf{k}|}\right)^2\right].$$

This expression will be used in Eq. (9) together with various forms of $K(\theta)$ to calculate the dispersive relationship and the scattering function numerically.

For each value of θ ($\theta=0°$ to 90° at 30° intervals) the dispersive relationships to the first- and third-order approximation, the phase speeds and the mean squared values of scatter are given in Figs. 1–4 for wind speeds of 10 and 15 m s^{-1}.

It is seen that since the surface wave spectrum is anisotropic, the above relationship and quantities are dependent on the direction of the component wave under consideration. Furthermore, the effects of nonlinearity become more pronounced in higher wavenumber ranges. In this connection it should be mentioned that, due to the small wave slope assumption that underlies the entire analysis, a cutoff wavenumber is selected in the computations so that the asymptotic expansion condition

$$|\mathbf{k}|\zeta\ll1$$

is strongly satisfied to ensure rapid convergence of the results.

Similar calculations are performed for two other directional energy distribution functions. The first one is simply using

$$K(\theta)=\begin{Bmatrix}\frac{1}{2}\cos\theta, & |\theta|<\pi/2\\ 0, & |\theta|>\pi/2\end{Bmatrix}. \qquad (12)$$

The results are presented in Figs. 5–8.

The second represents a class of directional distribution functions that changes with wind speed and frequency as suggested by Longuet-Higgins *et al.* (1963). Under this approach, the directional distribution

