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The ideal Craik–Leibovich equations

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Abstract

We show that the Craik–Leibovich (CL) theory of Langmuir circulation in an ideal incompressible fluid driven by rapidly fluctuating surface waves due to the wind may be formulated in terms of Eulerian mean fluid variables as a Hamiltonian system. This formulation is facilitated by first determining Hamilton's principle for the CL equations. The CL Hamilton's principle is similar to that for a fluid plasma, driven by a rapidly varying external electromagnetic field via "$\boldsymbol{J}\cdot\boldsymbol{A}$" minimal coupling, after averaging the plasma action over the fast phase of the (single frequency) driving field. This similarity leads to a precise analogy between the CL vortex force and the Lorentz force on an electrically charged fluid due to an exernally imposed electromagnetic field. We determine the effect of this force on the inflection point criterion and the Richardson number criterion for stability of planar CL flows. The Noether symmetries of Hamilton's principle for the CL equations (under fluid particle relabeling) lead to conservation laws for Eulerian mean potential vorticity and helicity, and generate the steady Eulerian mean flows as canonical transformations. The generalized Lagrangian mean theory is discussed from the same viewpoint.

Keywords: Wave mean-flow interaction; Hamilton's principle; Geophysical fluid dynamics

1. Introduction

The Craik–Leibovich (CL) equations [8,9,15–19] describe the dynamics of the Eulerian mean fluid velocity $\boldsymbol{u}$ depending on time t and spatial position $\boldsymbol{x}$ in three dimensions, when the fluid motion is driven by rapidly oscillating surface waves due to the wind. These circumstances may generate Langmuir circulations – sets of vortices with axes nearly parallel to the wind direction which sometimes occur in the upper layers of lakes and oceans. Here we recast the ideal (nondissipative) CL equations as a Hamiltonian system. We discuss the implications of this Hamiltonian formulation for the steady flows, circulation theorems, and conservation laws for the CL equations.

In the CL theory, the rapidly oscillating waves at the surface are assumed to be unaffected by the more slowly changing currents below. The effect of the waves on the Eulerian mean flow is parameterized in the CL theory by introducing into the Navier–Stokes equations a "vortex force," expressed in terms of a prescribed Stokes drift velocity, $\bar{\boldsymbol{u}}_{\mathrm{S}}(\boldsymbol{x},t)$. The CL equations are given by

$$\frac{\partial\bar{\boldsymbol{u}}}{\partial t}+(\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}}+\nabla\varpi=\bar{\boldsymbol{u}}_{\mathrm{S}}\times\operatorname{curl}\bar{\boldsymbol{u}}+\nu_{\mathrm{T}}\nabla^2\bar{\boldsymbol{u}},\quad \nabla\cdot\bar{\boldsymbol{u}}=0,\quad \varpi=p+\tfrac{1}{2}|\bar{\boldsymbol{u}}+\bar{\boldsymbol{u}}_{\mathrm{S}}|^2-\tfrac{1}{2}|\bar{\boldsymbol{u}}|^2. \tag{1.1}$$

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Here ϖ is a modified pressure term that includes the mean pressure p as well as the increase of the kinetic energy of the fluid due to the waves. The term $\bar{\boldsymbol{u}}_{\mathrm{S}} \times \operatorname{curl} \bar{\boldsymbol{u}}$ is the "vortex force" of the CL theory of Langmuir circulation and ν_{T} is the constant eddy viscosity. The Eulerian mean fluid velocity $\bar{\boldsymbol{u}}$ is assumed to be divergenceless and is required to vanish on the fixed mean boundaries of the domain of flow for $\nu_{\mathrm{T}} \neq 0$, and be tangential to them for $\nu_{\mathrm{T}} = 0$.

If the Eulerian mean velocity $\bar{\boldsymbol{u}}$ is initially divergenceless, it will remain so according to (1.1), provided the mean pressure p solves the Poisson equation,

$$-\boldsymbol{\nabla}^2 \left(p + \tfrac{1}{2}|\bar{\boldsymbol{u}} + \bar{\boldsymbol{u}}_{\mathrm{S}}|^2 - \tfrac{1}{2}|\bar{\boldsymbol{u}}|^2 \right) = \operatorname{div}[(\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla})\bar{\boldsymbol{u}} - \bar{\boldsymbol{u}}_{\mathrm{S}} \times \operatorname{curl} \bar{\boldsymbol{u}}], \tag{1.2}$$

obtained by taking the divergence of the CL motion equation (1.1) and setting $\boldsymbol{\nabla} \cdot \bar{\boldsymbol{u}} = 0$. A Neumann boundary condition for the pressure in the Poisson equation is found by evaluating the velocity boundary condition in the motion equation. The Poisson equation for the pressure closes the CL equations, and shows how the vortex force and increased kinetic energy due to the mean Stokes drift velocity contribute to the mean pressure. Thus, the CL theory preserves the form of the Navier–Stokes equations, while introducing inhomogeneous forcing terms involving $\bar{\boldsymbol{u}}_{\mathrm{S}}$. These forcing terms are both ponderomotive (proportional to $\boldsymbol{\nabla}|\bar{\boldsymbol{u}}_{\mathrm{S}}|^2$) and parametric (proportional to $\bar{\boldsymbol{u}}$ and $\boldsymbol{\nabla}\bar{\boldsymbol{u}}$). The most important effect is vortex stretching along $\bar{\boldsymbol{u}}_{\mathrm{S}}$. The curl of the motion equation in (1.1) gives

$$\frac{\partial \bar{\boldsymbol{\omega}}}{\partial t} = -(\bar{\boldsymbol{u}} + \bar{\boldsymbol{u}}_{\mathrm{S}}) \cdot \boldsymbol{\nabla}\bar{\boldsymbol{\omega}} + \bar{\boldsymbol{\omega}} \cdot \boldsymbol{\nabla}(\bar{\boldsymbol{u}} + \bar{\boldsymbol{u}}_{\mathrm{S}}) + \bar{\boldsymbol{\omega}} \operatorname{div} \bar{\boldsymbol{u}}_{\mathrm{S}} + \nu_{\mathrm{T}} \boldsymbol{\nabla}^2 \bar{\boldsymbol{\omega}}, \tag{1.3}$$

where $\bar{\boldsymbol{\omega}} = \operatorname{curl} \bar{\boldsymbol{u}}$ is the Eulerian mean vorticity. Thus, the vortex force in Eq. (1.1) adds vorticity transport, stretching and creation terms to the Navier–Stokes vorticity equation. These terms are proportional to $\bar{\boldsymbol{u}}_{\mathrm{S}}$ and its gradients. The additional vortex stretching term $\bar{\boldsymbol{\omega}} \cdot \boldsymbol{\nabla}\bar{\boldsymbol{u}}_{\mathrm{S}}$ tends to convert Eulerian mean vorticity in the vertical direction into Langmuir circulations oriented along the mean Stokes drift velocity. Of course, the vorticity source term $\bar{\boldsymbol{\omega}} \operatorname{div} \bar{\boldsymbol{u}}_{\mathrm{S}}$ vanishes when $\bar{\boldsymbol{u}}_{\mathrm{S}}$ has no divergence.

The Stokes drift velocity $\bar{\boldsymbol{u}}_{\mathrm{S}}(\boldsymbol{x}, t)$ is related to the velocity $\boldsymbol{u}_{\mathrm{w}}$ of the surface waves treated in the CL theory by [9]

$$\bar{\boldsymbol{u}}_{\mathrm{S}} = \overline{\left(\int^{t} \boldsymbol{u}_{\mathrm{w}} \, \mathrm{d}\tau \right) \cdot \boldsymbol{\nabla}\boldsymbol{u}_{\mathrm{w}}}, \tag{1.4}$$

where overbar denotes average over fast time variations at fixed Eulerian position, e.g.,

$$\bar{f} = \frac{1}{T} \int_{-T}^{T} f \, \mathrm{d}t, \tag{1.5}$$

and the time T is assumed to be long compared to the period of the surface wave oscillations.

The *unaveraged* Eulerian velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ is supposed to satisfy the original Navier–Stokes equations (Eqs. (1.1) with $\bar{\boldsymbol{u}}_{\mathrm{S}}$ absent) and is expressed as

$$\boldsymbol{u}(\boldsymbol{x}, t) = \bar{\boldsymbol{u}}(\boldsymbol{x}, t) + \boldsymbol{u}'(\boldsymbol{x}, t) \quad \text{with } \overline{\boldsymbol{u}'} = 0. \tag{1.6}$$

Here the mean Eulerian velocity $\bar{\boldsymbol{u}}(\boldsymbol{x}, t)$ satisfies the CL equations (1.1) and $\boldsymbol{u}'(\boldsymbol{x}, t)$ represents the fluctuations in velocity caused by the surface waves at a certain Eulerian position. The CL theory is based on the observation that these velocity fluctuations are irrotational to leading order in an asymptotic expansion in the surface wave slope, ϵ, which is assumed to be small compared to unity. An associated small $\mathrm{O}(\epsilon)$ oscillatory displacement from the reference path of a fluid particle is assumed to be given by a prescribed vector field $\boldsymbol{\xi}(\boldsymbol{x}, t)$. This assumption leads to an alternative expression for the Stokes drift velocity which may be calculated in this scaling as [17]

$$\bar{\boldsymbol{u}}_{\mathrm{S}} = \overline{\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\xi},t)} - \overline{\boldsymbol{u}(\boldsymbol{x},t)} = \overline{\xi^i \frac{\partial}{\partial x^i}\boldsymbol{u}'} + \frac{1}{2}\overline{\xi^i\xi^j}\frac{\partial^2 \bar{\boldsymbol{u}}_{\mathrm{L}}}{\partial x^i \partial x^j} + \mathrm{O}(\epsilon^4 c)$$
$$= \overline{(\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\boldsymbol{u}'} + \mathrm{O}(\epsilon^4 c), \tag{1.7}$$

where we have used Eq. (1.6) and Taylor expanded in ξ. The quantity c is a typical wave speed. Driven by waves, the Stokes drift velocity $\bar{\boldsymbol{u}}_{\mathrm{S}}$ is supposed to achieve its maximum at the upper mean boundary (the mean wave height), to which it may be taken to be tangential. It decreases steadily with depth in a prescribed fashion, until it vanishes at a certain depth, below which the Navier–Stokes equations hold.

Leibovich [17] shows under the assumptions of small surface wave slope, ϵ, and nearly irrotational rapid oscillations, $\operatorname{curl}\boldsymbol{u}' = \mathrm{O}(\epsilon)$, that the sum of the Eulerian mean velocity $\bar{\boldsymbol{u}}$ and the Stokes drift velocity $\bar{\boldsymbol{u}}_{\mathrm{S}}$ is equal to the Lagrangian mean velocity $\bar{\boldsymbol{u}}_{\mathrm{L}}$ discussed by Andrews and McIntyre [3], to order $\mathrm{O}(\epsilon^4 c)$,

$$\bar{\boldsymbol{u}} + \bar{\boldsymbol{u}}_{\mathrm{S}} = \bar{\boldsymbol{u}}_{\mathrm{L}} + \mathrm{O}(\epsilon^4 c). \tag{1.8}$$

In this relation and in Eq. (1.7) $\bar{\boldsymbol{u}}_{\mathrm{L}}(\boldsymbol{x},t) = \overline{\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\xi},t)}$, so that

$$\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\xi},t) = \bar{\boldsymbol{u}}_{\mathrm{L}}(\boldsymbol{x},t) + \boldsymbol{u}^l(\boldsymbol{x},t), \tag{1.9}$$

where the Lagrangian fluctuation velocity $\boldsymbol{u}^l$ satisfies [3]

$$\boldsymbol{u}^l = \left(\frac{\partial}{\partial t} + \bar{\boldsymbol{u}}_{\mathrm{L}}\cdot\boldsymbol{\nabla}\right)\xi \equiv \frac{\mathrm{d}\xi}{\mathrm{d}t} \quad \text{and} \quad \overline{\boldsymbol{u}^l} = 0. \tag{1.10}$$

In this paper, we shall assume the following expression for the Stokes mean drift velocity in terms of the displacement field $\boldsymbol{\xi}$:

$$\bar{\boldsymbol{u}}_{\mathrm{S}} \equiv \overline{(\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t}} \quad \text{with } \operatorname{curl}\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = 0. \tag{1.11}$$

As we shall see, this form of $\bar{\boldsymbol{u}}_{\mathrm{S}}$ brings the CL theory into agreement with the generalized Lagrangian mean formulation discussed in [3].

In their discussion of the generalized Lagrangian mean formulation, Andrews and McIntyre [3] point out that the divergence of the Lagrangian mean velocity does not vanish in general, since [3 Eq. (9.4)] (summing on repeated indices)

$$\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}_{\mathrm{L}} = \left(\frac{\partial}{\partial t} + \bar{\boldsymbol{u}}_{\mathrm{L}}\cdot\boldsymbol{\nabla}\right)\overline{(\xi_i\xi_j)_{,ij}} + \mathrm{O}(\epsilon^3). \tag{1.12}$$

However, being quadratic in the small displacement $\boldsymbol{\xi}$, the divergence of the Lagrangian mean velocity is neglected in the CL theory. Thus, the Lagrangian mean velocity $\bar{\boldsymbol{u}}_{\mathrm{L}}$ is taken in the CL theory at the appropriate order in ϵ to be divergenceless and equal to the sum $\bar{\boldsymbol{u}} + \bar{\boldsymbol{u}}_{\mathrm{S}}$, which we assume to be tangential to the mean boundary $\overline{\partial\mathcal{D}}$, i.e.,

$$\bar{\boldsymbol{u}}_{\mathrm{L}} = \bar{\boldsymbol{u}} + \bar{\boldsymbol{u}}_{\mathrm{S}} \quad \text{and } \boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}_{\mathrm{L}} = 0 \ \text{ with } \hat{\boldsymbol{n}}\cdot\bar{\boldsymbol{u}}_{\mathrm{L}} = 0 \qquad \text{at } \overline{\partial\mathcal{D}}. \tag{1.13}$$

As a consequence of these relations, $\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}} = 0$ implies $\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}_{\mathrm{S}} = 0$, as well [16]. So the Stokes drift velocity $\bar{\boldsymbol{u}}_{\mathrm{S}}$ is divergenceless at this order in ϵ. (We discuss modifications of the ideal CL theory to accomodate nonvanishing $\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}_{\mathrm{L}}$ in Section 9.)

The Lagrangian mean velocity is a useful construct in interpreting the role of the vortex force in the CL theory. In terms of $\bar{\boldsymbol{u}}_{\mathrm{L}}$, the CL equations (1.1) may be rewritten as

$$\begin{aligned}&\frac{\partial\bar{\boldsymbol{u}}_{\mathrm{L}}}{\partial t} + (\bar{\boldsymbol{u}}_{\mathrm{L}}\cdot\boldsymbol{\nabla})\bar{\boldsymbol{u}}_{\mathrm{L}} + \boldsymbol{\nabla}p - \nu_{\mathrm{T}}\boldsymbol{\nabla}^2\bar{\boldsymbol{u}}_{\mathrm{L}} = \frac{\partial\bar{\boldsymbol{u}}_{\mathrm{S}}}{\partial t} - \bar{\boldsymbol{u}}_{\mathrm{L}}\times\operatorname{curl}\bar{\boldsymbol{u}}_{\mathrm{S}} - \nu_{\mathrm{T}}\boldsymbol{\nabla}^2\bar{\boldsymbol{u}}_{\mathrm{S}},\\ &\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}_{\mathrm{L}} = 0 = \boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}_{\mathrm{S}}.\end{aligned} \tag{1.14}$$

In this form, the Stokes drift terms drive the dynamics of the Lagrangian mean velocity with the "Lorentz force" of an externally imposed "electromagnetic field" with vector potential given by $-\bar{\boldsymbol{u}}_{\mathrm{S}}$; implying an electric field, $\partial\bar{\boldsymbol{u}}_{\mathrm{S}}/\partial t$, and a magnetic field, $-\mathrm{curl}\,\bar{\boldsymbol{u}}_{\mathrm{S}}$. The plasma physics analogy suggested by this form of the CL equations will be a useful guide at several points in the present discussion. Of course, for a steady prescribed $\bar{\boldsymbol{u}}_{\mathrm{S}}$ the "Lorentz force" is also equivalent to a Lagrangian mean Coriolis force, $\bar{\boldsymbol{u}}_{\mathrm{L}} \times 2\bar{\boldsymbol{\Omega}}$, with $2\bar{\boldsymbol{\Omega}} = -\mathrm{curl}\,\bar{\boldsymbol{u}}_{\mathrm{S}}$.

