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The fourth-order evolution equation for deep-water gravity-capillary waves

By S. J. Hogan
Mathematical Institute, University of Oxford, St Giles, Oxford OX1 3LB, U.K.

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The stability of a train of nonlinear gravity-capillary waves on the surface of an ideal fluid of infinite depth is considered. An evolution equation is derived for the wave envelope, which is correct to fourth order in the wave steepness. The derivation is made from the Zakharov equation under the assumption of a narrow band of waves, and including the full form of the interaction coefficient for gravity-capillary waves. It is assumed that conditions are away from subharmonic resonant wavelengths.

Just as was found by K. B. Dysthe (*Proc. R. Soc. Lond.* A **369** (1979)) for pure gravity waves, the main difference from the third-order evolution equation is, as far as stability is concerned, the introduction of a mean flow response. There is a band of waves that remains stable to fourth order. In general the mean flow effects for pure capillary waves are of opposite sign to those of pure gravity waves.

The second-order corrections to first-order stability properties are shown to depend on the interaction between the mean flow and the envelope frequency-dispersion term in the governing equation.

The results are shown to be in agreement with some recent computations of the full problem for sufficiently small values of the wave steepness.

1. Introduction

The problem of the stability of a wave train on water has attracted a great deal of interest ever since a uniform train of gravity waves was shown to be unstable in the (inevitable) presence of side bands (Benjamin & Feir 1967). This and other analyses were analytic and involved an assumption of weak nonlinearity or very long and slow perturbations. For small amplitudes, the nonlinear Schrödinger equation (Benney & Newell 1967; Zakharov 1968) was derived to describe the evolution of a wave train. This equation is correct to third order in the wave steepness. It was not until the work of Longuet-Higgins (1978*a, b*) on the normal mode perturbations of the fully nonlinear problem that the Schrödinger equation was found to give an inadequate description for all but the smallest wave steepness. Later Dysthe (1979) made a perturbation analysis and derived the evolution equation to fourth order in the amplitude of the wave potential. This produced a dramatic improvement in agreement with Longuet-Higgins (1978*a, b*). But curiously, of the several extra terms that were included, only one, corresponding to the mean flow response to radiation stress non-uniformity in the modulated nonlinear wave, actually influences the stability characteristics. These other terms are nevertheless relevant in other areas of nonlinear wave propagation (Lo & Mei

1985), in particular the asymmetric development over long times. But the nonlinear Schrödinger equation in three dimensions is known to be unsuitable in another, more serious, aspect, namely that energy can leak to arbitrarily high modes outside its range of validity (Martin & Yuen 1980). The fourth-order equation does not possess this problem to such an extent. Nevertheless energy can leak out to a finite range of higher modes.

Zakharov (1968) derived the third-order nonlinear Schrödinger equation from an integral equation of considerable complexity, under the assumption of a narrow spectrum. In fact it was only necessary to include second-order terms in the bandwidth. Taking this procedure a step further, Stiassnie (1984) has shown that the fourth-order evolution equation in the wave amplitude is directly obtainable from Zakharov's integral equation by assuming a narrow spectrum and retaining terms up to third order in the bandwidth. Incidentally, Stiassnie (1984) refers to the fourth-order evolution equation as 'the modified nonlinear Schrödinger equation' and Dysthe (1979) hints at this description. Following Janssen (1983), the expression 'fourth-order evolution equation' is preferred here. This is because it is quite possible to modify the Schrödinger equation even further by adding fifth and higher order terms. But then the adjective could not specify the precise equation being used.

In §2, Stiassnie (1984) is followed and the fourth-order evolution equation is derived for a train of gravity-capillary waves on the surface of an ideal fluid of infinite depth, starting with Zakharov's integral equation. It is necessary to use the full form of the interaction coefficient. Previous authors have omitted a term directly proportional to surface tension. In §3, it is shown that as far as the wave stability is concerned, a simplified fourth-order equation is a sufficient description, as shown by Dysthe (1979) for pure gravity waves. In general the new stability properties at this order are of opposite sign for pure capillary waves to the case of pure gravity waves. The corrections are shown to depend on the interaction between the mean flow and the frequency dispersion of the wave envelope. Comparison is made with some recent computations of the full problem.

2. Derivation of the fourth-order equation

$$\mathrm{i}\frac{\partial B}{\partial t}(\boldsymbol{k},t) = \iiint_{-\infty}^{\infty} T(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\,B^*(\boldsymbol{k}_1,t)\,B(\boldsymbol{k}_2,t)\,B(\boldsymbol{k}_3,t)\,\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)$$
$$\times \exp\{\mathrm{i}[w(\boldsymbol{k})+w(\boldsymbol{k}_1)-w(\boldsymbol{k}_2)-w(\boldsymbol{k}_3)]\,t\}\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3, \quad (2.1)$$

where $B(\boldsymbol{k},t)$ is related to the free surface $\eta(\boldsymbol{x},t)$ by

$$\eta(\boldsymbol{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{|\boldsymbol{k}|}{2w(\boldsymbol{k})}\right)^{\frac{1}{2}}\{B(\boldsymbol{k},t)\exp\{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-w(\boldsymbol{k})\,t)\}+\text{c.c.}\}\,\mathrm{d}\boldsymbol{k}. \quad (2.2)$$

The complex conjugate is denoted by c.c., $\boldsymbol{k} = (k, l)$ is the wave vector, $\boldsymbol{x} = (x, y)$ is the horizontal space vector, z is the vertical coordinate, and the origin of coordinates is in the undisturbed surface. In addition, $w(\boldsymbol{k}) = (g|\boldsymbol{k}|+S|\boldsymbol{k}|^3)^{\frac{1}{2}}$ is the linear dispersion relation for gravity-capillary waves, g is the acceleration due to gravity and S is the surface tension coefficient divided by the density of the bulk fluid. $T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ is a real function and was first given for free surface gravity-capillary waves by Zakharov (1968). An earlier paper by the same author (Zakharov 1967) gave the form of the function for surface gravity waves and ion-sound waves.

