On the Influence of Viscosity on Waves and Currents. *By* S. S. Hough, M.A., Isaac Newton Student in the University of Cambridge. Received December 9th, 1896. Read December 10th, 1896.

In the following paper my aim has been to present the solution of certain problems illustrative of the effects of viscosity on the motions of the sea. With this end in view, I have therefore had no hesitation in introducing such approximations as would be applicable in the case presented by nature. The loss of generality resulting from these approximations will be compensated for by corresponding simplicity of the analysis, while the results are more readily intelligible, in that they admit of being expressed in a form which may be at once converted into numbers.

The motions dealt with may be divided into the following classes: (1) large-scale currents, of which the most familiar illustration is to be found in the circulatory system of the North Atlantic Ocean; (2) tidal oscillations, either of the nature of stationary vibrations or consisting of progressive wave-motions with a wave-length large in comparison with the depth; and (3) deep-sea waves, in which the wave-length is very short compared with the depth. Each of these motions, if once started and then left free from external maintaining cause, would slowly subside under the influence of dissipative forces, and my object has been to evaluate for the various types of motion the modulus of decay, that is, the period in which the velocities in the current motions and the amplitudes of vibration in the periodic motions would be reduced in the ratio $1 : e$, due to the combined action of internal viscosity and friction at the ocean bed. It might be anticipated *à priori*, and it is established in the present paper, that in the two former classes of motion the friction of the ocean bed is by far the more important influence in destroying the motion, whereas in the case of short waves at the surface of deep water the friction of the ocean bed is of no moment in comparison with internal viscosity.

To deal with bottom-friction, it has been necessary to introduce some hypothesis as to the nature of the action between the water and the solid bed with which it is in contact. The most probable hypothesis, and that which I have adopted, is that no slipping at all is possible; but, if this be not the true law, the effects stated may at least be regarded as the maximum results which could be produced with the assigned degree of internal viscosity, and the moduli of decay obtained may be treated as inferior limits to the moduli of decay which would appear from a more general supposition as to the action in question. As it is the large order of magnitude of these moduli, rather than their actual numerical values, on which the physical application of the results turns, the practical value of these results is therefore by no means diminished.

I hope to enter more fully into the physical bearings of the problems solved in a later paper; but it will not be out of place to make the following remarks as illustrative of my purpose. The existence of ocean currents has been variously attributed to the tendency of the winds in certain regions (*e.g.*, the trade-wind regions) to set in particular directions, and to differences of density arising from differences of temperature, salinity, &c. The opponents of each theory have urged that the energy derivable from these sources is totally inadequate to *generate* the large motions known to exist in the ocean. Were the ocean free from viscosity, however, and initially at rest, it follows that currents arising from the sources in question must inevitably be set up, and, the causes being continuous in their action, it only requires lapse of time for the motions to become sensible or even very large. The same will be true when there is a small amount of viscosity; but in the latter case a limit will ultimately be attained when the rate at which currents are generated by the causes in question is on the average equal to that at which they are destroyed by friction. After this state has been attained the motion will remain steady, but we see that no estimate of the amounts of currents that could be set up can be obtained from considerations of the amounts of energy involved in the sources apart from considerations of the rate at which energy is dissipated by friction. The extremely large values we have obtained for the moduli of decay of the current-motions imply that energy is dissipated very slowly, and thus, though no doubt an extremely long time would be necessary for the currents, starting from rest, to acquire their present magnitude, there appears no difficulty in supposing that the causes suggested are quite adequate to *maintain* these motions when once set up.

As regards the tidal oscillations it appears that in a system comparable with the actual Earth the moduli of decay of the principal free oscillations will be very large compared with the periods of the disturbing forces due to the Sun and Moon. Hence it follows that the tidal forces will produce their full dynamical effect, and that the conclusions derived from an equilibrium theory, except in so far as they coincide with those derived from a dynamical theory, are without foundation. In the case of the long-period tides, which have usually been supposed to follow the equilibrium law in consequence of viscosity, it is known that the equilibrium theory and the dynamical theory lead to different results,* and therefore it follows that an equilibrium theory must be at fault even for such tides as the solar semi-annual tide.

1. *On the Rate of Decay of Current-Motions.*

The equation of motion of a viscous liquid moving everywhere parallel to the axis of x with velocity u, and subject to uniform gravity parallel to the axis of z, is

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2}, \tag{1}$$

where ν is the kinematic coefficient of viscosity.

If we suppose $u \propto e^{-at}$, and put $k^2 = a/\nu$, this equation becomes

$$\frac{\partial^2 u}{\partial z^2} + k^2 u = 0;$$

from which we determine u as a function of z in the form

$$u = A \cos kz + B \sin kz,$$

where A, B are functions of the time alone.

Expressing the time factor e^{-at}, we obtain as the general solution of (1) of the assumed type

$$u = (A \cos kz + B \sin kz)\, e^{-at}, \tag{2}$$

where A, B are now arbitrary constants to be determined from the boundary-conditions.

Let $z = 0$, $z = h$ be the equations to the ocean bed and to the free surface respectively.

At the former we suppose that no slipping is possible, so that $u = 0$ when $z = 0$; this leads to

$$A = 0. \tag{3}$$

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* Lamb, *Hydrodynamics*, § 210.

At the free surface the condition to be satisfied is that the tangential stress must vanish; or that $\frac{\partial u}{\partial z} = 0$ when $z = h$. Therefore

$$kB \cos kh = 0. \tag{4}$$

If k or B vanish, u will be everywhere zero, and no motion will be involved. Hence the admissible values of k, which determine the various types of free "laminar" motion of which the system is capable, are the roots of the equation

$$\cos kh = 0;$$

the roots of this equation are of the form $(2n+1)\,\pi/2h$, where n is integral, and therefore the appropriate values of a are found by giving n integral values in the formula

$$a = \frac{(2n+1)^2 \pi^2 \nu}{4h^2}.^*$$

A particular solution of (1) satisfying the assigned boundary-conditions is therefore

$$u = Be^{-\frac{(2n+1)^2\pi^2}{4h^2}\nu t} \sin \frac{(2n+1)\,\pi z}{2h}, \tag{5}$$

where B is an arbitrary constant.