One of the issues raised here is the distinction between "flow" and "particle" properties of ideal CL flows, as distinguished by whether a given property (e.g., a conservation law, or a stability condition) is expressed in terms of only the Eulerian mean velocity, or only the Lagrangian mean velocity, respectively. The kinetic energy, for example, is conserved and turns out to be a particle property – being expressed as the integral of the square of the Lagrangian mean velocity. On the other hand, the conserved helicity is a flow property: this is the spatially integrated scalar product of the Eulerian mean velocity with its curl. There are, of course, also mixed expressions, such as the conditions required for an ideal CL solution to be steady (cf. Eq. (3.1)). In Sections 3 and 8 we show that the stability conditions for steady planar CL solutions are expressed either as flow properties (e.g., inflection point of Eulerian mean velocity), or as particle properties (Richardson number criterion, written in terms of Lagrangian mean velocity), depending on the situation being investigated.

The distinction between flow and particle properties helps in interpreting the effects of the vortex force introduced in the CL theory. As expected from the form of equation (1.14), this force turns out to be closely analogous to the Lorentz force on a charged particle in an external electromagnetic field. Moreover, Hamilton's principle for the ideal CL equations is similar to that for a fluid plasma which is driven by a rapidly varying external electromagnetic field via "$\boldsymbol{J} \cdot \boldsymbol{A}$". minimal coupling, after performing in the plasma action a two-time-scale average over the fast phase of the (single frequency) driving field [23,24].

This paper is organized as follows. Section 2 discusses the energy, helicity and enstrophy balances for the CL equations (1.1). We then specialize to the ideal case ($\nu_{\mathrm{T}} = 0$) for the remainder of the paper, in order to focus on the nonlinear effects of introducing the Stokes drift velocity terms into Euler's equations. Section 3 discusses the effects of $\bar{\boldsymbol{u}}_{\mathrm{S}}$ on steady flows, circulation theorems and conservation laws for the ideal CL equations. This discussion is elucidated by recovering the ideal CL equations from a constrained Hamilton's principle in Section 4. Section 4 shows that the $\bar{\boldsymbol{u}}_{\mathrm{S}}$ terms in the CL theory preserve the particle relabeling symmetry of its Hamilton's principle. We pass to the Hamiltonian formulation of the ideal CL theory in Section 5 and discuss the implications of preserving this symmetry from the Hamiltonian viewpoint in Section 6. Section 7 extends the treatment of the previous sections to include rotation and buoyancy due to density stratification in the Boussinesq approximation. Section 8 discusses the linearized stability conditions for planar steady ideal CL flows with density stratification, and derives the modifications due to Stokes drift of both the inflection point criterion and the Richardson number criterion for stability of planar steady ideal Euler flows. Section 9 extends the analysis presented for the ideal CL theory to the case of fluctuations described by the generalized Lagrangian mean theory [3,21].

2. Energy, helocity and enstrophy balances

We begin by rewriting the dissipative CL motion equation (1.1) as

$$\frac{\partial \bar{\boldsymbol{u}}}{\partial t} - \bar{\boldsymbol{u}}_{\mathrm{L}} \times \bar{\boldsymbol{\omega}} + \boldsymbol{\nabla}\big(p + \tfrac{1}{2}|\bar{\boldsymbol{u}}_{\mathrm{L}}|^2\big) = \nu_{\mathrm{T}}\boldsymbol{\nabla}^2\bar{\boldsymbol{u}}, \tag{2.1}$$

where $\bar{\boldsymbol{\omega}} = \mathrm{curl}\,\bar{\boldsymbol{u}}$ is the Eulerian mean vorticity. We obtain a local balance law for the Lagrangian mean kinetic energy by taking the scalar product of (2.1) with the Lagrangian mean velocity $\bar{\boldsymbol{u}}_{\mathrm{L}}$. Namely,

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}|\bar{\boldsymbol{u}}_{\mathrm{L}}|^2\right) = -\operatorname{div}\left[\bar{\boldsymbol{u}}_{\mathrm{L}}\left(p + \tfrac{1}{2}|\bar{\boldsymbol{u}}_{\mathrm{L}}|^2\right) - \tfrac{1}{2}\nu_{\mathrm{T}}\nabla|\bar{\boldsymbol{u}}|^2\right] + \bar{\boldsymbol{u}}_{\mathrm{L}} \cdot \frac{\partial}{\partial t}\bar{\boldsymbol{u}}_{\mathrm{S}} + \nu_{\mathrm{T}}\bar{\boldsymbol{u}}_{\mathrm{S}} \cdot \nabla^2\bar{\boldsymbol{u}} - \nu_{\mathrm{T}}\bar{u}_{i,j}\bar{u}_{i,j}, \tag{2.2}$$

where we have used the divergence-free condition, $\nabla \cdot \bar{\boldsymbol{u}}_{\mathrm{L}} = 0$ from (1.13). Hence, the integrated Lagrangian mean kinetic energy will satisfy a balance relation obtained by integrating (2.2) over the mean domain of flow $\overline{\mathcal{D}}$ and using the divergence theorem. Upon using the tangential boundary condition for $\bar{\boldsymbol{u}}_{\mathrm{L}}$ in (1.13) to remove a boundary term, this integration of (2.2) yields the balance relation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\overline{\mathcal{D}}} \mathrm{d}^3x\, \tfrac{1}{2}|\bar{\boldsymbol{u}}_{\mathrm{L}}|^2 = \int_{\overline{\mathcal{D}}} \mathrm{d}^3x \left[\bar{\boldsymbol{u}}_{\mathrm{L}} \cdot \frac{\partial}{\partial t}\bar{\boldsymbol{u}}_{\mathrm{S}} + \nu_{\mathrm{T}}\bar{\boldsymbol{u}}_{\mathrm{S}} \cdot \nabla^2\bar{\boldsymbol{u}} - \nu_{\mathrm{T}}\bar{u}_{i,j}\bar{u}_{i,j}\right], \tag{2.3}$$

provided $\bar{\boldsymbol{u}}$ vanishes on the mean boundary, $\overline{\partial\mathcal{D}}$, which is required in the case that $\nu_{\mathrm{T}} \neq 0$. This energy balance relation is *not* definite in sign, even for $\nu_{\mathrm{T}} = 0$, since the terms involving the mean Stokes drift velocity $\bar{\boldsymbol{u}}_{\mathrm{S}}$ may have either sign. The indefinite middle term in (2.3) would be removed, though, if the turbulent viscous force were taken to be $\nu_{\mathrm{T}}\nabla^2\bar{\boldsymbol{u}}_{\mathrm{L}}$, instead of $\nu_{\mathrm{T}}\nabla^2\bar{\boldsymbol{u}}$ in (2.1). This change would result in the standard form for dissipation of (Lagrangian mean) kinetic energy by viscosity when the mean Stokes drift velocity is steady. However, this change would have other implications for helicity balance, which is addressed next.

Eq. (2.1) and its curl yield the equation

$$\frac{\partial}{\partial t}(\bar{\boldsymbol{u}} \cdot \bar{\omega}) = -\operatorname{div}[(\bar{\boldsymbol{u}} \cdot \bar{\omega})\bar{\boldsymbol{u}}_{\mathrm{L}} - \nabla\varpi' \times \bar{\boldsymbol{u}}] + \nu_{\mathrm{T}}[\bar{\omega} \cdot \nabla^2\bar{\boldsymbol{u}} + \bar{\boldsymbol{u}} \cdot \nabla^2\bar{\omega}], \tag{2.4}$$

where we have defined another modified pressure

$$\varpi' = p + \tfrac{1}{2}|\bar{\boldsymbol{u}}_{\mathrm{S}}|^2 - \tfrac{1}{2}|\bar{\boldsymbol{u}}|^2. \tag{2.5}$$

Hence, the Eulerian mean helicity, the total "knottedness" of the flow lines of $\bar{\boldsymbol{u}}$,

$$\Lambda = \int_{\overline{\mathcal{D}}} \mathrm{d}^3x\, \bar{\boldsymbol{u}} \cdot \bar{\omega}, \tag{2.6}$$

satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\overline{\mathcal{D}}} \mathrm{d}^3x\, \bar{\boldsymbol{u}} \cdot \bar{\omega} = -\oint_{\overline{\partial\mathcal{D}}} \mathrm{d}^2x\, [(\bar{\boldsymbol{u}} \cdot \bar{\omega})\hat{\boldsymbol{n}} \cdot \bar{\boldsymbol{u}}_{\mathrm{L}} + \nabla\varpi' \times \hat{\boldsymbol{n}} \cdot \bar{\boldsymbol{u}}] + \nu_{\mathrm{T}}\int_{\overline{\mathcal{D}}} \mathrm{d}^3x\, [\bar{\omega} \cdot \nabla^2\bar{\boldsymbol{u}} + \bar{\boldsymbol{u}} \cdot \nabla^2\bar{\omega}]. \tag{2.7}$$

The volume integral proportional to ν_{T} is indefinite in sign, so viscosity may either create or destroy Eulerian mean helicity in CL flows. The boundary term is also indefinite, although it does vanish for steady flows. If ν_{T} vanishes and the modified pressure ϖ' is *constant* on the mean boundary (so that the cross product $\nabla\varpi' \times \hat{\boldsymbol{n}}$ vanishes in the surface integral), then the Eulerian mean helicity (2.6) is conserved. Had the turbulent viscous force in (2.1) been $\nu_{\mathrm{T}}\nabla^2\bar{\boldsymbol{u}}_{\mathrm{L}}$, it would have contributed an additional volume source of helicity in (2.7). Thus, the choice of turbulent eddy viscosity modeling ($\nu_{\mathrm{T}}\nabla^2\bar{\boldsymbol{u}}$ versus $\nu_{\mathrm{T}}\nabla^2\bar{\boldsymbol{u}}_{\mathrm{L}}$) is quite important to the relative balances of energy and helicity.

The Eulerian mean vorticity equation (1.3) with $\operatorname{div}\bar{\boldsymbol{u}}_{\mathrm{S}} = 0$ gives an equation for transport, production and dissipation of the square of the Eulerian mean vorticity,

$$\frac{\partial}{\partial t}\frac{1}{2}|\bar{\omega}|^2 = -\bar{\boldsymbol{u}}_{\mathrm{L}} \cdot \nabla\tfrac{1}{2}|\bar{\omega}|^2 + \bar{\omega}_j\bar{\omega}^i\bar{u}^j_{\mathrm{L},i} + \nu_{\mathrm{T}}(\bar{\omega}_i\bar{\omega}^i_{,j})_{,j} - \nu_{\mathrm{T}}(\bar{\omega}_{i,j}\bar{\omega}^i_{,j}). \tag{2.8}$$

Consequently, we have the following equation for Eulerian mean enstrophy production:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{D}} \mathrm{d}^3x|\bar{\boldsymbol{\omega}}|^2 = -\oint_{\partial\mathcal{D}} \mathrm{d}^2x\left[-\tfrac{1}{2}|\bar{\boldsymbol{\omega}}|^2\hat{\boldsymbol{n}}\cdot\bar{\boldsymbol{u}}_{\mathrm{L}} + \nu_{\mathrm{T}}(\bar{\omega}_i\hat{\boldsymbol{n}}\cdot\nabla\bar{\omega}^i)\right]$$
$$-\nu_{\mathrm{T}}\int_{\mathcal{D}} \mathrm{d}^3x|\nabla\bar{\boldsymbol{\omega}}|^2 + \int_{\mathcal{D}} \mathrm{d}^3x\bar{\omega}_j\bar{\omega}^i\bar{u}^j_{\mathrm{L},i}. \tag{2.9}$$

The (indefinite) last term shows that $\bar{\boldsymbol{u}}_{\mathrm{S}}$ contributes to Eulerian mean enstrophy production, as a result of its contribution to vortex stretching in the vorticity equation (1.3). We leave the dissipative aspects of the CL theory at this point and focus on the nonlinear effects of introducing the mean Stokes drift velocity terms into Euler's equations.

3. Steady flows, circulation theorems, and conservation laws for the ideal CL equations

3.1. Steady ideal CL flows

The curl of the ideal CL motion equation, (2.1) with $\nu_{\mathrm{T}} = 0$, implies a "frozen-in" relation for the Eulerian mean vorticity, $\bar{\boldsymbol{\omega}} = \operatorname{curl}\bar{\boldsymbol{u}}$. Namely (cf. Eq. (1.3))

$$\frac{\partial}{\partial t}\bar{\boldsymbol{\omega}} = \operatorname{curl}(\bar{\boldsymbol{u}}_{\mathrm{L}}\times\bar{\boldsymbol{\omega}}) = -(\bar{\boldsymbol{u}}_{\mathrm{L}}\cdot\nabla)\bar{\boldsymbol{\omega}} + (\bar{\boldsymbol{\omega}}\cdot\nabla)\bar{\boldsymbol{u}}_{\mathrm{L}} = -[\bar{\boldsymbol{u}}_{\mathrm{L}}, \bar{\boldsymbol{\omega}}], \tag{3.1}$$

where $[\cdot,\cdot]$ denotes the Lie bracket between divergenceless vector fields. As discussd earlier. the vorticity stretching term $(\bar{\boldsymbol{\omega}}\cdot\nabla)\bar{\boldsymbol{u}}_{\mathrm{L}}$ contains the *Lagrangian* mean strain rate, $\nabla\bar{\boldsymbol{u}}_{\mathrm{L}}$, rather than the corresponding Eulerian mean quantity. From (3.1), we see that steady ideal CL vorticity solutions (denoted with subscript e for "equilibrium") are characterized by the symmetry relation $[\bar{\boldsymbol{u}}_{\mathrm{Le}}, \bar{\boldsymbol{\omega}}_{\mathrm{e}}] = 0$. Thus, in steady ideal CL flows the Lagrangian mean velocity $\bar{\boldsymbol{u}}_{\mathrm{Le}}$ generates a volume preserving spatial transformation that leaves invariant the Eulerian mean *vorticity* $\bar{\boldsymbol{\omega}}_{\mathrm{e}}$.

3.1.1. Planar flows

The simplest example of the situation $[\bar{\boldsymbol{u}}_{\mathrm{Le}}, \bar{\boldsymbol{\omega}}_{\mathrm{e}}] = 0$ occurs when the Stokes drift velocity is time-independent and the Lagrangian mean velocity is invariant under a spatial translation. There are the planar ideal CL flows, e.g., taking place in the x–z plane when $\partial\bar{\boldsymbol{u}}_{\mathrm{L}}/\partial y$ vanishes and $\bar{\boldsymbol{u}}_{\mathrm{S}} = U_{\mathrm{S}}(z)\hat{\boldsymbol{x}}$, with $\operatorname{curl}\bar{\boldsymbol{u}}_{\mathrm{S}} = U_{\mathrm{S}}'(z)\hat{\boldsymbol{y}}$, for a function U_{S} and its derivative $U_{\mathrm{S}}' = \mathrm{d}U_{\mathrm{S}}/\mathrm{d}z$. In this case, a stream function $\bar{\psi}(x,z,t)$ may be introduced, and used to express the Lagrangian mean velocity and Eulerian mean vorticity as

$$\begin{aligned}\bar{\boldsymbol{u}}_{\mathrm{L}} &= \hat{\boldsymbol{y}}\times\nabla\bar{\psi} = (\bar{\psi}_z, 0, -\bar{\psi}_x),\\ \bar{\boldsymbol{\omega}} &= \operatorname{curl}(\bar{\boldsymbol{u}}_{\mathrm{L}} - \bar{\boldsymbol{u}}_{\mathrm{S}}) = \hat{\boldsymbol{y}}(\Delta\bar{\psi} - U_{\mathrm{S}}'(z)) = \hat{\boldsymbol{y}}\bar{\omega},\end{aligned} \tag{3.2}$$

where $\Delta\bar{\psi} = \bar{\psi}_{xx} + \bar{\psi}_{zz}$ is the planar Laplacian of the stream function $\bar{\psi}$. Thus, the Eulerian mean vorticity equation takes the familiar form

$$\frac{\partial\bar{\omega}}{\partial t} = \hat{\boldsymbol{y}}\cdot\nabla\bar{\omega}\times\nabla\bar{\psi} = \mathcal{J}(\bar{\omega}, \bar{\psi}), \tag{3.3}$$

where $\mathcal{J}(g,h) = g_z h_x - h_z g_x$ is the Jacobian of the functions $g(x,z)$ and $h(x,z)$. The steady planar flows then satisfy

$$\mathcal{J}(\bar{\psi}_{\mathrm{e}}, \bar{\omega}_{\mathrm{e}}) = 0. \tag{3.4}$$

So there is a functional dependence at equilibrium, which we write as

$$\bar{\psi}_{\rm e} = \Phi'(\bar{\omega}_{\rm e}) \quad \text{with } \bar{\omega}_{\rm e} = \Delta\bar{\psi}_{\rm e} - U'_{\rm S}(z) \tag{3.5}$$

for a function we denote as Φ'.

