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Wave propagation in random media

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This paper discusses a general theory of wave propagation through a random medium whose random inhomogeneities are confined to small deviations from the mean. The theory is initially worked out in detail for the propagation of transverse waves along an infinite stretched string whose density is a random function of position. The manner in which the mean wave profile is modified by scattering from the density inhomogeneities is discussed in great detail, with particular emphasis on physical interpretation. The general theory of wave propagation in arbitrary dispersive or non-dispersive media is then discussed, and it is shown how the theory may be extended to wave propagation problems involving scattering from rough boundaries.

1. Introduction

The study of wave propagation through media with inhomogeneous random properties is of increasing technological importance, as well as being richly endowed with notions of a fundamental and academic nature. Such problems arise quite naturally in radiophysics, for example, where atmospheric turbulence can manifest itself as atmospheric 'noise', causing fluctuations in the parameters of wave propagation through the atmosphere; in other cases the atmospheric turbulence behaves like a source of inhomogeneities which produce scattering. Reviews of some of the recent work in these fields are to be found in the books of Tatarski (1961) and Chernov (1960). The effect of atmospheric turbulence on 'sonic bangs' is perhaps a more topical and, in view of the imminence of large supersonic transports, a somewhat more pressing problem. These latter effects were first considered in detail by Lighthill (1953) as an example of the Born approximation in scattering theory. The same method has recently been applied to sonic boom N-waves by Crow (1969).

It is becoming increasingly clear, however, that there is a general need for a reappraisal of many of the equations of mathematical physics. Inhomogeneity is a characteristic property of every real medium, and problems where these inhomogeneities are assumed to be known with any degree of precision are tending to be recognized as the exceptions rather than the rule. It must therefore be accepted that in reality one deals with *stochastic* equations of motion whose solutions will in general depend on the statistical properties of the medium and of the boundary conditions.

In the case of wave propagation problems, it is customary to speak of the signal present at any point in the random medium as consisting of the *coherent* field, i.e. the mean field, this being an ensemble average over all probabilistically possible media, together with a fluctuating or random component. Physically one expects the statistics of the fluctuations of the wave field to be linked intimately with those defining the random medium. Since the precise form of the medium in any particular experiment is only known in probability, the problem of determining the precise solution for waves propagating through that medium has no meaning. Indeed, such a solution would be quite useless without some knowledge of its realization probability!

A notable advance in the method of treatment of these problems was made by Keller (1964) and Karal & Keller (1964). They give a concise, if somewhat cumbersome, derivation of the equations governing the *coherent* component of the wave field. However, their treatment tends to obscure the physics of the various approximations involved. A fuller discussion is needed, which includes all aspects of the mean and random components of the wave field, together with a rigorous consideration of the approximations and their physical interpretation.

The present paper presents a coherent theory applicable to wave propagation problems in which the random deviations of the medium from the mean are small. The theory is initially worked out in detail for the simple case of transverse waves propagating along an infinite stretched string whose density is a random function of position. The object is to illustrate the methods and implications of the more general theory applicable to arbitrary media, in an arbitrary number of dimensions, which is discussed later. The motivation of the present work lies in a desire to give a consistent treatment of the sonic bang problem mentioned above. This work, undertaken in collaboration with Professor J. E. Ffowcs-Williams, has now been completed and will appear in a future publication.

2. Transverse waves on a random string

Consider the problem of the propagation of transverse waves of amplitude ϕ, say, along a stretched string of randomly variable mass density. More precisely let us consider a string of mean mass per unit length ρ_0, and let ρ' be a random function of position x on the string which represents the fluctuations of the actual density about the mean. By a random string we understand a family of strings each with a well-defined value of $\rho'(x)$ at each x, and each with a well-defined probability of being realized in an experiment.

Since the precise form of the density at any point is only known in probability, there is no point in trying to calculate the progress of a wave along the string *exactly*. It is more realistic to inquire into what happens to such a wave *on an average*. A sensible procedure, therefore, would appear to involve a separation of the wave field into two distinct parts: $\overline{\phi}(x,t)$, the mean wave profile at a given time t, say, after the initial generation of the wave; and $\phi'(x,t)$, the fluctuations of the field about $\overline{\phi}$. In other words, we imagine a sequence of identical experiments performed on each member of the family of strings comprising the random string. Each experiment involves the generation of a given initial wave; at time t after the start of each experiment, the wave profile is measured, and the average form

$\bar{\phi}(x,t)$ taken over all the experiments. The field $\bar{\phi}(x,t)$ is thus an ensemble average over all probabilistically possible strings, and $\phi'(x,t)$ represents the deviation of the exact field from $\bar{\phi}(x,t)$ in any particular experimental realization. Thus,

$$\phi(x,t) = \bar{\phi}(x,t) + \phi'(x,t), \tag{2.1}$$

where it will generally happen that $\phi' \ll \bar{\phi}$, but this is in no way a necessary prerequisite for the validity of the present theory. Let us now obtain equations governing the evolution of $\bar{\phi}$ and ϕ'.

The exact linearized equation for transverse vibrations of a stretched string has the well-known, simple form,

$$\frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial x^2} = 0. \tag{2.2}$$

The 'wave speed' c is a function of the tension T of the string, and the density ρ of the form

$$c^2 = T/\rho. \tag{2.3}$$

When $\rho = \rho_0 + \rho'$, it is meaningful to speak of a mean square wave speed a^2, say,

$$a^2 = \overline{c^2} = \overline{\{T/(\rho_0+\rho')\}}, \tag{2.4}$$

where the overbar denotes the ensemble average discussed above, and then to define the random function $\xi(x)$, say, by

$$a^2(1+\xi) = \frac{T}{\rho_0+\rho'}. \tag{2.5}$$

This means that $\overline{\xi(x)} = 0$.