To determine the motion resulting from assigned initial circumstances we may express the initial velocity u_0 by means of a Fourier's series in the form

$$u_0 = \sum_{n=0}^{n=\infty} A_n \sin \frac{(2n+1)\,\pi z}{2h}. \tag{6}$$

The subsequent motion will then be given by

$$u = \sum_{n=0}^{n=\infty} A_n e^{-\frac{(2n+1)^2\pi^2\nu t}{4h^2}} \sin \frac{(2n+1)\,\pi z}{2h}. \tag{7}$$

For example, if the velocity is initially constant and equal to u_0, the series (6) becomes

$$u_0 = \frac{4u_0}{\pi} \Sigma \frac{1}{2n+1} \sin \frac{(2n+1)\,\pi z}{2h},$$

and the motion at time t is given by

$$u = \frac{4u_0}{\pi} \sum_{n=0}^{n=\infty} \frac{1}{2n+1} e^{-\frac{(2n+1)^2\pi^2}{4h^2}\nu t} \sin \frac{(2n+1)\,\pi z}{2h}.$$

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* *Cf.* Helmholtz, *Werke*, Vol. III., p. 289.

After a sufficiently long interval all the types of motion except that corresponding to $n = 0$ may be supposed to have subsided, and the ultimate state of motion will be expressed by

$$u = \frac{4u_0}{\pi} e^{-\frac{\pi^2 \nu t}{4h^2}} \sin \frac{\pi z}{2h}.$$

The moduli of decay of the various types of free motion are

$$\frac{4h^2}{\pi^2 \nu}, \; \frac{4h^2}{3^2 \pi^2 \nu}, \; \dots \; \frac{4h^2}{(2n+1)^2 \pi^2 \nu}, \; \dots .$$

For water the value of ν referred to C.G.S. units is about ·0178, and, if the depth be taken as 1 metre, we find that the modulus of decay for that type of motion which subsides least rapidly is about 63 hours.

For depths at all comparable with the depth of the ocean, the moduli of decay will be extremely large. Thus, if we take the depth as 4,000 metres, which is probably less than the true mean depth, we find a modulus of decay for the type $n = 0$ slightly exceeding 100,000 years, while, even for the type $n = 100$, the modulus will be nearly 3 years.

2. *Dynamical Equations for Wave-Motions in Two Dimensions.*

The equations of motion of a viscous liquid oscillating in two dimensions, under uniform gravity parallel to the axis of z, can be expressed in the form

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial \psi}{\partial x} + \nu \nabla^2 u, \\ \frac{\partial w}{\partial t} &= \frac{\partial \psi}{\partial z} + \nu \nabla^2 w, \\ \frac{\partial u}{\partial x} &+ \frac{\partial w}{\partial z} = 0, \end{aligned}\right\} \tag{8}$$

where u, w denote the velocity-components parallel to the axes of x, z respectively; ν is the kinematic coefficient of viscosity, and

$$\psi = \text{const.} - gz - p/\rho, \tag{9}$$

g denoting the acceleration due to gravity, p the pressure, and ρ the density.

To obtain solutions of these equations, suppose that u, w, ψ are each proportional to $e^{im(x-Vt)}$; they then become

$$\left.\begin{aligned} \left\{\frac{\partial^2}{\partial z^2}-m^2+imV/\nu\right\} u &= -\frac{im}{\nu}\psi, \\ \left\{\frac{\partial^2}{\partial z^2}-m^2+imV/\nu\right\} w &= -\frac{1}{\nu}\frac{\partial\psi}{\partial z}, \\ imu+\frac{\partial w}{\partial z} &= 0; \end{aligned}\right\} \quad (10)$$

whence, if we eliminate u, w, we obtain

$$\left(\frac{\partial^2}{\partial z^2}-m^2\right)\psi = 0. \quad (11)$$

The solution of (11) is

$$\psi = Ae^{mz}+Be^{-mz},$$

where A, B are functions of x and t.

Introducing this value of ψ into the right-hand members of (10), we find at once the particular integrals

$$\left.\begin{aligned} u &= -\frac{\psi}{V} &&= -\frac{1}{V}\left\{Ae^{mz}+Be^{-mz}\right\}, \\ w &= -\frac{1}{imV}\frac{\partial\psi}{\partial z} &&= \frac{i}{V}\left\{Ae^{mz}-Be^{-mz}\right\}. \end{aligned}\right\} \quad (12)$$

To these must be added complementary functions which satisfy the equations

$$\left.\begin{aligned} \left(\frac{\partial^2}{\partial z^2}-k^2\right) u &= 0, \\ \left(\frac{\partial^2}{\partial z^2}-k^2\right) w &= 0, \\ imu+\frac{\partial w}{\partial z} &= 0, \end{aligned}\right\} \quad (13)$$

where $$k^2 = m^2-imV/\nu. \quad (14)$$

From the second of equations (13), we have

$$w = Ce^{kz}+De^{-kz}, \quad (15)$$

and therefore, by means of the third,

$$u = \frac{ik}{m}(Ce^{kz} - De^{-kz}). \tag{16}$$

Combining these complementary functions with the particular integrals (12), and expressing the factor $e^{im(x-Vt)}$, we obtain as the general solution of the equations of motion of the assumed type

$$\left.\begin{aligned} \psi &= (Ae^{mz} + Be^{-mz})\, e^{im(x-Vt)}, \\ u &= \left[-\frac{1}{V}(Ae^{mz} + Be^{-mz}) + \frac{ik}{m}(Ce^{kz} - De^{-kz})\right] e^{im(x-Vt)}, \\ w &= \left[\frac{i}{V}(Ae^{mz} - Be^{-mz}) + (Ce^{kz} + De^{-kz})\right] e^{im(x-Vt)}, \end{aligned}\right\} \tag{17}$$

where A, B, C, D are arbitrary constants, to be determined by the boundary-conditions.

If F, H denote the components of traction parallel to the axes of x, z across any plane $z = \text{const.}$, we have

$$F = \rho\nu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right); \quad H = -p + 2\rho\nu\frac{\partial w}{\partial z};$$

and, therefore,

$$F/\rho = \nu\left[-\frac{2m}{V}(Ae^{mz} - Be^{-mz}) + \frac{i(k^2+m^2)}{m}(Ce^{kz} + De^{-kz})\right] e^{im(x-Vt)}, \tag{18}$$

$$\begin{aligned} H/\rho &= \psi + gz + \text{const.} \\ &\quad + 2\nu\left[\frac{im}{V}(Ae^{mz} + Be^{-mz}) + k(Ce^{kz} - De^{-kz})\right] e^{im(x-Vt)}, \end{aligned}$$

the latter of which, on introducing the value of ψ from (17) becomes

$$H/\rho = \text{const.} + \left[\left(1 + \frac{2\nu im}{V}\right)(Ae^{mz} + Be^{-mz}) + k\nu\,(Ce^{kz} - De^{-kz})\right] e^{im(x-Vt)}. \tag{19}$$

3. *The Boundary-Conditions.*

Let ζ denote the height of the free surface above the plane $z = h$, and suppose that ζ is expressible in the form

$$ae^{im(x-Vt)}.$$

Then at the surface $z = h$ we must satisfy the kinematical condition

$$w = \frac{\partial \zeta}{\partial t},$$

which requires $\frac{i}{V}(Ae^{mh} - Be^{-mh}) + Ce^{kh} + De^{-kh} = -imVa.$ (20)