The stability of these flows may be understood by noticing that they are critical points of the following conserved functional (cf. [5] and references in [13])

$$H_{\rm C} = \int {\rm d}x\ {\rm d}z\big[\tfrac{1}{2}|\nabla\bar{\psi}|^2 + \Phi(\bar{\omega})\big]. \tag{3.6}$$

This is the Lagrangian mean kinetic energy, constrained by the quantity,

$$C_\Phi = \int {\rm d}x\ {\rm d}z\, \Phi(\bar{\omega}), \tag{3.7}$$

which is conserved when the velocity is tangential on the boundary. The first variation of $H_{\rm C}$ is expressible (after integration by parts) as

$$\delta H_{\rm C} = \int {\rm d}x\, {\rm d}z\, \delta\bar{\omega}[-\bar{\psi} + \Phi'(\bar{\omega})], \tag{3.8}$$

where Φ' is the derivative of the function Φ. Thus, the integrand of (3.8) vanishes for steady flows of Eq. (3.3), by virtue of (3.5).

The second variation of $H_{\rm C}$, given by

$$\delta^2 H_{\rm C} = \int {\rm d}x\, {\rm d}z[\delta\bar{\omega}(-\Delta^{-1})\delta\bar{\omega} + \Phi''(\bar{\omega}_{\rm e})(\delta\bar{\omega})^2], \tag{3.9}$$

is (positive) definite, and thus the equilibrium is linearly Lyapunov stable [5], provided

$$c_- \le \Phi''(\bar{\omega}_{\rm e}) = \frac{{\rm d}\bar{\psi}_{\rm e}}{{\rm d}\bar{\omega}_{\rm e}} \le c_+, \tag{3.10}$$

for finite positive constants c_+ and c_-, with $c_+ \ge c_-$. (There is also a negative-definite case, analogous to the corresponding case for Euler flows discussed in [5].) For a plane-parallel flow, with $\bar{\boldsymbol{u}}_{\rm Le}(z) = ({\rm d}\bar{\psi}_{\rm e}/{\rm d}z, 0)$, condition (3.10) becomes

$$c_- \le \frac{{\rm d}\bar{\psi}_{\rm e}}{{\rm d}\bar{\omega}_{\rm e}} = \frac{{\rm d}\bar{\psi}_{\rm e}/{\rm d}z}{{\rm d}\bar{\omega}_{\rm e}/{\rm d}z} = \frac{\bar{\boldsymbol{u}}_{\rm Le}}{{\rm d}^2[\bar{\boldsymbol{u}}_{\rm Le} - U_{\rm S}(z)]/{\rm d}z^2} = \frac{\bar{\boldsymbol{u}}_{\rm Le}}{{\rm d}^2\bar{\boldsymbol{u}}_{\rm e}(z)/{\rm d}z^2} \le c_+. \tag{3.11}$$

Consequently, plane-parallel ideal CL flows with no inflection point in the *Eulerian* mean velocity within the region of flow are stable. This is the analog for planar CL flows of Rayleigh's inflection point theorem for planar Euler flows. Actually, this is a slight extension of Rayleigh's theorem, since it refers to Lyapunov stability, besides linearized stability (cf. [5,13]).

3.1.2. Beltrami flows

Another, more complex, example of the steady flow condition $[\bar{\boldsymbol{u}}_{\rm Le}, \bar{\boldsymbol{\omega}}_{\rm e}] = 0$ is the case $\bar{\boldsymbol{u}}_{\rm Le} = \lambda^{-1}\bar{\boldsymbol{\omega}}_{\rm e}$ with $\bar{\boldsymbol{u}}_{\rm e}\cdot\nabla\lambda = 0$. In this case, the steady Lagrangian mean velocity shifts the fluid particles along the lines of steady Eulerian mean vorticity. This situation reduces to the Beltrami flows for Euler's equations when $\bar{\boldsymbol{u}}_{\rm S}$ is absent. The Beltrami flows are truly three-dimensional when λ is a constant. (See, e.g., [4] for discussions and references to Beltrami flows for Euler's equations.)

Steady flows for the ideal CL equations may be obtained from Beltrami flows with constant eigenvalue λ for the Euler equations, by solving

$$\operatorname{curl}\bar{\boldsymbol{u}}_{\mathrm{e}} = \lambda\bar{\boldsymbol{u}}_{\mathrm{e}} + \lambda\bar{\boldsymbol{u}}_{\mathrm{S}}, \tag{3.12}$$

given the prescribed functional form of the Stokes drift. In the case that $\bar{\boldsymbol{u}}_{\mathrm{S}} = U_{\mathrm{S}}(z)\hat{\boldsymbol{x}}$, so that $\operatorname{curl}\bar{\boldsymbol{u}}_{\mathrm{S}} = U_{\mathrm{S}}'(z)\hat{\boldsymbol{y}}$, we obtain

$$-\nabla^2\bar{\boldsymbol{u}}_{\mathrm{e}} = \lambda^2\bar{\boldsymbol{u}}_{\mathrm{e}} + \lambda^2\bar{\boldsymbol{u}}_{\mathrm{S}} + \lambda\operatorname{curl}\bar{\boldsymbol{u}}_{\mathrm{S}} = \lambda^2\bar{\boldsymbol{u}}_{\mathrm{e}} + \lambda^2 U_{\mathrm{S}}(z)\hat{\boldsymbol{x}} + \lambda U_{\mathrm{S}}'(z)\hat{\boldsymbol{y}}, \tag{3.13}$$

by taking the curl of (3.12). Consequently, the inhomogeneous terms (those involving $U_{\mathrm{S}}(z)$) in the Helmholtz equation (3.13) have no influence on the $\hat{\boldsymbol{z}}$-component of $\bar{\boldsymbol{u}}_{\mathrm{e}}$ in this case. The inhomogeneous solutions of the $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$-components of (3.13) satisfy decoupled ordinary differential equation in the depth, z. These are each driven harmonic oscillator equations, that may be solved by Green's function method to give

$$\bar{\boldsymbol{u}}_{\mathrm{e}} = \bar{\boldsymbol{u}}_{\mathrm{Bel}} + \tilde{U}_{\mathrm{S}}(z)\hat{\boldsymbol{x}} + \tilde{U}_{\mathrm{S}}'(z)\hat{\boldsymbol{y}}, \tag{3.14}$$

where the homogeneous solution $\bar{\boldsymbol{u}}_{\mathrm{Bel}}$ satisfies the Euler–Beltrami condition, $\operatorname{curl}\bar{\boldsymbol{u}}_{\mathrm{Bel}} = \lambda\bar{\boldsymbol{u}}_{\mathrm{Bel}}$, and $\hat{U}_{\mathrm{S}}$ denotes the sine transform of U_{S},

$$\tilde{U}_{\mathrm{S}}(z) = \int^{z} \mathrm{d}z' \sin\lambda(z'-z)U_{\mathrm{S}}(\lambda z'), \tag{3.15}$$

satisfying

$$\frac{\mathrm{d}}{\mathrm{d}z}\tilde{U}_{\mathrm{S}} = \lambda\tilde{U}_{\mathrm{S}}' \quad \text{and} \quad -\frac{\mathrm{d}}{\mathrm{d}z}\tilde{U}_{\mathrm{S}}'(z) = \lambda\tilde{U}_{\mathrm{S}}(z) + \lambda U_{\mathrm{S}}(z), \tag{3.16}$$

so that (cf. Eq. (3.13))

$$-\frac{\mathrm{d}^2\tilde{U}_{\mathrm{S}}}{\mathrm{d}z^2} = \lambda^2[\tilde{U}_{\mathrm{S}}(z) + U_{\mathrm{S}}(z)]. \tag{3.17}$$

Consequently, given a Beltrami flow $\bar{\boldsymbol{u}}_{\mathrm{Bel}}$ for the Euler equations, we may obtain a corresponding steady flow of the ideal CL equations, by substituting $\bar{\boldsymbol{u}}_{\mathrm{S}} = U_{\mathrm{S}}(z)\hat{\boldsymbol{x}}$ into (3.14).

The stability of the steady ideal CL flows satisfying (3.12) with constant λ may be ascertained by noticing that these flows are critical points of the conserved functional,

$$H_{\Lambda} = \int_{\mathcal{D}} \mathrm{d}^3x \left[\frac{1}{2}|\bar{\boldsymbol{u}}_{\mathrm{L}}|^2 - \frac{1}{2\lambda}\bar{\boldsymbol{u}}\cdot\bar{\boldsymbol{\omega}}\right], \tag{3.18}$$

which is the Lagrangian mean kinetic energy, constrained by the Eulerian mean helicity. The first variation of H_{Λ} is expressible (after integration by parts) as

$$\delta H_{\Lambda} = \int \mathrm{d}^3x\,(\bar{\boldsymbol{u}}_{\mathrm{L}} - \lambda^{-1}\bar{\boldsymbol{\omega}})\cdot\delta\bar{\boldsymbol{u}}. \tag{3.19}$$

Consequently, the first variation δH_{Λ} vanishes for steady ideal CL Beltrami flows satisfying (3.12). Now, the theory of Lyapunov stability of conservative dynamical systems (see, e.g., [5,13]) implies stability of equilibrium solutions which are critical points of a conserved quantity that is convex at the equilibrium. However, the quadratic conserved quantity H_{Λ} is *not* convex at the equilibrium. This becomes clear upon examining the second variation of H_{Λ},

$$\delta^2 H_{\Lambda} = \int \mathrm{d}^3x\,(|\delta\bar{\boldsymbol{u}}|^2 - \lambda^{-1}\delta\bar{\boldsymbol{u}}\cdot\operatorname{curl}\delta\bar{\boldsymbol{u}}), \tag{3.20}$$

which is conserved, even for *finite* perturbations of the CL Beltrami equilibrium. Being indefinite in sign, this conserved quantity imposes no restrictions on the growth of perturbations, independently of the choice of $\bar{\boldsymbol{u}}_{\mathrm{S}}$. Consequently, the critical points of H_Λ corresponding to the CL Beltrami flows satisfying (3.12) may possess unstable directions, independently of the choice of $\bar{\boldsymbol{u}}_{\mathrm{S}}$. The same situation applies, of course, in Beltrami flows of the Euler equations, when $\bar{\boldsymbol{u}}_{\mathrm{S}}$ is absent.

3.2. Kelvin circulation theorem and vorticity advection

From the ideal CL motion equation (cf. equation (2.1))

$$\frac{\partial \bar{\boldsymbol{u}}}{\partial t} - \bar{\boldsymbol{u}}_{\mathrm{L}} \times \operatorname{curl} \bar{\boldsymbol{u}} + \boldsymbol{\nabla}\left(p + \tfrac{1}{2}|\bar{\boldsymbol{u}}_{\mathrm{L}}|^2\right) = 0, \tag{3.21}$$

a Kelvin circulation theorem may be found for the time-averaged contour integral, with $\boldsymbol{X} = \boldsymbol{x} + \boldsymbol{\xi}$,

$$\begin{aligned} I(t) &= \overline{\oint_{\gamma(t)} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{X}} = \oint_{\bar{\gamma}(t)} (\bar{\boldsymbol{u}}_{\mathrm{L}} \cdot \mathrm{d}\boldsymbol{x} + \overline{\boldsymbol{u}^l \cdot \mathrm{d}\boldsymbol{\xi}}) \\ &= \oint_{\bar{\gamma}(t)} \left[\bar{\boldsymbol{u}}_{\mathrm{L}} - \overline{(\boldsymbol{\xi} \cdot \boldsymbol{\nabla}) \frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t}} - \overline{\boldsymbol{\xi} \times \operatorname{curl} \frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t}} + \boldsymbol{\nabla} \overline{\left(\boldsymbol{\xi} \cdot \frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t}\right)} \right] \cdot \mathrm{d}\boldsymbol{x} \\ &= \oint_{\bar{\gamma}(t)} (\bar{\boldsymbol{u}}_{\mathrm{L}} - \bar{\boldsymbol{u}}_{\mathrm{S}}) \cdot \mathrm{d}\boldsymbol{x} = \oint_{\bar{\gamma}(t)} \bar{\boldsymbol{u}} \cdot \mathrm{d}\boldsymbol{x}, \end{aligned} \tag{3.22}$$

where we have used Eq. (1.11). The closed contour $\bar{\gamma}(t)$ moves with the *Lagrangian* mean velocity $\bar{\boldsymbol{u}}_{\mathrm{L}} = \bar{\boldsymbol{u}} + \bar{\boldsymbol{u}}_{\mathrm{S}}$, since it follows the fluid particles as the time average is taken. The circulation of $\bar{\boldsymbol{u}}$ around such a contour is conserved by the CL equations (1.1), as shown by direct calculation,

$$\begin{aligned} \frac{\mathrm{d}I}{\mathrm{d}t} &= \oint_{\bar{\gamma}(t)} \left[\frac{\partial \bar{\boldsymbol{u}}}{\partial t} + (\bar{\boldsymbol{u}}_{\mathrm{L}} \cdot \boldsymbol{\nabla}) \bar{\boldsymbol{u}} + \bar{u}_j \boldsymbol{\nabla} \bar{u}_{\mathrm{L}}^j \right] \cdot \mathbf{d}\boldsymbol{x} \\ &= \oint_{\bar{\gamma}(t)} \left[\frac{\partial \bar{\boldsymbol{u}}}{\partial t} - \bar{\boldsymbol{u}}_{\mathrm{L}} \times \operatorname{curl} \bar{\boldsymbol{u}} + \boldsymbol{\nabla}(\bar{\boldsymbol{u}}_{\mathrm{L}} \cdot \bar{\boldsymbol{u}}) \right] \cdot \mathbf{d}\boldsymbol{x} \\ &= \oint_{\bar{\gamma}(t)} \boldsymbol{\nabla}\left(- p - \tfrac{1}{2}|\bar{\boldsymbol{u}}_{\mathrm{L}}|^2 + \bar{\boldsymbol{u}}_{\mathrm{L}} \cdot \bar{\boldsymbol{u}}\right) \cdot \mathbf{d}\boldsymbol{x} \\ &= - \oint_{\bar{\gamma}(t)} \boldsymbol{\nabla} \varpi' \cdot \mathbf{d}\boldsymbol{x} = 0, \end{aligned} \tag{3.23}$$

where the third line is obtained by using the ideal CL motion equation (3.21).So the time-averaged Kelvin theorem is satisfied by the ideal CL equations. In fact, we could have *derived* the correct form of the ideal CL equations by requiring that the time-averaged Kelvin theorem be satisfied.

In terms of the Lagrangian mean, then, the Kelvin circulation equation (3.23) expresses itself as a "flux rule." The Lorentz-force version of this is (cf. (1.14))

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\bar{\gamma}(t)} \bar{\boldsymbol{u}}_{\mathrm{L}} \cdot \mathbf{d}\boldsymbol{x} = \frac{\mathrm{d}}{\mathrm{d}t} \oint_{\bar{\gamma}(t)} \bar{\boldsymbol{u}}_{\mathrm{S}} \cdot \mathbf{d}\boldsymbol{x} = \oint_{\bar{\gamma}(t)} \left(\frac{\partial \bar{\boldsymbol{u}}_{\mathrm{S}}}{\partial t} - \bar{\boldsymbol{u}}_{\mathrm{L}} \times \operatorname{curl} \bar{\boldsymbol{u}}_{\mathrm{S}} \right) \cdot \mathbf{d}\boldsymbol{x}. \tag{3.24}$$

This relation rephrases the Kelvin circulation theorem as an electrical-circuit analogy, in which Lagrangian mean "current" is induced by the "electromotive force" of the mean Stokes drift velocity, $\bar{\boldsymbol{u}}_{\mathrm{S}}$, regarded as the negative of the electromagnetic vector potential. The difference is that the "electrical circuit" deforms to follow the fluid as it moves.

