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A study of Coriolis effects on long waves propagating in an unbounded ocean, channel and basin

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Abstract

The effects of Coriolis force on long waves have been discussed based on gravity waves propagating in an unbounded ocean, channel and basin. In case of ocean, results show that the Coriolis effect will be significant and negligible, when the wave period is comparable to $2\pi/f$ and much shorter, respectively. Results also show in a channel, the wave amplitude and water particle velocity decrease exponentially in the positive y direction in the northern hemisphere (where f is positive). Moreover, in a basin, the Cotidal lines have been found as curves and rotate counterclockwise around the origin.

Keywords: Coriolis force; Scale analysis; Long waves; Kelvin waves

1. Introduction

Recently, long waves with periods of several minutes have been recognized as important and exciting component to beach erosion, sedimentation in harbors, and oscillation of water. Due to earth rotation, the Coriolis force is continuously generated which affects the long wave. The motion of the atmosphere and the ocean is naturally studied in a coordinate frame rotating with the earth. The wave frequency ω is the same order as f, the Coriolis parameter defined as $2\Omega \sin\phi$, where ϕ is the earth's latitude (Fig. 1) measured positive and negative in the northern and southern hemisphere, respectively, while Ω is the earth rotation speed (7.28×10^{-5} rad/s). As a particle moving forward will tend to be deflected to the right by the Coriolis force under the wave crest, but this motion is resisted by the crest elevation gradient. The relevant work has discussed extensively for simple flows in a paper of Leblanc and Cambon (1998).

Knowledge of the long waves (Fig. 2) is of great importance in a number of coastal engineering problems. Various forcing factors such as the bottom friction, wind stress and Coriolis force are involved in the changes of long waves generation. Clarification of these individual forces and evaluation of their mutual interactions are useful for the prediction of long waves generation. The present study regards the influence of Coriolis force on long waves. Although great efforts have been devoted to obtain the influence of different pertinent variables affecting the long waves, the effects of Coriolis force on long waves have not been well investigated yet.

Previous studies (Sorensen, 1978; Dean and Dalrymple, 1984; Fovell, 1991; Durran, 1993) on long waves generation were commonly performed based on the earth's rotation. Maa (1990) used implicit finite-difference scheme to solve the depth-averaged equations of motion and the continuity equation while including Coriolis force and his model was successfully worked. Neumann (1984) emphasized that the observed rate of rotation of direction is not constant over the diurnal cycle as it should be only the earth rotation were operative. Bishop (1979) suggested that there is only weak coupling between Coriolis force and Stokes drift.

Some recent works (Hsiung and Aboul-Azm, 1982; Mousseau et al., 2002; Li, 2004) on long waves including Coriolis force have been considerable attention to the researchers. Hsiung and Aboul-Azm (1982) developed a

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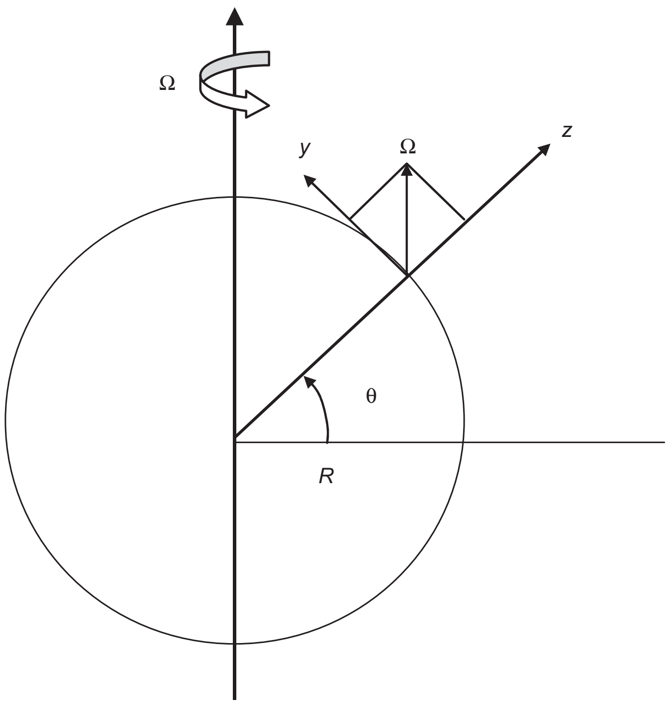

Fig. 1. Definition sketch of Coriolis force in local Cartesian coordinates.