Again the stress-conditions at this surface may be expressed by equating the stress across the plane $z = h$ to a normal stress equal to the weight of the harmonic inequalities. This requires $F = 0$, $H = -g\rho\zeta$, when $z = h$. The non-periodic constant on the right of (19) must vanish identically when $z = h$, and we find

$$\left.\begin{aligned} -\frac{2m}{V}(Ae^{mh} - Be^{-mh}) + \frac{i(k^2+m^2)}{m}(Ce^{kh} + De^{-kh}) &= 0, \\ \left(1 + \frac{2i\nu m}{V}\right)(Ae^{mh} + Be^{-mh}) + 2\nu k(Ce^{kh} - De^{-kh}) &= -ga. \end{aligned}\right\} \quad (21)$$

If $z = 0$ be the equation to the bottom, the conditions to be satisfied at this surface will depend on the assumption we make as to the nature of the action between the water and the ocean bed. If we assume that no slipping is possible, we must have $u = 0$, $w = 0$ when $z = 0$, and therefore

$$\left.\begin{aligned} -\frac{1}{V}(A+B) + \frac{ik}{m}(C-D) &= 0, \\ \frac{i}{V}(A-B) + (C+D) &= 0. \end{aligned}\right\} \quad (22a)$$

If, on the other hand, we suppose that the bottom is perfectly smooth, we require $w = 0$, $F = 0$ when $z = 0$; these conditions lead to

$$\left.\begin{aligned} \frac{i}{V}(A-B) + (C+D) &= 0, \\ -\frac{2m}{V}(A-B) + \frac{i(k^2+m^2)}{m}(C+D) &= 0. \end{aligned}\right\} \quad (22b)$$

The elimination of the constants A, B, C, D, a from the equations (20), (21), and (22a) or (22b) will lead to an equation connecting V, the velocity of wave-propagation, with $2\pi/m$, the wave-length. The character of the resulting motion will depend on the nature of the roots of this equation. When ν is absolutely zero, the roots will be of the form

$$V = V_0,$$

where V_0 is a real quantity; in this case we have

$$\zeta = ae^{im(x-V_0t)},$$

or, retaining only the real part,

$$\zeta = a \cos m(x - V_0 t).$$

The motion will therefore consist of a train of simple harmonic waves propagated with velocity V_0 in direction parallel to the axis of x.

On the other hand, if V be purely imaginary, say

$$V = -\frac{i}{m\tau},$$

where τ is real, we have

$$\zeta = ae^{im(x+it/m\tau)} = ae^{-t/\tau}e^{imx},$$

whence, discarding the imaginary part,

$$\zeta = ae^{-t/\tau} \cos mx.$$

The surface will at any instant be of the form of a curve of sines, subsiding without displacement of the nodal lines until it ultimately takes the equilibrium form $\zeta = 0$. This is the case which might be expected to occur for very large values of ν.

Lastly, if V is complex, it may be expressed in the form

$$V_1 - \frac{i}{m\tau},$$

and we shall have

$$\zeta = ae^{-t/\tau} \cos m(x - V_1 t).$$

The motion will then consist of a train of waves of length $2\pi/m$ propagated with velocity V_1, the amplitude of vibration slowly declining and being reduced in the ratio $1 : e$ in a period τ. This is the case which may be expected to occur when the viscosity is very small; further, we may anticipate that V_1 will differ but slightly from V_0, its value when $\nu = 0$, and that τ will be very large, when the value of ν is very small.

We have assumed that the values of τ will be positive, which is a necessary consequence of the stability of the equilibrium in the zero configuration.

4. *Approximate Solution when the Viscosity is Small.*

We propose for the future to confine ourselves to the case where the viscosity is very small; we see from (14) that k will then be a large quantity of the order $\nu^{-\frac{1}{2}}$.

Let k denote that root of (14) which has its real part positive. Then e^{-hk} will be a very small quantity; but from (22*a*) we see that D cannot become large, while from (22*b*) we see that D cannot become large unless at the same time C becomes large of the same order but with opposite sign. In either case we may neglect De^{-hk} in comparison with Ce^{hk}, and therefore the first of equations (21) is approximately equivalent to

$$Ce^{kh} = -\frac{2im^2}{V(k^2+m^2)}(Ae^{mh}-Be^{-mh}). \tag{23}$$

On substituting this value for Ce^{kh} and neglecting De^{-kh} in (20), we obtain

$$Ae^{mh}-Be^{-mh} = -\frac{k^2+m^2}{k^2-m^2}mV^2a; \tag{24}$$

while from the second of equations (21) we obtain, with errors of the order $\nu^{\frac{3}{2}}$ only,

$$Ae^{mh}+Be^{-mh} = -\frac{gaV}{V+2im\nu}. \tag{25}$$

On eliminating a from (24), (25), we have

$$\frac{Ae^{mh}-Be^{-mh}}{Ae^{mh}+Be^{-mh}} = \frac{mV^2}{g}\frac{k^2+m^2}{k^2-m^2}\frac{V+2im\nu}{V}.$$

But, by (14), $\dfrac{k^2+m^2}{k^2-m^2} = \dfrac{2m^2-imV/\nu}{-imV/\nu} = \dfrac{V+2im\nu}{V}$;

whence
$$\frac{Ae^{mh}-Be^{-mh}}{Ae^{mh}+Be^{-mh}} = \frac{mV^2}{g}(1+2im\nu/V)^2. \tag{26}$$

Take first the case where the bottom is perfectly smooth. We then have from (22*b*)

$$A = B, \quad C = -D;$$

and, therefore, from (26)

$$\frac{mV^2}{g}(1+2im\nu/V)^2 = \tanh mh,$$

an equation which may be readily solved by successive approximation. If, as a first approximation, we neglect ν, we find

$$V_0^2 = \frac{g}{m}\tanh mh; \tag{27}$$

thus verifying the well-known formula for the velocity of wave-propagation in a frictionless liquid.* Replacing V by V_0 in the small terms which contain ν, we deduce, as a second approximation,

$$V^2 = \frac{g}{m}\tanh mh\left(1-\frac{4im\nu}{V_0}\right);$$

whence $$V = V_0 - 2im\nu.$$

To the order of approximation considered the velocity of wave-propagation is therefore unaltered by friction, while the modulus of decay is given by the formula

$$\tau = \frac{1}{2m^2\nu} = \frac{\lambda^2}{8\pi^2\nu}, \tag{28}$$

where λ denotes the wave-length. This agrees with the formula given by Prof. Lamb† for the case of waves in deep water. We see now that it holds for waves of any wave-length in water of any depth, provided that the bottom is perfectly smooth and that the internal viscosity is sufficiently small to allow of our approximations.