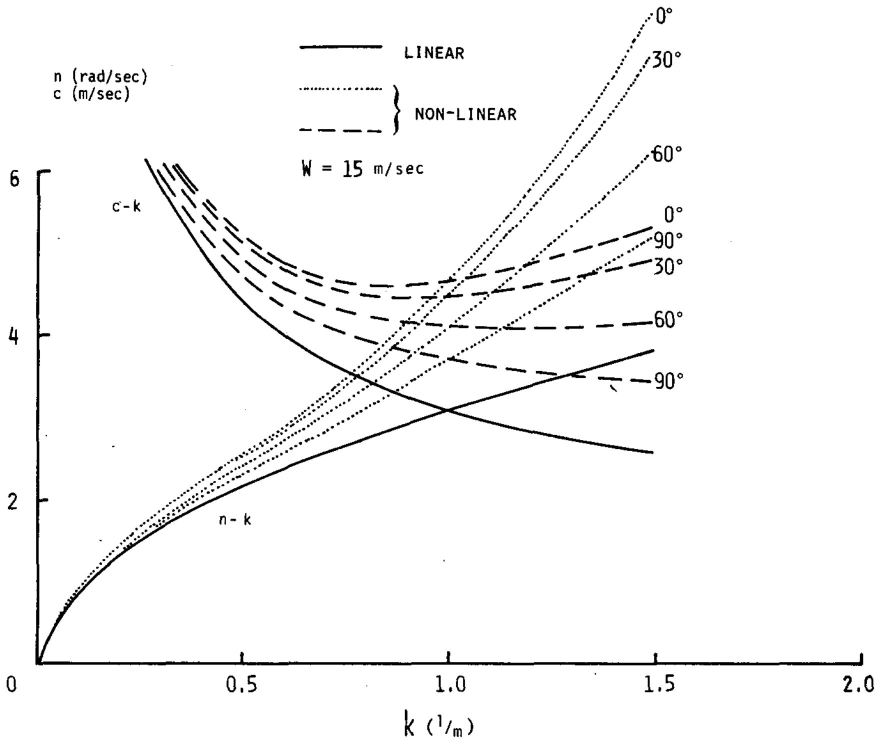

FIG. 2. As in Fig. 1 except for $W=15$ m s^{-1}

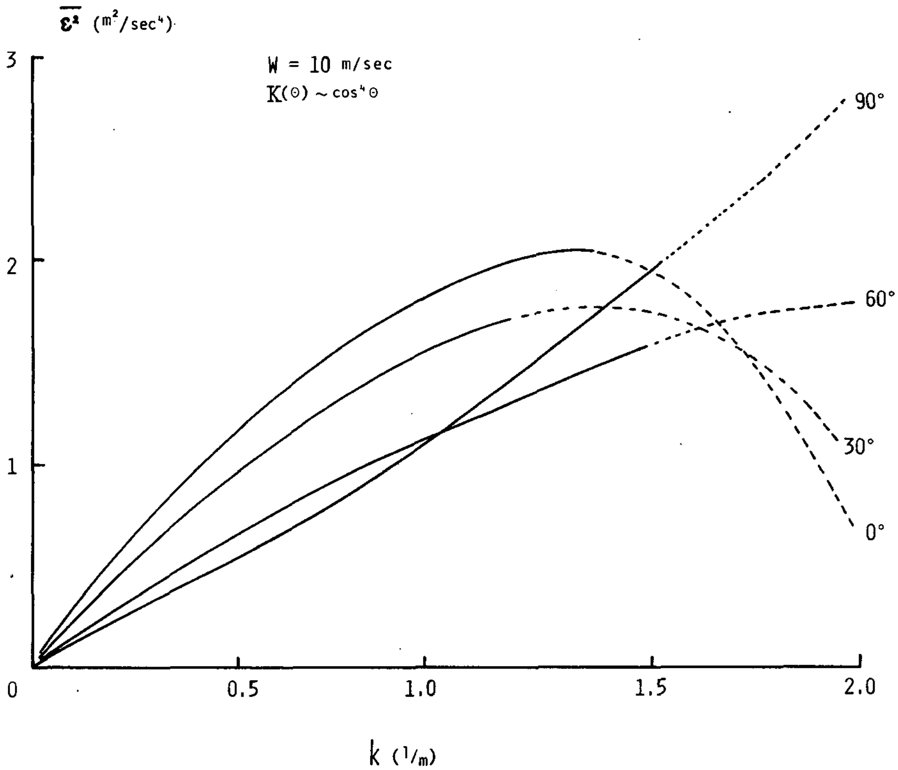

FIG. 3. The random scatter of the dispersion relation from the mean value for $K(\theta)=\cos^4\theta$ for $W=10$ m s^{-1}.

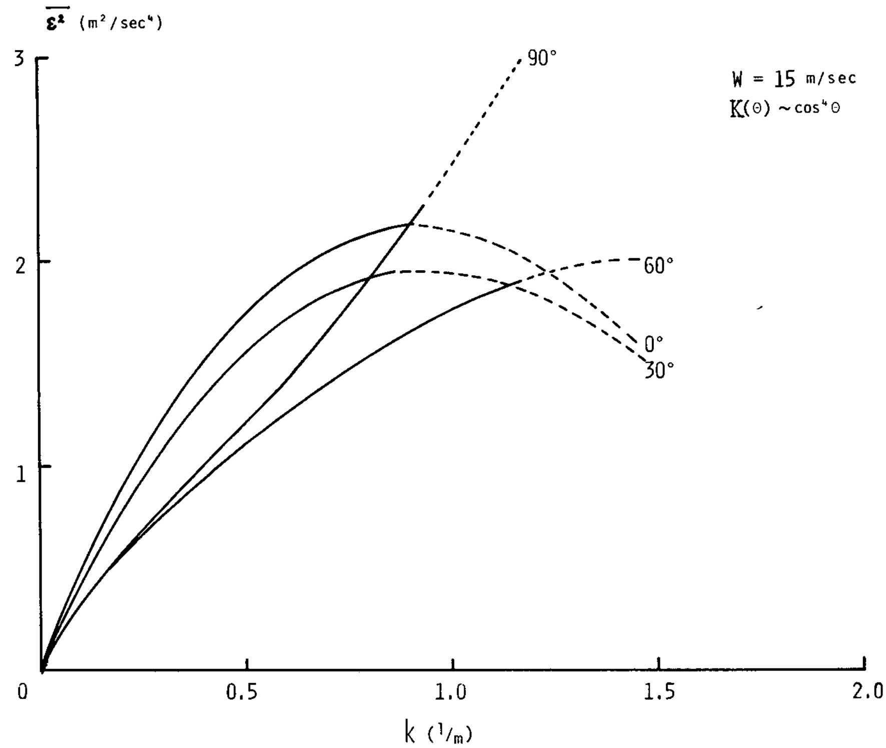

FIG. 4. As in Fig. 3 except for $W=15$ m s^{-1}.