The fluid particle labels $l^A(\boldsymbol{x}, t)$, $A = 1, 2, 3$, that parameterize the contour $\bar{\gamma}(t)$ in the CL Kelvin theorem (3.23) follow the Lagrangian mean velocity and, therefore, satisfy

$$\frac{\mathrm{d}l^A}{\mathrm{d}t} = \frac{\partial l^A}{\partial t} + \bar{\boldsymbol{u}}_{\mathrm{L}} \cdot \boldsymbol{\nabla} l^A = 0, \quad A = 1, 2, 3. \tag{3.25}$$

Hence, we may write the components of the Lagrangian mean velocity in terms of derivatives of l^A as

$$\bar{u}^i_{\mathrm{L}} = -(D^{-1})^i_A \frac{\partial l^A}{\partial t}, \tag{3.26}$$

where we sum on repeated indices, as usual, and $(D^{-1})^i_A$ is the inverse of $D^A_i = (\partial l^A / \partial x^i)$, the 3×3 Jacobian matrix for the map from Eulerian coordinates to Lagrangian fluid labels, $l^A(\boldsymbol{x}, t)$, $A = 1, 2, 3$. This inverse exists, provided the determinant $D = \det(D^A_i)$ does not vanish. This determinant is equal to unity for incompressible flow. In fact, as a consequence of (3.25), D satisfies the continuity equation

$$\frac{\partial D}{\partial t} = -\boldsymbol{\nabla} \cdot D\bar{\boldsymbol{u}}_{\mathrm{L}}. \tag{3.27}$$

Thus, if D is initially equal to unity, it will remain so, according to (3.27), provided $\boldsymbol{\nabla} \cdot \bar{\boldsymbol{u}}_{\mathrm{L}} = 0$ at all times. This is ensured by imposing $\boldsymbol{\nabla} \cdot \bar{\boldsymbol{u}}_{\mathrm{S}} = 0$ and solving for the pressure p via the Poisson equation (1.2).

From the mean vorticity equation (3.1), we find by using (3.25) and (3.27) an advection relation

$$\frac{\mathrm{d}\Omega^A}{\mathrm{d}t} = \frac{\partial \Omega^A}{\partial t} + \bar{\boldsymbol{u}}_{\mathrm{L}} \cdot \boldsymbol{\nabla} \Omega^A = 0, \quad A = 1, 2, 3, \tag{3.28}$$

for the quantities Ω^A given by

$$\Omega^A = \frac{\boldsymbol{\nabla} l^A \cdot \operatorname{curl} \bar{\boldsymbol{u}}}{D}, \quad A = 1, 2, 3. \tag{3.29}$$

That is, the CL equations preserve each component of the Eulerian mean flow vorticity, relative to the coordinate frame of the Lagrangian mean flow. This is expressible as the "frozen-in" condition, or Cauchy solution for the mean vorticity,

$$\operatorname{curl} \bar{\boldsymbol{u}}(t) \cdot \mathbf{d}\boldsymbol{S}(t) = \operatorname{curl} \bar{\boldsymbol{u}}(0) \cdot \mathbf{d}\boldsymbol{S}(0), \tag{3.30}$$

where $\mathbf{d}\boldsymbol{S}(t)$ is a surface element composed at time t of line elements that flow under the Lagrangian mean velocity, $\bar{\boldsymbol{u}}_{\mathrm{L}}(t)$. In coordinates, (3.30) is expressible as

$$\operatorname{curl} \bar{\boldsymbol{u}}(t) = \bar{\boldsymbol{u}}_{C,B}(0) \boldsymbol{\nabla} l^B \times \boldsymbol{\nabla} l^C, \tag{3.31}$$

from which we see that

$$\Omega^A = \epsilon^{ABC} \bar{\boldsymbol{u}}_{C,B}(0), \quad A = 1, 2, 3. \tag{3.32}$$

Thus, the quantities Ω^A, $A = 1, 2, 3$, represent the initial Eulerian mean vorticity components, which are then frozen into the Lagrangian mean $\bar{\boldsymbol{u}}_{\mathrm{L}}(t)$ flow. As a consequence of this, the following infinite family of integrals is conserved under the CL dynamics:

$$C_\Phi = \int \mathrm{d}^3x \, D\Phi(\Omega^A), \quad A = 1, 2, 3, \tag{3.33}$$

for *any* function Φ, provided $\bar{\boldsymbol{u}}_{\mathrm{L}}$ is tangent to the boundary. To explain how these conservation laws, as well as the conservation of helicity Λ in (2.6), arise from Noether symmetries, we will write Hamilton's principle for the ideal CL equations in terms of Lagrangian path variations, then project the resulting Hamiltonian formulation onto the Eulerian variables. The conserved quantities C_Φ and Λ will be seen to generate canonical transformations of the Lagrangian mean fluid particle coordinates and their canonically conjugate momenta that leave invariant the Eulerian fluid variables appearing in Hamilton's principle. In fact, these canonical transformations shift the Lagrangian mean fluid particles along the streamlines of the steady ideal CL flows. So the Noether symmetries generated by C_Φ and Λ are precisely the motions of the fluid particles, under steady CL flows in the Eulerian fluid variables. This is a general property of Eulerian fluid dynamics.

4. Hamilton's principle

The ideal CL equations arise from stationarity of a constrained Hamilton's principle $\delta\mathcal{L} = 0$, under variations of the Lagrangian particle paths $l^A(\boldsymbol{x}, t)$, $A = 1, 2, 3$, at constant Eulerian position. The constrained Hamilton's principle is given by

$$\mathcal{L} = \int \mathrm{d}t \int \mathrm{d}^3x\big[\tfrac{1}{2}D|\bar{\boldsymbol{u}}_{\mathrm{L}}|^2 - D\bar{\boldsymbol{u}}_{\mathrm{L}}\cdot\bar{\boldsymbol{u}}_{\mathrm{S}} - p(D-1)\big], \tag{4.1}$$

where $D = \det(\boldsymbol{\nabla} l^A)$, the Lagrangian mean velocity $\bar{\boldsymbol{u}}_{\mathrm{L}}$ is given in (3.26) in terms of derivatives of l^A, and p is a Lagrange multiplier that enforces incompressibility. (The integrand in the action (4.1) is expressible compactly in terms of the pressure, as $p - D\varpi'$.) Varying the action (4.1) with respect to the Lagrange coordinate l^A at fixed $\boldsymbol{x}$ and t gives

$$\delta\mathcal{L} = \int \mathrm{d}t \int \mathrm{d}^3x\,\big[\delta\bar{\boldsymbol{u}}_{\mathrm{L}}\cdot(D\bar{\boldsymbol{u}}_{\mathrm{L}} - D\bar{\boldsymbol{u}}_{\mathrm{S}}) + \delta D\big(\tfrac{1}{2}|\bar{\boldsymbol{u}}_{\mathrm{L}}|^2 - \bar{\boldsymbol{u}}_{\mathrm{L}}\cdot\bar{\boldsymbol{u}}_{\mathrm{S}} - p\big) - \delta p(D-1)\big] \tag{4.2}$$

with definitions

$$\delta D = D(D^{-1})^i_A\delta l^A_{,i} \qquad \delta\bar{u}^i_{\mathrm{L}} = -(D^{-1})^i_{\mathrm{B}}\bar{u}^j_{\mathrm{L}}\delta l^B_{,j} - (D^{-1})^i_{\mathrm{B}}\delta l^B_{,t}. \tag{4.3}$$

Substituting the definition of $\delta\bar{u}^i_{\mathrm{L}}$ into the variational formula (4.2) and using $\bar{\boldsymbol{u}} = \bar{\boldsymbol{u}}_{\mathrm{L}} - \bar{\boldsymbol{u}}_{\mathrm{S}}$ gives the momentum π_A canonically conjugate to l^A as

$$\pi_A = \frac{\delta L}{\delta l^A_{,t}} = -D\bar{u}_i(D^{-1})^i_A. \tag{4.4}$$

Hence, the ith component of the Eulerian mean velocity is related to the canonical variables l^A and π_A by

$$\bar{u}_i = -D^{-1}\pi_A l^A_{,i}. \tag{4.5}$$

Upon integrating by parts and using the tangency conditions on the boundary, the variation of the action (4.2) becomes

$$\begin{aligned}\delta\mathcal{L} = \int \mathrm{d}t \int \mathrm{d}^3x\,\Big\{&\delta l^A[\partial_t(D(D^{-1})^i_A\bar{u}_i) + \partial_j(D\bar{u}^j_{\mathrm{L}}(D^{-1})^i_A\bar{u}_i)]\\ &+\delta l^A\partial_i[D(D^{-1})^i_A(p - \tfrac{1}{2}|\bar{\boldsymbol{u}}|^2 + \tfrac{1}{2}|\bar{\boldsymbol{u}}_{\mathrm{S}}|^2)] - \delta p(D-1)\Big\}.\end{aligned} \tag{4.6}$$

Rearrangement of formula (4.6) using the continuity equation (3.27) for D and the identities

$$\partial_j D = D(D^{-1})^i_A \partial_j D^A_i, \qquad (D(D^{-1})^i_A)_{,i} = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}(D^{-1})^i_A = \bar{u}^i_{\mathrm{L},j}(D^{-1})^j_A \tag{4.7}$$

gives the following expression for the variation of the action:

$$\delta\mathcal{L} = \int \mathrm{d}t \int \mathrm{d}^3x \, \{D(D^{-1})^i_A \delta l^A [(\partial_t + \bar{u}^j_{\mathrm{L}}\partial_j)\bar{u}_i + \bar{u}_j \partial_i \bar{u}^j_{\mathrm{L}} + \partial_i(p - \tfrac{1}{2}|\bar{\boldsymbol{u}}|^2 + \tfrac{1}{2}|\bar{\boldsymbol{u}}_{\mathrm{S}}|^2)] - \delta p(D-1)\}, \tag{4.8}$$

Vanishing of $\delta\mathcal{L}$ for arbitrary variations δl^A and δp within the domain of flow now implies, upon using the fundamental vector identify of fluid mechanics,

$$-\bar{\boldsymbol{u}}_{\mathrm{L}} \times \operatorname{curl} \bar{\boldsymbol{u}} = (\bar{\boldsymbol{u}}_{\mathrm{L}} \cdot \boldsymbol{\nabla})\bar{\boldsymbol{u}} - \bar{u}^j_{\mathrm{L}} \boldsymbol{\nabla} \bar{u}_j, \tag{4.9}$$

that

$$\frac{\partial \bar{\boldsymbol{u}}}{\partial t} - \bar{\boldsymbol{u}}_{\mathrm{L}} \times \operatorname{curl} \bar{\boldsymbol{u}} + \boldsymbol{\nabla}(p + \tfrac{1}{2}|\bar{\boldsymbol{u}}_{\mathrm{L}}|^2) = 0, \tag{4.10}$$

and

$$D = 1, \tag{4.11}$$

which recover the ideal CL equations (3.21).

In [12], the Euler equations for an incompressible fluid in three dimensions are derived from stationarity of the constrained action (4.1) with $\bar{\boldsymbol{u}}_{\mathrm{S}}$ absent. The action $\mathcal{L}$ appearing in (4.1) in terms of averaged quantities may be interpreted as the kinetic energy of the fluid particles, coupled to the "external field" $\bar{\boldsymbol{u}}_{\mathrm{S}}$ through the particle "current" $D\bar{\boldsymbol{u}}_{\mathrm{L}}$ and constrained by incompressibility. This is the same as the action for a fluid plasma, driven by a rapidly varying external electromagnetic field via "$\boldsymbol{J} \cdot \boldsymbol{A}$" coupling, after averaging in Hamilton's principle over the fast phase of the electromagnetic field in the action as in [23,24]. The further approximation is made for the CL equations that the fluid motion does not act back on the surface wave field, since this field is assumed to produce a prescribed Stokes drift velocity, $\bar{\boldsymbol{u}}_{\mathrm{S}}$. Thus, the plasma physics analogy encountered earlier upon writing the CL equations in the form (1.14) is exact in the case of prescribed wave motion.

We next pass to the Hamiltonian formulation via the Legendre transformation, using the relation obtained from (3.26) and (4.4),

$$\pi_A l^A_{,t} = -D\bar{u}_i (D^{-1})^i_A l^A_{,t} = D\bar{\boldsymbol{u}} \cdot \bar{\boldsymbol{u}}_{\mathrm{L}}, \tag{4.12}$$

to find the Hamiltonian. Then we transform the Poisson bracket from the canonically conjugate variables π_A and $l^A(\boldsymbol{x},t)$, $A = 1,2,3$, to the noncanonical Eulerian fluid variables, $D = \det \boldsymbol{\nabla} l^A$ and $\boldsymbol{m} = D\bar{\boldsymbol{u}} = -\pi_A \boldsymbol{\nabla} l^A$, by using the chain rule for functional derivatives. This will yield the ideal CL equations in Lie–Poisson Hamiltonian form [12] in terms of the noncanonical Eulerian fluid variables, thereby allowing us to investigate the Noether symmetries of the CL theory as canonical transformations and the equilibrium solutions of the CL theory as critical points of conserved quantities.

5. Hamiltonian structure

Passing from the constrained Lagrangian (4.1) for the CL equations via the Legendre transformation yields the following constrained CL Hamiltonian (actually this is a Routhian; the pressure p is not Legendre-transformed, since it has no canonically conjugate momentum, see e.g., [12] for the analogous situation in the case of the incompressible Euler equations):

$$H = \int \mathrm{d}^3x \big[\tfrac{1}{2}D|\bar{\boldsymbol{u}} + \bar{\boldsymbol{u}}_\mathrm{S}|^2 + p(D-1)\big]. \tag{5.1}$$

Evaluating this Hamiltonian at $D = 1$ gives the Lagrangian mean kinetic energy. The definitions $D = \det \boldsymbol{\nabla} l^A$ and $\boldsymbol{m} = -\pi_A \boldsymbol{\nabla} l^A = D\bar{\boldsymbol{u}}$ allow one to use the chain rule to transform the canonical Poisson bracket in terms of π_A and l^A, that follows from Hamilton's principle with Lagrangian (4.1),

$$\{F, G\}(\pi_A, l^A) = -\int \mathrm{d}^3x \left[\frac{\delta F}{\delta \pi_A}\frac{\delta G}{\delta l^A} - \frac{\delta G}{\delta \pi_A}\frac{\delta F}{\delta l^A}\right], \tag{5.2}$$

into the Lie–Poisson bracket in terms of variables $\boldsymbol{m}$ and D that is discussed in [12]. Namely,

$$\{F, G\}(\boldsymbol{m}, D) = -\int \mathrm{d}^3x \left[\frac{\delta F}{\delta m_i}\left((\partial_j m_i + m_j \partial_i)\frac{\delta G}{\delta m_j} + D\partial_i \frac{\delta G}{\delta D}\right) + \frac{\delta F}{\delta D}\partial_j\left(D\frac{\delta G}{\delta m_j}\right)\right], \tag{5.3}$$

where $\partial_j = \partial/\partial x^j$, $j = 1, 2, 3$, operates on all terms it multiplies to its right. This Lie–Poisson bracket satisfies the Jacobi identity,

$$\{E, \{F, G\}\} + \{F, \{G, E\}\} + \{G, \{E, F\}\} = 0, \tag{5.4}$$

for any functionals E, F and G of $\boldsymbol{m}$ and D, simply because (5.4) is a variable transform of the Jacobi identity for the canonical Poisson bracket.

In terms of the variables $\boldsymbol{m}$ and D, the Hamiltonian (5.1) is expressible as

$$H = \int \mathrm{d}^3x \left[\frac{1}{2D}|\boldsymbol{m}|^2 + \boldsymbol{m}\cdot\bar{\boldsymbol{u}}_\mathrm{S} + \tfrac{1}{2}D|\bar{\boldsymbol{u}}_\mathrm{S}|^2 + p(D-1)\right] \tag{5.5}$$

with variational derivatives

$$\frac{\delta H}{\delta \boldsymbol{m}} = \boldsymbol{m}/D + \bar{\boldsymbol{u}}_\mathrm{S} = \bar{\boldsymbol{u}}_\mathrm{L}, \qquad \frac{\delta H}{\delta D} = p + \tfrac{1}{2}|\bar{\boldsymbol{u}}_\mathrm{S}|^2 - \tfrac{1}{2}|\bar{\boldsymbol{u}}|^2 = \varpi'. \tag{5.6}$$

The corresponding equations of motion are given in Hamiltonian form by

$$\begin{aligned} \frac{\partial m_i}{\partial t} &= \{m_i, H\} = -(\partial_j m_i + m_j \partial_i)\bar{u}^j_\mathrm{L} - D\partial_i \varpi' \\ \frac{\partial D}{\partial t} &= \{D, H\} = -\partial_j D \bar{u}^j_\mathrm{L}. \end{aligned} \tag{5.7}$$

These are the CL equations (1.1) in Lie–Poisson Hamiltonian form in terms of $\boldsymbol{m}$ and D. These equations imply the CL motion equation in the form that appears in the Kelvin theorem calculation, (3.23),

$$\frac{\partial \bar{\boldsymbol{u}}}{\partial t} = -(\bar{\boldsymbol{u}}_\mathrm{L}\cdot\boldsymbol{\nabla})\bar{\boldsymbol{u}} - \bar{u}_j \boldsymbol{\nabla}\bar{u}^j_\mathrm{L} - \boldsymbol{\nabla}\varpi'. \tag{5.8}$$

As discussed after Eq. (3.27), if D is initially equal to unity, it will remain so under these dynamics, provided the Lagrange multiplier p in (5.5) satisfies the Poisson equation (1.2) and the Stokes drift velocity is taken to be divergenceless, $\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}_\mathrm{S} = 0$.