Then equation (2.2) becomes

$$\frac{\partial^2\phi}{\partial t^2} - a^2\frac{\partial^2\phi}{\partial x^2} = a^2\xi\frac{\partial^2\phi}{\partial x^2}. \tag{2.6}$$

Take the ensemble average of this to obtain

$$\frac{\partial^2\bar{\phi}}{\partial t^2} - a^2\frac{\partial^2\bar{\phi}}{\partial x^2} = a^2\overline{\xi\frac{\partial^2\phi'}{\partial x^2}}, \tag{2.7}$$

since $\overline{\xi\partial^2\bar{\phi}/\partial x^2} = \bar{\xi}\,.\,\partial^2\bar{\phi}/\partial x^2 \equiv 0$. Equation (2.7) is exact and shows that, since both ξ and ϕ' may generally be regarded as first-order quantities, changes in the mean field are generally expected to be of the second order in these quantities. This was only to be expected, since second-order quantities are the first to exhibit non-vanishing mean values.

To obtain the equation for ϕ' we subtract (2.7) from the full equation (2.6):

$$\frac{\partial^2\phi'}{\partial t^2} - a^2\frac{\partial^2\phi'}{\partial x^2} = a^2\xi\frac{\partial^2\bar{\phi}}{\partial x^2} + a^2\left\{\xi\frac{\partial^2\phi'}{\partial x^2} - \overline{\xi\frac{\partial^2\phi'}{\partial x^2}}\right\}. \tag{2.8}$$

This equation describes the continuous generation and modification of the random field ϕ' by means of the following interactions: (i) between the density fluctuation $\xi(x)$ and the mean field $\bar{\phi}(x,t)$ (this creates new $\phi'(x,t)$); (ii) between the density fluctuation $\xi(x)$ and those component waves of $\phi'(x,t)$ which are *not* correlated with $\xi(x)$.

Equations (2.7) and (2.8) constitute an *exact* pair of coupled equations governing the evolution of the mean field $\overline{\phi}$ and the random fluctuations ϕ'. Given an initial wave form, and assuming that initially there is no random field, those equations are to be solved as an initial value problem. Before considering such an analysis, however, it is of interest to interpret the right-hand members of both equations in somewhat greater detail.

The first equation, (2.7), describes the modification of the mean field $\overline{\phi}$ resulting from interactions between correlated components of $\xi(x)$ and $\phi'(x,t)$. In other words, only those component waves of $\phi'(x,t)$ arriving at the point x which are strongly correlated with $\xi(x)$ will contribute to this interaction. Equation (2.8) shows that such a mean interaction product does exist. The interaction (i) generates components of the random field ϕ' by direct 'collision' between the mean field $\overline{\phi}$ and the density fluctuation represented by ξ. Hence, the main contribution to the mean interaction product on the right of (2.7) will be from those wave components of $\phi'(x,t)$ which were initially scattered out of the mean field at points lying within a distance λ, say, from the point x, where λ is the correlation length for the density fluctuations $\xi(x)$. In this sense, the gradual modification of the mean wave profile may be described as a *local* effect.

Actually, it is more instructive to think of the interactions as being between density fluctuations $\xi(x)$ lying at different points. The modification of the mean field is due entirely to the presence of inhomogeneous fluctuations ξ in the density of the string. When such fluctuations lie within a correlation length of one another they interact in a concertive manner to produce a change in the mean field. The interaction is represented by the passage of a secondary wave ϕ' between the two components of inhomogeneity. Higher-order 'collisions' between density fluctuations ξ involve the passage of a secondary wave from $\xi(x_1)$, say, to $\xi(x_2)$, the generation of a tertiary wave at x_2, and its subsequent transmission to $\xi(x_3)$. Such multiple scattering effects appear in the interaction term of (2.7) when the interactions (ii) are taken into account. However, because (ii) essentially represents interactions between non-correlated density fluctuations $\xi(x)$, it may be interpreted as representing interactions between density fluctuations whose distance apart exceeds a correlation length λ, so that their ultimate effect on the mean interaction product of (2.7) might be expected to be small.

An analogy may perhaps be drawn with Boltzmann's equation in kinetic theory (Chapman & Cowling 1939). Neglect of (ii) is equivalent to adopting the *binary collision* approximation to that equation.

3. Analysis of the stretched string equations

We have seen that the evolution of an initially well-defined wave form on the stretched string is governed by the pair of equations

$$\left.\begin{aligned} \frac{\partial^2\overline{\phi}}{\partial t^2} - a^2\frac{\partial^2\phi}{\partial x^2} &= a^2\overline{\xi\frac{\partial^2\phi'}{\partial x^2}}, \\ \frac{\partial^2\phi'}{\partial t^2} - a^2\frac{\partial^2\phi'}{\partial x^2} &= a^2\xi\frac{\partial^2\overline{\phi}}{\partial x^2} + a^2\left\{\xi\frac{\partial^2\phi'}{\partial x^2} - \overline{\xi\frac{\partial^2\phi'}{\partial x^2}}\right\}. \end{aligned}\right\} \quad (3.1)$$

The scattered field ϕ' is determined as soon as $\xi(x)$ and $\overline{\phi}(x,t)$ are specified. Formally, we may suppose these to be known, then the second of equations (3.1) may be solved to any degree of accuracy by the method of successive approximations.

The approximation scheme would generate a sequence of solutions ϕ'_n by means of the system of equations:

$$\left.\begin{aligned}\frac{\partial^2\phi'_n}{\partial t^2}-a^2\frac{\partial^2\phi'_n}{\partial x^2}=a^2\xi\frac{\partial^2\overline{\phi}}{\partial x^2}+a^2\left\{\xi\frac{\partial^2\phi'_{n-1}}{\partial x^2}-\overline{\xi\frac{\partial^2\phi'_{n-1}}{\partial x^2}}\right\}\quad(n>1),\\ \frac{\partial^2\phi'_1}{\partial t^2}-a^2\frac{\partial^2\phi'_1}{\partial x^2}=a^2\xi\frac{\partial^2\overline{\phi}}{\partial x^2}.\end{aligned}\right\}\tag{3.2}$$

Having determined ϕ' in terms of ξ and $\overline{\phi}$, the solution may be substituted into the mean wave equation to give an equation for $\overline{\phi}$ alone.