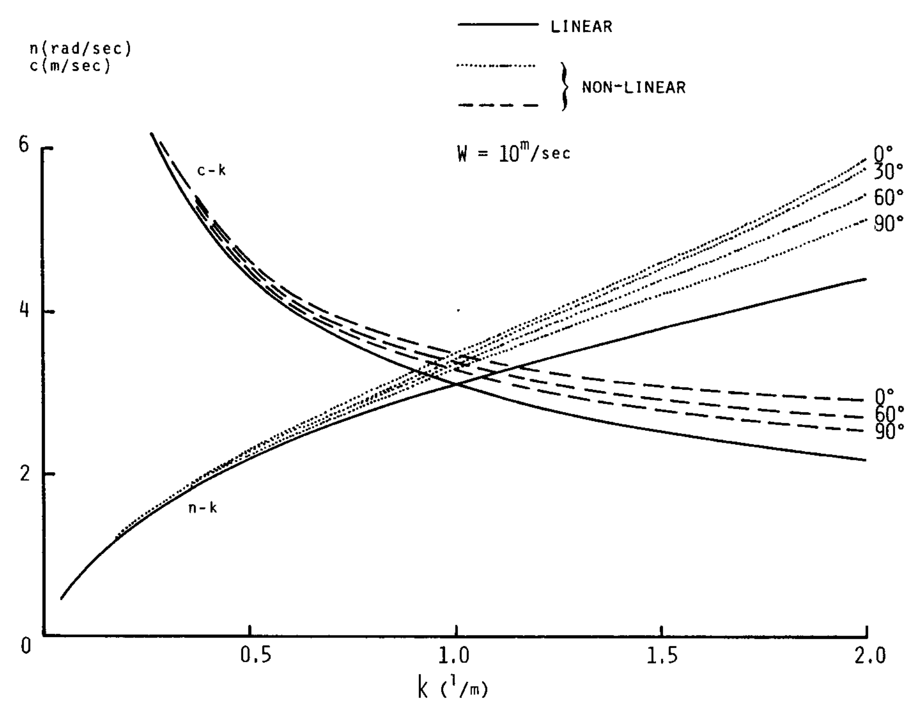

FIG. 5. Comparisons of the nonlinear and linear dispersion relations and the phase velocity using Kitaigorodskii's spectral function with $K(\theta)=\cos\theta$ for $W=10$ m s^{-1}. The wave components under consideration are propagating at 0°, 30°, 60° and 90° with respect to the wind direction.

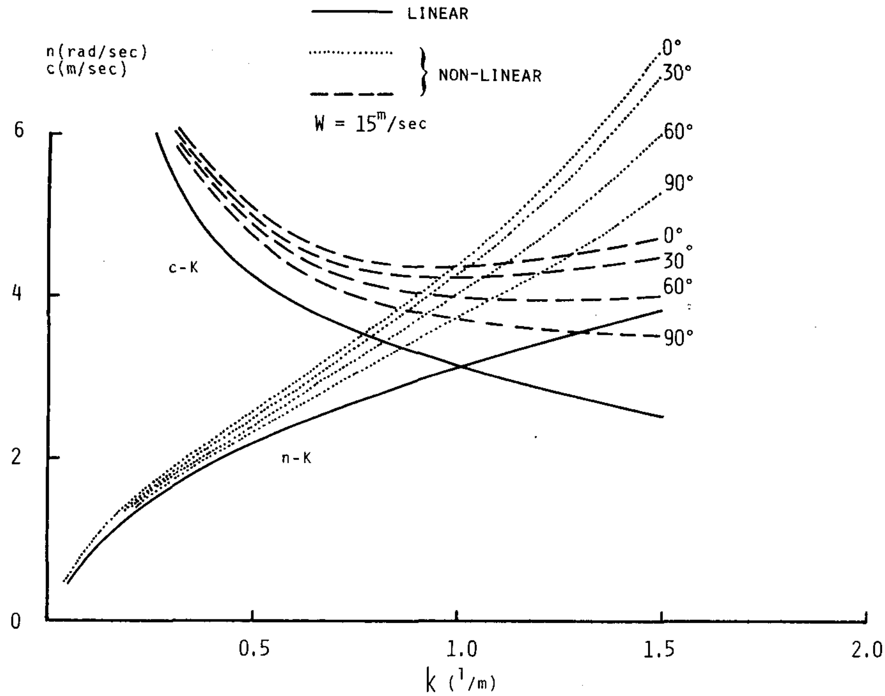

FIG. 6. As in Fig. 5 except for $W=15$ m s^{-1}.

function $K(\theta)$ becomes

$$K(\theta;W,n)=\frac{\Gamma(S+1)}{4\sqrt{\pi}\Gamma(S+\frac{1}{2})}(1+\cos\theta)^S$$

$$=\frac{\Gamma(S+1)}{4\sqrt{\pi}\Gamma(S+\frac{1}{2})}\cos^{2S}\frac{\theta}{2}, \quad (13)$$

$$|\theta|<\pi,$$

where S is a function of wind speed and frequency. From the data collected by Longuet-Higgins *et al.* (1963), Chen (1972) produced a best-fitted curve for S as

$$S=10^{-0.38}\frac{W}{c}+1.06. \quad (14)$$

When (13) and (14) are used, the results are shown in Figs. 9–12.

By comparing the above three models, it becomes obvious that the nonlinear influence decreases as the spreading of the energy increases. It should be mentioned here that the exact form of the energy spreading function is not known. Physical and intuitive arguments nevertheless suggest that directional energy spreading should be frequency and wind dependent. The directional energy spreading function advanced by Longuet-Higgins *et al.* (1963) is therefore a more likely candidate in which case the mean deviation and the random scatter are expected to be smaller than would be the case if simple power law of cosine functions are assumed for $K(\theta)$. However, the emphasis of this paper is to indicate the influence of energy spreading on the dispersion relation, not on exhaustive testing of energy spreading functions. The discussion on various energy spreading functions will not be explored further here. Since there is no generally accepted form for the energy spreading function, *ad hoc* formulas will be used in the next section as suggested by the particular author when comparison is made with that set of data.