Fig. 2. Photograph of long waves (Cox's bazar, Bangladesh).

mathematical model for iceberg drift based on Coriolis and geostrophic effect. Mousseau et al. (2002) solved the two-dimensional shallow water equation including Coriolis parameter numerically. Li (2004) presented the horizontal length scale of wave profile is proportional to the square root of the product of amplitude and gravity and is inversely proportional to the Coriolis parameter. Polton et al. (2005) investigated that how the Coriolis–Stokes forcing affects the mean current profile in a wind-driven mixed using simple model and found good agreement to the experimental data.

The objective of this paper is to characterize the long waves including Coriolis parameter in an unbounded ocean, channel and basin.

2. Scale analysis

Scale analysis, or scaling, is a convenient technique for estimating the magnitudes of various terms in the governing for a particular type of motion. In scaling, typical expected values of the following quantities are specified: (1) the magnitudes of the field variables, (2) the amplitudes of fluctuations in the field variables, and (3) the characteristic length, depth and time scales on which these fluctuating occur. These typical values are then used to compare the magnitudes of various terms in the governing equations.

For synoptic scale motions, we define the following characteristic scales of the field variables based on observed values for midlatitude synoptic systems:

$U \sim 10\,\mathrm{m\,s^{-1}}$	horizontal velocity scale
$W \sim 1\,\mathrm{cm\,s^{-1}}$	vertical velocity scale
$L \sim 10^6\,\mathrm{m}$	length scale
$D \sim 10^6\,\mathrm{m}$	depth scale
$\Delta P/\rho \sim 10^3\,\mathrm{m^2\,s^{-2}}$	horizontal pressure fluctuation scale
$L/U \sim 10^5\,\mathrm{s}$	time scale

3. Mathematical formulation

In an inertial reference frame, Newton's second law of motion may be written symbolically as

$$\frac{\mathrm{d}_a \bar{U}_a}{\mathrm{d}t} = \sum \bar{F}. \tag{1}$$

In order to transform this expression to rotating coordinates, we must first find a relationship between $\bar{U}_a$ and the velocity $\bar{U}$ relative to the rotating system. This relationship can be written as

$$\bar{U}_a = \bar{U} + \bar{\Omega} \times \bar{r}. \tag{2}$$

Letting $\bar{A}$ be an arbitrary vector and $\mathrm{d}_a\bar{A}/\mathrm{d}t$ be the total derivative of $\bar{A}$, we can write

$$\frac{\mathrm{d}_a \bar{A}}{\mathrm{d}t} = \frac{\mathrm{d}\bar{A}}{\mathrm{d}t} + \bar{\Omega} \times \bar{A}. \tag{3}$$

Next, we apply Eq. (2) to the velocity vector $\bar{U}_a$ and obtain

$$\frac{\mathrm{d}_a \bar{U}_a}{\mathrm{d}t} = \frac{\mathrm{d}\bar{U}_a}{\mathrm{d}t} + \bar{\Omega} \times \bar{U}_a. \tag{4}$$

Substituting from Eq. (2) into right-hand side of Eq. (4) gives

$$\frac{\mathrm{d}_a \bar{U}_a}{\mathrm{d}t} = \frac{\mathrm{d}\bar{U}}{\mathrm{d}t} + 2\bar{\Omega} \times \bar{U} - \Omega^2 \bar{R}, \tag{5}$$

where Ω is assumed to be constant and $\bar{R}$ is a vector perpendicular to the axis of rotation. Eq. (5) states that the motion in an inertial system equals the acceleration following the relative motion in a rotating system plus the Coriolis acceleration plus the centripetal acceleration.