Dealing next with the case where no slipping is allowed at the bottom, we see from (23) that, since e^{kh} is large, C must be excessively small. Hence the equations (22a) which are applicable under these circumstances take the approximate forms

$$\left.\begin{aligned} A+B &= -\frac{ik}{m}VD,\\ A-B &= iVD, \end{aligned}\right\}$$

from which we deduce

$$\frac{A}{B} = \frac{k-m}{k+m},$$

and hence from (26)

$$\frac{mV^2}{g}(1+2im\nu/V)^2 = \frac{k\sinh mh - m\cosh mh}{k\cosh mh - m\sinh mh}. \tag{29}$$

But from (14) we have, with errors of the orders $\nu^{\frac{3}{2}}$,

$$k = \pm\sqrt{\{-imV/\nu\}} = \pm(1-i)\sqrt{(mV/2\nu)},$$

and, since by hypothesis the real part of k is positive, we must take the upper sign; we therefore find

$$\frac{m}{k} = \frac{(1+i)\sqrt{(m\nu)}}{\sqrt{(2V)}}. \tag{30}$$

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* Lamb, *Hydrodynamics*, p. 372.
† *L.c.*, p. 545; or *Proc. Lond. Math. Soc.*, Vol. XIII., p. 62.

The right-hand member of (29) is approximately equal to

$$\tanh mh\left(1-\frac{m}{k}\coth mh\right)\left(1+\frac{m}{k}\tanh mh+\frac{m^2}{k^2}\tanh^2 mh+\ldots\right)$$

$$=\tanh mh\left(1-\frac{2m}{k}\frac{1}{\sinh 2mh}-\frac{m^2}{k^2}\frac{1}{\cosh^2 mh}+\ldots\right).$$

Hence equation (29) may be written

$$V^2(1+2im\nu/V)^2 = V^2\left(1-\frac{2m}{k\sinh 2mh}\right);$$

or

$$V\ (1+2im\nu/V) = V_0\left(1-\frac{m}{k\sinh 2mh}\right);$$

where we have retained only the most important terms on the right. As a first approximation we find, as before, on omitting all small terms involving ν or $\nu^{\frac{1}{2}}$,

$$V=V_0.$$

Using this value of V in small terms, we obtain as a further approximation

$$V = V_0\left\{1-\frac{(1+i)\sqrt{(m\nu)}}{\sqrt{(2V_0)}\sinh 2mh}\right\}/(1+2im\nu/V_0)$$

$$= V_0\left\{1-\frac{(1+i)\sqrt{(m\nu)}}{\sqrt{(2V_0)}\sinh 2mh}\right\}(1-2im\nu/V_0)$$

$$= V_0\left\{1-\frac{\sqrt{(m\nu)}}{\sqrt{(2V_0)}\sinh 2mh}\right\}-2im\nu-\frac{i\sqrt{(mV_0\nu)}}{\sqrt{2}\sinh 2mh}$$

$$= V_1-\frac{i}{m\tau},$$

where

$$\left.\begin{aligned} V_1 &= V_0\left\{1-\frac{\sqrt{(m\nu)}}{\sqrt{(2V_0)}\sinh 2mh}\right\}, \\ \frac{1}{\tau} &= 2m^2\nu+\frac{\sqrt{(m^3V_0\nu)}}{\sqrt{2}\sinh 2mh}. \end{aligned}\right\} \quad (31)$$

From these equations we see that the velocity of wave-propagation is slightly retarded by friction with the ocean bed. Of the two terms in $1/\tau$ the first alone appears when we neglect bottom-friction. Hence we may attribute these terms to internal viscosity and to bottom-friction respectively. In general for small values of ν the second term, involving $\nu^{\frac{1}{2}}$, will be far more important than the first, which involves only ν; in other words, the friction at the bottom will be far more efficacious than internal viscosity in destroying the motion. But, if the depth be large in comparison with the wave-length, $\sinh 2mh$ will assume a very large value, and therefore the second term of $1/\tau$ becomes relatively unimportant. For waves of short wave-length (in comparison with the depth) the nature of the action at the bottom will be of no account, and we shall obtain the same formula (28) whatever assumption we make as to this action.

For "long waves" we may replace $\tanh mh$ by mh, and $\sinh 2mh$ by $2mh$. The formulæ (27), (31) then give

$$\left.\begin{aligned} V_0^2 &= gh, \\ V_1 &= \sqrt{gh} - \frac{\nu^{\frac{1}{2}} g^{\frac{1}{4}}}{2\sqrt{2}\, m^{\frac{1}{2}} h^{\frac{3}{4}}}, \\ \tau &= \frac{2\sqrt{2}\, h^{\frac{3}{4}}}{m^{\frac{1}{2}} g^{\frac{1}{4}} \nu^{\frac{1}{2}}}. \end{aligned}\right\} \quad (32)$$

The approximations we have used require m to be small compared with V_0/ν and e^{-kh} small compared with unity.

The first condition requires ν to be small compared with V_0/m; the second requires ν to be small compared with $m^2h^2(V_0/m)$.

In the case of deep sea waves, both conditions will be satisfied provided ν is small compared with V_0/m. Taking

$$\nu = \cdot 0178,$$

the value for water, the requisite condition will be satisfied provided the wave-length is large compared with ·04 of a centimetre.

On the other hand, if the waves are of the nature of "long waves," so that mh is small, the two conditions will be satisfied provided ν is small compared with

$$m^2h^2(V_0/m).$$

With the same value of ν this requires the wave-length to be small compared with

$$10^4 h^{\frac{5}{2}} \text{ centimetres},$$

where the depth h is to be expressed in centimetres.

We see, then, that no serious limitations are introduced on the range of applicability of our results, provided we are dealing with a liquid of such a small degree of viscosity as water.

For waves of 100 metres in a depth of 1 metre, the formulæ (32) will give a modulus of decay of about 1 hour 20 minutes, while there will be no sensible retardation in the velocity of wave propagation.

If there were no friction at the bottom, the modulus of decay in this case would be about $2\frac{1}{4}$ years.

Except in the case of deep sea waves, where it is necessary to take into account the tangential forces in the neighbourhood of the surface, we may put

$$Ce^{kh} = 0,$$

and *à fortiori* $C = 0$. Referring back to equations (17), we therefore have, with a high degree of approximation,

$$\psi = (Ae^{mz} + Be^{-mz})\, e^{im(x-Vt)},$$

$$u = \left[-\frac{1}{V}(Ae^{mz} + Be^{-mz}) - \frac{ik}{m} De^{-kz}\right] e^{im(x-Vt)},$$

$$w = \left[\frac{i}{V}(Ae^{mz} - Be^{-mz}) + De^{-kz}\right] e^{im(x-Vt)}.$$

These equations indicate that the motion is the same as if there were no friction, except through the region in the immediate neighbourhood of the bottom over which the term De^{-kz} remains sensible. If there is no friction at the bottom, D will be small of the same order as C, and hence the motion will be sensibly the same throughout as if there were no friction.