4. Comparisons with observational results

Comparisons of observations with analytic results are essential for the verification of a model. In order to make meaningful comparisons, both directional spectra and phase velocities must be measured simultaneously. Unfortunately, such data are almost nonexistent. The first set of data used here is taken from Yefimov *et al.* (1972). In that paper values of $\phi(n)$ are given. It was further assumed that (7) holds and that

$$\left.\begin{aligned} K(\theta)&\sim\cos^p m\theta, & |\theta|&<\pi/2m\\ &=0, & |\theta|&>\pi/2m \end{aligned}\right\}, \quad (15)$$

in which m and p are fixed constants. Although no explicit dispersion relation was given, the measured average phase speed $C(n)$, defined as

$$C(n)=\frac{ln}{\mu(n)}, \quad (16)$$

was presented graphically. In (16), l is the distance

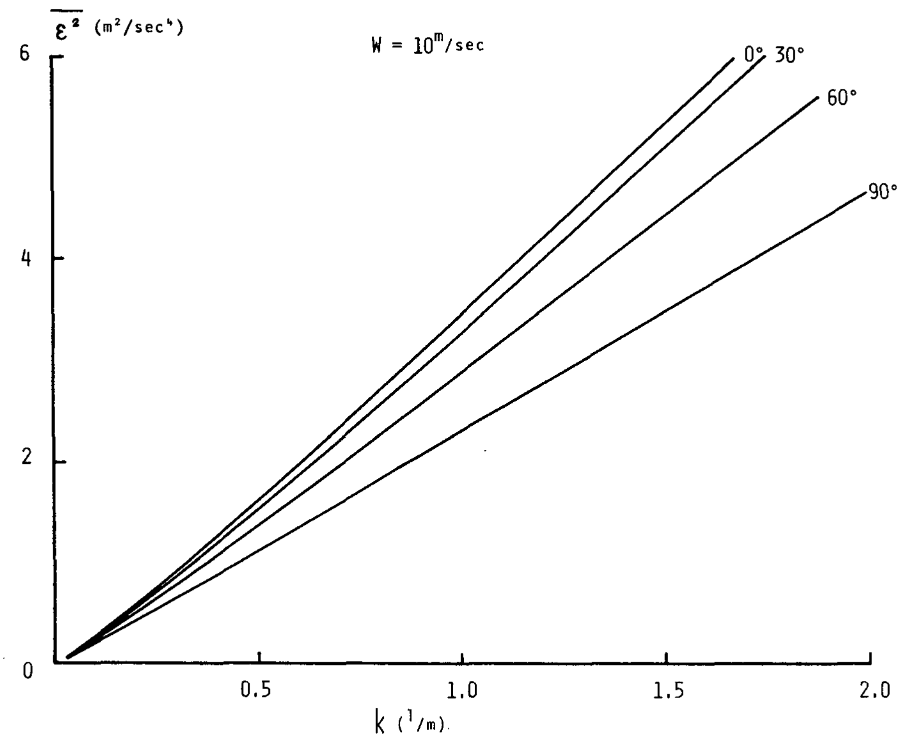

FIG. 7. The random scatter of the dispersion relation from the mean value for $K(\theta)=\cos\theta$ for $W=10$ m s^{-1}.

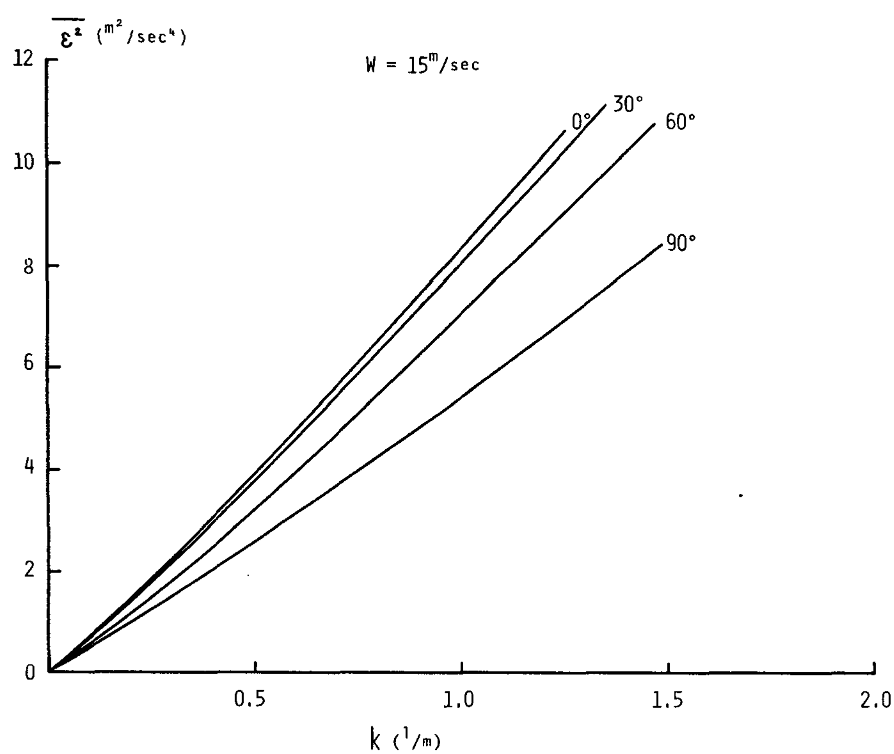

FIG. 8. As in Fig. 7 except for $W=15$ m s^{-1}.