6. Noether symmetries

The Lagrangian (4.1) and the Hamiltonian (5.1) are invariant under transformations of the fluid-particle labels $l^A(\boldsymbol{x}, t)$, $A = 1, 2, 3$, and their canonically conjugate momenta π_A that leave invariant the Eulerian fluid variables $\boldsymbol{m}$ and D. Among these symmetry transformations are those generated by the helicity Λ in (2.6) and the frozen-in vorticity conservation laws C_Φ in (3.33). This may be checked by computing the infinitesimal canonical transformations generated by Λ and C_Φ according to either bracket, (5.2) or (5.3). First, the helicity Λ satisfies the Poisson bracket relations (ignoring surface terms)

$$\{\Lambda, \boldsymbol{m}\} = 0, \qquad \{\Lambda, D\} = 0 \qquad \{\Lambda, l^A\} = (1/D)\overline{\boldsymbol{\omega}} \cdot \nabla l^A, \tag{6.1}$$

in which the Eulerian variables are invariant and the particle labels are shifted along the mean vorticity. Thus, the helicity is conserved, since it Poisson-commutes with the Hamiltonian (5.1) depending only on the Eulerian fluid variables. Moreover, the Noether symmetry that the helicity generates as an infinitesimal canonical transformation via the Poisson bracket relations (6.1) is a shift of the fluid particles along the streamlines of a steady CL *Beltrami* flow. This explains why the CL Beltrami flows are associated with critical points of the sum of energy and helicity: the CL Beltrami flows are *relative* equilibria. That is, they are stationary relative to the frame of motion of the fluid particles generated by the conserved helicity the Lie–Poisson bracket (5.3).

Next we compute the infinitesimal canonical transformations generated by C_Φ, via the Poisson bracket relations,

$$\begin{aligned} &\{C_\Phi, \boldsymbol{m}\} = 0, \qquad \{C_\Phi, D\} = 0, \\ &\{C_\Phi, l^A\} = \tilde{\boldsymbol{v}} \cdot \nabla l^A, \quad \text{with } \tilde{\boldsymbol{v}} = D^{-1}\nabla l^B \times \nabla \frac{\partial \Phi}{\partial \Omega^B}, \end{aligned} \tag{6.2}$$

obtained using $\{l^A(\boldsymbol{x}), m_i(\boldsymbol{x}')\} = l^A_{,i}\delta(\boldsymbol{x} - \boldsymbol{x}')$ and the chain rule. Thus, the symmetry generated by C_Φ is a shift in the particle labels l^A by an amount $\tilde{\boldsymbol{v}}$ depending on the function Φ. Under this canonical transformation of the particle labels, the Hamiltonian H in (5.5) is invariant, since the Eulerian fluid variables $\boldsymbol{m}$ and D are invariant. Hence, the conserved integrals C_Φ are Noether symmetries that shift the fluid particle labels along the vector field $\tilde{\boldsymbol{v}}$ in (6.2) without changing the Eulerian fluid variables. This implies that $\overline{\boldsymbol{u}}_{\mathrm{Le}} = \tilde{\boldsymbol{v}}$ in (6.2) is a steady solution of the CL equations. Note that $\tilde{\boldsymbol{v}}$ in (6.2) satisfies the symmetry relation $[\tilde{\boldsymbol{v}}, \overline{\boldsymbol{\omega}}_{\mathrm{e}}] = 0$ required by (3.1) for steady solutions. Since they leave the Eulerian fluid variables invariant, the spatial transformations (volume-preserving diffeomorphisms) of the fluid labels generated by the vector fields of the steady flows may be regarded as the "gauge transformations" of fluid dynamics.

Having cast the CL equations into Hamiltonian form, we now may interpret the energy balance relation (2.3) simply as

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t}, \tag{6.3}$$

when $\nu_{\mathrm{T}} = 0$. This is the usual relation for time-dependent Hamiltonian systems. If $\partial\overline{\boldsymbol{u}}_{\mathrm{S}}/\partial t = 0$, as well, the Lagrangian mean kinetic energy is preserved and the CL system enjoys all the properties of Lie–Poisson Hamiltonian fluid system, including classification of its (relative) equilibrium solutions as critical points of the constrained energy $H + C_\Phi$ and a method for determining Lyapunov stability conditions for these equilibria, as in, e.g., [13] Thus, the vortex forcing introduced by Craik and Leibovich to parameterize the effects due to rapidly fluctuating surface waves which may produce Langmuir circulations has the advantage of preserving the fundamental Hamiltonian structure of the Euler equations for an incompressible fluid.

7. Boussinesq approximation with rotation

7.1. The averaged equations

The motion of a rotating continuously stratified ideal incompressible fluid is governed by the adiabatic inviscid Euler equations, in which the effects of buoyancy are treated in the Boussinesq approximation and the Coriolis parameter $2f$ is allowed to vary spatially. Euler's equations in the Boussinesq approximation (EB equations) for such a fluid are

$$\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u}+2f\hat{\boldsymbol{z}}\times\boldsymbol{u}+g\rho\hat{\boldsymbol{z}}+\boldsymbol{\nabla}p=0,\qquad \left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\rho=0,\qquad \boldsymbol{\nabla}\cdot\boldsymbol{u}=0, \tag{7.1}$$

where ρ is the buoyancy and f is the local rotation frequency about the vertical direction, $\hat{\boldsymbol{z}}$.

The EB motion equation in (7.1) implies the following Kelvin circulation theorem for any closed curve $\gamma(t)$ moving with the fluid:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma(t)}(\boldsymbol{u}+\boldsymbol{R}(\boldsymbol{x}))\cdot\mathbf{dx}&=\oint_{\gamma(t)}\left[\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)(\boldsymbol{u}+\boldsymbol{R})+(u_j+R_j)\boldsymbol{\nabla}u^j\right]\cdot\mathbf{dx}\\ &=-g\oint_{\gamma(t)}\rho\hat{\boldsymbol{z}}\cdot\mathbf{dx}+\oint_{\gamma(t)}\boldsymbol{\nabla}\left(-p+\tfrac{1}{2}|\boldsymbol{u}|^2+\boldsymbol{u}\cdot\boldsymbol{R}\right)\cdot\mathbf{dx}\\ &=-g\oint_{\gamma(t)}\rho\,\mathrm{d}z,\end{aligned} \tag{7.2}$$

where $\operatorname{curl}\boldsymbol{R}(\boldsymbol{x})=2f(\boldsymbol{x})\hat{\boldsymbol{z}}$. We may derive the form of the time-averaged EB motion equation by first decomposing the fluid velocity into its Eulerian mean plus irrotational fluctutions, $\boldsymbol{u}(\boldsymbol{x},t)=\bar{\boldsymbol{u}}(\boldsymbol{x},t)+\boldsymbol{u}'(\boldsymbol{x},t)$, as in Eq. (1.6). Substituting this decomposition into the Kelvin circulation theorem (7.2) for the EB equations and averaging gives (cf. Eq. (3.22))

$$\overline{\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma(t)}(\bar{\boldsymbol{u}}+\boldsymbol{u}'+\boldsymbol{R}(\boldsymbol{x}))\cdot\,\mathrm{d}\boldsymbol{x}}=\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\bar{\gamma}(t)}(\bar{\boldsymbol{u}}+\boldsymbol{R}(\boldsymbol{x}))\cdot\,\mathrm{d}\boldsymbol{x}=-g\oint_{\bar{\gamma}(t)}\rho\,\mathrm{d}z. \tag{7.3}$$

Thus, the time-averaged EB motion equation takes the following form (cf. Eqs. (3.23) and (7.2)):

$$\frac{\mathrm{d}}{\mathrm{d}t}(\bar{\boldsymbol{u}}+\boldsymbol{R})+(\bar{u}_j+R_j)\boldsymbol{\nabla}\bar{u}^j_{\mathrm{L}}=-g\rho\hat{\boldsymbol{z}}-\boldsymbol{\nabla}\bar{p}, \tag{7.4}$$

where $\mathrm{d}/\mathrm{d}t=\partial/\partial t+\bar{\boldsymbol{u}}_{\mathrm{L}}\cdot\boldsymbol{\nabla}$, as usual, and $\bar{p}$ is an undetermined function. A slight rearrangement of Eq. (7.4) gives

$$\frac{\partial\bar{\boldsymbol{u}}}{\partial t}-\bar{\boldsymbol{u}}_{\mathrm{L}}\times(\operatorname{curl}\bar{\boldsymbol{u}}+2f\hat{\boldsymbol{z}})=-g\rho\hat{\boldsymbol{z}}-\boldsymbol{\nabla}(\bar{p}+(\bar{\boldsymbol{u}}+\boldsymbol{R})\cdot\bar{\boldsymbol{u}}_{\mathrm{L}}). \tag{7.5}$$

Next, we rederive this form of the time-averaged EB equations by modifying the action (4.1) in Hamilton's principle for the CL theory to include effects of rotation and stratification, then pass to its Hamiltonian formulation and study its equilibrium solutions and their stability using the Hamiltonian framework.

7.2. Hamilton's principle

Hamitlon's principle (4.1) may be modified to incorporate buoyancy effects due to density stratification in the Boussinesq approximation and Coriolis force due to rotation, by accounting for gravitational potential energy and rotation as follows:

$$\mathcal{L}_{\mathrm{B}} = \int \mathrm{d}t \int \mathrm{d}^3x \big[\tfrac{1}{2}D|\bar{\boldsymbol{u}}_{\mathrm{L}}|^2 - \rho D g z - D\bar{\boldsymbol{u}}_{\mathrm{L}} \cdot \bar{\boldsymbol{u}}_{\mathrm{S}} + D\bar{\boldsymbol{u}}_{\mathrm{L}} \cdot \boldsymbol{R}(\boldsymbol{x}) - p(D-1)\big]. \tag{7.6}$$

This action is the difference of the kinetic and potential energy, plus "$\boldsymbol{J} \cdot \boldsymbol{A}$" coupling to external driving and rotation, along with the incompressibility constraint imposed by the pressure as a Lagrange multiplier. The canonical momentum π_A now includes the prescribed "vector potential" $\boldsymbol{R}$, according to

$$\pi_A = \frac{\delta L}{\delta l^A_{,t}} = -D(\bar{u}_i + R_i)(D^{-1})^i_A. \tag{7.7}$$

The action $\mathcal{L}_{\mathrm{B}}$ depends on the fluid variables l^A only through the quantities $\bar{\boldsymbol{u}}_{\mathrm{L}}$, D and ρ, and incompressibility is imposed by $-p(D-1)$. In this case, Hamilton's principle gives

$$\begin{aligned}\delta\mathcal{L}_{\mathrm{B}} = {} & \int \mathrm{d}t \int \mathrm{d}^3x \left\{ D(D^{-1})^i_A \delta l^A \left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{D}\frac{\delta \mathcal{L}_{\mathrm{B}}}{\delta \bar{u}^i_{\mathrm{L}}} + \frac{1}{D}\frac{\delta \mathcal{L}_{\mathrm{B}}}{\delta \bar{u}^j_{\mathrm{L}}}\bar{u}^j_{\mathrm{L},i} - \left(\frac{\delta \mathcal{L}_{\mathrm{B}}}{\delta D}\right)_{,i} + \frac{1}{D}\rho_{,i}\frac{\delta \mathcal{L}_{\mathrm{B}}}{\delta \rho} \right] - \delta p (D-1) \right\} \\ & - \int \mathrm{d}t \int \mathrm{d}^3x \left\{ \frac{\partial}{\partial t}\left[D(D^{-1})^i_A \delta l^A \frac{1}{D}\frac{\delta \mathcal{L}_{\mathrm{B}}}{\delta \bar{u}^i_{\mathrm{L}}} \right] \right. \\ & \left. + \frac{\partial}{\partial x^j}\left[D(D^{-1})^i_A \delta l^A \left(-\frac{\delta \mathcal{L}_{\mathrm{B}}}{\delta D}\delta^j_i + \frac{1}{D}\frac{\delta \mathcal{L}_{\mathrm{B}}}{\delta \bar{u}^i_{\mathrm{L}}}\bar{u}^j_{\mathrm{L}} \right) \right] \right\}.\end{aligned} \tag{7.8}$$

Vanishing of the coefficient of δl^A gives the motion equation, while vanishing of the coefficient of δp gives volume preservation. Vanishing of the exact derivatives gives the "natural" boundary conditions. We note that the coefficient of δl^A in square brackets in (7.8) is expressible as

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\bar{\gamma}(t)} \frac{1}{D}\frac{\delta \mathcal{L}_{\mathrm{B}}}{\delta \bar{\boldsymbol{u}}_{\mathrm{L}}} \cdot \mathbf{d}\boldsymbol{x} = -\oint_{\bar{\gamma}(t)} \frac{1}{D}\frac{\delta \mathcal{L}_{\mathrm{B}}}{\delta \rho}\,\mathrm{d}\rho, \tag{7.9}$$

where the contour $\bar{\gamma}(t)$ moves with velocity $\bar{\boldsymbol{u}}_{\mathrm{L}}$. This is Kelvin's circulation theorem, which thus holds for *any* action whose dependence on l^A is expressed only in terms of the Eulerian variables $\bar{\boldsymbol{u}}_{\mathrm{L}}$, D and ρ.

The variation of the action (7.6) at fixed x and t give (cf. Eq. (4.2))

$$\begin{aligned}\delta\mathcal{L}_{\mathrm{B}} = \int \mathrm{d}t \int \mathrm{d}^3x \big[& \delta\bar{\boldsymbol{u}}_{\mathrm{L}} \cdot (D\bar{\boldsymbol{u}} - D\boldsymbol{R}(\boldsymbol{x})) + \delta D\big(\tfrac{1}{2}|\bar{\boldsymbol{u}}_{\mathrm{L}}|^2 - \bar{\boldsymbol{u}}_{\mathrm{L}} \cdot \bar{\boldsymbol{u}}_{\mathrm{S}} + \bar{\boldsymbol{u}}_L \cdot \boldsymbol{R}(\boldsymbol{x}) - p - \rho g z\big) \\ & - \delta\rho(gzD) - \delta p(D-1)\big],\end{aligned} \tag{7.10}$$

with definitions (cf. (4.3))

$$\delta D = D(D^{-1})^i_A \delta l^A_{,i}, \quad \delta\rho = \frac{\partial \rho}{\partial l^A}\delta l^A, \quad \delta\bar{u}^i_{\mathrm{L}} = -(D^{-1})^i_{\mathrm{B}}\bar{u}^j_{\mathrm{L}}\delta l^{\mathrm{B}}_{,j} - (D^{-1})^i_{\mathrm{B}}\delta l^{\mathrm{B}}_{,t}. \tag{7.11}$$

The equations resulting from Hamilton's principle with the modified action $\mathcal{L}_{\mathrm{B}}$ in (7.6) may be obtained either by following the same route as in (4.6)–(4.10), or by substituting the variational derivatives in Eq. (7.10) into the general form (7.8) for $\delta\mathcal{L}_{\mathrm{B}}$. These equations are:

$$\begin{aligned}\frac{\partial \bar{\boldsymbol{u}}}{\partial t} &= \bar{\boldsymbol{u}}_{\mathrm{L}} \times \operatorname{curl}(\bar{\boldsymbol{u}} + \boldsymbol{R}) - \rho g \hat{\boldsymbol{z}} - \boldsymbol{\nabla}\big(p + \tfrac{1}{2}|\bar{\boldsymbol{u}}_{\mathrm{L}}|^2\big), \\ \frac{\partial \rho}{\partial t} &= -\bar{\boldsymbol{u}}_{\mathrm{L}} \cdot \boldsymbol{\nabla}\rho, \qquad \boldsymbol{\nabla} \cdot \bar{\boldsymbol{u}} = 0.\end{aligned} \tag{7.12}$$

These CL equations in the Boussinesq approximation (CLB equations) first appear in [16]. Of course, these equations are in the form (7.5) obtained by averaging Kelvin's circulation theorem for the EB equations and assuming that the Eulerian velocity fluctuations are irrotational.