We shall illustrate this procedure by taking the 'binary collision' approximation, i.e. the approximation $\phi'=\phi'_1$. Now the particular integral of the equation,

$$\frac{\partial^2 G}{\partial t^2}-a^2\frac{\partial^2 G}{\partial x^2}=\delta(x)\,\delta(t)\tag{3.3}$$

$(-\infty<x<\infty,\ -\infty<t<\infty)$, which satisfies the radiation condition is well known to have the simple form,

$$G(x,t)=\frac{1}{2a}H(at-|x|),\tag{3.4}$$

where $H(x)$ is the Heaviside unit function. The required particular integral of the second of equations (3.2) is now given by the convolution product of G and $a^2\xi(x)\,\partial^2\overline{\phi}(x,t)/\partial x^2$; viz.

$$\phi'_1=a^2G*\xi\,\partial^2\overline{\phi}/\partial x^2.$$

In other words,

$$\phi'_1=a^2\int\int_{-\infty}^{\infty}G(x-x_0,t-t_0)\xi(x_0)\frac{\partial^2\overline{\phi}}{\partial x_0^2}(x_0,t_0)\,dx_0dt_0.\tag{3.5}$$

This result becomes useful as soon as the form for $\overline{\phi}$ is known, for then one can determine the statistics of ϕ'_1 in terms of those of $\xi(x)$. For example, by squaring (3.5) and taking the ensemble average, an expression for $\overline{(\phi'_1)^2}$ is obtained.

To determine $\overline{\phi}$ under the present approximation, we now substitute for ϕ' in the first of equations (3.1) from (3.5). This gives

$$\frac{\partial^2\overline{\phi}}{\partial t^2}-a^2\frac{\partial^2\overline{\phi}}{\partial x^2}=a^4\int\int_{-\infty}^{\infty}\overline{\xi(x)\xi(x_0)}\frac{\partial^2 G}{\partial x^2}(x-x_0,t-t_0)\frac{\partial^2\overline{\phi}}{\partial x_0^2}(x_0,t_0)\,dx_0dt_0.\tag{3.6}$$

At this stage, to proceed further we require a knowledge of the statistical properties of $\xi(x)$. We shall assume for simplicity of exposition that $\xi(x)$ is a *stationary* random function of x, in consequence of which the correlation product $\overline{\xi(x)\xi(x_0)}$ is an even function of $x-x_0$ alone. Let us define, in fact,

$$R(x-x_0)=\overline{\xi(x)\xi(x_0)},\tag{3.7}$$

with $R(0) = \overline{\xi^2}$ = constant; then (3.6) may be expressed in the form,

$$\frac{\partial^2 \overline{\phi}}{\partial t^2} - a^2 \frac{\partial^2 \overline{\phi}}{\partial x^2} = a^4 \int \int_{-\infty}^{\infty} R(\eta) \frac{\partial^2 G}{\partial \eta^2}(\eta, t-t_0) \frac{\partial^2 \overline{\phi}}{\partial x^2}(x-\eta, t_0)\, d\eta\, dt_0. \tag{3.8}$$

But

$$\frac{\partial^2 G}{\partial \eta^2} = -\frac{\delta(\eta)\,\delta(t-t_0)}{a^2} + \frac{1}{2a^3}\delta'(t-t_0-|\eta|/a), \tag{3.9}$$

where $\delta(\eta)$ is the Dirac δ-function, so that the final form of the equation for $\overline{\phi}$ is

$$\frac{\partial^2 \overline{\phi}}{\partial t^2} - a^2 \frac{\partial^2 \overline{\phi}}{\partial x^2} = -a^2 R(0) \frac{\partial^2 \overline{\phi}}{\partial x^2} + \frac{a}{2} \int_{-\infty}^{\infty} R(\eta) \frac{\partial^3 \overline{\phi}}{\partial t\, \partial x^2}(x-\eta, t-|\eta|/a)\, d\eta,$$

or, since $R(0) = \overline{\xi^2}$,

$$\frac{\partial^2 \overline{\phi}}{\partial t^2} - a^2(1-\overline{\xi^2}) \frac{\partial^2 \overline{\phi}}{\partial x^2} = \frac{a}{2} \int_{-\infty}^{\infty} R(\eta) \frac{\partial^3 \overline{\phi}}{\partial t\, \partial x^2}(x-\eta, t-|\eta|/a)\, d\eta. \tag{3.10}$$

This is the required integro-differential equation for the mean field $\overline{\phi}$.

4. Simplified forms of the mean field equation. The dispersion relation

To discuss the implications of (3.10), let us first consider the case in which the correlation length λ of the density fluctuations is *small* compared to a typical wavelength of $\overline{\phi}$. Mathematically, this means that the operators,

$$\lambda \frac{\partial}{\partial x} \quad \text{and} \quad \frac{\lambda}{a} \frac{\partial}{\partial t}, \tag{4.1}$$

are *small* when applied to $\overline{\phi}$. Now, if we introduce a new variable of integration into (3.10) by

$$z = \eta/\lambda,$$

we shall obtain

$$\frac{\partial^2 \overline{\phi}}{\partial t^2} - a^2(1-\overline{\xi^2}) \frac{\partial^2 \overline{\phi}}{\partial x^2} = \frac{a\lambda}{2} \int_{-\infty}^{\infty} R(z) \frac{\partial^3 \overline{\phi}}{\partial t\, \partial x^2}(x-\lambda z,\, t-\lambda|z|/a)\, dz. \tag{4.2}$$