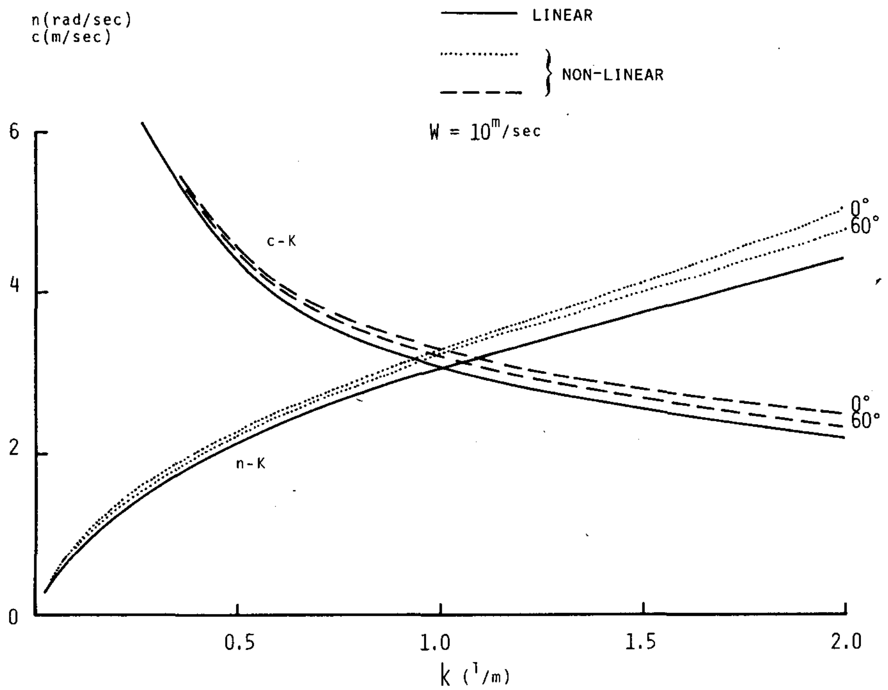

FIG. 9. Comparisons of the nonlinear and linear dispersion relations and the phase velocity using the Longuet-Higgins *et al.* (1963) model of the directional energy distribution with $W=10$ m s^{-1}. The wave components under consideration are propagating at 0° and 60° with respect to the wind.

between two wave observation stations and $\mu(n)$ is the phase lag of wave signals at the two stations. Thus

$$\mu(n)=\arctan\frac{Q(n)}{Co(n)}, \tag{17}$$

with $Q(n)$ and $Co(n)$ as the quadrature spectrum and cospectrum, respectively. Theoretically, the average phase velocity of a spectral component propagating along the x axis is given by

$$C(n)=\frac{n}{k_x}=\frac{n\int_\theta \phi_0(n,\theta)d\theta}{\int_\theta |\mathbf{k}|\cos\theta\phi_0(n,\theta)d\theta}. \tag{18}$$

The two expressions given in (16) and (18) are similar, and their values would be identical for small separations of the two measuring stations. The numerical calculations of $C(n)$ given in this paper are all based on (18). For nonlinear cases, (2) is used in lieu of (1). It should be noted that if the first-order dispersive relationship is used, as was done in Yefimo *et al.* (1972), then the relation between $|\mathbf{k}|$ and n is independent of θ. Then

$$C(n)=\gamma\frac{g}{n}, \tag{19}$$

where γ is a constant depending only on the functional forms of $K(\theta)$. When the third-order dispersion relation is used, $|\mathbf{k}|$ is dependent on θ. The values of average phase speed computed according to (18) are plotted along with the measured values of Yefimov *et al.* (1972) in Fig. 13. The nonlinear results compare favorably with the observed values not only in the trends but also in magnitudes.

The second set of data used here is taken from Ramamonjiarisoa and Coantic (1976). The data are taken in a wind-wave channel without a suggested directional spreading function; therefore a unidirectional energy spreading function was assumed, although the same computational method using (18) was employed. Comparisons of the nonlinear results computed from the spectral data with the observed phase velocity are shown in Figs. 14 and 15. The agreements are not as good as that in Fig. 13. It is noted that there is a remarkable discrepancy at the low-frequency end where the measured data show a decrease in the phase velocity. This trend obviously contradicts the established characteristics of water wave motion.