An interesting feature of the CLB equations (7.12) is that the rotating frame couples to the shear of the mean Stokes drift velocity to produce vortex stretching in the direction of $\bar{\boldsymbol{u}}_{\mathrm{S}}$. Taking the curl of the CLB motion equation gives (cf. Eq. (1.3))

$$\frac{\partial \bar{\boldsymbol{\omega}}}{\partial t} = -\bar{\boldsymbol{u}}_{\mathrm{L}} \cdot \boldsymbol{\nabla}(\bar{\boldsymbol{\omega}} + 2f\hat{\boldsymbol{z}}) + (\bar{\boldsymbol{\omega}} + 2f\hat{\boldsymbol{z}}) \cdot \boldsymbol{\nabla}(\boldsymbol{u} + \bar{\boldsymbol{u}}_{\mathrm{S}}). \tag{7.13}$$

If $\bar{\boldsymbol{u}}_{\mathrm{S}}$ is time-independent, the term $2f\hat{\boldsymbol{z}} \cdot \boldsymbol{\nabla}\bar{\boldsymbol{u}}_{\mathrm{S}}$ is a *steady* source of Eulerian mean vorticity in the direction of $\bar{\boldsymbol{u}}_{\mathrm{S}}$. Thus, in the CLB equations, a vertical gradient of $\bar{\boldsymbol{u}}_{\mathrm{S}}$ conspires with the Coriolis force to produce a steady source of Eulerian mean vorticity along $\bar{\boldsymbol{u}}_{\mathrm{S}}$, by the mechanism of vortex stretching. In reality (or, in a higher order theory) this apparent difficulty would be removed by a self-consistent dependence of $\bar{\boldsymbol{u}}_{\mathrm{S}}$ on the mean Lagrangian flow velocity $\bar{\boldsymbol{u}}_{\mathrm{L}}$, which would allow feedback between the fluctuations and the mean flow. See [11] for details.

7.3. Hamiltonian structure

We may now pass from Hamilton's principle (7.6) to the corresponding noncanonical Hamiltonian theory for the CLB equations, (7.12). See, e.g., [12,1,2] for descriptions of this step in the case of the usual Boussinesq approximation, without the vortex forcing of the CL theory. The Hamiltonian that results from (7.6) via the Legendre transformation is

$$H_{\mathrm{B}} = \int \mathrm{d}^3x\, \big[\tfrac{1}{2} D|\bar{\boldsymbol{u}} + \bar{\boldsymbol{u}}_{\mathrm{S}}|^2 + D\rho g z + p(D-1)\big]. \tag{7.14}$$

Upon defining

$$\rho = \rho(l^A), \qquad D = \det \boldsymbol{\nabla} l^A, \qquad \boldsymbol{\mu} = -\pi_A \boldsymbol{\nabla} l^A = D(\bar{\boldsymbol{u}} + \boldsymbol{R}) = \frac{\delta \mathcal{L}_{\mathrm{B}}}{\delta \bar{\boldsymbol{u}}_{\mathrm{L}}}, \tag{7.15}$$

the canonical Poisson bracket (5.2) in terms of π_A and l^A transforms into the following Lie–Poisson bracket in terms of variables $\boldsymbol{\mu}$, D and ρ:

$$\begin{aligned}\{F, G\}(\boldsymbol{\mu}, D) = -\int \mathrm{d}^3x \bigg[&\frac{\delta F}{\delta \mu_i}\left((\partial_j \mu_i + \mu_j \partial_i)\frac{\delta G}{\delta \mu_j} + D\partial_i \frac{\delta G}{\delta D} - \rho_{,i}\frac{\delta G}{\delta \rho}\right) \\ &+ \frac{\delta F}{\delta D}\partial_j D \frac{\delta G}{\delta \mu_j} + \frac{\delta F}{\delta \rho}\rho_{,j}\frac{\delta G}{\delta \mu_j}\bigg],\end{aligned} \tag{7.16}$$

where, as before, $\partial_j = \partial/\partial x^j$, $j = 1, 2, 3$, operates on all terms it multiplies to its right.

In terms of the variables $\boldsymbol{\mu}$, D and ρ, the Hamiltonian H_{B} in (7.14) is expressible as

$$H_{\mathrm{B}} = \int \mathrm{d}^3x \left[\frac{D}{2}\left|\frac{\boldsymbol{\mu}}{D} - \boldsymbol{R} + \bar{\boldsymbol{u}}_{\mathrm{S}}\right|^2 + D\rho g z + p(D-1)\right] \tag{7.17}$$

with variational derivatives

$$\begin{aligned} \frac{\delta H_{\mathrm{B}}}{\delta \boldsymbol{\mu}} &= \boldsymbol{\mu}/D - \boldsymbol{R} + \bar{\boldsymbol{u}}_{\mathrm{S}} = \bar{\boldsymbol{u}}_{\mathrm{L}}, \qquad \frac{\delta H_{\mathrm{B}}}{\delta \rho} = Dgz, \\ \frac{\delta H_{\mathrm{B}}}{\delta D} &= p + \tfrac{1}{2}|\bar{\boldsymbol{u}}_{\mathrm{S}}|^2 - \tfrac{1}{2}|\bar{\boldsymbol{u}}|^2 - \bar{\boldsymbol{u}}_{\mathrm{L}}\cdot\boldsymbol{R} + \rho g z = -\frac{\delta \mathcal{L}_{\mathrm{B}}}{\delta D} = \varpi' - \bar{\boldsymbol{u}}_{\mathrm{L}}\cdot\boldsymbol{R} + \rho g z. \end{aligned} \tag{7.18}$$

The CLB equations (7.12) are then given in Hamiltonian form by

$$\frac{\partial \mu_i}{\partial t} = \{\mu_i, H_{\mathrm{B}}\}, \qquad \frac{\partial \rho}{\partial t} = \{\rho, H_{\mathrm{B}}\}, \qquad \frac{\partial D}{\partial t} = \{D, H_{\mathrm{B}}\} \tag{7.19}$$

in terms of the Hamiltonian H_{B} and Lie–Poisson bracket (7.16). Namely,

$$\begin{aligned} \frac{\partial \mu_i}{\partial t} &= -\partial_j \mu_i \bar{u}^j_{\mathrm{L}} - \mu_j \partial_i \bar{u}^j_{\mathrm{L}} - D\partial_i(\varpi' - \bar{\boldsymbol{u}}\cdot\boldsymbol{R} + \rho g z) + Dgz\rho_{,i}, \\ \frac{\partial D}{\partial t} &= -\partial_j D\bar{u}^j_{\mathrm{L}}, \qquad \frac{\partial \rho}{\partial t} = -\rho_{,j}\bar{u}^j_{\mathrm{L}}. \end{aligned} \tag{7.20}$$

Rearranging gives (cf. Eq. (7.12))

$$\frac{\partial \bar{\boldsymbol{u}}}{\partial t} = \bar{\boldsymbol{u}}_{\mathrm{L}} \times (\mathrm{curl}\,\bar{\boldsymbol{u}} + 2f\hat{\boldsymbol{z}}) - \boldsymbol{\nabla}\big(p + \tfrac{1}{2}|\bar{\boldsymbol{u}}_{\mathrm{L}}|^2\big) - g\rho\hat{\boldsymbol{z}}. \tag{7.21}$$

According to these dynamics, if D is initially equal to unity, it will remain so, provided $\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}_{\mathrm{S}} = 0$ and the Lagrange multiplier p satisfies the Poisson equation obtained by taking the divergence of (7.21) and requiring that $\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}} = 0$.

7.4. Nother symmetries

Taking the scalar product of $\boldsymbol{\nabla}\rho$ with the curl of the CLB motion equation (7.21) gives conservation of potential vorticity $q = \boldsymbol{\nabla}\rho\cdot(\mathrm{curl}\,\bar{\boldsymbol{u}} + 2f\hat{\boldsymbol{z}})/D$ along flow lines of the Lagrangian mean velocity $\bar{\boldsymbol{u}}_{\mathrm{L}}$. That is,

$$\frac{\partial q}{\partial t} = -\bar{\boldsymbol{u}}_{\mathrm{L}}\cdot\boldsymbol{\nabla} q, \quad \text{for } q = \frac{\boldsymbol{\nabla}\rho\cdot\bar{\boldsymbol{\Omega}}}{D} \ \text{ with } \bar{\boldsymbol{\Omega}} = \mathrm{curl}\,\bar{\boldsymbol{u}} + 2f\hat{\boldsymbol{z}}, \quad D = 1. \tag{7.22}$$

Consequently, the following infinite family of integrals is conserved under the CLB dynamics:

$$C_\Phi = \int \mathrm{d}^3x\, D\Phi(\rho, q), \tag{7.23}$$

for any function Φ of its two arguments.

Under the Lie–Poisson bracket (7.16) for the CLB dynamics, the infinitesimal transformation generated by the conserved quantities C_Φ leaves invariant the Eulerian fluid variables $\boldsymbol{\mu}$, D and ρ. That is (cf. (6.2)),

$$\{C_\Phi, \boldsymbol{\mu}\} = 0 = \{C_\Phi, D\} = \{C_\Phi, \rho\}. \tag{7.24}$$

The corresponding infinitesimal canonical transformation (gauge transformation) of the Lagrangian fluid labels $l^A(\boldsymbol{x}, t)$, $A = 1, 2, 3$, is given by (cf. (6.2))

$$\{C_\Phi, l^A\} = \tilde{\boldsymbol{v}}\cdot\boldsymbol{\nabla} l^A \quad \text{with } \tilde{\boldsymbol{v}} = D^{-1}\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}\frac{\partial\Phi}{\partial q}, \quad D = 1. \tag{7.25}$$

Thus, C_Φ generates a volume-preserving shift in the Lagrangian fluid labels along intersections of level surfaces of density ρ and potential vorticity q, that leaves invariant the fluid's Eulerian momentum density, $\boldsymbol{\mu}$. The corresponding relative equilibrium flow of the CLB equations is given by $\bar{\boldsymbol{u}}_{\mathrm{Le}} = \tilde{\boldsymbol{v}}$ with $\tilde{\boldsymbol{v}}$ from (7.25). The relative equilibrium solutions of the CLB dynamics are critical points of the sum $H_C = H_B + C_\Phi$, in which the function Φ is related to the Bernoulli function for the equilibrium solution. The stability of these equilibrium solutions may be investigated by using constrained energy methods similar to those developed for Euler's equations in the Boussinesq approximation in [2].

Remark. From (7.16), the helicity Λ satisfies the Poisson bracket relation $\{\Lambda, \rho\} = (2/D)\bar{\omega} \cdot \nabla\rho \neq 0$. Hence, the helicity Λ is not conserved in CLB flows, because the Hamiltonian (7.17) depends upon ρ explicitly. In fact, $\partial\Lambda/\partial t = \{\Lambda, H_B\} = 2gz\bar{\omega} \cdot \nabla\rho \neq 0$, in general.

7.5. Equilibrium solutions

The equilibrium states $(\rho_e, \bar{\boldsymbol{u}}_e)$ of the dynamical system (7.12) are the three-dimensional steady CLB flows. For such steady flows, there are three "streamline relations" for the equilibrium Lagrangian mean velocity, $\bar{\boldsymbol{u}}_{Le}$,

$$\bar{\boldsymbol{u}}_{Le} \cdot \nabla\rho_e = 0, \qquad \bar{\boldsymbol{u}}_{Le} \cdot \nabla q_e = 0, \qquad \bar{\boldsymbol{u}}_{Le} \cdot \nabla\big(p_e + \tfrac{1}{2}|\bar{\boldsymbol{u}}_{Le}|^2 + \rho_e g z\big) = 0. \tag{7.26}$$

The first two of these relations follow by Lagrangian-mean advection of buoyancy ρ in (7.12) and potential vorticity q in (7.22), while the last one is the Bernoulli Law, obtained by taking the scalar product of $\bar{\boldsymbol{u}}_{Le}$ with (7.21) and using $\bar{\boldsymbol{u}}_{Le} \cdot \nabla\rho_e = 0$ for steady solutions.

At points where $\bar{\boldsymbol{u}}_{Le} \neq 0$, the three streamline relations (7.26) imply that the quantities ρ_e, q_e and $\big(p_e + \frac{1}{2}|\bar{\boldsymbol{u}}_{Le}|^2 + \rho_e g z\big)$ are functionally dependent. We *assume* we may express this dependence explicity by solving for

$$p_e + \tfrac{1}{2}|\bar{\boldsymbol{u}}_{Le}|^2 + \rho_e g z = K(\rho_e, q_e), \tag{7.27}$$

where $K(\rho_e, q_e$ is called the Bernoulli function. We also assume that $\nabla\rho_e \times \nabla q_e \neq 0$, so that level surfaces of ρ_e, q_e divide the volume of flow into "cells." We now show that if $q_e \neq 0$, then (cf. (7.25))

$$\bar{u}_{Le} = \frac{1}{q_e} K_q(\rho_e, q_e)\nabla\rho_e \times \nabla q_e, \tag{7.28}$$

which automatically satisfies the three streamline relations (7.26). In (7.28), the subscript notation denotes partial derivative, e.g., $K_q = \partial K/\partial q$. The CLB motion equation (7.21) for steady flows and the relation (7.27) lead to

$$\bar{\boldsymbol{u}}_{Le} \times \bar{\boldsymbol{\Omega}}_e = \nabla K(\rho_e, q_e) - gz\nabla\rho_e \tag{7.29}$$

Vector multiplication of this by $\nabla\rho_e$ produces

$$\bar{\boldsymbol{u}}_{Le}(\bar{\boldsymbol{\Omega}}_e \cdot \nabla\rho_e) - \bar{\boldsymbol{\Omega}}_e(\bar{\boldsymbol{u}}_{Le} \cdot \nabla\rho_e) = K_q(\rho_e, q_e)\nabla\rho_e \times \nabla q_e. \tag{7.30}$$

Relation (7.28) follows, since $\bar{\boldsymbol{\Omega}}_e \cdot \nabla\rho_e = q_e$ and the scalar product $\bar{\boldsymbol{u}}_{Le} \cdot \nabla\rho_e$ vanishes by (7.26)

Another useful relation for steady flows arises by scalar multiplication of (7.29) by $\bar{\boldsymbol{\Omega}}_e$, yielding

$$gz - (\bar{\boldsymbol{\Omega}}_e \cdot \nabla q_e)K_q(\rho_e, q_e)/q_e - K_\rho(\rho_e, q_e) = 0. \tag{7.31}$$

Relations (7.30) and (7.31) will be useful in developing a variational principle for steady CLB flows in three dimensions.

7.6. Variational principle for steady CLB flows

Steady CLB flows will now be sought as extrema of the conserved quantities $H_B + C_\Phi$ in (7.14) and (7.23). Let

$$H_C(\rho, \bar{\boldsymbol{u}}) = H_B + C_\Phi = \int \mathrm{d}^3x\big[\tfrac{1}{2}|\bar{\boldsymbol{u}} + \bar{\boldsymbol{u}}_S|^2 + \rho g z + \Phi(\rho, q)\big]. \tag{7.32}$$

We compute δH_C, the first variation of H_C with respect to ρ and $\bar{\boldsymbol{u}}$, i.e.,

$$\delta H_C = \mathcal{D} H_C(\rho, \bar{u}) \cdot (\delta\rho, \delta\bar{u}). \tag{7.33}$$

(We note that in taking this variation we may as well set $D = 1$ and $\delta D = 0$ before varying; since their contributions vanish after taking variations.) After integration by parts and use of the divergence theorem to remove inconsequential boundary terms, δH_C is expressible as

$$\delta H_C = \int \mathrm{d}^3x\, (gz + \Phi_\rho - \bar{\boldsymbol{\Omega}} \cdot \boldsymbol{\nabla}\Phi_q)\delta\rho + (\bar{\boldsymbol{u}}_{\mathrm{L}} - \Phi_{qq}\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla} q) \cdot \delta\bar{\boldsymbol{u}}. \tag{7.34}$$

Consequently, the first variation δH_C vanishes for steady CLB flows, provided $\Phi(\rho_e, q_e)$ is determined from $K(\rho_e, q_e)$ by

$$q_e\Phi_q(\rho_e, q_e) - \Phi(\rho_e, q_e) = K(\rho_e, q_e). \tag{7.35}$$

If (7.35) holds, then, e.g., $\Phi_{qq} = K_q/q$ and the coefficients of $\delta\rho$ and $\delta\bar{\boldsymbol{u}}$ in (7.34) will vanish, in view of (7.31), the velocity relation (7.28) and the definition $q = \boldsymbol{\nabla}\rho \cdot \bar{\boldsymbol{\Omega}}/D$ with $D = 1$ in (7.22). Solving relation (7.35) gives the constraint function Φ in terms of the Bernoulli function for the steady flow, as

$$\Phi(\rho_e, q_e) = q_e \left[\int^{q_e} \frac{\mathrm{d}s}{s^2} K(\rho_e, s) + \kappa(\rho_e) \right], \tag{7.36}$$

where the integration "constant" $\kappa(\rho_e)$ is an arbitrary function of ρ_e. Thus, a steady CLB flow whose Bernoulli function K is expressible in the form (7.27) may be identified as a critical point of H_C in (7.32), with constraint function Φ given in terms of K by (7.36). Note that $\bar{\boldsymbol{u}}_{\mathrm{Le}}$ in (7.28) and $\tilde{\boldsymbol{v}}$ in (7.25) are identical when the critical point condition (7.36) is satisfied.