Equation (2.1) has been used with considerable success in the study of the stability of progressive gravity waves by Crawford *et al.* (1980), Crawford *et al.* (1981), Stiassnie & Shemer (1984) and others and for standing gravity waves by Okamura (1984). These authors quote Zakharov (1968) but curiously omit that part of $T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ directly proportional to S. Thus in the Appendix of Crawford *et al.*(1981) a modification is required in the third-order interaction coefficient $W_{0,1,2,3}$, which should be replaced by

$$W'_{0,1,2,3} = W_{0,1,2,3} - \frac{3S}{32\pi^2} \frac{(k_0 k_1 k_2 k_3)^{\frac{1}{2}}}{[w(\boldsymbol{k}_0)\, w(\boldsymbol{k}_1)\, w(\boldsymbol{k}_2)\, w(\boldsymbol{k}_3)]^{\frac{1}{2}}} (\boldsymbol{k}_0 \cdot \boldsymbol{k}_1)(\boldsymbol{k}_2 \cdot \boldsymbol{k}_3) \qquad (2.3)$$

with the second-order interaction coefficients unchanged.

As pointed out by Zakharov (1968), there are potential difficulties in applying (2.1) to gravity-capillary waves. This is because unlike gravity waves, these waves, as well as pure capillary waves, can satisfy equations of the sort

$$\left.\begin{aligned} w(\boldsymbol{k}) &= w(\boldsymbol{k}_1) + w(\boldsymbol{k}_2), \\ \boldsymbol{k} &= \boldsymbol{k}_1 + \boldsymbol{k}_2 \end{aligned}\right\} \qquad (2.4)$$

(see McGoldrick (1965) for a full discussion of these triad solutions). This is precisely the condition that will give a zero denominator in one of the terms of $T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ corresponding to the second-order interaction. But if the wave packet is sufficiently narrow, that is $|\boldsymbol{k}-\boldsymbol{k}_0| \ll \boldsymbol{k}_0$ for some $\boldsymbol{k}_0$, then (2.4) can not be satisfied. Zakharov (1968) then proceeded to derive the nonlinear Schrödinger equation (which is correct to third order in the wave steepness) from (2.1) under the assumption of a narrow spectrum of waves (he went to second order in spectral width). He then considered the stability properties of gravity-capillary waves by using this equation. Stiassnie (1984) derived the fourth-order evolution equation from (2.1) for gravity waves by going to third order in spectral width.

In this section the fourth-order evolution equation for a group of gravity-capillary waves is derived from (2.1) in the manner of Stiassnie (1984). Thus we consider a narrow spectrum centred on $\boldsymbol{k}_0 = (k_0, 0)$ and set

$$\boldsymbol{k}_i = \boldsymbol{k}_0 + \boldsymbol{\chi}_i \quad (i = 1, 2, 3), \quad \boldsymbol{\chi} = (\chi, \lambda), \qquad (2.5)$$

where the measure of the spectral width is given by $|\boldsymbol{\chi}|/k_0 = o(1)$.

We define a new function $A(\boldsymbol{\chi}, t)$ as

$$A(\boldsymbol{\chi}, t) = B(\boldsymbol{k}, t) \exp\{-\mathrm{i}[w(\boldsymbol{k}) - w_0]\, t\}, \qquad (2.6)$$

where $w_0 = w(\boldsymbol{k}_0)$, and substitute this into (2.1) and (2.2) to obtain

$$\mathrm{i}\frac{\partial A}{\partial t}(\chi, t) - [w(\boldsymbol{k}) - w_0]\,A(\chi, t)$$

$$= \iiint_{-\infty}^{\infty} T(\boldsymbol{k}_0+\chi, \boldsymbol{k}_0+\chi_1, \boldsymbol{k}_0+\chi_2, \boldsymbol{k}_0+\chi_3)$$

$$\times A^*(\chi_1)\,A(\chi_2)\,A(\chi_3)\,\delta(\chi+\chi_1-\chi_2-\chi_3)\,\mathrm{d}\chi_1\,\mathrm{d}\chi_2\,\mathrm{d}\chi_3 \tag{2.7}$$

and

$$\eta(\boldsymbol{x}, t) = \frac{1}{2\pi}\left[\frac{w_0}{2g(1+\kappa)}\right]^{\frac{1}{2}}\left\{\mathrm{e}^{\mathrm{i}(k_0 x - w_0 t)}\int_{-\infty}^{\infty}\left[1+\frac{\chi}{4k_0}\frac{(1-\kappa)}{(1+\kappa)}\right]A(\chi, t)\,\mathrm{e}^{\mathrm{i}\chi\cdot\boldsymbol{x}}\,\mathrm{d}\chi + \text{c.c.}\right\}, \tag{2.8}$$

where

$$\kappa = Sk_0^2/g. \tag{2.9}$$

We can rewrite (2.8) as

$$\eta(\boldsymbol{x}, t) = \operatorname{Re}\{a(\boldsymbol{x}, t)\,\mathrm{e}^{\mathrm{i}(k_0 x - w_0 t)}\}, \tag{2.10}$$

where

$$a(\boldsymbol{x}, t) = \frac{1}{2\pi}\left[\frac{2w_0}{g(1+\kappa)}\right]^{\frac{1}{2}}\int_{-\infty}^{\infty}\left[1+\frac{\chi}{4k_0}\frac{(1-\kappa)}{(1+\kappa)}\right]A(\chi, t)\,\mathrm{e}^{\mathrm{i}\chi\cdot\boldsymbol{x}}\,\mathrm{d}\chi. \tag{2.11}$$

In (2.7), we express $[w(\boldsymbol{k})-w_0]$ as a Taylor series expansion in powers of $|\chi|/k_0$. We find

$$w(\boldsymbol{k}) - w_0 = \frac{1}{2}\left(\frac{g}{k_0(1+\kappa)}\right)^{\frac{1}{2}}\left\{\chi(1+3\kappa) + \frac{\chi^2}{4k_0}\left(\frac{-1+6\kappa+3\kappa^2}{1+\kappa}\right) + \frac{\lambda^2}{2k_0}(1+3\kappa)\right.$$
$$\left. + \frac{\chi^3}{8k_0^2}\left[\frac{(1-\kappa)(1+6\kappa+\kappa^2)}{(1+\kappa)^2}\right] - \frac{\chi\lambda^2}{4k_0^2}\left(\frac{3+2\kappa+3\kappa^2}{1+\kappa}\right) + O\left(\frac{|\chi|^4}{k_0^3}\right)\right\}. \tag{2.12}$$