Some possible reasons are advanced here to explain the different characteristics of the discrepancy between theoretical and observed data in Figs. 13–15. Fig. 13 is based on field data, while Figs. 14 and 15 are based on laboratory data. Even under the same mean wind velocity, the wave field will be quite different. In the laboratory the constraints of the top and sides of the

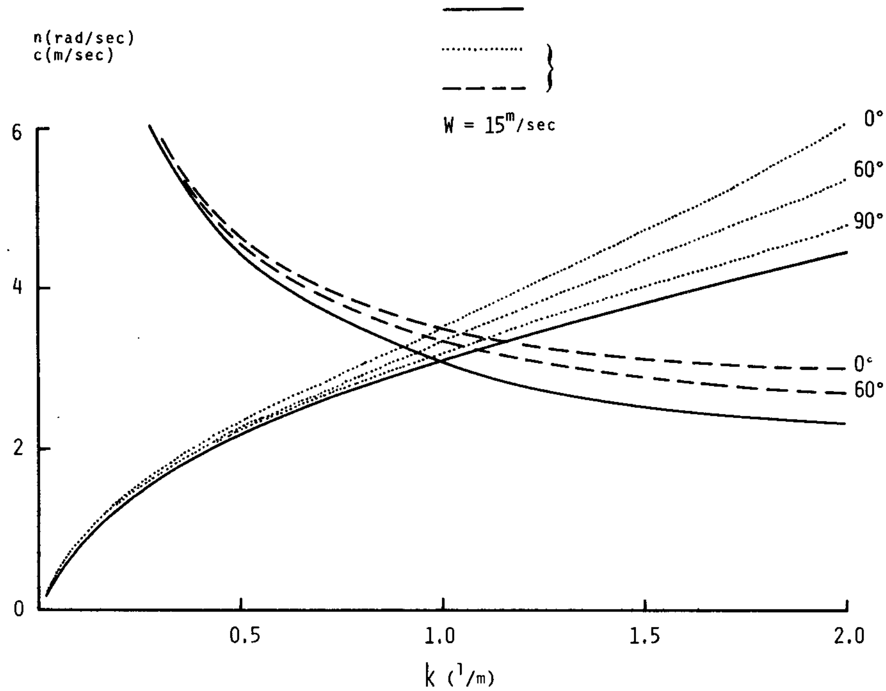

FIG. 10. As in Fig. 9 except for $W=15$ m s^{-1}.

tank will tend to exaggerate the shearing effect of the wind. Consequently, the waves are definitely forced waves rather than free ones as assumed in the theoretical model here. This probably contributes to the larger discrepancy seen in Figs. 14 and 15. Incorporating surface stresses is definitely an urgent extension of the present study. Other possibilities are uncertainties in the directional energy distribution functions, the discrepancy between the quantities given in (16) and (18) due to finite separation, and last, if not the least, bias in the measurements. In addition to the data used here, there is another set of data collected by Longuet-

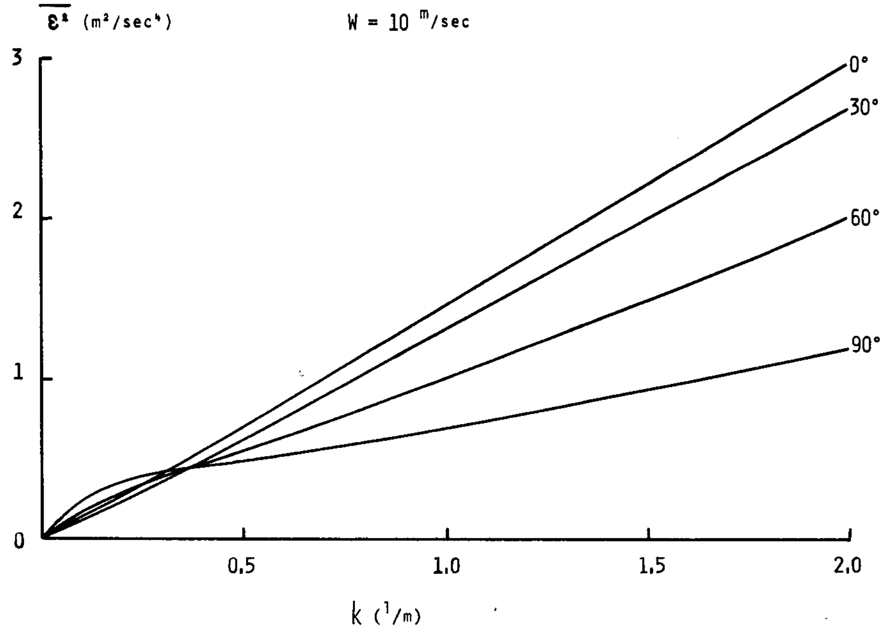

FIG. 11. The random scatter of the dispersion relation from the mean for the Longuet-Higgins *et al.* (1963) energy distribution model for $W=10$ m s^{-1}.