$R(z)$ is a function of the dimensionless variable z alone, and in terms of this variable the correlation length is equal to unity. Hence, if $\partial^3\overline{\phi}/\partial t\, \partial x^2(x-\lambda z, t-\lambda|z|/a)$ is expanded in a Taylor series about (x, t), the integration in (4.2) may then be performed, and the result will be a power series expansion in terms of the operators (4.1). If only the first-order terms in such an expansion are retained, then, since $R(z)$ is an even function of z, we obtain

$$\frac{\partial^2 \overline{\phi}}{\partial t^2} - a^2(1-\overline{\xi^2}) \frac{\partial^2 \overline{\phi}}{\partial x^2} = ab_0 \lambda \frac{\partial^3 \overline{\phi}}{\partial t\, \partial x^2} - b_1 \lambda^2 \frac{\partial^4 \overline{\phi}}{\partial t^2\, \partial x^2}, \tag{4.3}$$

where

$$b_0 = \int_0^{\infty} R(z)\, dz, \quad b_1 = \int_0^{\infty} zR(z)\, dz,$$

are both independent of λ.

Equation (4.3) illustrates several important points of theory. The first is that the wave speed of the mean wave, viz. $a(1-\overline{\xi^2})^{\frac{1}{2}}$ is *less* than the wave speed in a uniform string; and, secondly, that the terms on the right of (4.3) vanish when the random fluctuations on the string have zero correlation length. Now, it is precisely these terms which are responsible for the change in form of the mean wave profile due to scattering. A vanishing correlation length really means that on an average the effects of scattering of the mean field by different elements ξ cancel each other.

A detailed analysis of the terms on the right of (4.3) reveals that the first term behaves as a sink of mean field energy, which is fed into the random field ϕ'. The remaining term, however, merely serves to *disperse* the harmonic components of the mean wave field without loss of energy. Further, the coefficients b_0 and b_1 can in principle vanish independently, even when $\lambda \neq 0$. Hence, we see that the size of the integral scale of the density fluctuations, b_0, determines the degree of 'scrambling' of the mean field due to differential energy losses of its harmonic components to the random field, whereas the first integral moment, b_1, determines the degree of scrambling due to the dispersion of these harmonic components.

In order to be able to make rather more precise statements about the propagation of the mean field we shall assume that the correlation function R has a Gaussian form:

$$R(\eta) = \overline{\xi^2}\exp(-\eta^2/\lambda^2). \tag{4.4}$$

It is now readily shown that

$$b_0 = \frac{\sqrt{\pi}}{2}\overline{\xi^2} \quad \text{and} \quad b_1 = \tfrac{1}{2}\overline{\xi^2}, \tag{4.5}$$

but we shall not dwell on this approach. Indeed, it is more instructive to determine the *dispersion relation* of (3.10) governing the propagation of elementary wave packets. Having done this, further approximate forms of the mean wave equation are easily obtained.

First, however, let us define the Fourier transform $F(k,\omega)$ of a function $F(x,t)$ by means of the reciprocal relations

$$\left.\begin{aligned} F(k,\omega) &= \frac{1}{(2\pi)^2}\int\!\!\int_{-\infty}^{\infty} F(x,t)\exp[-i(kx-\omega t)]\,dx\,dt, \\ F(x,t) &= \int\!\!\int_{-\infty}^{\infty} F(k,\omega)\exp[i(kx-\omega t)]\,dk\,d\omega. \end{aligned}\right\} \tag{4.6}$$

Next, take the Fourier transform of (3.10) and then divide through by $\overline{\phi}(k,\omega)$:

$$a^2(1-\overline{\xi^2})k^2-\omega^2 = \frac{a}{2}\int_{-\infty}^{\infty} i\omega k^2 R(\eta)\exp\left[i\left(\frac{\omega|\eta|}{a}-k\eta\right)\right]d\eta. \tag{4.7}$$

Hence, when $R(\eta)$ has the Gaussian form (4.4), we obtain for the general binary collision dispersion relation

$$a^2(1-\overline{\xi^2})k^2-\omega^2 = \frac{a\lambda\overline{\xi^2}}{4}\omega k^2\left\{Z\left[\frac{\lambda}{2}\left(\frac{\omega}{a}+k\right)\right]+Z\left[\frac{\lambda}{2}\left(\frac{\omega}{a}-k\right)\right]\right\}, \tag{4.8}$$

where $Z(\zeta)$ is the so-called *plasma dispersion function*, which has been discussed and extensively tabulated by Fried & Conte (1961). It is defined by

$$Z(\zeta) = 2ie^{-\zeta^2}\int_{-\infty}^{i\zeta} e^{-\tau^2}d\tau. \tag{4.9}$$

Alternatively, we have the power series expansion,

$$Z(\zeta) = i\sqrt{\pi}\, e^{-\zeta^2} - \zeta \sum_{n=0}^{\infty} \frac{(-\zeta^2)^n \sqrt{\pi}}{(n+\frac{1}{2})!}, \tag{4.10}$$

and the asymptotic form

$$Z(\zeta) = i\sqrt{\pi}\sigma\, e^{-\zeta^2} - \sum_{n=0}^{\infty} \zeta^{-(2n+1)} \frac{(n-\frac{1}{2})!}{\sqrt{\pi}}, \tag{4.11}$$

where $\sigma = 0, 1, 2$ according as $\mathrm{Im}(\zeta)$ is greater than, equal to, or less than zero, respectively.

These expansions may be used to obtain limiting forms for (4.8) when the wavelength becomes large or small compared with the correlation length λ. First let us consider the long wavelength limit.