We substitute (2.12) into (2.7), multiply both sides by

$$\left[\frac{2w_0}{g(1+\kappa)}\right]^{\frac{1}{2}}\left[1+\frac{\chi}{4k_0}\frac{(1-\kappa)}{(1+\kappa)}\right]$$

and take the inverse Fourier transform to find, by using equation (2.11),

$$\mathrm{i}a_t + \frac{1}{2}\left[\frac{g}{k_0(1+\kappa)}\right]^{\frac{1}{2}}\left[\mathrm{i}(1+3\kappa)\,a_x + \frac{(-1+6\kappa+3\kappa^2)}{4k_0(1+\kappa)}\,a_{xx}\right.$$
$$\left. + \frac{(1+3\kappa)}{2k_0}\,a_{yy} - \frac{\mathrm{i}(1-\kappa)(1+6\kappa+\kappa^2)}{8k_0^2(1+\kappa)^2}\,a_{xxx} + \frac{\mathrm{i}(3+2\kappa+3\kappa^2)}{4k_0^2(1+\kappa)}\,a_{xyy}\right]$$
$$= \frac{1}{2\pi}\left[\frac{2w_0}{g(1+\kappa)}\right]^{\frac{1}{2}}\iiint_{-\infty}^{\infty}\left[1+\frac{(\chi_2+\chi_3-\chi_1)}{4k_0}\frac{(1-\kappa)}{(1+\kappa)}\right]$$
$$\times T(\boldsymbol{k}_0+\chi_2+\chi_3-\chi_1, \boldsymbol{k}_0+\chi_1, \boldsymbol{k}_0+\chi_2, \boldsymbol{k}_0+\chi_3)$$
$$\times A^*(\chi_1)\,A(\chi_2)\,A(\chi_3)\exp\{\mathrm{i}(\chi_2+\chi_3-\chi_1)\cdot\boldsymbol{x}\}\,\mathrm{d}\chi_1\,\mathrm{d}\chi_2\,\mathrm{d}\chi_3. \tag{2.13}$$

To evaluate the integral on the right side of (2.13) we need the Taylor expansion of T for gravity-capillary waves, correct to first-order in the spectral width. Using the form quoted by Crawford *et al.* (1981) as amended by our equation (2.3) we find

$$T(\boldsymbol{k}_0+\boldsymbol{\chi}_2+\boldsymbol{\chi}_3-\boldsymbol{\chi}_1, \boldsymbol{k}_0+\boldsymbol{\chi}_1, \boldsymbol{k}_0+\boldsymbol{\chi}_2, \boldsymbol{k}_0+\boldsymbol{\chi}_3) = \frac{k_0^3}{8\pi^2}\left[\frac{(8+\kappa+2\kappa^2)}{4(1+\kappa)(1-2\kappa)}+\frac{3(\chi_2+\chi_3)}{8k_0}\frac{(8-\kappa+9\kappa^2-4\kappa^3-4\kappa^4)}{(1+\kappa)^2(1-2\kappa)^2} - \frac{(\chi_3-\chi_1)^2}{k_0|\boldsymbol{\chi}_1-\boldsymbol{\chi}_3|}-\frac{(\chi_2-\chi_1)^2}{k_0|\boldsymbol{\chi}_1-\boldsymbol{\chi}_2|}+O\left(\frac{|\boldsymbol{\chi}|^2}{k_0^2}\right)\right]. \tag{2.14}$$

By using this form of T, we find by using (2.11) that the right side of (2.13) becomes on integration

$$\frac{g}{16w_0}\left[\frac{k_0^3(8+\kappa+2\kappa^2)}{(1-2\kappa)}|a|^2 a-\frac{\mathrm{i}k_0^2}{2}\frac{(1-\kappa)(8+\kappa+2\kappa^2)}{(1+\kappa)(1-2\kappa)}a^2a_x^* - 3\mathrm{i}k_0^2\frac{(8-\kappa+9\kappa^2-4\kappa^3-4\kappa^4)}{(1+\kappa)(1-2\kappa)^2}|a|^2a_x\right]-\frac{k_0^2}{4\pi^2}aI, \tag{2.15}$$

where

$$I=\iint_{-\infty}^{\infty}\frac{(\chi_1-\chi_2)^2}{|\boldsymbol{\chi}_1-\boldsymbol{\chi}_2|}A^*(\boldsymbol{\chi}_1)A(\boldsymbol{\chi}_2)\exp\{\mathrm{i}(\boldsymbol{\chi}_2-\boldsymbol{\chi}_1)\cdot\boldsymbol{x}\}\,\mathrm{d}\boldsymbol{\chi}_1\,\mathrm{d}\boldsymbol{\chi}_2. \tag{2.16}$$

In a manner identical to Stiassnie (1984), this can be shown to reduce to

$$I=\left[\frac{g(1+\kappa)}{2w_0}\right]2\pi\int_{-\infty}^{\infty}\frac{\partial}{\partial\xi}(|a|^2)\frac{(x-\xi)}{|\boldsymbol{x}-\boldsymbol{\xi}|^3}\,\mathrm{d}\boldsymbol{\xi}. \tag{2.17}$$

We can relate I to the mean flow potential $\bar{\phi}(x,y,z,t)$, which arises naturally out of the kinematic condition at the nonlinear free surface, which does not explicitly depend on κ. An analysis identical to that by Dysthe (1979), based on the kinematic condition (his equation (2.3)), shows that

$$\bar{\phi}_z=\tfrac{1}{2}w_0\frac{\partial}{\partial x}(|a|^2)\quad\text{on}\quad z=0. \tag{2.18}$$

This is the same form as that obtained when surface tension is neglected (Dysthe (1979) equation (4.2), Stiassnie (1984)). It is possible, to show further, that (Stiassnie 1984)

$$\bar{\phi}_x=\frac{w_0}{4\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial\xi}(|a|^2)\frac{(\xi-x)}{|\boldsymbol{x}-\boldsymbol{\xi}|^3}\,\mathrm{d}\boldsymbol{\xi}\quad\text{on}\quad z=0, \tag{2.19}$$

Thus I is directly proportional to $\bar{\phi}_x$.