If we assume that the only real forces acting on the atmosphere are the pressure gradient, gravitation and friction force, we can write Eq. (1) with the help of Eq. (5) as

$$\frac{d\bar{U}}{dt} = -2\bar{\Omega} \times \bar{U} - \frac{1}{\rho}\nabla p + \bar{g} + \bar{F}_r. \quad (6)$$

The components of equation of motion for a thin shell on a rotating earth can be written as

$$\frac{du}{dt} - \frac{uv \tan\phi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v \sin\phi - 2\Omega w \cos\phi + F_x, \quad (7a)$$

$$\frac{dv}{dt} + \frac{u^2 \tan^2\phi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u \sin\phi + F_y, \quad (7b)$$

$$\frac{dw}{dt} - \frac{u^2 + v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega u \cos\phi + F_z, \quad (7c)$$

where $d/dt = (\partial u/\partial t) + u(\partial u/\partial x) + v(\partial u/\partial y) + w(\partial u/\partial z)$.

Which are the eastward, northward, and vertical component momentum equations, respectively. These arise from transformation of the accelerations in the Cartesian system of equation of motions to the non-Cartesian system. These equations are nonlinear, so they are difficult to handle in theoretical analyses.

In order to simplify Eqs. (7a) and (7b) for synoptic scale motions, we use the characteristics scales. It should be pointed out here that the synoptic scale vertical velocity is not a directly measurable quantity. We can now estimate the magnitude of each term in Eqs. (7a) and (7b) for synoptic scale motions at the given latitude $\phi = 45^\circ$.

Table 1 shows the characteristic magnitude of each term in Eqs. (7a) and (7b) based on the above scaling considerations (Section 2). We obtain, as an excellent approximation for the horizontal equilibrium is a balance between the Coriolis force (E) and the pressure gradient (D) while other terms are not similar:

$$-fv \approx -\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad fu \approx -\frac{1}{\rho}\frac{\partial p}{\partial y}. \quad (8)$$

To retain the acceleration terms, the approximation horizontal momentum equations can be found as

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad (9a)$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad (9b)$$

where the nonlinear advective terms have been neglected under the small amplitude assumptions.

For long-wave theory, if we define the pressure at height z from the bottom, being hydrostatic, and the velocity components are given, respectively,

$$p = \rho g(h + \eta - z),$$

$$U = \frac{1}{h+\eta}\int_{-h}^{\eta} u\,dz \quad \text{and} \quad V = \frac{1}{h+\eta}\int_{-h}^{\eta} v\,dz.$$

The linearized continuity equation for long waves can be written as

$$\frac{\partial \eta}{\partial t} + h\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) = 0 \quad (10)$$

while the equation of continuity with the time average depth, h, constant and $\eta \ll h$. Substituting the value of p, u and v into Eq. (9) and taking the time average depth, we get

$$\frac{\partial U}{\partial t} - fV = -g\frac{\partial \eta}{\partial x}, \quad (11)$$

$$\frac{\partial V}{\partial t} + fU = -g\frac{\partial \eta}{\partial y}. \quad (12)$$

The above Eqs. (10)–(12) are now linear and it is possible to solve the equations analytically.

4. Solutions

4.1. Gravity wave propagating in a horizontally unbounded ocean

For a horizontally infinite ocean with $f =$ constant, it can be shown that the free modified gravity waves have periods $T < 2\pi/f$ for both surface and internal types of gravity wave. Taking $f =$ constant, a solution for a wave traveling in the x-direction is

$$(U, V, \eta) = (\hat{U}, \hat{V}, \hat{\eta})e^{i(kx+ly-\omega t)}, \quad (13)$$

where $\hat{U}, \hat{V}, \hat{\eta}$ are the complex amplitudes and the real part of the right side is meant. After simplification of Eqs. (10)–(12) with the help of Eq. (13), we have