(i) *The long wavelength limit*

This means that we are considering the case

$$\lambda k \ll 1, \quad \lambda\omega/a \ll 1. \tag{4.12}$$

Hence, the arguments of the dispersion functions in (4.8) are both small, and we may therefore use the power series (4.10) to obtain the long wave approximation,

$$a^2(1-\overline{\xi^2})\,k^2 - \omega^2 = \frac{i\sqrt{\pi}}{2}\lambda a \overline{\xi^2}\omega k^2 - \frac{\lambda^2 \overline{\xi^2}}{2}\omega^2 k^2. \tag{4.13}$$

But this is simply the dispersion relation of the long wavelength approximation to the mean field equation, (4.3), already derived. It may be used to derive the frequency correction due to the presence of the random inhomogeneities. The unperturbed frequency ω_0 is given by

$$\omega_0 = \pm ak, \tag{4.14}$$

The first correction is $O(\overline{\xi^2})$, and is obtained by substituting $\omega = \pm ak$ in the right-hand side of (4.13). The solution of the resulting quadratic in ω may then be expanded in a power series in $\overline{\xi^2}$ to yield the first correction, viz.

$$\omega = \pm ak\left\{1 - \frac{\overline{\xi^2}}{2} + \frac{\lambda^2\overline{\xi^2}}{2}k^2\right\} - \frac{i\sqrt{\pi}}{4}\overline{\xi^2} a\lambda k^2. \tag{4.15}$$

This result illustrates the reduction in the wave speed and the dispersive effect of a finite correlation length due to the presence of inhomogeneities. The *negative* complex part of ω represents the *damping* of the mean field (cf. the definition (4.6)), i.e. the generation of random waves at the expense of mean field energy.

The long wavelength limit considered above is likely to be of some practical interest. For completeness of illustration, however, we shall now consider the perhaps less important short wavelength limit.

(ii) *Short wavelength limit*

This case requires

$$\lambda k \gg 1, \quad \lambda\omega/a \gg 1. \tag{4.16}$$

Carrying through the procedure which led to (4.15) we obtain (using the asymptotic expansion (4.11)) the following correction to the frequency:

$$\omega = \pm ak(1 - \tfrac{3}{8}\overline{\xi^2}) - \frac{i\sqrt{\pi}}{8}\overline{\xi^2} a\lambda k^2. \tag{4.17}$$

We may extend this result somewhat by recognizing that, correct to terms of $O(\overline{\xi^2})$, these frequencies are the roots of the dispersion equation,

$$\omega^2 + \frac{i\sqrt{\pi}}{4} a\lambda\overline{\xi^2}\omega k^2 - \left(1 - \frac{3\overline{\xi^2}}{4}\right) a^2 k^2 = 0, \tag{4.18}$$

which leads to the following short wavelength approximation to the mean field equation:

$$\frac{\partial^2\overline{\phi}}{\partial t^2} - a^2(1 - \tfrac{3}{4}\overline{\xi^2}) \frac{\partial^2\overline{\phi}}{\partial x^2} = \frac{\sqrt{\pi}}{4} \lambda a \overline{\xi^2} \frac{\partial^3\overline{\phi}}{\partial t\, \partial x^2}. \tag{4.19}$$

Thus, we have seen that in the case of long and short waves limiting forms of the dispersion equation (4.8) may be obtained. These enable us to formulate approximate equations governing the mean field in their respective limits. Further, they illustrate the damping of the mean field due to scattering by inhomogeneities, the damping rate for long waves being precisely twice that for short waves.

Physically, we anticipate a gradual transition between these two extremes, and that in all the intermediate cases the roots of the dispersion equation (4.8) will have *negative* imaginary parts. To show that this is indeed the case, we shall conclude this discussion of the stretched string by proving that all the roots of the more general dispersion relation (4.7) must lie in the lower half of the complex ω-plane.

To do this, consider, for each fixed real wave-number k, the analytic function of ω defined by

$$z(\omega) = \omega^2 - a^2(1 - \overline{\xi^2})\, k^2 + \frac{ia\omega k^2}{2} \int_{-\infty}^{\infty} R(\eta) \exp\left[i\left(\frac{\omega|\eta|}{a} - k\eta\right)\right] d\eta. \tag{4.20}$$

The *eigenfrequencies* of the dispersion relation (4.7) are given by the roots of

$$z(\omega) = 0. \tag{4.21}$$

The function $z(\omega)$ is *regular* in the upper complex ω-plane, and, for large ω, is given by

$$z(\omega) = \omega^2 - a^2k^2 + O(1/\omega). \tag{4.22}$$

When ω is real, remembering that $R(\eta)$ is an *even* function of η, it follows that

$$\operatorname{Im}(z) = \frac{\pi a\omega k^2}{2}\left\{\Phi\left(\frac{\omega}{a} - k\right) + \Phi\left(\frac{\omega}{a} + k\right)\right\}, \tag{4.23}$$

where Φ is the Fourier transform of the correlation function $R(\eta)$, and is therefore *non-negative*. Hence, $\operatorname{Im}(z)$ is an *odd* function of real ω. Also, for real ω,

$$\operatorname{Re}(z) = \omega^2 - a^2k^2(1 - \overline{\xi^2}) - a^2\omega k^2 \int_0^{\infty} R(\eta) \cos k\eta \sin(\omega\eta/a)\, d\eta \tag{4.24}$$

is an *even* function of ω.

Now, consider a closed contour in the complex ω-plane (figure 1), consisting of the real interval $(-L, L)$ and the semicircle C of radius L in the upper half-plane. Since $z(\omega)$ is regular in the upper half plane, it also has no zeros there if, for arbitrary large L, arg $z(\omega)$ returns to its initial value when ω moves once around the contour (Titchmarsh 1960, p. 116).

To prove that $\arg z$ is single-valued on the contour, we construct the image contour in the complex z-plane. (The 'Nyquist diagram', figure 2.)

(*a*) Provided L is sufficiently large the semicircle C is mapped onto a circle a in the z-plane, traversed once in the anti-clockwise sense as ω, moves from $+L$ to $-L$ along C.

(*b*) The interval $0 < \omega \leqslant L$ is mapped onto a curve b above the real axis, since (4.23) is positive, traversed in the direction indicated in figure 2.