We now collect (2.13), (2.17) and (2.19) together with the expression (2.15) to find the fourth-order evolution equation. If we then make the scaling transformation $t=t'/w_0$, $(x,y)=(x',y')/k_0$, $\bar{\phi}=w_0\bar{\phi}'/2k_0^2$ and $a=a'/k_0$ and then drop the primes, we find

$$2\mathrm{i}(a_t+c_g a_x)+pa_{xx}+qa_{yy}-\gamma|a|^2a = -\mathrm{i}sa_{xyy}-\mathrm{i}ra_{xxx}-\mathrm{i}ua^2a_x^*+\mathrm{i}v|a|^2a_x+a\bar{\phi}_x, \tag{2.20}$$

where c_g is the group velocity given by

$$c_g = \frac{w_0}{2k_0}\frac{(1+3\kappa)}{(1+\kappa)}$$

and the other constants are given by

$$p = \frac{3\kappa^2+6\kappa-1}{4(1+\kappa)^2}, \quad q = \frac{1+3\kappa}{2(1+\kappa)},$$

$$r = -\frac{(1-\kappa)(1+6\kappa+\kappa^2)}{8(1+\kappa)^3}, \quad s = \frac{3+2\kappa+3\kappa^2}{4(1+\kappa)^2},$$

$$\gamma = \frac{8+\kappa+2\kappa^2}{8(1-2\kappa)(1+\kappa)}, \quad u = \frac{(1-\kappa)(8+\kappa+2\kappa^2)}{16(1-2\kappa)(1+\kappa)^2},$$

$$v = \frac{3(4\kappa^4+4\kappa^3-9\kappa^2+\kappa-8)}{8(1+\kappa)^2(1-2\kappa)^2}.$$

Equation (2.20) is the fourth-order evolution equation for deep-water gravity-capillary waves, in terms of the complex wave amplitude $a(\boldsymbol{x}, t)$. When $\kappa = 0$, it is not directly comparable with Dysthe's (1979) result (his equation (2.19)) without deriving an expression for a in terms of the complex potential amplitude Ψ correct to second order (see Appendix).

Stiassnie (1984) quotes $u = -\frac{1}{2}$ when $\kappa = 0$ in his equation (14), which is in terms of the wave amplitude. We obtain $u = \frac{1}{2}$ in this case. Stiassnie's equation (14) is identical to Dysthe's equation (2.19), with the sign change in the third term on the right side as noted by Janssen (1983), even though the two amplitudes are different at second order (see Appendix for details).

For pure capillary waves we require κ infinite and (2.20) becomes

$$2\mathrm{i}(a_t+\tfrac{3}{2}a_x)+\tfrac{3}{4}a_{xx}+\tfrac{3}{2}a_{yy}+\tfrac{1}{8}|a|^2 a = -\tfrac{3}{4}\mathrm{i}a_{xyy}-\tfrac{1}{8}\mathrm{i}a_{xxx}-\tfrac{1}{16}\mathrm{i}a^2a_x^*+\tfrac{3}{8}\mathrm{i}|a|^2a_x+a\bar{\phi}_x. \quad (2.21)$$

By ignoring the right side of (2.20), we recover the third-order envelope equation for gravity-capillary waves on water of infinite depth. In this way we can obtain the result of Djordjevic & Redekopp (1977), equation (2.20).

A similar treatment of (2.21), which additionally ignores the y derivatives, recovers the result of Matsuuchi (1974) for the third-order evolution equation of pure capillary waves on water of infinite depth.

This analysis breaks down at $\kappa = \frac{1}{2}$ where γ and v are singular. In fact higher order analysis produces further breakdowns at $\kappa = 1/n$, where n is an integer greater than 1. These wavelengths correspond to harmonic resonance where the wave and one of its harmonics both travel at the same speed. In that case an evolution equation is required for both amplitudes. See, for example, the work by McGoldrick (1970) on the case $\kappa = \frac{1}{2}$. This is a special case of (2.4) with $\boldsymbol{k}_1 = \boldsymbol{k}_2$ (see also Ma 1982).

In general, Zakharov's integral equation (2.1) fails to completely describe the evolution of $B(\boldsymbol{k}, t)$, the spectral component of the wave envelope, when the dominant interactions are triad-like in nature. If P denotes a typical wave period, (2.1) is valid for interactions whose timescale is $(ak)^{-2}P$. Other interactions similar to those described by (2.4) occur over the shorter timescale $(ak)^{-1}P$. It is necessary to reconsider the problem to include these interactions, beginning with Zakharov's (1968) equation (1.15). The relative importance of the triad- and quartet-like interactions is a function of the wave steepness and the parameter κ. Our equation (2.1) is valid for quartet-like interactions and for small steepness. Equation (2.20) is valid with the additional assumption of a narrow band of waves.

3. Stability of a nonlinear wavetrain

Dysthe (1979) pointed out that as far as stability properties were concerned, it is sufficient to deal with a shortened version of (2.20); other terms do not contribute. A similar situation holds here, but we shall not simplify the equations *a priori*. The neglected terms, corresponding to setting r, s, u and v to zero, are, however, important in other aspects of wave propagation, as shown by Lo & Mei (1985).

It is straightforward to show that one solution to (2.20), together with Laplace's equation for $\bar{\phi}$, is given by

$$\bar{\phi} = 0, \quad a = a_0 \exp\{-\tfrac{1}{2}\mathrm{i}a_0^2\gamma t\}, \tag{3.1}$$

where a_0 is the wave steepness. Note that the phase of this solution is different depending on the sign of $2\kappa - 1$.

Following Dysthe (1979), we perturb this solution by taking

$$a = a_0(1+a')\exp\{\mathrm{i}(\theta' - \tfrac{1}{2}a_0^2\gamma t)\}. \tag{3.2}$$

We substitute (3.2) into (2.20) and find

$$\mathrm{L}a' + (u-v)\,a_0^2 a'_x + \mathrm{M}\theta' = 0, \tag{3.3}$$

$$\mathrm{L}\theta' - (u+v)\,a_0^2 a'_x - \mathrm{M}a' + 2a_0^2\gamma a' = -\bar{\phi}_x, \tag{3.4}$$

where the operators L and M are defined by

$$\mathrm{L} \equiv 2\left(\frac{\partial}{\partial t} + c_g\frac{\partial}{\partial x}\right) + r\frac{\partial^3}{\partial x^3} + s\frac{\partial^3}{\partial x\,\partial y^2}, \tag{3.5}$$

$$\mathrm{M} \equiv p\frac{\partial^2}{\partial x^2} + q\frac{\partial^2}{\partial y^2}. \tag{3.6}$$