$$-i\omega\hat{\eta} + ih(k\hat{U} + l\hat{V}) = 0, \quad (14)$$

Table 1
Scale analysis of the horizontal momentum equations

Moment. equation	(A)	(B)	(C)		(D)	(E)	(F)	(G)
x-com. eq.→	$\frac{du}{dt}$	$-\frac{uv\tan\phi}{a}$	$\frac{uw}{a}$	=	$-\frac{1}{\rho}\frac{\partial p}{\partial x}$	$2\Omega v \sin\phi$	$-2\Omega w\cos\phi$	F_x
y-com. eq.→	$\frac{dv}{dt}$	$\frac{u^2\tan^2\phi}{a}$	$\frac{vw}{a}$	=	$-\frac{1}{\rho}\frac{\partial p}{\partial y}$	$-2\Omega u\sin\phi$	0	F_y
Scales→	$\frac{U^2}{L}$	$\frac{U^2}{L}$	$\frac{UW}{a}$	=	$\frac{P}{L\rho}$	fU	fW	
Magni.→	10^{-4}	10^{-5}	10^{-8}	=	10^{-3}	10^{-3}	10^{-6}	10^{-12}

$$-\mathrm{i}\omega\hat{U} - f\hat{V} = -\mathrm{i}kg\hat{\eta}, \tag{15}$$

$$-\mathrm{i}\omega\hat{V} + f\hat{U} = -\mathrm{i}gl\hat{\eta}. \tag{16}$$

Solving Eqs. (15) and (16), yielding

$$\hat{U} = \frac{g\hat{\eta}}{\omega^2 - f^2}(\omega k + \mathrm{i}fl),$$

$$\hat{V} = \frac{g\hat{\eta}}{\omega^2 - f^2}(-\mathrm{i}fk + \omega l).$$

Substituting these in Eq. (14), we get

$$\omega^2 = f^2 + ghK^2, \tag{17}$$

where $K = \sqrt{k^2 + l^2}$ is the magnitude of the horizontal wave number. Eq. (17) represents the dispersion relation of gravity waves in the presence of Coriolis force. This relation shows that the waves can propagate in any horizontal direction and have $\omega > f$.

A plot of Eq. (17) is shown in Fig. 3. It is seen that waves are dispersive except for $\omega \gg f$ and Eq. (17) gives $\omega^2 = ghK^2$, so that the wave propagation speed C becomes $C = \omega/K = \sqrt{gh}$. Finally, we may conclude that the surface gravity waves unaffected by Coriolis forces in the case of high-frequency limit.

4.2. Gravity wave propagating parallel in a channel

If a vertical boundary exists, then the solution to Eqs. (10)–(12) is also possible. Consider the propagation of long progressive waves in an infinitely long straight canal in the x-direction with flat bottom. With transverse velocity $V = 0$, the x-momentum equation and continuity equation are unchanged. That means, these are not affected by the presence of the Coriolis force. Eqs. (10)–(12) give

$$\frac{\partial \eta}{\partial t} + h\frac{\partial U}{\partial x} = 0, \tag{18a}$$

$$\frac{\partial U}{\partial t} = -g\frac{\partial \eta}{\partial x}, \tag{18b}$$

$$fU = -g\frac{\partial \eta}{\partial y}. \tag{18c}$$

The above equations are linear, a solution can be assumed as

$$\eta = \hat{\eta}(y)\cos(kx - \sigma t),$$

$$U = \frac{\omega\hat{\eta}(y)}{kh}\cos(kx - \sigma t).$$

Then a nontrivial solution is therefore possible only if $\omega = \pm k\sqrt{gh}$, so that the wave propagates with a non-dispersive speed $C = \sqrt{gh}$.