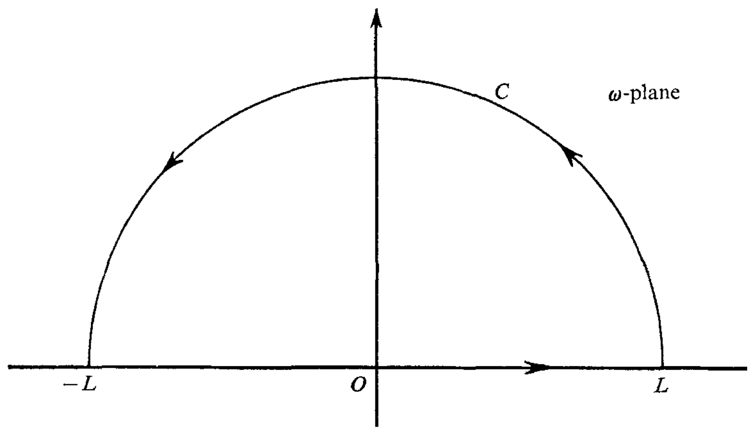

FIGURE 1. The contour described by ω in the complex ω-plane.

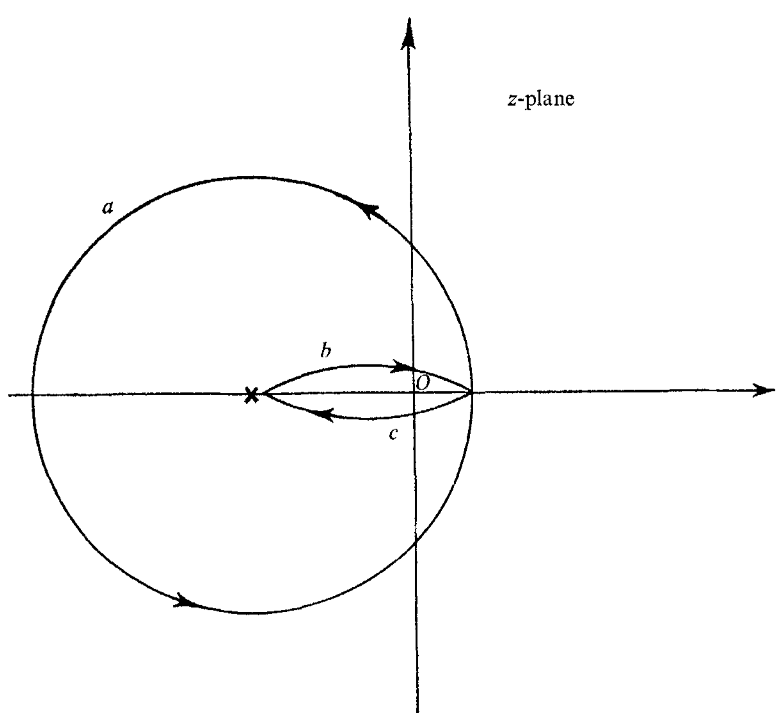

FIGURE 2. The image contour in the complex z-plane. (The 'Nyquist diagram'.)

(*c*) The interval $-L \leqslant \omega < 0$ is mapped below the real axis onto c.

(*d*) The point $\omega = 0$ is mapped onto the negative real axis.

It now follows that the image contour does not enclose the origin O of the z-plane, so that arg z is single-valued on the contour. This completes the proof.

5. Wave propagation in an arbitrary random dispersive or non-dispersive medium

We now consider the general theory of wave propagation through a medium whose random inhomogeneities are confined to small deviations from the mean. Let L be a linear wave operator such that, in the absence of inhomogeneities, the waves ϕ satisfy the equation,

$$L\phi = 0. \tag{5.1}$$

When random inhomogeneities are present in the medium, let equation (5.1) take the modified form,

$$L\phi = G\phi, \tag{5.2}$$

where G is a random linear operator.

Normal methods of treating the scattering problem defined by (5.2) involve a decomposition of the wave field into an incident field ϕ_I together with a scattered ϕ_S. In practice, one has then to make an assumption regarding the relative magnitudes of these fields, which, under favourable circumstances, enables one to adopt the Born approximation. Here the scattered field is given by the particular integral of the equation,

$$L\phi_S = G\phi_I. \tag{5.3}$$

This is the approximation used by Lighthill (1953) in his treatment of the scattering of sound by turbulence, and more recently by Crow (1969), who considered the problem of sonic boom 'spikes'. Having determined ϕ_S from (5.3) one can then go on to compute the rate of decay of the incident field, ϕ_I, due to the scattering.

The validity of the Born approximation, when the scattering takes place over a large region of space, may be criticized for at least two reasons. The first is that, in using it to calculate the rate of decay of the incident wave, we are in fact trying to calculate the decay rate of the *coherent* part of the wave field. However, because of the random 'buffeting' experienced during its passage through the medium, it is not normally valid to assume that this wave propagates at the velocity it would have if it were moving in the homogeneous, non-random medium (cf. (4.15), (4.17)). Secondly, and more significantly, the assumption that all the energy scattered out of the coherent field according to (5.3) constitutes a loss of coherent field energy is a poor approximation to the actual state of affairs.

To discuss these points in rather more detail, we adopt the notation of § 2, which represents the wave field in the random medium by

$$\phi = \overline{\phi} + \phi', \tag{5.4}$$

where $\overline{\phi}$ denotes the mean, or coherent, component of the field, in the sense of an ensemble average. Then ϕ' represents the fluctuations of the actual field about this mean in any particular realization.