In addition, we have the linearized form of equation (2.18)

$$\bar{\phi}_z = 2a_0^2 a'_x. \tag{3.7}$$

We seek solutions of (3.3), (3.4) and (3.7) of the form $\exp\{\mathrm{i}(\lambda x + \mu y - \Omega t)\}$ for a' and θ' and of the form $\exp\{Kz + \mathrm{i}(\lambda x + \mu y - \Omega t)\}$ for $\bar{\phi}$, where $K^2 = \lambda^2 + \mu^2$, to satisfy Laplace's equation for $\bar{\phi}$. Equation (3.7) gives us the magnitude of $\bar{\phi}$ in terms of λ, K, a_0 and the magnitude of a'. We use this in (3.4) and eventually obtain the dispersion relation

$$[\bar{L} - (u-v)\,a_0^2\lambda][\bar{L} + (u+v)\,a_0^2\lambda] = \bar{M}[\bar{M} - 2a_0^2(\gamma - \lambda^2/K)], \tag{3.8}$$

where
$$\bar{L} = 2(\Omega - c_g \lambda) + \lambda(r\lambda^2 + s\mu^2), \tag{3.9}$$
$$\bar{M} = -(p\lambda^2 + q\mu^2). \tag{3.10}$$

A consistent solution of (3.8) is given by
$$\bar{L} = -v\lambda a_0^2 \pm \{\bar{M}[\bar{M} - 2a_0^2(\gamma - \lambda^2/K)]\}^{\frac{1}{2}}. \tag{3.11}$$
We note that the first term is independent of u. Nevertheless the stability characteristics only depend on quantities within the square root and hence only on the terms $a\bar{\phi}_x$ on the right side of equation (2.20), as stated at the beginning of this section.

To examine the stability of the wavetrain, let us first consider perturbations in the x direction only, that is, take $\mu = 0$. Then we shall have instability when
$$\lambda < \left(-\frac{2\gamma}{p}\right)^{\frac{1}{2}} a_0 + \frac{a_0^2}{p}. \tag{3.12}$$
As in the rest of this section, we can recover Dysthe's (1979) results for pure gravity waves by setting $\kappa = 0$. For pure capillary waves, (3.12) reduces to
$$\lambda < \frac{1}{\sqrt{3}} a_0 + \tfrac{4}{3} a_0^2. \tag{3.13}$$
We note from (3.12) that instability to this order is impossible in a range of κ where p and γ are both negative. Inspection of the relevant expressions gives this range as
$$\frac{2}{\sqrt{3}} - 1 < \kappa < \tfrac{1}{2}. \tag{3.14}$$
The lower value corresponds to the minimum group velocity and the upper value to the first Wilton (1915) ripple, where the wave has the same phase velocity as its first harmonic. Zakharov (1968) showed that instability was impossible in this range at third order. Lighthill (1965) shows that this range corresponds to an elliptic dispersion equation. Outside the range, it is hyperbolic.

At marginal stability, there is equality on the right side of equation (3.12), and we can show that in this case
$$\Omega = c_g \left[\left(-\frac{2\gamma}{p}\right)^{\frac{1}{2}} + \frac{a_0}{p}\right] a_0. \tag{3.15a}$$
When κ is infinite, this becomes
$$\Omega = \tfrac{1}{2}\sqrt{3}\, a_0 + 2a_0^2 \tag{3.15b}$$
and for small κ we find
$$\Omega = (1 + \tfrac{105}{16}\kappa)\sqrt{2}\, a_0 - 2(1 + 10\kappa)\, a_0^2. \tag{3.15c}$$
The maximum growth rate δ_m is given by
$$\delta_m = \frac{|\gamma|}{2}\left[1 - \frac{1}{\gamma}\left(\frac{-\gamma}{p}\right)^{\frac{1}{2}} a_0\right] a_0^2, \tag{3.16a}$$
which reduces to
$$\delta_m = \tfrac{1}{16}[1 + 4(\tfrac{2}{3})^{\frac{1}{2}} a_0]\, a_0^2 \tag{3.16b}$$
for pure capillary waves and to
$$\delta_m = \tfrac{1}{2}[(1 + \tfrac{9}{8}\kappa) - 2(1 + \tfrac{73}{16}\kappa)\, a_0]\, a_0^2 \tag{3.16c}$$

when κ is small. This occurs at a wavenumber given by λ_{m}, where

$$\lambda_{\mathrm{m}} = \left[\left(\frac{-\gamma}{p}\right)^{\frac{1}{2}} + \frac{3a_0}{4p}\right] a_0, \tag{3.17a}$$

which becomes, for κ infinite,

$$\lambda_{\mathrm{m}} = \frac{1}{\sqrt{6}} a_0 + a_0^2 \tag{3.17b}$$

and for small κ,

$$\lambda_{\mathrm{m}} = 2(1+\tfrac{73}{16}\kappa)\, a_0 - 3(1+8\kappa)\, a_0^2. \tag{3.17c}$$

The value of the real part of Ω corresponding to λ_{m} is given by

$$\operatorname{Re}\Omega_{\mathrm{m}} = c_{\mathrm{g}}\left[\left(\frac{-\gamma}{p}\right)^{\frac{1}{2}} + \frac{3a_0}{4p}\right] a_0. \tag{3.18a}$$

For pure capillary waves, this becomes

$$\operatorname{Re}\Omega_{\mathrm{m}} = (\tfrac{3}{8})^{\frac{1}{2}}\, a_0 + \tfrac{3}{2} a_0^2 \tag{3.18b}$$

and for small κ,

$$\operatorname{Re}\Omega_{\mathrm{m}} = (1+\tfrac{105}{16}\kappa)\, a_0 - \tfrac{3}{2}(1+10\kappa)\, a_0^2. \tag{3.18c}$$

Naturally each of equations (3.15*a*), (3.16*a*), (3.17*a*) and (3.18*a*) do not hold in the region of κ given in (3.14). In each expression the key elements are the sign of the product $p\gamma$ at $O(a_0)$ and the inverse of p at $O(a_0^2)$. The former can be shown to be equivalent to Lighthill's criterion and the latter is the generalized mean flow response.