The y-equation of motion is now

$$\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}y} + \frac{f\omega}{gkh}\hat{\eta} = 0. \tag{19}$$

The solution of Eq. (19) that decays away from the coast is

$$\hat{\eta} = \eta_0 \mathrm{e}^{-fy/C},$$

where η_0 is the amplitude at the coast. Therefore, the sea surface slope and the velocity field for a Kelvin wave have the form

$$\eta = \eta_0 \mathrm{e}^{-fy/C}\cos k(x - ct), \tag{20a}$$

$$U = \eta_0\sqrt{\frac{g}{h}}\mathrm{e}^{-fy/C}\cos k(x - ct). \tag{20b}$$

Solutions (20a) and (20b) show that at the wave crest, the wave amplitude and water particle velocity decrease exponentially in the positive y-direction in the northern hemi-sphere (where f is positive). This wave is called a Kelvin wave.

The Coriolis force wants to divert the particle motion to the right under the wave crest, but this motion is resisted by the crest elevation gradient. The opposite occurs under the wave trough. The crest and trough amplitudes are depicted in Fig. 4, where the wave is propagating along the x-axis.

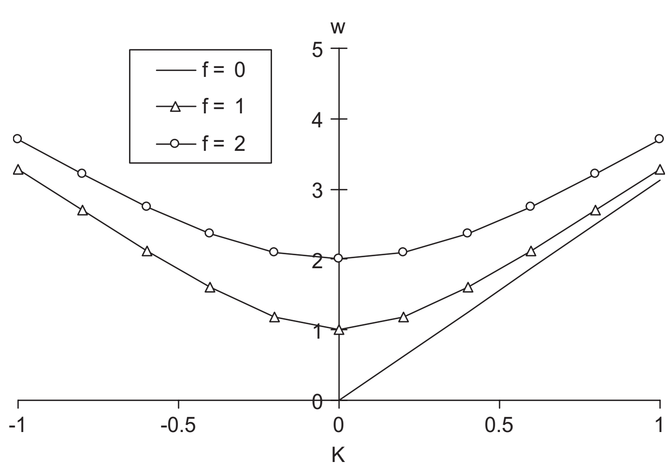

Fig. 3. Dispersion relation for different values of Coriolis force.

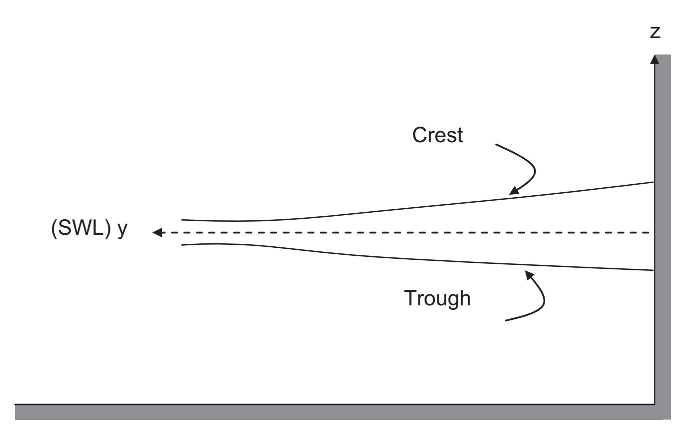

Fig. 4. Wave crest and trough profiles for a Kelvin wave propagating into the page.

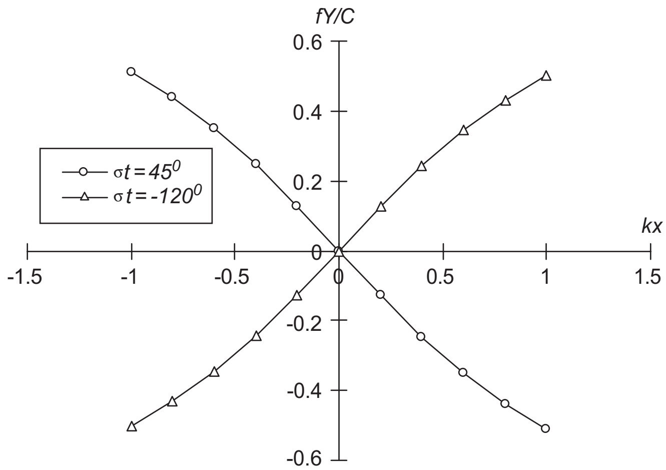

Fig. 5. Cotidal lines for a rectangular basin.