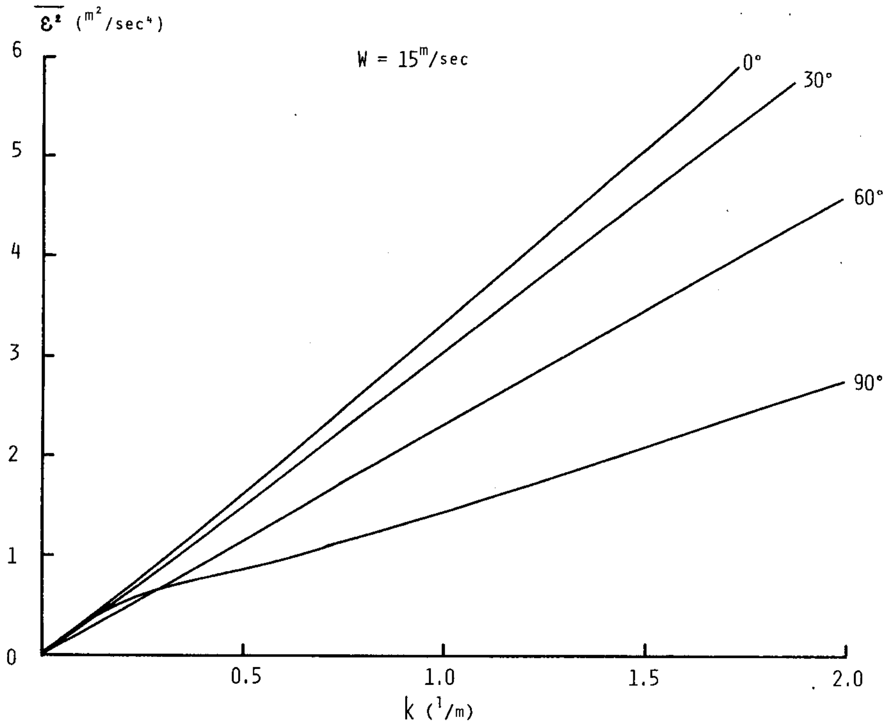

FIG. 12. As in Fig. 11 except for $W=15$ m s^{-1}.

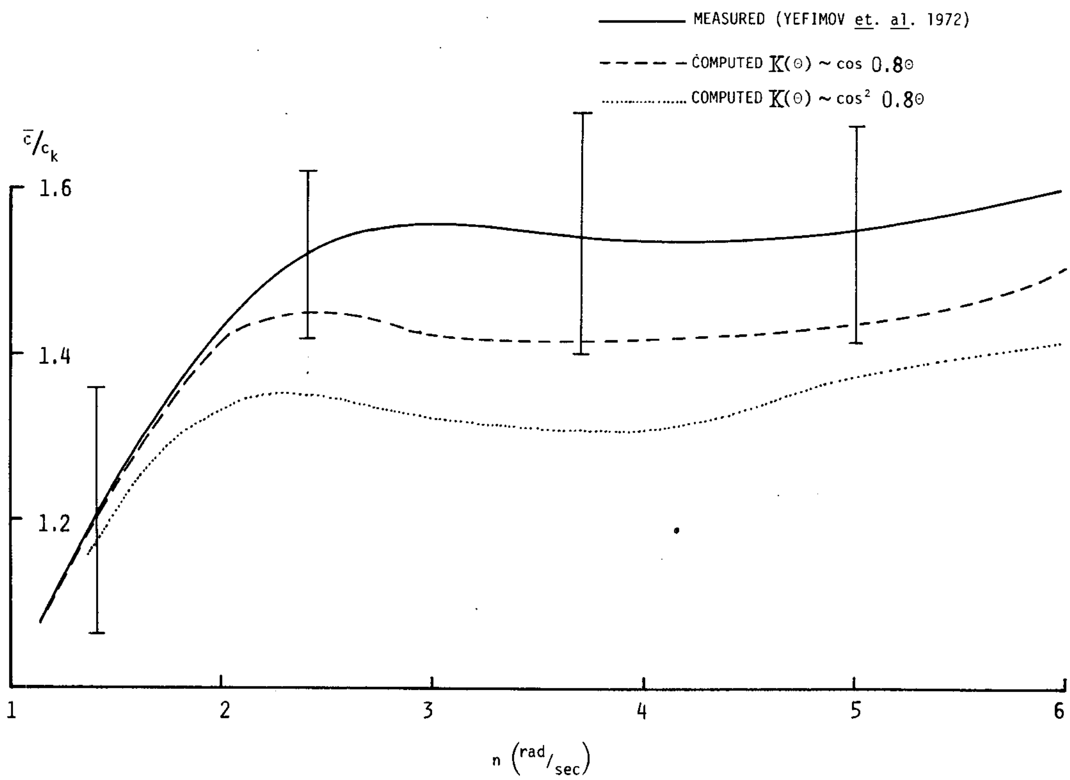

FIG. 13. Comparison of the nonlinear phase velocity with observational data by Yefimov *et al.* (1972).