If we set $\kappa = 0$ in each of equations (3.15*c*)–(3.18*c*), we recover Dysthe's (1979) results. In each case the effect of the variations of the mean flow on the wave stability are of opposite sign from the pure capillary case. The results to $O(a_0)$ can be shown to be identical to those obtained by Matsuuchi (1974) when κ is infinite. The recent work of Barakat (1984) does not concur with the present results or those of Matsuuchi. This is because his equation (2.13) is incorrect. On the left side the term $T\tilde{\eta}_{xx}$ should be replaced by

$$T\{(1+H_x^2)^{-\frac{3}{2}}\,\tilde{\eta}_{xx} - 3(1+H_x^2)^{-\frac{5}{2}}\, H_{xx}\, H_x\, \tilde{\eta}_x\}.$$

Lake & Yuen (1977) made a series of experiments and measured $\operatorname{Re}\Omega_{\mathrm{m}}$ (their figure 1). The waves they used were approximately 20 cm in length with a frequency of around 3 Hz. This corresponds to taking $\kappa \approx 0.01$. Substitution of this value into (3.18*c*) shows that it increases the value of $\operatorname{Re}\Omega_{\mathrm{m}}$ when $\kappa = 0$, over the range of steepness used by the authors. From Dysthe's figure 3, this indicates a slightly better agreement with their results, but still different from the experimental results.

When we allow for perturbations at any angle to the wave, that is, take $\mu \neq 0$, we find the growth rate is now given by

$$\delta = \tfrac{1}{2}\{\overline{M}[2a_0^2(\gamma - \lambda^2/K) - \overline{M}]\}^{\frac{1}{2}}. \tag{3.19}$$

With no mean-flow term λ^2/K, and if $\kappa < \frac{2}{\sqrt{3}} - 1$, the unstable perturbations have wavenumbers between the hyperbola $\overline{M} = 2a_0^2\gamma$ and the asymptotes $-p\lambda^2 = q\mu^2$.

If $\kappa > \frac{1}{2}$, the unstable wavenumbers lie on the interior of the ellipse $\overline{M} = 2a_0^2\gamma$. For small amplitudes, unstable wavenumbers lie close to the origin. As before the rest of the range of κ does not give instability. The maximum growth rate occurs for all wavevectors on the conic $\overline{M} = a_0^2\gamma$. If we include the mean flow term λ^2/K, all these regions of instability are modified slightly. But the maximum growth rate remains the same as for $\mu = 0$ and occurs at the same wavenumber.

We have an independent check on our results for $\mu = 0$ and very long wavelength perturbations. From equation (3.11),

$$\Omega = c_g\lambda - \tfrac{1}{2}v\lambda a_0^2 \pm i\lambda a_0(-\tfrac{1}{2}p\gamma)^{\frac{1}{2}}. \tag{3.20}$$

If we now let $\mathscr{L}(w,k)$ be the averaged Lagrangian of the waves, then Lighthill (1965, 1967) has shown that the perturbations of a wavetrain satisfy the dispersion equation

$$\left(\frac{\Omega}{\lambda}\right)^2\frac{\partial^2\mathscr{L}}{\partial w^2} + 2\left(\frac{\Omega}{\lambda}\right)\frac{\partial^2\mathscr{L}}{\partial w\,\partial k} + \frac{\partial^2\mathscr{L}}{\partial k^2} = 0, \tag{3.21}$$

where $w = w(a,k)$ is now the fully nonlinear form of the dispersion relation. Lighthill (1965) gives an exact expression for $\mathscr{L}$ for pure capillary waves. It is

$$\mathscr{L} = \mathscr{L}_c = 2S - \frac{w^2}{k^3} - \frac{S^2k^3}{w^2}. \tag{3.22}$$

For gravity-capillary waves he gives the approximate expression

$$\mathscr{L} = \mathscr{L}_{gc} = \frac{g}{k^2}\left[\frac{(1-2\kappa)}{(2\kappa^2+\kappa+8)}\left(\frac{w^2}{gk}-1-\kappa\right)^2 + \frac{(24\kappa^5-116\kappa^4-74\kappa^3+351\kappa^2-110\kappa+64)}{(2\kappa^2+\kappa+8)^3(3\kappa-1)}\left(\frac{w^2}{gk}-1-\kappa\right)^3+\dots\right]. \tag{3.23}$$

Using (3.21) and (3.22), we can show that

$$\frac{\Omega}{\lambda} = \tfrac{3}{2}(1-\tfrac{1}{8}a_0^2) \pm i\tfrac{\sqrt{3}}{8}a_0, \tag{3.24}$$

which is the form of (3.20) when κ is infinite. A much lengthier calculation by using (3.21) and (3.23) recovers the full equation (3.20). Thus we have a completely independent check on the quantities v, p and γ as derived in §2.

Recent calculations by Zhang & Melville (1985) support our conclusions in this section. They calculated stability boundaries in the (λ,μ)-plane for gravity-capillary waves with $\kappa = 3$ and $\kappa = 7$ over a wide range of wave steepness. For the quartet-like interactions they found that the elliptical region of instability became annular as the steepness increased and subsequently disappeared or became absorbed into the triad region.

For $a_0 = 0.05$ the position of the stability boundary agrees closely with our equation (3.12) for two-dimensional perturbations for both values of κ (see table 1 for details). Zhang & Melville (1985) give no information on growth rates for these interactions. For perturbations normal to the wavefronts, our calculations predict no departure from the weakly nonlinear result. This is in agreement with the computations.

Ma (1984) and Chen & Saffman (1985) have made computer calculations on the stability of pure capillary waves. It has proved difficult to compare our results with the former paper because some of the figures are not annotated. The latter paper is, however, more detailed. Both sets of results show that the maximum growth rate for two-dimensional disturbances to pure capillary waves is described almost exactly by the weakly nonlinear theory, even for large values of the wave steepness. The wavelength at which this occurs is also in striking agreement. Thus in their table 1, for $a_0 = (0.09425, 0.18850, 0.28274, 0.37699)$ they find $10^3\,\delta_{\mathrm{m}} = (0.54, 2.2, 4.9, 8.6)$ and $\lambda_{\mathrm{m}} = (0.04, 0.08, 0.12, 0.14)$. Weakly nonlinear theory gives $\lambda_{\mathrm{m}} = \frac{1}{16}a_0^2$ and so $10^3\,\delta_{\mathrm{m}} = (0.56, 2.2, 5.0, 8.9)$ and also $\lambda_{\mathrm{m}} = a_0/\sqrt{6} = (0.04, 0.08, 0.12, 0.16)$. The reason for such close agreement at large amplitudes remains unclear. The maximum growth rate for three-dimensional disturbances is derived from triad interactions and therefore is not comparable with our results.