To obtain the equation governing the evolution of the mean field, as in §2, we take the ensemble average (denoted by an over-bar) of (5.2). Note, however, that

$$G = \bar{G} + G', \quad \text{with} \quad \overline{G'} = 0.$$

Hence, $\overline{G\phi} = \bar{G}\bar{\phi} + \overline{G'\phi'}$, and the equation for the mean field becomes

$$\mathscr{L}\bar{\phi} = \overline{G'\phi'}, \tag{5.5}$$

where

$$\mathscr{L} = L - \bar{G}. \tag{5.6}$$

Next, the equation for ϕ' is obtained by subtracting (5.5) from the full equation (5.2):

$$\mathscr{L}\phi' = G'\bar{\phi} + \{G'\phi' - \overline{G'\phi'}\}. \tag{5.7}$$

The second term on the right-hand side of (5.7) may be written in the compact form,

$$\{G' - \overline{G'}\}\,\phi',$$

where the operator $\overline{G'}$ is now defined by

$$\overline{G'}.\phi' = \overline{G'\phi'}.$$

Now, assume that it is permissible to solve (5.7) by iteration, the first approximation to the scattered field being given by

$$\phi' = \mathscr{L}^{-1}G'\bar{\phi}, \tag{5.8}$$

where $\mathscr{L}^{-1}$ is the Green's function operator inverse to $\mathscr{L}$. Compare this with (3.5), where $\mathscr{L}^{-1}$ is the retarded potential integral. Then, formally, we have

$$\begin{aligned}\phi' &= \mathscr{L}^{-1}G'\bar{\phi} + \{\mathscr{L}^{-1}G' - \mathscr{L}^{-1}\overline{G'}\}\,\mathscr{L}^{-1}G'\bar{\phi} + \{\mathscr{L}^{-1}G' - \mathscr{L}^{-1}\overline{G'}\}^2\,\mathscr{L}^{-1}G'\bar{\phi} + \dots \\ &= \sum_{n=0}^{\infty}\{\mathscr{L}^{-1}G' - \mathscr{L}^{-1}\overline{G'}\}^n\,\mathscr{L}^{-1}G'\bar{\phi}.\end{aligned} \tag{5.9}$$

This represents a multiple scattering solution for ϕ'.

To interpret this in more detail, in the manner of §2, consider the random wave scattered from the point P in figure 3. In that figure wavy directed lines represent scattered *random* wave packets of zero mean, whereas full directed lines represent wave packet components of the mean field.

The zeroth-order term on the right of (5.9), viz. $\mathscr{L}^{-1}G'\bar{\phi}$, represents the contribution to the random field scattered at points P directly out of the mean field, which is represented by the full line entering the vertex at P. Such scattering occurs at all points of the random medium. That random wave which is scattered from the point Q, say, may be supposed to experience second scattering at P, as illustrated. This is represented by the *binary collision* term on the right of (5.9) corresponding to $n = 1$, viz.

$$\{\mathscr{L}^{-1}G' - \mathscr{L}^{-1}\overline{G'}\}\,\mathscr{L}^{-1}G'\bar{\phi} \equiv \{\mathscr{L}^{-1}G'\mathscr{L}^{-1}G' - \overline{\mathscr{L}^{-1}G'\mathscr{L}^{-1}G'}\}\,\bar{\phi}, \tag{5.10}$$

which clearly has zero mean. Now $\overline{\mathscr{L}^{-1}G'\mathscr{L}^{-1}G'}$ involves the random operator G' twice, evaluated at the different points P and Q, and so is dependent on the *correlation* function of the fluctuations in the random medium. Thus, the binary collision expression (5.10) may be interpreted as the difference between the *total* second scattered field from P and the second scattered field of those waves which were initially scattered directly out of the mean field at points lying within a correlation length λ, say, of P. Hence, in effect, the second scattering term (5.10) would tend to involve points Q, whose distance from P exceeds this correlation length, and may in certain circumstances be regarded as a weak interaction. Similar arguments may be applied to discuss the higher-order random collision terms in (5.9), the total random scattered field from P being obtained by summation over all these contributions.

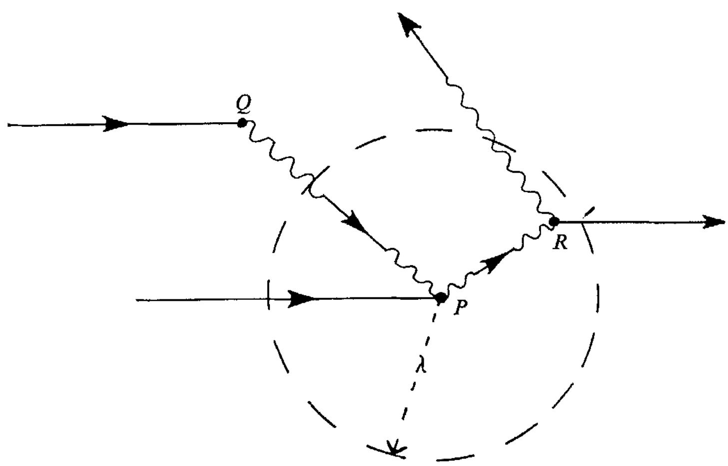

FIGURE 3. Binary interactions. Wavy lines represent random scattered wave packets of zero mean; full lines represent wave-packet components of the coherent field.

Evidently, the random wave packet scattered from P will eventually be re-scattered at a point R, say. Such a collision will produce a scattered random field represented by the wavy line leaving the vertex R in figure 3, and is obtained from (5.9) by an application of the operator $\{\mathscr{L}^{-1}G' - \mathscr{L}^{-1}\overline{G'}\}$, together with a mean field component. The latter represents the energy scattered *back* into the coherent wave field, and will be significant, provided that the fluctuations of medium at P and R are well correlated. In other words, provided that the distance between P and R is less than the correlation length λ. It is apparent, therefore, that modification of the mean field is an essentially *local* phenomenon. The effect at R is dependent only upon interactions with points P which lie within a correlation sphere centred on R.