These considerations put the ideal CLB theory exactly into the framework developed in [2] for determining sufficient conditions for nonlinear stability to steady three-dimensional stratified incompressible Euler flows in the Boussinesq approximation. Consequently, we may immediately take over the conditional stability results of [2] for the steady Euler flows to the case of steady ideal CLB flows. For these flows, the only difference from stability results expressed in terms of the Bernoulli function K for Euler flows is that the equilibrium pressure p_e depends on the Stokes drift velocity $\bar{\boldsymbol{u}}_{\mathrm{S}}$ through the Poisson equation obtained from the divergence of Eq. (7.21) (cf. also (1.2)).

8. Linear instability conditions for planar CLB flows

We first rewrite the CLB equations (7.12) in the "Lorentz force" form, as

$$\begin{aligned}
&\frac{\partial\bar{\boldsymbol{u}}_{\mathrm{L}}}{\partial t} + (\bar{\boldsymbol{u}}_{\mathrm{L}} \cdot \boldsymbol{\nabla})\bar{\boldsymbol{u}}_{\mathrm{L}} + \boldsymbol{\nabla} p + \rho g\hat{\boldsymbol{z}} - \bar{\boldsymbol{u}}_{\mathrm{L}} \times \operatorname{curl}\boldsymbol{R} = \frac{\partial\bar{\boldsymbol{u}}_{\mathrm{S}}}{\partial t} - \bar{\boldsymbol{u}}_{\mathrm{L}} \times \operatorname{curl}\bar{\boldsymbol{u}}_{\mathrm{S}},\\
&\frac{\partial\rho}{\partial t} + \bar{\boldsymbol{u}}_{\mathrm{L}} \cdot \boldsymbol{\nabla}\rho = 0,\\
&\boldsymbol{\nabla} \cdot \bar{\boldsymbol{u}}_{\mathrm{L}} = 0 = \boldsymbol{\nabla} \cdot \bar{\boldsymbol{u}}_{\mathrm{S}}.
\end{aligned} \tag{8.1}$$

These equations admit two-dimensional solutions in the x–z plane when the Coriolis force and $\partial\bar{\boldsymbol{u}}_{\mathrm{L}}/\partial y$ both vanish, and $\bar{\boldsymbol{u}}_{\mathrm{S}} = U_{\mathrm{S}}(z)\hat{\boldsymbol{x}}$ with $\operatorname{curl}\bar{\boldsymbol{u}}_{\mathrm{S}} = U_{\mathrm{S}}'(z)\hat{\boldsymbol{y}}$ for a function U_{S} and its derivative $U_{\mathrm{S}}' = \mathrm{d}U_{\mathrm{S}}/\mathrm{d}z$. As in Section 3.1.1, for two-dimensional motion we ignore $\partial\bar{\boldsymbol{u}}_{\mathrm{S}}/\partial t$ and find that the CLB equations reduce to

$$\frac{\partial\bar{\omega}}{\partial t} = -\bar{\boldsymbol{u}}_{\mathrm{L}} \cdot \boldsymbol{\nabla}\bar{\omega} + g\frac{\partial\rho}{\partial x}, \qquad \frac{\partial\rho}{\partial t} = -\bar{\boldsymbol{u}}_{\mathrm{L}} \cdot \boldsymbol{\nabla}\rho, \tag{8.2}$$

or, in terms of the steam function $\bar{\psi}$,

$$\frac{\partial \bar{\omega}}{\partial t} = \mathcal{J}(\bar{\omega}, \bar{\psi}) + \mathcal{J}(gz, \rho), \qquad \frac{\partial \rho}{\partial t} = \mathcal{J}(\rho, \bar{\psi}), \qquad \bar{\omega} = \Delta\bar{\psi} - U_S'(z), \tag{8.3}$$

where $\Delta\bar{\psi} = \bar{\psi}_{xx} + \bar{\psi}_{zz}$ and $\mathcal{J}(g, h) = g_z h_x - h_z g_x$ is the Jacobian of the functions $g(x, z)$, $h(x, z)$. Thus, certain functional relationships must hold among the equilibrium solutions $\bar{\psi}_e$, ρ_e and $\bar{\omega}_e$, for which the motion (8.3) is time-independent.

We set $\bar{\psi}_e = \bar{\psi}(\rho_e)$, in order to satisfy the steady buoyancy equation, $\mathcal{J}(\rho_e, \bar{\psi}_e) = 0$. The steady vorticity equation is given by

$$\mathcal{J}(\bar{\omega}_e, \bar{\psi}_e) + \mathcal{J}(gz, \rho_e) = 0. \tag{8.4}$$

This implies

$$\mathcal{J}\left(\bar{\omega}_e + \frac{d\rho_e}{d\bar{\psi}_e} gz, \bar{\psi}_e\right) = 0, \tag{8.5}$$

which will be satisfied by the relation

$$\bar{\omega}_e + \frac{d\rho_e}{d\bar{\psi}_e} gz = L(\bar{\psi}_e), \tag{8.6}$$

for a function L, or written in terms of the equilibrium stream function only,

$$\Delta\bar{\psi}_e - U_S'(z) + \frac{d\rho_e}{d\bar{\psi}_e} gz = L(\bar{\psi}_e). \tag{8.7}$$

This is the modification of Long's equation [10,20,26] which is necessary for a planar Boussinesq equilibrium to accommodate the CL vortex force.

We investigate the linearized stability conditions for a plane-parallel CLB flow, with Lagrangian mean velocity

$$\bar{\boldsymbol{u}}_{Le} = (U(z), 0, 0) = \left(\frac{d\bar{\psi}_e}{dz}, 0, 0\right), \tag{8.8}$$

and buoyancy $\rho_e = \rho_e(z)$. For this, we follow [14,22] and return to the velocity equation (8.1). We set

$$\bar{\boldsymbol{u}}_{Le} = (U(z) + u, 0, w) \quad \text{and} \quad \rho = \rho_e - \rho_e'\eta, \quad \text{with } \rho_e' < 0, \tag{8.9}$$

where u and w denote the x- and z-components of the perturbation velocity and η denotes the vertical displacement of a particle from its initial position. The planar CLB equations linearize to give

$$\begin{aligned} &\frac{Du}{Dt} + wU' = -(p - p_e)_x + wU_S', \qquad \frac{Dw}{Dt} - g\eta\rho_e' = -(p - p_e)_z - uU_S', \\ &\frac{D\eta}{Dt} = w, \qquad u_x + w_z = 0, \qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + U(z)\frac{\partial}{\partial x}, \end{aligned} \tag{8.10}$$

where subscripts denote partial derivatives and primes denote derivatives with respect to depth, z. We assume the separated-variable solution for a streamwise perturbation in η,

$$\eta(x, z, t) = F(z)e^{ik(x-ct)}, \tag{8.11}$$

where k is real, but c may be complex. Consequently, we obtain

$$w = ik(U - c)\eta, \qquad u = -[(U - c)\eta]', \qquad p - p_e = (U - c)\eta U_S' + (U - c)^2\eta', \tag{8.12}$$

and

$$[(U-c)FU_S'+(U-c)^2F']'-[(U-c)F]'U_S'-k^2(U-c)^2F-g\rho_e'F=0. \tag{8.13}$$

The boundary conditions are that F vanish at fixed $z=z_1$ and $z=z_2$. The flow is unstable if (8.13) with these boundary conditions has nontrivial solutions with $\operatorname{Im} c>0$. Set $c=c_r+\mathrm{i}c_i$, and let $W=U-c$. If F is an unstable solution, then $c_i>0$, so $W\neq 0$. Now set $G=W^{1/2}F$, and replace the variable F in (8.13) by G. Then Eq. (8.13) becomes (cf. [14, Eq. (2.2)])

$$(WG')'-\left[\tfrac{1}{2}U''+k^2W+\frac{1}{W}\left(\tfrac{1}{4}U'^2-N^2(z)\right)\right]G+(GU_S')'-U_S'G'=0, \tag{8.14}$$

where $N^2(z)=-g\rho_e'$ is the buoyancy frequency. Multiplication by the complex conjugate $\overline{G}$ of G and integration over (z_1,z_2) then leads to

$$\int_{z_1}^{z_2}\mathrm{d}z\,[W(|G'|^2+k^2|G|^2)+\tfrac{1}{2}U''|G|^2+\left(\tfrac{1}{4}U'^2-N^2(z)\right)\overline{W}|G/W|^2]$$
$$+\int_{z_1}^{z_2}\mathrm{d}z\,U_S'(\overline{G}'G+G'\overline{G})=0. \tag{8.15}$$

The last term in *real*, so it does not contribute to the imaginary part of (8.15), which gives

$$c_i\int_{z_1}^{z_2}\mathrm{d}z\,[|G'|^2+k^2|G|^2+|G/W|^2(N^2(z)-\tfrac{1}{4}U'^2)]=0. \tag{8.16}$$

Hence, the familiar Richardson number criterion follows, but expressed in terms of the *Langrangian* mean velocity for planar steady CLB flows. Namely, $N^2(z)-\left(\tfrac{1}{2}U'\right)^2$ being everywhere nonnegative is sufficient for linearized stability of the plane-parallel flow $\bar{\boldsymbol{u}}_{\mathrm{Le}}=U(z)\hat{\boldsymbol{x}}$ to streamwise perturbations.

Transforming Eq. (8.13) to the vertical velocity function,

$$\varphi(z)=(U-c)F=WF, \tag{8.17}$$

gives the equation

$$\varphi''-k^2\varphi+\left[-\frac{U''-U_S''}{W}+\frac{N^2}{W^2}\right]\varphi=0. \tag{8.18}$$

Multiplication of this by $\overline{W}$ and integration over (z_1,z_2) gives the integral relation,

$$\int_{z_1}^{z_2}\mathrm{d}z\left[|\varphi'|^2+k^2|\varphi|^2-\left(\frac{N^2}{W^2}-\frac{U''-U_S''}{W}\right)|\varphi|^2\right]=0, \tag{8.19}$$

whose imaginary part is

$$c_i\int_{z_1}^{z_2}\left[U''-U_S''-\frac{2N^2}{|W|^2}(U-c_r)\right]\frac{|\varphi|^2}{|W|^2}=0. \tag{8.20}$$

Thus, a necessary condition for instability is that the quantity

$$U'' - U''_S - \frac{2N^2}{|W|^2}(U - c_r) \tag{8.21}$$

must change sign somewhere in the flow. This is the inflection point theorem for planar steady CLB flows, which has the same form as Synge's generalization of Rayleigh's theorem for steady Euler–Boussinesq flows [14] when expressed in terms of the *Eulerian* mean velocity (cf. Section 3.1.1) in the case $N^2 = 0$.

Thus, the linearized stability analysis of Miles [22 and Howard 14] for planar steady stratified ideal Euler flows passes over almost unchanged to the linearized analysis of wave-induced instability of planar steady ideal CLB flows. The difference is in the distinction between particle criteria and flow criteria for stability. For planar steady CLB flows, the Richardson number criterion is unchanged for the Lagrangian mean velocity (a particle criterion) and the inflection point criterion is unchanged for the Eulerian mean velocity (a flow criterion).

The Howard semicircle theorem for planar steady CLB flows also follows from Eq. (8.13), by multiplying by F and integrating over (z_1, z_2) as in the derivation of the stability criteria. The inflection-point criterion and the semicircle theorem for CLB flows have been discussed by Craik [6] from the viewpoint of the generalized Lagrangian mean (GLM) formulation [3]. Leibovich [17] and Craik [7] discuss linearized stability analysis of spanwise perturbations of planar steady ideal CLB flows from the GLM viewpoint.

9. Generalized Lagrangian Mean (GLM) equations

The GLM theory of Andrews and McIntyre [3] is a hybrid Eulerian–Lagrangian description in which Lagrangian-mean flow quantities satisfy equations expressed in Eulerian form. The GLM description associates to an Eulerian velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ a unique "related velocity field" $\boldsymbol{v}(\boldsymbol{x}, t)$, such that when a fluid parcel at $\boldsymbol{X} = \boldsymbol{x} + \boldsymbol{\xi}$ moves with its velocity $\boldsymbol{u}(\boldsymbol{x} + \boldsymbol{\xi}, t)$, a fictional parcel at $\boldsymbol{x}$ is moving with velocity $\boldsymbol{v}(\boldsymbol{x}, t)$. That is,

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla\right)[\boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x}, t)] = \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{\xi}, t). \tag{9.1}$$

The displacement $\boldsymbol{\xi}(\boldsymbol{x}, t)$ associated with the waves has zero Eulerian mean, $\overline{\boldsymbol{\xi}(\boldsymbol{x}, t)} = 0$, and $\boldsymbol{v}(\boldsymbol{x}, t)$ is defined to be a mean quantity, i.e. $\overline{\boldsymbol{v}(\boldsymbol{x}, t)} = \boldsymbol{v}(\boldsymbol{x}, t)$. In fact $\boldsymbol{v}(\boldsymbol{x}, t)$ is the Lagrangian mean velocity, denoted in this section as $\bar{\boldsymbol{u}}^{\mathrm{L}}(\boldsymbol{x}, t)$, following the notation of Ref [3]. The GLM operator $\overline{(\ )}^{\mathrm{L}}$ involves averaging over particles at the displaced positions $\boldsymbol{X} = \boldsymbol{x} + \boldsymbol{\xi}$, e.g.,

$$\overline{\rho(\boldsymbol{x}, t)}^{\mathrm{L}} = \overline{\rho^{\xi}(\boldsymbol{x}, t)} \quad \text{with } \rho^{\xi}(\boldsymbol{x}, t) = \rho(\boldsymbol{x} + \boldsymbol{\xi}, t), \tag{9.2}$$

where $\overline{(\ )}$ denotes a suitably defined Eulerian average. For example, $\overline{(\ )}$ may denote the average over the rapid oscillation phase of a single-frequency wave displacement.