5. *Preliminary Analysis applicable to the Case of a Spherical Sheet of Water.*

The differential equations for the three-dimensional oscillations of a mass of viscous liquid can be expressed in the form

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial \psi}{\partial x} + \nu \nabla^2 u, \\ \frac{\partial v}{\partial t} &= \frac{\partial \psi}{\partial y} + \nu \nabla^2 v, \\ \frac{\partial w}{\partial t} &= \frac{\partial \psi}{\partial z} + \nu \nabla^2 w, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &+ \frac{\partial w}{\partial z} = 0, \end{aligned}\right\} \qquad (33)$$

where

$$\psi = V - p/\rho + \text{const.}, \qquad (34)$$

the notation used being the usual notation for such problems.

The general solution of these equations applicable for satisfying boundary-conditions at concentric spherical surfaces has been frequently discussed.* There is an inconvenience, however, attached to the use of the functions usually employed in the applications which we propose to make here, arising from the fact that these functions will always appear with large arguments. I have found it possible, however, to considerably simplify the analysis by utilizing a modified form of these functions applicable in the case where the total range of the arguments involved is small in comparison with the actual values of these arguments. Such will be the case if the distance between the internal and external bounding spheres is small in comparison with the radii of these spheres. We will commence by a recapitulation of the analysis used in the papers cited above, introducing where necessary the modifications referred to.

Take first the equation

$$(\nabla^2+k^2)\,\Phi = 0, \tag{35}$$

and suppose Φ is of the form $R\phi_n$, where R is a function of r only, and ϕ_n a solid harmonic of degree n. On substituting this form for Φ in (35), we deduce

$$\frac{d^2R}{dr^2}+\frac{2(n+1)}{r}\frac{dR}{dr}+k^2R = 0. \tag{36}$$

Now, suppose $$r = a+h\xi,$$

where h is small compared with a, and in the region to which our solution is required to apply ξ lies between 0 and 1. Changing the variable from r to ξ and neglecting terms of the order h^2/a^2, the above equation becomes

$$\frac{d^2R}{d\xi^2}+2(n+1)\frac{h}{a}\frac{dR}{d\xi}+h^2k^2R = 0;$$

or, with errors of the same order of magnitude,

$$\frac{d^2R}{d\xi^2}+2(n+1)\frac{h}{a}\frac{dR}{d\xi}+\left\{(n+1)^2\frac{h^2}{a^2}+h^2k^2\right\}R = 0. \tag{37}$$

The rigorous solution of this equation is

$$R = A\xi_n+B\Xi_n,$$

* Lamb, *Proc. Lond. Math. Soc.*, Vol. XIII., pp. 51, 189; Love, *ibid.*, Vol. XIX., p. 170, &c.

where $$\xi_n = e^{hk\xi i - \frac{(n+1)h\xi}{a}}, \quad \Xi_n = e^{-hk\xi i - \frac{(n+1)h\xi}{a}}; \tag{38}$$

and therefore a solution of (35) is

$$\Phi = (A\xi_n + B\Xi_n)\,\phi_n. \tag{39}$$

From (38), we have

$$\left.\begin{aligned} \frac{d\xi_n}{dr} &= \frac{1}{h}\frac{d\xi_n}{d\xi} = \quad \{ki - (n+1)/a\}\,\xi_n, \\ \frac{d\Xi_n}{dr} &= \frac{1}{h}\frac{d\Xi_n}{d\xi} = -\{ki + (n+1)/a\}\,\Xi_n; \end{aligned}\right\} \tag{40}$$

while, with errors of order $\dfrac{h^2}{a^2}$,

$$\left.\begin{aligned} \left(\frac{r}{a}\right)^n \xi_n &= e^{hk\xi i - h\xi/a} = \xi_0, \\ \left(\frac{r}{a}\right)^n \Xi_n &= e^{-hk\xi i - h\xi/a} = \Xi_0. \end{aligned}\right\} \tag{41}$$

Consider next the equations

$$\left.\begin{aligned} (\nabla^2 + k^2)\,u = 0, \quad (\nabla^2 + k^2)\,v = 0, \quad (\nabla^2 + k^2)\,w = 0, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \end{aligned}\right\} \tag{42}$$

By the preceding we see that, if ϕ_{n+1}, χ_n denote spherical solid harmonics of degree indicated by their suffixes, since

$$\frac{\partial\phi_{n+1}}{\partial x}, \quad \frac{\partial\phi_{n+1}}{\partial y}, \quad \frac{\partial\phi_{n+1}}{\partial z}, \quad y\frac{\partial\chi}{\partial z} - z\frac{\partial\chi_n}{\partial y}, \text{ \&c.}$$

are solid harmonics of degree n, the following will satisfy the first three of equations (42),

$$\left.\begin{aligned} u &= \xi_n\left[\frac{\partial\phi_{n+1}}{\partial x} + y\frac{\partial\chi_n}{\partial z} - z\frac{\partial\chi_n}{\partial y}\right], \\ v &= \xi_n\left[\frac{\partial\phi_{n+1}}{\partial y} + z\frac{\partial\chi_n}{\partial x} - x\frac{\partial\chi_n}{\partial z}\right], \\ w &= \xi_n\left[\frac{\partial\phi_{n+1}}{\partial z} + x\frac{\partial\chi_n}{\partial y} - y\frac{\partial\chi_n}{\partial x}\right]. \end{aligned}\right\} \tag{43}$$

These do not satisfy the last of equations (42), but, in virtue of (40), they make

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\frac{(n+1)(kai-n-1)}{ar}\xi_n\phi_{n+1}.$$

Another set of solutions of the first three of (42) is

$$\left.\begin{aligned} u &= C\xi_{-n-3}\frac{\partial}{\partial x}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right),\\ v &= C\xi_{-n-3}\frac{\partial}{\partial y}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right),\\ w &= C\xi_{-n-3}\frac{\partial}{\partial z}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right),\end{aligned}\right\} \qquad (44)$$

from which we deduce

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=-C\frac{(n+2)(kai+n+2)}{ar^{2n+4}}\xi_{-n-3}\phi_{n+1}$$

$$=-C\frac{(n+2)(kai+n+2)}{a^{2n+4}r}\xi_n\phi_{n+1},$$

in virtue of (41).