TABLE 1. VALUES OF PERTURBATION WAVENUMBER λ CORRESPONDING TO ONSET OF STABILITY OF GRAVITY-CAPILLARY WAVES OF STEEPNESS $a_0 = 0.05$

source	$\kappa = 3$	$\kappa = 7$
equation (3.12) (to $O(a_0)$)	0.036	0.030
equation (3.12) (to $O(a_0^2)$)	0.040	0.034
full calculation (Zhang & Melville 1985)	0.042	0.037

Equations (3.16*b*) and (3.17*b*), here appear to overestimate the results of the exact computations. Nevertheless at high values of a_0, this also occurs when comparing with Zhang & Melville (1985). So it appears that the starting value of steepness used by Chen & Saffman (1985), that is $a_0 = 0.09425$, is too high for comparison with the theory for this value of κ, and more terms are needed in the expansions. For $\kappa = 0$, Dysthe (1979) found agreement up to $a_0 = 0.2$. This provides clear evidence that the usefulness of (2.20) is a function of both a_0 and κ.

4. CONCLUSIONS

In general the conclusions are similar to those of Dysthe (1979), namely that the fourth-order evolution equation can be simplified greatly if we only wish to consider stability. The physical picture also remains the same.

The full equation (2.20) was derived from Zakharov's equation (2.1), together with the full form of the interaction coefficient, under the assumption that the wave packet under consideration is sufficiently narrow. A broad spectrum would admit triad-like resonances, which act on a shorter time scale than is consistent with the fourth-order equation.

A band of stable gravity-capillary waves, known to exist at third order, is shown to exist at this higher order too. The effect of the mean flow variations is generally to decrease key quantities relating to the stability of long gravity waves whereas for short capillary waves these quantities are increased.

Nevertheless there is still an energy leakage to higher modes that lie outside the range of validity of the evolution equation when $\kappa < (\frac{2}{\sqrt{3}}-1)$. It is possibly the case that the full Zakharov integral equation should be used here, just as it was by Crawford *et al.* (1981) for pure gravity waves. Presumably, as in that paper, energy leakage would be confined. When $\kappa > \frac{1}{2}$, the recent numerical work of Zhang & Melville (1985) indicates that the elliptic region becomes annular at finite amplitude. As the steepness increases, the width of the region appears to grow smaller and possibly vanish at high values of a_0.

Physically we see two types of interaction influencing the results given here. First, as is well known (Lighthill 1965), the relative signs of the envelope frequency dispersion term (pa_{xx}) and the nonlinear term ($-\gamma|a|^2a$) govern the overall stability of the solution. There exists a band of stable waves, where the product $p\gamma$ is positive. Secondly, the corrections to the stability properties that occur at higher order can now be seen to arise from an interaction between the mean flow term ($a\bar{\phi}_x$) and the frequency dispersion term (pa_{xx}). This is clear from (3.15a), (3.17a) and (3.18a). The correction to the maximum growth-rate equation (3.16a) is of higher order and is more complicated, involving all three terms. Since the frequency-dispersion term for gravity waves is of opposite sign for capillary waves, so too are the corrections to the stability properties.

Put another way, for gravity waves the detuned nonlinear terms are of the form $A(|A|^2+\bar{\phi}_x)$ and they must balance the dispersion term $-\frac{1}{4}A_{xx}$ (see Dysthe (1979), equation (4.1)). In this way a resonant quartet is maintained and energy transferred to growing sidebands. For capillary waves, the nonlinearity is of the form $a(\frac{1}{8}|a|^2-\bar{\phi}_x)$ and the dispersion is represented by $-\frac{3}{4}a_{xx}$. Clearly the detuning arising from the mean flow has the opposite sign.

Finally, significance has been attached to the term $\mathrm{i}v|a|^2a_x$ of (2.20). From (3.20) we see that it provides the real $O(a_0^2)$ correction to the frequency of very long plane perturbations to the waveform.

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References

Barakat, R. 1984 Instability of periodic capillary-gravity waves on deep water. *Wave Motion* **6**, 155–165.

Benjamin, T. B. & Feir, J. E. 1967 The disintegration of wave trains on deep water. Part 1. Theory. *J. Fluid Mech.* **27**, 417–430.

Benney, D. J. & Newell, A. C. 1967 The propagation of nonlinear wave envelopes. *J. Math. Phys.* **46**, 133–139.

Chen, B. & Saffman, P. G. 1985 Three-dimensional stability and bifurcation of capillary and gravity waves on deep water. *Stud. appl. Math.* **72**, 125–147.

Crapper, G. D. 1957 An exact solution for progressive capillary waves of arbitrary amplitude. *J. Fluid Mech.* **2**, 532–540.

Crawford, D. R., Lake, B. M., Saffman, P. G. & Yuen, H. C. 1981 Stability of weakly nonlinear deep water waves in two and three dimensions. *J. Fluid Mech.* **105**, 177–191.

Crawford, D. R., Saffman, P. G. & Yuen, H. C. 1980 Evolution of a random inhomogeneous field of nonlinear deep water gravity waves. *Wave Motion* **2**, 1–16.

Djordjevic, V. D. & Redekopp, L. G. 1977 On two-dimensional packets of capillary-gravity waves. *J. Fluid Mech.* **79**, 703–714.