Finally, the equation which describes this modification of the coherent field is obtained by formal substitution of (5.9) into the mean field equation (5.5):

$$\mathscr{L}\bar{\phi} = \overline{G'} \sum_{n=0}^{\infty} \{\mathscr{L}^{-1}G' - \mathscr{L}^{-1}\overline{G'}\}^n \mathscr{L}^{-1}G'\bar{\phi}. \tag{5.11}$$

This equation is to be solved subject to boundary conditions which specify the initial form of the coherent wave $\bar{\phi}$. The solution is then substituted into (5.9), from which the statistical properties of the random field may be computed. In practice, however, (5.11) is too complicated to be treated in its full generality. In the case of the stretched string equation considered above, only the zeroth-order term on the right, $\overline{G'\mathscr{L}^{-1}G'}\bar{\phi}$, was retained to give the *binary collision* approximation (3.6).

Detailed properties of the mean wave field may be derived from (5.11) by taking Fourier transforms (cf. §4). This leads to the dispersion relation governing the propagation of elementary wave packets. When the operator $\mathscr{L}$ is wave bearing *without* dissipation, the effect of scattering by inhomogeneities will manifest itself by the appearance of a negative imaginary part in the frequency ω (using the definition (4.6)). Damping which is already present in the operator $\mathscr{L}$ will otherwise be enhanced. For example, sound waves propagating through turbulence will tend to lose energy not only because of the effects of ordinary viscous dissipation, but also because of the appearance of an *eddy viscosity* term in the mean wave equation.

6. The effect of rough boundaries

It is convenient to conclude this discussion of wave propagation in random media by illustrating how the above theory may be extended to cover the case in which the random fluctuations occur in the boundary conditions. Surface waves propagating over a stretch of water with a rough, or 'pebbly', bottom afford an interesting application of the theory. The presence of the pebbly bottom serves to scatter energy out of the mean wave field, and therefore, from a macroscopic viewpoint, tends to behave as a damping agent. This type of 'cascade' damping might be an important feature in the problem of the tidal bore (Lighthill 1957), where it has long been recognized that there is a need for a damping mechanism additional to turbulent dissipation.

Let us consider the case in which the propagation of disturbances is governed by a *non-random* equation of the form,

$$\mathscr{L}\phi = 0, \tag{6.1}$$

in a region D, say. This means that both the mean field $\bar{\phi}$ and the random fluctuations ϕ' satisfy the *same* propagation equation. However, ϕ' is generated by the interaction between the mean field and the rough boundary. Let us suppose that this coupling is described by a boundary condition of the form,

$$\mathscr{G}\phi = 0, \tag{6.2}$$

on part of the boundary, ∂D say, of D. In (6.2) $\mathscr{G}$ is a random linear operator. For example, in the water wave case mentioned above, if the bottom be represented by

$$z = \zeta(x, y), \tag{6.3}$$

where $\zeta(x, y)$ is a random function of its arguments, and if ϕ denotes the velocity potential, then the boundary condition (6.2) would have the form,

$$\left\{\frac{\partial}{\partial z} - \frac{\partial \zeta}{\partial x}.\frac{\partial}{\partial x} - \frac{\partial \zeta}{\partial y}.\frac{\partial}{\partial y}\right\}\phi = 0, \tag{6.4}$$

on the bottom.

First, take the ensemble average of (6.2):

$$\overline{\mathscr{G}}.\overline{\phi}+\overline{\mathscr{G}'\phi'} = 0. \tag{6.5}$$

Subtract this from (6.2) to obtain,

$$\overline{\mathscr{G}}\phi' = -\mathscr{G}'\overline{\phi}-[\mathscr{G}'\phi'-\overline{\mathscr{G}'\phi'}]. \tag{6.6}$$

The method of solution is as follows. Assume that $\overline{\phi}$ is known. Then determine ϕ' by the method of successive approximations by means of the approximating sequence ϕ'_n satisfying

$$\mathscr{L}\phi'_n = 0 \quad \text{in} \quad D, \tag{6.7}$$

and

$$\left.\begin{aligned}\overline{\mathscr{G}}\phi'_n &= -\mathscr{G}'\overline{\phi}-[\mathscr{G}'\phi'_{n-1}-\overline{\mathscr{G}'\phi'_{n-1}}] \quad (n>1),\\ \overline{\mathscr{G}}\phi'_1 &= -\mathscr{G}'\overline{\phi},\end{aligned}\right\} \tag{6.8}$$

on ∂D.

Having determined ϕ' to the desired degree of approximation, the solution is substituted into condition (6.5) to give the mean field boundary condition, in terms of $\overline{\phi}$ alone, on ∂D. This may be used in conjunction with

$$\mathscr{L}\overline{\phi} = 0 \tag{6.9}$$

to derive the properties of the mean field.

In the case of water waves, one also has to satisfy certain non-random boundary conditions on the free surface. By using these in conjunction with (6.5) and (6.9) one may derive the dispersion equation governing the propagation of mean field wave packets. This calculation is currently being undertaken and will appear in a future publication.

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REFERENCES

CHAPMAN, S. & COWLING, T. G. 1939 *The Mathematical Theory of Non-Uniform Gases.* Cambridge University Press.

CHERNOV, L. A. 1960 *Wave Propagation in a Random Medium.* New York: McGraw-Hill.

CROW, S. C. 1969 *J. Fluid Mech.* **37**, 529.

FRIED, B. D. & CONTE, S. D. 1961 *The Plasma Dispersion Function.* New York: Academic.

KARAL, F. C. & KELLER, J. B. 1964 *J. Math. Phys.* **5**, 537.

KELLER, J. B. 1964 *Proc. Symp. appl. Math.* **16**, 145. Providence, R.I.: Amer. Math. Soc.

LIGHTHILL, M. J. 1953 *Proc. Camb. Phil. Soc.* **49**, 531.

LIGHTHILL, M. J. 1957 *Naval Hydrodynamics*, ch. 2. Washington, D.C.: Publ. 515, Nat. Acad. Sci., Nat. Res. Coun.

TATARSKI, V. I. 1961 *Wave Propagation in a Turbulent Medium.* New York: McGraw-Hill.

TITCHMARSH, E. C. 1960 *The Theory of Functions* (2nd edn.). Oxford University Press.