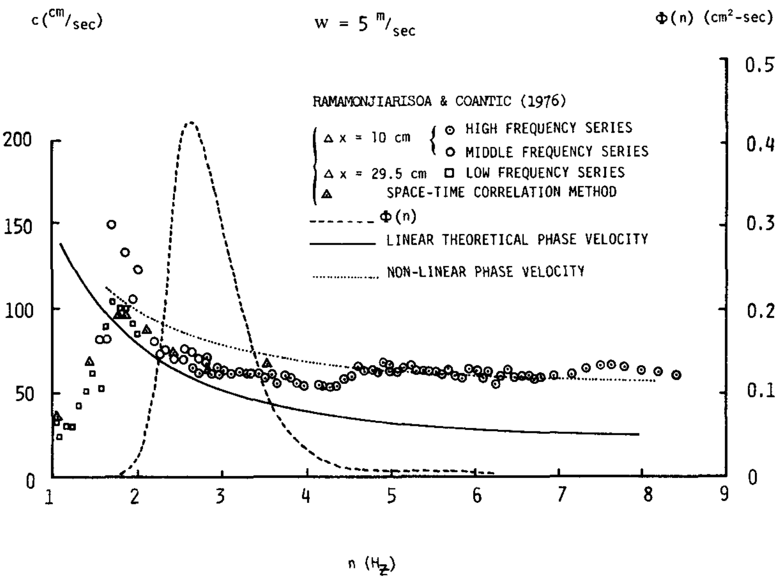

FIG. 14. Comparison of the nonlinear and linear phase velocities with observational data by Ramamonjiarisoa and Coantic (1976) for $W=5$ m s^{-1}.

Higgins *et al.* (1963), which includes both directional spectra and some measure of the dispersion relation. Even though the general trend of the data is similar to what the theoretical analysis predicted, the dispersion relation in that paper appears to be opposite to that of the classical Stokes (1847) result in the single wave train case. Longuet-Higgins *et al.* (1963) reported

$$\frac{gk}{n^2}>1, \tag{19}$$

but the classical results indicate

$$n^2=gk[1+a^2k^2+O(a^4k^4)]; \tag{20}$$

hence,

$$\frac{gk}{n^2}=\frac{1}{1+a^2k^2+O(a^4k^4)}<1.$$

Since this basic contradiction seems to suggest the possibility of a calibration problem in absolute values, no attempt is made to compare the analytic results with the data. Theoretical studies as well as additional

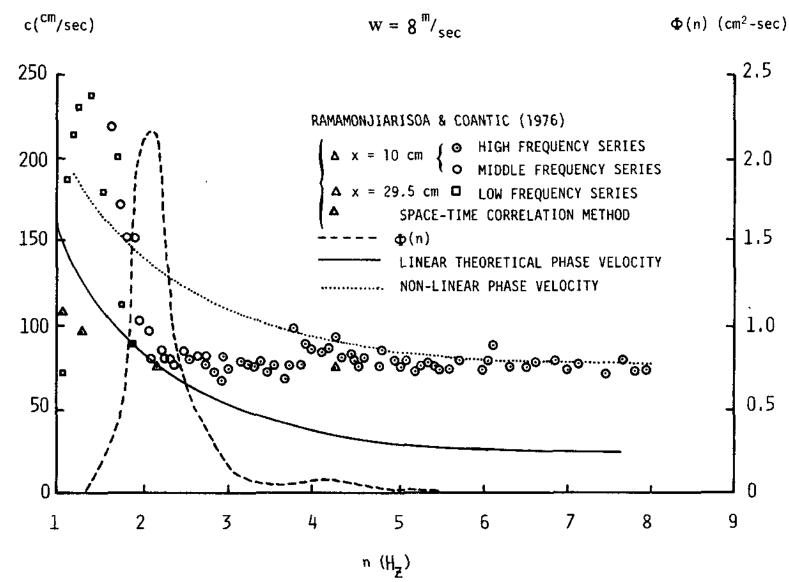

FIG. 15. As in Fig. 14 except for $W=8$ m s^{-1}.

observations both in the laboratory and in the field are urgently needed.

5. Concluding remarks

Numerical results are obtained for the dispersion relation of nonlinear random gravity waves. Results indicate that the nonlinear influence decreases as the directional spreading of wave energy increases. Comparison of computed results with field and laboratory measurements show that there is general agreement except that field observed "phase speed" invariably exceeds computed values. This discrepancy may be partially attributable to lack of knowledge and inappropriateness in the modelling of the directional spectra of the waves. It is also recognized that under persistent wind, current is induced in the surface layer by wind stresses; this is known to increase the propagation speeds of the waves. These two factors are deemed to have major influences on the dispersion relationship and merits further investigations.

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