Dysthe, K. B. 1979 Note on a modification to the nonlinear Schrödinger equation for application to deep water waves. *Proc. R. Soc. Lond.* A **369**, 105–114.
Janssen, P. A. E. M. 1983 On a fourth-order envelope equation for deep water waves. *J. Fluid Mech.* **126**, 1–11.
Lake, B. M. & Yuen, H. C. 1977 A note on some nonlinear water wave experiments and the comparison of data with theory. *J. Fluid Mech.* **83**, 75–81.
Lighthill, M. J. 1965 Contributions to the theory of waves in non-linear dispersive systems. *J. Inst. Math. Applics* **1**, 269–306.
Lighthill, M. J. 1967 Some special cases treated by the Whitham theory. *Proc. R. Soc. Lond.* A **299**, 28–53.
Lo, E. & Mei, C. C. 1985 A numerical study of water wave modulation based on a higher order nonlinear Schrödinger equation. *J. Fluid Mech.* **150**, 395–416.
Longuet-Higgins, M. S. 1978*a* The instabilities of gravity waves of finite amplitude in deep water. I. Superharmonics. *Proc. R. Soc. Lond.* A **360**, 471–488.
Longuet-Higgins, M. S. 1978*b* The instabilities of gravity waves of finite amplitude in deep water. II. Subharmonics. *Proc. R. Soc. Lond.* A **360**, 489–506.
Ma, Y.-C. 1982 Weakly nonlinear steady gravity-capillary waves. *Physics Fluids* **25**, 945–948.
Ma, Y.-C. 1984 A note on the instability of capillary waves. In *Advances in nonlinear waves, vol. I.* (ed. L. Debnath), pp. 64–74. Research notes in mathematics no. 95. Boston: Pitman.
Martin, D. U. & Yuen, H. C. 1980 Spreading of energy in solutions of the nonlinear Schrödinger equation. *Physics Fluids* **23**, 1269–1271.
Matsuuchi, K. 1974 Modulational instability of nonlinear capillary waves on thin liquid sheet. *J. phys. Soc. Japan* **37**, 1680–1687.
McGoldrick, L. F. 1965 Resonant interactions among capillary-gravity waves. *J. Fluid Mech.* **21**, 305–331.
McGoldrick, L. F. 1970 On Wilton's ripples: a special case of resonant interactions. *J. Fluid Mech.* **42**, 193–200.
Okamura, M. 1984 Instabilities of weakly nonlinear standing gravity waves. *J. phys. Soc. Japan* **53**, 3788–3796.
Stiassnie, M. 1984 Note on the modified nonlinear Schrödinger equation for deep water waves. *Wave Motion* **6**, 431–433.
Stiassnie, M. & Shemer, L. 1984 On modifications of the Zakharov equation for surface gravity waves. *J. Fluid Mech.* **143**, 47–67.
Wilton, J. R. 1915 On ripples. *Phil. Mag.* **29**, 688–700.
Zakharov, V. E. 1967 The instability of waves in nonlinear dispersive media. *Soviet Phys. JETP* **24**, 740–744.
Zakharov, V. E. 1968 Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. appl. Mech. tech. Phys.* **2**, 190–194.
Zhang, J. & Melville, W. K. 1985 On the stability of gravity-capillary waves. (Submitted to *Physics Fluids.*)

Appendix

Although Stiassnie (1984) obtains $u = -\frac{1}{2}$ when $\kappa = 0$ for (2.20) in this paper, this appears to be a misprint in both his equations (10) and (14). Thus working backwards from his equation (10) as printed, with $k_0 = 1$, we find

$$|a|^2 a + \tfrac{1}{2}ia^2 a_x^* - 3i\,|a|^2 a_x$$

$$\propto \iiint_{-\infty}^{\infty} [1 + \tfrac{1}{4}(\chi_2 + \chi_3 + 3\chi_1) + \tfrac{3}{2}(\chi_2 + \chi_3) + O(|\boldsymbol{\chi}|^2/k_0^2)]$$

$$+ A^*(\boldsymbol{\chi}_1)\, A(\boldsymbol{\chi}_2)\, A(\boldsymbol{\chi}_3) \exp\{i(\boldsymbol{\chi}_2 + \boldsymbol{\chi}_3 - \boldsymbol{\chi}_1)\cdot \boldsymbol{x}\}\, d\boldsymbol{\chi}_1\, d\boldsymbol{\chi}_2\, d\boldsymbol{\chi}_3. \quad \text{(A 1)}$$

On the other hand, if we substitute his equation (9) into equation (8), we find the term in square brackets in (A 1) has become

$$[1 + \tfrac{1}{4}(\chi_2 + \chi_3 - \chi_1) + \tfrac{3}{2}(\chi_2 + \chi_3) + O(|\boldsymbol{\chi}|^2/k_0^2)].$$

This is precisely what is obtained if the right-hand side of Stiassnie's equation (10) contained the (scaled) terms

$$|a|^2 a - \tfrac{1}{2}\mathrm{i}\, a^2 a_x^* - 3\mathrm{i}\,|a|^2 a_x.$$

Another proof of the correct sign has been provided by Janssen (personal communication). It is given here for completeness. Equation (2) of Janssen (1983) is the fourth-order evolution equation, in terms of the dimensionless complex potential amplitude, Ψ, which can be written in the form

$$\mathcal{M}\Psi = |\Psi|^2\Psi + T_4, \tag{A 2}$$

where
$$\mathcal{M} = 2\mathrm{i}\left(\frac{\partial}{\partial t}+\frac{1}{2}\frac{\partial}{\partial x}\right)-\frac{1}{4}\frac{\partial^2}{\partial x^2}+\frac{1}{2}\frac{\partial^2}{\partial y^2} \tag{A 3}$$

and T_4 represents the fourth-order terms.

For pure gravity waves, the dimensionless elevation a is related to Ψ by (Janssen 1983, equation (19))

$$a = \mathrm{i}(\Psi - \tfrac{1}{2}\mathrm{i}\Psi_x). \tag{A 4}$$

Thus
$$\mathcal{M}a = \mathrm{i}\,|\Psi|^2\Psi + \tfrac{1}{2}[\,|\Psi|^2\Psi_x + \Psi(|\Psi|^2)_x] + \mathrm{i}T_4 \tag{A 5}$$

since
$$\mathcal{M}(\Psi_x) = |\Psi|^2\Psi_x + \Psi(|\Psi|^2)_x \tag{A 6}$$

with higher order terms. From (A 4) we can show that

$$|\Psi|^2\Psi = -\mathrm{i}[\,|a|^2 a - \tfrac{1}{2}\mathrm{i}a^2 a_x^* + \mathrm{i}\,|a|^2 a_x] \tag{A 7}$$

with higher order terms. This we substitute in (A 5). The second and third terms on the right side of (A 5) being of fourth order only need the substitution $\Psi = -\mathrm{i}a$. Thus we find

$$\mathcal{M}a = |a|^2 a - \mathrm{i}a^2 a_x^* + \mathrm{i}T_4 \tag{A 8}$$

and the relevant terms from T_4 are readily seen to be

$$\mathrm{i}T_4 = \ldots\ldots + \tfrac{1}{2}\mathrm{i}a^2 a_x^* - 3\mathrm{i}\,|a|^2 a_x + \ldots. \tag{A 9}$$

Thus on combining (A 8) and (A 9) we find

$$\mathcal{M}a = |a|^2 a - \tfrac{1}{2}\mathrm{i}a^2 a_x^* - 3\mathrm{i}\,|a|^2 a_x + \ldots. \tag{A 10}$$

This is in exact agreement with (2.20) when $\kappa = 0$.

The fact that $u = \frac{1}{2}$ when $\kappa = 0$ has recently been confirmed by Stiassnie (personal communication).