4.3. Gravity wave propagating in a basin

If we superpose two-Kelvin waves having opposite directions of travel along the x-axis to form a standing wave with the same height, the water surface elevation is given by

$$\eta = \frac{H}{2} e^{-fy/C} \cos(kx - \sigma t) - \frac{H}{2} e^{fy/C} \cos(kx + \sigma t). \tag{21}$$

At x, $y = 0$ the water surface elevation is zero at all times. Lines of maximum water surface elevation may be found when

$$\frac{\partial \eta}{\partial t} = 0.$$

Eq. (21) becomes

$$e^{-fy/C} \sin(kx - \sigma t) + e^{fy/C} \sin(kx + \sigma t) = 0$$

which, after some manipulation yields

$$\tanh \frac{fy}{C} = -\frac{\tan kx}{\tan \sigma t}. \tag{22}$$

For given standing wave period and water depth, the x, y coordinate positions of Cotidal lines can be plotted as a function of time from Eq. (22). With Coriolis effects, the Cotidal line pattern will be as shown in Fig. 5, which is plan view of this basin. Cotidal lines are curved and rotate counterclockwise around the amphidromic point (origin).

5. Conclusions

A theoretical analysis was performed to establish the effects of Coriolis force for a gravity wave propagating in a horizontally unbounded ocean, channel and basin. The dispersion relation shows that when the wave period is comparable to $2\pi/f$, then the Coriolis force will be important for horizontal unbounded ocean. Moreover, for high-frequency limit, the surface gravity wave is unaffected by the Coriolis force and will be consistent with the Kelvin wave.

In a channel, the wave amplitude and water particle velocity decreased exponentially in the positive y-direction in the northern hemi-sphere due to the effects of Coriolis force. Where as in a basin, the Cotidal lines were found as curve and rotate counter clockwise around the origin.

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References

Bishop, J.M., 1979. A note on surface wind-driven flow. Ocean Engineering 6, 273–284.

Dean, R.G., Dalrymple, R.A., 1984. Water Waves Mechanics for Engineers and Scientists. Prentice-Hall, Englewood Cliffs, NJ.

Durran, D.R., 1993. Is the Coriolis force really responsible for the inertial oscillation? Bulletin of the American Meteorological Society 74, 2179–2184.

Fovell, R.G., 1991. Influence of the Coriolis force on two-dimensional model storms. Monthly Weather Review 119, 606–630.

Hsiung, C.C., Aboul-Azm, A.F., 1982. Iceberg drift affected by wave action. Ocean Engineering 9, 433–439.

Leblanc, S., Cambon, C., 1998. Effects of Coriolis force on the stability of Stuart vortices. Journal of Fluid Mechanics 356, 353–379.

Li, C., 2004. A geosolitary wave solution on an f plane. Journal of Physical Oceanography 34, 856–864.

Maa, J.P., 1990. An efficient horizontal two-dimensional hydrodynamic model. Coastal Engineering 14, 1–18.

Mousseau, V.A., Knoll, D.A., Reisner, J.M., 2002. An implicit nonlinearity consistent method for the two-dimensional shallow water equations with Coriolis force. Monthly Weather Review 130, 2611–2625.

Neumann, J., 1984. The Coriolis force in relation to the sea and land breezes—A historical note. Bulletin of the American Meteorological Society 65, 24–26.

Polton, J., Lewis, D.M., Belcher, S.E., 2005. The role of wave-induced Coriolis–Stokes forcing on the wind-driven mixed layer. Journal of Physical Oceanography 35, 444–457.

Sorensen, R.M., 1978. Basic Coastal Engineering. Wiley, New York.