The ideal compressible (homentropic) GLM momentum and continuity equations in the absence of rotation are given by [3]

$$\begin{aligned} &\bar{D}^{\mathrm{L}}(\bar{\boldsymbol{u}}^{\mathrm{L}} - \boldsymbol{p}) + (\bar{u}_k^{\mathrm{L}} - p_k)\nabla \bar{u}_k^{\mathrm{L}} + \frac{1}{\rho^{\xi}}\nabla p_{\mathrm{glm}} = 0, \\ &\bar{D}^{\mathrm{L}}\tilde{\rho} + \tilde{\rho}\nabla \cdot \bar{\boldsymbol{u}}^{\mathrm{L}} = 0, \qquad \bar{D}^{\mathrm{L}} \equiv \frac{\partial}{\partial t} + \bar{\boldsymbol{u}}^{\mathrm{L}} \cdot \nabla. \end{aligned} \tag{9.3}$$

The vector $\boldsymbol{p}(\boldsymbol{x}, t)$ is the "pseudomomentum" (per unit mass), defined as

$$\boldsymbol{p}(\boldsymbol{x}, t) = -\overline{u_j^l \nabla \xi_j}, \tag{9.4}$$

and satisfying curl $(\boldsymbol{p} - \bar{\boldsymbol{u}}_{\mathrm{S}}) = 0$ provided $\boldsymbol{u}^l$ is irrotational, since [cf. Eq. (1.7)]

$$\boldsymbol{p} - \bar{\boldsymbol{u}}_{\mathrm{S}} = -\overline{u^l_j \nabla \xi_j} - \overline{(\boldsymbol{\xi} \cdot \nabla)\boldsymbol{u}^l} = -\overline{(\mathrm{curl}\,\boldsymbol{u}^l) \times \boldsymbol{\xi}} - \overline{\nabla(\boldsymbol{u}^l \cdot \boldsymbol{\xi})}. \tag{9.5}$$

Here symbols with label $(\ )^l$ are "wave quantities" with zero mean, such that, e.g.,

$$u^l_j = u^{\xi}_j - \overline{u^{\xi}_j} = u^{\xi}_j - \bar{u}^{\mathrm{L}}_j, \tag{9.6}$$

and we have neglected the difference between $\boldsymbol{u}'$ and $\boldsymbol{u}^l$ as being of higher order. Also, the wave fields $\boldsymbol{\xi}(\boldsymbol{x}, t)$ and $\boldsymbol{u}^l(\boldsymbol{x}, t)$ are related by the kinematic condition,

$$\boldsymbol{u}^l(\boldsymbol{x}, t) = \overline{D}^{\mathrm{L}} \boldsymbol{\xi}. \tag{9.7}$$

The physical significance of the pseudomomentum $\boldsymbol{p}$ can be understood from the contour integral appearing in Kelvin's circulation theorem (cf. Eq. (3.22)). Namely,

$$\begin{aligned} I(t) &= \overline{\oint_{\gamma(t)} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{X}} = \overline{\oint_{\gamma(t)} (\bar{\boldsymbol{u}}_{\mathrm{L}} + \boldsymbol{u}^l) \cdot (\mathrm{d}\boldsymbol{x} + \mathrm{d}\boldsymbol{\xi})} \\ &= \oint_{\bar{\gamma}(t)} (\bar{\boldsymbol{u}}_{\mathrm{L}} + \overline{u^l_j \nabla \xi_j}) \cdot \mathrm{d}\boldsymbol{x} = \oint_{\bar{\gamma}(t)} (\bar{\boldsymbol{u}}_{\mathrm{L}} - \boldsymbol{p}) \cdot \mathrm{d}\boldsymbol{x}, \end{aligned} \tag{9.8}$$

where the contour $\bar{\gamma}(t)$ moves with velocity $\boldsymbol{u}_{\mathrm{L}}$, since it follows the fluid parcels as the average is taken. Consequently, Kelvin's theorem gives (cf. Eq. (3.23)),

$$0 = \frac{\mathrm{d}I}{\mathrm{d}t} = \oint_{\bar{\gamma}(t)} [\overline{D}^{\mathrm{L}}(\bar{\boldsymbol{u}}^{\mathrm{L}} - \boldsymbol{p}) + (\bar{u}^{\mathrm{L}}_k - p_k)\nabla \bar{u}^{\mathrm{L}}_k] \cdot \mathrm{d}\boldsymbol{x}, \tag{9.9}$$

upon using Eq. (9.3) and $(1/\rho^{\xi})\nabla p_{\mathrm{glm}} = \nabla h_{\mathrm{glm}}(\rho^{\xi})$, which is implied by the homentropic equation of state. Thus, in the case that the fluctuation velocity is irrotational, the "relative mean velocity" $(\bar{\boldsymbol{u}}^{\mathrm{L}} - \boldsymbol{p})$ plays the same role in the GLM theory as played by the Eulerian mean velocity $\bar{\boldsymbol{u}}$ in the CL theory and these two quantities differ by a gradient. In this case, the formal analogy between $(\bar{\boldsymbol{u}}^{\mathrm{L}} - \boldsymbol{p})$ and $(\bar{\boldsymbol{u}}_{\mathrm{L}} - \bar{\boldsymbol{u}}_{\mathrm{S}})$ provides many parallels between the two theories which allow ready transfer of results from one to the other.

The density of the related flow $\bar{\boldsymbol{u}}^{\mathrm{L}}(\boldsymbol{x}, t)$ is defined so as to satisfy the continuity equation in (9.3), and is connected to the actual fluid density $\rho^{\xi}(\boldsymbol{x}, t) = \rho(\boldsymbol{x} + \boldsymbol{\xi}, t)$ by

$$\tilde{\rho} = \rho^{\xi} J, \quad \text{where } J = \det(\nabla(\boldsymbol{x} + \boldsymbol{\xi})). \tag{9.10}$$

Note that constant density flows, $\rho^{\xi} = \mathrm{const}$, do not necessarily produce a constant related density $\tilde{\rho}$. However, a simplification is that the ratio $\tilde{\rho}/\rho^{\xi}$ is *prescribed.* To complete the GLM formulation, $\tilde{\rho}$ in (9.10) is defined to be a mean quantity, $\overline{\tilde{\rho}(\boldsymbol{x}, t)} = \tilde{\rho}(\boldsymbol{x}, t) = \rho^{\xi} \overline{J}$.The GLM equations are then closed by choosing an equation of state which links pressure p_{glm} and density ρ^{ξ}, and specifying the wave field $\boldsymbol{\xi}(\boldsymbol{x}, t)$.

Leibovich [17] explains that the CL equations result from an asymptotic expansion of the compressible GLM equations (9.3) for small-amplitude waves and relatively larger mean shear, provided the Eulerian velocity fluctuations $\boldsymbol{u}'$ are assumed to be irrotational. (See also [6–8].) In order to compare with the CL theory, we write the GLM equations for an *incompressible* relative mean velocity, $\bar{\boldsymbol{u}}^{\mathrm{L}} - \boldsymbol{p}$, with constant ρ^{ξ} and irrotational $\boldsymbol{u}^l$.

In this case, rearranging the motion equation in (9.3) for the incompressible case with constant ρ^{ξ} leads to (cf. Eq. (1.14))

$$
\begin{aligned}
&\frac{\partial \bar{\boldsymbol{u}}^{\mathrm{L}}}{\partial t}+(\bar{\boldsymbol{u}}^{\mathrm{L}}\cdot\nabla)\bar{\boldsymbol{u}}^{\mathrm{L}}+\nabla\left(p_{\mathrm{glm}}/\rho^{\xi}+\tfrac{1}{2}|\bar{\boldsymbol{u}}^{\mathrm{L}}-\boldsymbol{p}|^{2}-\tfrac{1}{2}|\boldsymbol{p}|^{2}\right)=\frac{\partial \boldsymbol{p}}{\partial t}-\boldsymbol{u}^{\mathrm{L}}\times\operatorname{curl}\boldsymbol{p},\\
&\bar{D}^{\mathrm{L}}\tilde{\rho}+\tilde{\rho}\nabla\cdot\bar{u}^{\mathrm{L}}=0\quad\text{with }\bar{D}^{\mathrm{L}}=\frac{\partial}{\partial t}+\bar{\boldsymbol{u}}^{\mathrm{L}}\cdot\nabla,\\
&\nabla\cdot(\bar{\boldsymbol{u}}^{\mathrm{L}}-\boldsymbol{p})=0.
\end{aligned}
\tag{9.11}
$$

Here, the GLM pressure p_{glm} is solved from the Poisson equation,

$$
-\nabla^{2}\left(\frac{p_{\mathrm{glm}}}{\rho^{\xi}}\right)=\nabla\cdot[(\bar{\boldsymbol{u}}^{\mathrm{L}}\cdot\nabla)(\bar{\boldsymbol{u}}^{\mathrm{L}}-\boldsymbol{p})+(\bar{u}_{k}^{\mathrm{L}}-p_{k})\nabla\bar{u}_{k}^{\mathrm{L}}],
\tag{9.12}
$$

obtained by taking the divergence of the GLM motion equation (9.11) and requiring the relative mean velocity to be divergenceless, i.e.,

$$
\nabla\cdot(\bar{\boldsymbol{u}}^{\mathrm{L}}-\boldsymbol{p})=0.
\tag{9.13}
$$

The appropriate boundary conditions for the Poisson equation (9.12) come from evaluating the motion equation (9.11) on the boundary and requiring the Lagrangian mean velocity to be tangential there.

Thus, in the case that the velocity fluctuations $\boldsymbol{u}^{l}$ are assumed to be irrotational, the incompressible GLM equations (9.11) correspond formally to the CL equations (5.8), upon identifying

$$
\bar{\boldsymbol{u}}^{\mathrm{L}}=\bar{\boldsymbol{u}}_{\mathrm{L}},\quad \boldsymbol{p}=\bar{\boldsymbol{u}}_{\mathrm{S}},\quad \bar{\boldsymbol{u}}^{\mathrm{L}}-\boldsymbol{p}=\bar{\boldsymbol{u}},\quad p_{\mathrm{glm}}/\rho^{\xi}+\tfrac{1}{2}|\bar{\boldsymbol{u}}^{\mathrm{L}}-\boldsymbol{p}|^{2}-\tfrac{1}{2}|\boldsymbol{p}|^{2}=p.
\tag{9.14}
$$

Making these formal replacements allows us to transfer many of the results derived for the CL equations in the previous sections over to the GLM equations. For example, the action principle for the GLM equations is expressible as (cf. Eq. (4.1))

$$
\mathcal{L}_{\mathrm{GLM}}=\int \mathrm{d}t\int \mathrm{d}^{3}x\left[\tfrac{1}{2}D|\bar{\boldsymbol{u}}^{\mathrm{L}}|^{2}-D\bar{\boldsymbol{u}}^{\mathrm{L}}\cdot\boldsymbol{p}-\pi(D/\rho^{\xi}-J)\right],
\tag{9.15}
$$

where $D=\det(\nabla l^{A})$, and the Lagrangian mean velocity $\bar{\boldsymbol{u}}^{\mathrm{L}}$ is given in terms of derivatives of mean particle labels l^{A} as (cf. (3.26))

$$
\bar{u}^{\mathrm{L}i}=-(D^{-1})^{i}_{A}\frac{\partial l^{A}}{\partial t}.
\tag{9.16}
$$

The variation of the GLM action (9.15) with respect to the mean Lagrange Labels l^{A} at fixed $\boldsymbol{x}$ and t gives

$$
\delta\mathcal{L}_{\mathrm{GLM}}=\int \mathrm{d}t\int \mathrm{d}^{3}x\left[\delta\bar{\boldsymbol{u}}^{\mathrm{L}}\cdot(D\bar{\boldsymbol{u}}^{\mathrm{L}}-D\boldsymbol{p})+\delta D\left(\frac{1}{2}|\bar{\boldsymbol{u}}^{\mathrm{L}}|^{2}-\bar{\boldsymbol{u}}^{\mathrm{L}}\cdot\boldsymbol{p}-\frac{\pi}{\rho^{\xi}}\right)-\delta\pi\left(\frac{D}{\rho^{\xi}}-J\right)\right]
\tag{9.17}
$$

with definitions (cf. (4.3))

$$
\delta D=D(D^{-1})^{i}_{A}\delta l^{A}_{,i}\quad\text{and}\quad \delta\bar{u}^{\mathrm{L}i}=-(D^{-1})^{i}_{\mathrm{B}}\bar{u}^{\mathrm{L}j}\delta l^{\mathrm{B}}_{,j}-(D^{-1})^{i}_{\mathrm{B}}\delta l^{\mathrm{B}}_{,t}.
\tag{9.18}
$$

The action $\mathcal{L}_{\mathrm{GLM}}$ depends on the fluid variables l^{A} only through the quantities u_{L} and D, and incompressibility is imposed by $-\pi(D/\rho^{\xi}-J)$. in this case, Hamilton's principle gives

$$\delta\mathcal{L}_{\rm GLM} = \int \mathrm{d}t \int \mathrm{d}^3x \left\{ D(D^{-1})^i_A \delta l^A \left[\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{D} \frac{\delta\mathcal{L}_{\rm GLM}}{\delta u^i_{\rm L}} + \frac{1}{D}\frac{\delta\mathcal{L}_{\rm GLM}}{\delta u^j_{\rm L}} u^j_{{\rm L},i} - \left(\frac{\delta\mathcal{L}_{\rm GLM}}{\delta D}\right)_{,i} \right] - \delta\pi \left(\frac{D}{\rho^\xi} - J\right) \right\}$$
$$- \int \mathrm{d}t \int \mathrm{d}^3x \left\{ \frac{\partial}{\partial t}\left[D(D^{-1})^i_A \delta l^A \frac{1}{D}\frac{\delta\mathcal{L}_{\rm GLM}}{\delta u^i_{\rm L}} \right] \right.$$
$$\left. + \frac{\partial}{\partial x^j}\left[D(D^{-1})^i_A \delta l^A \left(-\frac{\delta\mathcal{L}_{\rm GLM}}{\delta D}\delta^j_i + \frac{1}{D}\frac{\delta\mathcal{L}_{\rm GLM}}{\delta u^i_{\rm L}} u^j_{\rm L} \right) \right] \right\}. \tag{9.19}$$

Vanishing of the coefficient of δl^A gives the motion equation, while vanishing of the exact-derivative terms gives the boundary conditions. Thus, either by following the lines of Eq. (4.2)–(4.8), or by substituting from (9.17) into (9.19), we find that vanishing of $\delta\mathcal{L}_{\rm GLM}$ implies

$$\begin{aligned} &\delta l^A\colon\ \overline{D}^{\rm L}(\bar{\boldsymbol{u}}^{\rm L} - \boldsymbol{p}) + (\bar{u}^{\rm L}_k - p_k)\nabla\bar{u}^{\rm L}_k + \nabla\left(\pi/\rho^\xi - \tfrac{1}{2}|\bar{\boldsymbol{u}}^{\rm L} - \boldsymbol{p}|^2 + \tfrac{1}{2}|\boldsymbol{p}|^2\right) = 0 \\ &\delta\pi\colon\ \frac{D}{\rho^\xi} = J, \end{aligned} \tag{9.20}$$

where J is given in (9.10). Now identifying D as $\tilde{\rho}$ in (9.10) and setting

$$\frac{\pi}{\rho^\xi} - \frac{1}{2}|\bar{\boldsymbol{u}}^{\rm L} - \boldsymbol{p}|^2 + \frac{1}{2}|\boldsymbol{p}|^2 = \frac{p_{\rm glm}}{\rho^\xi} \tag{9.21}$$

recovers the ideal GLM equations from Hamilton's principle for the action $\mathcal{L}_{\rm GLM}$ in (9.15).

The action $\mathcal{L}_{\rm GLM}$ is invariant under the spatial transformations of l^A that leave the Eulerian quantities D and $\bar{\boldsymbol{u}}^{\rm L}$ unchanged (the particle-relabeling gauge transformations). These Noether symmetries endow the GLM theory with the various fundamental properties we have discussed for the CL theory, including Kelvin circulation theorem, Lie–Poisson Hamiltonian structure, and conservation of potential vorticity and Eulerian mean helicity. Stability results for the GLM equilibria analogous to those discussed for the CL theory are also available, including both Lyapunov stability conditions and linearized stability results. Inclusion of rotational effects in the GLM theory parallels that of the CLB theory discussed in the previous sections.

10. Discussion

The subject of three-dimensional fluid dynamics driven at a free surface is, of course, very complex. Approximate theories such as the CLB theory and the GLM formulation that preserve the underlying structure of the original Euler equations are more likely to produce acceptable results than approximations which would destroy this structure. The structure to which we refer for the Euler equations is their Kelvin circulation theorem and Hamilton's principle, which are linked by the Noether symmetry of fluid particle relabeling. This symmetry, in turn, generates the steady motions as canonical transformations in the Hamiltonian framework and associates the steady flows to critical points of a constrained energy. The nature of these critical points determines whether the steady flows are stable to perturbations. Our discussion places both the ideal Craik–Leibovich theory of wave–current interactions at widely different time scales and the hybrid Eulerian–Lagrangian GLM theory into this classical framework. Although the presence of the Stokes drift velocity $\bar{\boldsymbol{u}}_{\rm S}$ in the CL theory introduces inhomogeneous terms into the equilibrium flow relations and their stability conditions, the CL theory preserves all of the structure emanating from particle-relabeling symmetry in Hamilton's principle. Moreover, the GLM formulation for currents driven by prescribed wave forcing also preserves this structure.

These examples suggest that the CLB equations as well as other approximations derived by averaging could be obtained by applying the averaging techniques directly to Hamilton's principle for the Euler equations themselves.

One benefit of meeting this challenge is that the resulting averaged theories for solutions varying at slow time scales retain as much as possible of the structure of the original theory, including conservation laws arising from symmetries. This, of course, is an old idea [25] which is seeing renewed interest in the context of the present example. Another benefit of averaging directly in Hamilton's principle is that such an approach leads to a self-consistent theory which includes both the rectified effect of fluctuations on the mean state and the influence of the developing mean flow on certain aspects of the fluctuating part of the motion. For details, see [11].

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