Hence, if we put $C=a^{2n+3}\dfrac{(n+1)}{(n+2)}\dfrac{kai-n-1}{kai+n+2}$,

and add together the particular solutions (43), (44), we obtain a set of values of u, v, w which satisfy all four of equations (42).

Another set of solutions may be found by using the functions Ξ_n instead of ξ_n; we therefore obtain solutions of the following types, which may be treated independently:—

$$\left.\begin{aligned} u &= \xi_n\left(y\frac{\partial\chi_n}{\partial z}-z\frac{\partial\chi_n}{\partial y}\right)+\Xi_n\left(y\frac{\partial X_n}{\partial z}-z\frac{\partial X_n}{\partial y}\right),\\ v &= \xi_n\left(z\frac{\partial\chi_n}{\partial x}-x\frac{\partial\chi_n}{\partial z}\right)+\Xi_n\left(z\frac{\partial X_n}{\partial x}-x\frac{\partial X_n}{\partial z}\right),\\ w &= \xi_n\left(x\frac{\partial\chi_n}{\partial y}-y\frac{\partial\chi_n}{\partial z}\right)+\Xi_n\left(x\frac{\partial X_n}{\partial y}-y\frac{\partial X_n}{\partial z}\right);\end{aligned}\right\} \qquad (45)$$

$$\left.\begin{aligned}
u &= \xi_n \frac{\partial \phi_{n+1}}{\partial x} + \frac{(n+1)}{(n+2)} \frac{kai-n-1}{kai+n+2} \xi_{-n-3} a^{2n+3} \frac{\partial}{\partial x}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right) \\
&\quad + \Xi_n \frac{\partial \Phi_{n+1}}{\partial x} + \frac{(n+1)}{(n+2)} \frac{kai+n+1}{kai-n-2} a^{2n+3} \Xi_{-n-3} \frac{\partial}{\partial x}\left(\frac{\Phi_{n+1}}{r^{2n+3}}\right), \\
v &= \xi_n \frac{\partial \phi_{n+1}}{\partial y} + \frac{(n+1)}{(n+2)} \frac{kai-n-1}{kai+n+2} \xi_{-n-3} a^{2n+3} \frac{\partial}{\partial y}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right) \\
&\quad + \Xi_n \frac{\partial \Phi_{n+1}}{\partial y} + \frac{(n+1)}{(n+2)} \frac{kai+n+1}{kai-n-2} a^{2n+3} \Xi_{-n-3} \frac{\partial}{\partial y}\left(\frac{\Phi_{n+1}}{r^{2n+3}}\right), \\
w &= \xi_n \frac{\partial \phi_{n+1}}{\partial z} + \frac{(n+1)}{(n+2)} \frac{kai-n-1}{kai+n+2} a^{2n+3} \xi_{-n-3} \frac{\partial}{\partial z}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right) \\
&\quad + \Xi_n \frac{\partial \Phi_{n+1}}{\partial z} + \frac{(n+1)}{(n+2)} \frac{kai+n+1}{kai-n-2} a^{2n+3} \Xi_{-n-3} \frac{\partial}{\partial z}\left(\frac{\Phi_{n+1}}{r^{2n+3}}\right),
\end{aligned}\right\} \quad (46)$$

where χ_n, X_n, ϕ_n, Φ_n denote solid harmonics of degree n.

6. *On the Rate of Decay of Slow Currents in a Spherical Sheet of Water.*

Returning now to equations (33), suppose u, v, w, ψ each proportional to e^{-at}, and put

$$k^2 = a/\nu.$$

Then (33) reduce to

$$\left.\begin{aligned}
(\nabla^2+k^2)\, u &= -\frac{1}{\nu} \frac{\partial \psi}{\partial x}, \\
(\nabla^2+k^2)\, v &= -\frac{1}{\nu} \frac{\partial \psi}{\partial y}, \\
(\nabla^2+k^2)\, w &= -\frac{1}{\nu} \frac{\partial \psi}{\partial z}, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0.
\end{aligned}\right\} \quad (47)$$

A set of particular integrals will be furnished by (45), provided

$$\psi = 0.$$

If F, G, H denote the components of surface-traction across any sphere $r =$ constant, we have

$$Fr = -px+\rho\nu\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)+\rho\nu\left(x\frac{\partial u}{\partial x}+y\frac{\partial v}{\partial x}+z\frac{\partial w}{\partial x}\right)$$

$$= -px+\rho\nu\left(r\frac{\partial}{\partial r}-1\right)u+\frac{\partial}{\partial x}(ux+vy+wz),$$

&c. &c.

But the particular integrals in question make

$$ux+vy+wz = 0;$$

and therefore

$$\left.\begin{aligned} Fr &= -px+\rho\nu\left(r\frac{\partial}{\partial r}-1\right)u,\\ Gr &= -py+\rho\nu\left(r\frac{\partial}{\partial r}-1\right)v,\\ Hr &= -pz+\rho\nu\left(r\frac{\partial}{\partial r}-1\right)w;\end{aligned}\right\}$$

or, from (45), with the aid of (40),

$$Fr = -px+\rho\nu\left\{(n-1)+\frac{r}{a}(kai-n-1)\right\}\xi_n\left(y\frac{\partial\chi_n}{\partial z}-z\frac{\partial\chi_n}{\partial y}\right)$$

$$+\rho\nu\left\{(n-1)-\frac{r}{a}(kai+n+1)\right\}\Xi_n\left(y\frac{\partial X_n}{\partial z}-z\frac{\partial X_n}{\partial y}\right),$$

&c. &c.

If there be no slipping at the bottom, we must suppose $u = 0$, $v = 0$, $w = 0$, when $\xi = 0$. These conditions will be satisfied if

$$\chi_n+X_n = 0, \tag{48}$$

since ξ_n, Ξ_n both reduce to unity when $\xi = 0$.

If there be no stress at the free surface, we must have $F = 0$, $G = 0$, $H = 0$, when $\xi = 1$ or $r = a+h$. This condition will be satisfied if

$$p = 0,$$

$$\left[(n-1)+\frac{a+h}{a}(kai-n-1)\right]e^{hki-\frac{(n+1)h}{a}}\chi_n$$

$$+\left[(n-1)-\frac{a+h}{a}(kai+n+1)\right]e^{-hki-\frac{(n+1)h}{a}}X_n = 0. \tag{49}$$

From (48), (49) we deduce, on eliminating χ_n, X_n,

$$e^{2hki} = \frac{a(n-1)-(a+h)(kai+n+1)}{a(n-1)+(a+h)(kai-n-1)}. \tag{50}$$

When the viscosity is small this equation may be solved by successive approximation. Putting k infinite, we deduce

$$e^{2hki} = -1,$$

or

$$2hk = (2n+1)\pi,$$

$$\alpha = k^2\nu = \frac{(2n+1)^2\pi^2\nu}{4h^2}, \tag{51}$$

which is the same as the value found in § 1.

We conclude that the rates of decay of the free current-motions are not sensibly affected by the curvature of the Earth's surface.

7. *On the Subsidence of Tidal Oscillations.*

The equations (47) will still be applicable if the motion is oscillatory. From them we deduce

$$\nabla^2\psi = 0, \tag{52}$$

with the particular integrals

$$u = -\frac{1}{\alpha}\frac{\partial\psi}{\partial x}, \quad v = -\frac{1}{\alpha}\frac{\partial\psi}{\partial y}, \quad w = -\frac{1}{\alpha}\frac{\partial\psi}{\partial z}. \tag{53}$$

A solution of (52) is

$$\psi = \psi_{n+1} + \Psi_{n+1}/r^{2n+3}, \tag{54}$$

where ψ_{n+1}, Ψ_{n+1} are solid harmonics of degree $n+1$; and the corresponding particular integrals are

$$\left.\begin{aligned} u &= -\frac{1}{\alpha}\frac{\partial\psi_{n+1}}{\partial x} - \frac{1}{\alpha}\frac{\partial}{\partial x}\left(\frac{\Psi_{n+1}}{r^{2n+3}}\right), \\ v &= -\frac{1}{\alpha}\frac{\partial\psi_{n+1}}{\partial y} - \frac{1}{\alpha}\frac{\partial}{\partial y}\left(\frac{\Psi_{n+1}}{r^{2n+3}}\right), \\ w &= -\frac{1}{\alpha}\frac{\partial\psi_{n+1}}{\partial z} - \frac{1}{\alpha}\frac{\partial}{\partial z}\left(\frac{\Psi_{n+1}}{r^{2n-3}}\right). \end{aligned}\right\} \tag{55}$$

To these must be added complementary functions of the type (46).

Now we have seen that in the case of long waves in two dimensions the tangential stresses are of no account except in the immediate neighbourhood of the bottom. In the present case we shall assume that the motion, except through a very thin layer, will be sensibly irrotational, and therefore will be represented by the particular integrals (55). The complementary functions selected should therefore be such that they vanish except for very small values of ξ. If we suppose that the real part of ik is negative, this condition will be satisfied provided we omit from the right-hand members of (46) the terms involving Ξ_n, Ξ_{-n-3} and retain only those involving ξ_n, ξ_{-n-3}. Thus we obtain as appropriate solutions of the equations of motion

$$\left.\begin{aligned} \psi &= \psi_{n+1}+\Psi_{n+1}/r^{2n+3}, \\ u &= -\frac{1}{a}\frac{\partial\psi_{n+1}}{\partial x}-\frac{1}{a}\frac{\partial}{\partial x}\left(\frac{\Psi_{n+1}}{r^{2n+3}}\right)+\xi_n\frac{\partial\phi_{n+1}}{\partial x}+\frac{(n+1)}{(n+2)}\frac{kai-n-1}{kai+n+2}a^{2n+3}\xi_{-n-3}\frac{\partial}{\partial x}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right), \\ v &= -\frac{1}{a}\frac{\partial\psi_{n+1}}{\partial y}-\frac{1}{a}\frac{\partial}{\partial y}\left(\frac{\Psi_{n+1}}{r^{2n+3}}\right)+\xi_n\frac{\partial\phi_{n+1}}{\partial y}+\frac{(n+1)}{(n+2)}\frac{kai-n-1}{kai+n+2}a^{2n+3}\xi_{-n-3}\frac{\partial}{\partial y}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right), \\ w &= -\frac{1}{a}\frac{\partial\psi_{n+1}}{\partial z}-\frac{1}{a}\frac{\partial}{\partial z}\left(\frac{\Psi_{n+1}}{r^{2n+3}}\right)+\xi_n\frac{\partial\phi_{n+1}}{\partial z}+\frac{(n+1)}{(n+2)}\frac{kai-n-1}{kai+n+2}a^{2n+3}\xi_{-n-3}\frac{\partial}{\partial z}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right). \end{aligned}\right\} \tag{56}$$

The conditions $u = 0$, $v = 0$, $w = 0$, when $r = a$, will therefore be satisfied if

$$\left.\begin{aligned} &-\psi_{n+1}/a+\phi_{n+1} = 0, \\ &-\Psi_{n+1}/a+\frac{(n+1)}{(n+2)}\frac{kai-n-1}{kai+n+2}a^{2n+3}\phi_{n+1} = 0. \end{aligned}\right\} \tag{57}$$

From equations (56) we find

$$ux+vy+wz = -\frac{n+1}{a}\psi_{n+1}+\frac{n+2}{a}\Psi_{n+1}/r^{2n+3}$$
$$+(n+1)\,\xi_n\phi_{n+1}-(n+1)\frac{kai-n-1}{kai+n+2}\frac{a^{2n+3}}{r^{2n+3}}\xi_{-n-3}\phi_{n+1},$$

which, by the aid of (57), reduces to

$$ux+vy+wz = (n+1)\left[(\xi_n-1)-\frac{kai-n-1}{kai+n+2}\frac{a^{2n+3}}{r^{2n+3}}(\xi_{-n-3}-1)\right]\phi_{n+1}.$$

At the free surface we may omit the small terms involving ξ_n, ξ_{-n-3}, and take

$$ux+vy+wz = -(n+1)\left[1-\frac{kai-n-1}{kai+n+2}\left\{1-\frac{(2n+3)\,h}{a}\right\}\right]\phi_{n+1}$$

$$= -(n+1)(2n+3)\left\{\frac{h}{a}\frac{kai-n-1}{kai+n+2}+\frac{1}{kai+n+2}\right\}\phi_{n+1}.$$

Let ζ denote the height of the waves at the free surface. Then we have the kinematical condition

$$r\frac{\partial\zeta}{\partial t} = ux + vy + wz,$$

when $r = a+h$; whence

$$a\zeta = \frac{(n+1)(2n+3)}{(kai+n+2)(a+h)}\left\{1+(kai-n-1)\frac{h}{a}\right\}\phi_{n+1}. \tag{58}$$

Finally, the pressure at the surface $r = a+h$ must be equated to $g\rho\zeta$, where g denotes the value of the acceleration due to gravity. But from (34) we have

$$\psi = \text{const.} + V - p/\rho,$$

and therefore

$$[\psi] = \text{const.} + [V] - g\zeta,$$

where the square brackets indicate that surface values are to be understood.

If there be no external disturbing force, the periodic part of V will be the potential due to the harmonic inequalities ζ. Denoting the surface-value of this potential by v' and equating periodic parts in the two members of the last equation, we obtain

$$[\psi] = v' - g\zeta. \tag{59}$$

But from (58) we see that ζ is a surface-harmonic of order $(n+1)$, and therefore we obtain at once

$$v' = \frac{4\pi\rho\,(a+h)}{2n+3}\zeta;$$

while, if σ denote the mean density of the system under consideration, including both solid nucleus and liquid surface layer, we have

$$g = \tfrac{4}{3}\pi\sigma\,(a+h).$$

Thus

$$v' = \frac{3\rho}{(2n+3)\,\sigma}g\zeta,$$

and, if we denote by g_{n+1} the expression

$$g\left\{1-\frac{3\rho}{(2n+3)\,\sigma}\right\}, \tag{60}$$

the equation (59) becomes

$$[\psi] = -g_{n+1}\zeta.$$

Introducing the values of ψ, ζ from (56), (58), we find

$$\psi_{n+1}+\Psi_{n+1}/r^{2n+3} = -\frac{g_{n+1}}{a}\frac{(n+1)(2n+3)}{(kai+n+2)(a+h)}\left\{1+(kai-n-1)\frac{h}{a}\right\}\phi_{n+1}$$

when $r = a+h$; thus, on using (57), we obtain

$$1+\frac{(n+1)}{(n+2)}\frac{kai-n-1}{kai+n+2}\{1-(2n+3)\,h/a\}$$

$$= -\frac{g_{n+1}}{a^2}\frac{(n+1)(2n+3)}{(kai+n+2)(a+h)}\left[1+(kai-n-1)\,h/a\right];$$

or
$$kai+1-(n+1)(kai-n-1)\,h/a$$

$$= -\frac{(n+1)(n+2)\,g_{n+1}}{a^2(a+h)}\left[1+(kai-n-1)\,h/a\right];$$

whence, finally,

$$\frac{a^2}{g_{n+1}} = -\frac{(n+1)(n+2)}{a+h}\frac{khi+1-(n+1)\,h/a}{kai-(n+1)\,khi+1+(n+1)^2 h/a}. \quad (61)$$

On putting k infinite it follows that, as a first approximation,

$$a^2 = -\frac{(n+1)(n+2)\,g_{n+1}\,h}{(a+h)\{a-(n+1)\,h\}}.$$

or, with sufficient accuracy,

$$a^2 = -\beta^2,$$

where
$$\beta^2 = \frac{(n+1)(n+2)\,g_{n+1}\,h}{a^2}.* \quad (62)$$

From this we obtain

$$k^2 = a/\nu = i\beta/\nu;$$

whence
$$k = \pm\,(1+i)\sqrt{(\beta/2\nu)},$$

or
$$ik = \pm\,(i-1)\sqrt{(\beta/2\nu)}.$$

The upper sign must be taken, since by hypothesis the real part of ik is negative. From (61) we therefore find, as a second approximation,

$$a^2 = -\beta^2\left[1+\frac{1}{khi}\right]$$

$$= -\beta^2\left[1-\frac{1+i}{\sqrt{2}}\sqrt{\frac{\nu}{\beta h^2}}\right],$$

whence
$$a = i\beta\left[1-\tfrac{1}{2}\sqrt{\frac{\nu}{2\beta h^2}}-\frac{i}{2}\sqrt{\frac{\nu}{2\beta h^2}}\right].$$

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* *Cf.* Lamb, *Hydrodynamics*, p. 315.

Hence the speed of the oscillation is reduced by friction in the ratio

$$1-\tfrac{1}{2}\sqrt{\frac{\nu}{2\beta h^2}} : 1,$$

and the modulus of decay is

$$\sqrt{(8h^2/\beta\nu)}. \qquad (63)$$

Using C.G.S. units and taking

$$\sigma/\rho = 5\tfrac{1}{2}, \quad \nu = \cdot 0178, \quad a = 6\cdot 357 \times 10^8, \quad h = 4 \times 10^5,$$

we deduce as the moduli of decay for the types corresponding to $n = 0$, $n = 1$, 42·6 and 31·7 years respectively, while, when $n = 100$, we find a modulus of decay of 4·8 years.

Even with considerably smaller depths, the moduli of decay of the principal types of oscillation will still be long. Thus, if we take

$$h = 2 \times 10^4,$$

which implies a depth of 200 metres, or rather more than 100 fathoms, the moduli of decay for the types $n = 0$, $n = 1$ are 4·5 and 3·3 years respectively.

These results indicate how slight can be the effects of viscosity on the motions of the sea except possibly in shallow confined waters. It seems that wherever the depth exceeds a very moderate amount, say 100 fathoms, the rise and fall of the waters due to the disturbing influence of the Sun and Moon will not be appreciably affected by friction. It may be urged that we have a direct contradiction of this statement in the fact that the phases of the tides even at islands in the open sea often differ widely from the phases of the corresponding equilibrium tides. According to Sir G. Airy,* the acceleration or retardation of the semi-diurnal tide on the Moon's transit "does not at one port in a hundred agree in any measure with the result of this [the equilibrium] theory," and the want of agreement is attributed by him entirely to the effects of friction. The explanation seems to have been generally accepted by his successors, and, if it be true, will entirely invalidate our present results. A different explanation of the phenomenon in question has, however, been given by Newton,† and, in spite of the criticisms to which this has been subjected by Airy,‡ it appears to me that the results as stated by Newton are

* *Encyc. Metrop.*, Art. "Tides and Waves," § 62.
† *Principia*, Book III., Prop. 24.
‡ *L.c.*, §§ 16, 19.

substantially correct, and that an agreement in phase at all places between the dynamical tides and the equilibrium tides, even without viscosity, could only be expected under very special circumstances as to the distribution of land and water on the globe. In support of this view, I may quote some examples worked out by Airy himself in later sections of the paper referred to above, the bearing of which does not seem to have been fully appreciated by the author. Thus, if we take the case of a continuous equatorial canal subjected to the disturbance of a luminary moving uniformly round the equator, it is known that the forced tide will consist of a progressive wave following the motion of the disturbing body, and that at any place high or low water will always accompany the transit of the luminary across the meridian. The circumstances will, however, be totally different if the canal, instead of being continuous so as to return into itself, is of limited extent. In the latter case,* to find the complete motion arising from the disturbing force, we must superpose on the primary wave, due directly to the attraction of the luminary, secondary positive and negative waves due to the repeated reflection of the former at the extremities of the canal. These secondary waves, which have, as it seems to me, been erroneously described by Airy as "free waves," will be co-periodic with the primary wave, but will give rise to an entire re-adjustment of phase. Even in the case of continuous canals the examples worked out by Airy† indicate that an agreement in phase with the phase of the equilibrium tide can only be regarded as fortuitous in character.

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* Airy, *l.c.*, § 296.
† *L.c.*, §§ 431 *et seq.*