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Effect of stratification due to suspended sand on velocity and concentration distribution in unidirectional flows

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[1] Sediment-induced stratification effects on velocity profiles and sediment concentration distribution in a steady, uniform turbulent flow are examined in this paper. The early work concerning sediment stratification relates this to the von Karman constant's variability. Subsequent attempts to account for stratification were based on the stratified flow analogy, introducing the parameters α and β, whose values were assumed to be those obtained for thermally stratified flows. Following these investigators, we assume stratification effects to be expressed through these parameters. We solve the governing equations for velocity and sediment concentration for a parabolic neutral eddy viscosity model. Analytically closed-form solutions are obtained. We run our model against experimental data to obtain the optimal set [α, β]. For neutral conditions, $\beta = 0$ by definition, and we obtain $\alpha = 1$. For stratified conditions we obtain [$\alpha = 0.8$, $\beta = 4.0$]. This is the first time both α and β have been obtained from sediment-laden flow observations. Accounting for stratification improves the prediction of velocity and concentration. For predictive purposes, we need to know the movable bed roughness and the reference concentration. Analyses of experimental data sets provide predictive relationships for these in terms of sediment and flow parameters.

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1. Introduction

[2] Sediment transport in steady river flows as well as in the continental shelf bottom boundary layer governs a wide range of processes and is of practical concern for many engineering applications. Many models for sediment transport processes have already been developed. Most of these models do not account for stratification due to concentration of suspended sediment. Exceptions to this are studies by *Vanoni* [1946], *Einstein and Chien* [1955], *Smith and McLean* [1977], followed by *Glenn and Grant* [1987] and *Styles and Glenn* [2000]. The early work concerning sediment stratification relates stratification with the von Karman constant's variability. *Vanoni* [1946] uses the Prandtl-von Karman velocity defect law for a two-dimensional, steady, uniform flow in an open channel to fit the log-profile of velocity measurements by a straight line, whose slope N is

$$N = \frac{\kappa}{2.3\sqrt{\frac{\tau_o}{\rho}}} \tag{1}$$

where κ is the von Karman constant, ρ is the fluid density, and τ_o is the bottom shear stress. From equation (1), he obtains values of κ. Plotting κ against $\overline{C}$, the mean concentration over the depth, he then concludes that an increase in the mean concentration $\overline{C}$ corresponds to a decrease in κ and a decrease in the eddy viscosity ν_T below their values for clear fluids. To explain the observed decrease in κ and the corresponding decrease in the momentum transfer coefficient when sediment is in suspension, he hypothesizes that the turbulence is damped by the sediment. According to *Vanoni* [1946], the sediment is actually kept in suspension by the vertical velocity fluctuations and the energy to do this must therefore come from the turbulence. He introduces the ratio P_s/P_f equal to the ratio of the energy required to support the sediment in the column of water to the energy required to overcome the friction. *Einstein* [1950] and *Einstein and Chien* [1955] correlated κ against P_s/P_f using data of several investigators.

[3] Later attempts to model the effects due to sediment in suspension in a turbulent flow have been based on the stratified flow analogy. By analogy with heat flow theory, following *Stull* [1988], the Flux Richardson number for a

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continuously sediment-stratified flow is defined as [e.g., *Styles and Glenn*, 2000]

$$R_f = -\frac{\frac{g}{\rho_T}\overline{\rho'_T w'}}{\overline{u'w'}\frac{\partial U}{\partial z}} \quad (2)$$

where g is the acceleration due to gravity, ρ_T is the mean density of the fluid-sediment suspension, ρ'_T is the turbulent density fluctuation, U is the horizontal component of the Reynolds-averaged velocity, u' is the horizontal and w' is the vertical component of turbulent velocity fluctuations, and an over-bar denotes Reynolds average. The x-axis is chosen parallel to the mean flow. Similarly, for sediment stratification problems, the Monin-Obukov length scale is defined as $L = \frac{z}{R_f}$ and z/L is referred to as the stability parameter. The Miles theorem, elegantly presented by *Howard* [1961], states that if the Richardson Number R_f is greater than 0.25 everywhere, then a stratified flow is stable. This suggests that the Richardson Number may be used as a measure of stratification: it measures the importance of flow stratification in inhibiting turbulent transfer of momentum and mass, with $R_{f,cr}$ = 0.25 being the critical value above which turbulence production is eliminated.

[4] *Smith and McLean* [1977], *Glenn and Grant* [1987], and *Styles and Glenn* [2000] examine the effects of suspended sediment-induced stable stratification and present procedures to compute the associated reduction in eddy diffusivity based on standard atmospheric boundary layer methods. They introduce an eddy viscosity and an eddy diffusivity of the form

$$\left.\begin{aligned} \nu_T &= \nu_{TN}\left(1 - \beta R_f\right) \\ \nu_S &= \frac{\nu_{TN}}{\alpha}\left(1 - \frac{\beta}{\alpha}R_f\right) = \nu_{SN}\left(1 - \frac{\beta}{\alpha}R_f\right) \end{aligned}\right\} \quad (3)$$

where ν_{TN} is the neutral eddy viscosity, ν_{SN} is the neutral eddy diffusivity, α is the ratio of the neutral eddy diffusivity of mass to that of momentum, i.e., the Schmidt number, and the damping coefficient β is a constant derived from the thermally stratified atmospheric boundary layer observations: α and β were found to be equal to 0.74 and 4.7 by *Businger et al.* [1971]. These authors then develop iterative procedures in order to compute concentration and velocity profiles. However, this analogy between thermally stratified boundary layers and sediment-induced stratification was not justified in these procedures and the validity of the theoretical framework for this application is open to question and must be established from experimental data from flows carrying sediment in suspension.

[5] *Villaret and Trowbridge* [1991] analyze concentration and velocity profiles obtained from laboratory measurements in order to test the applicability of the stratified flow analogy to dilute suspensions of sand in turbulent flows of water. Their model includes sand grading and wake effects. They examine the difference between velocity profiles in neutral and stratified flows, following *Smith and McLean*'s [1977] formulation of the eddy viscosity and the eddy diffusivity. They assume stratification corrections to be small and derive approximate solutions for the velocity and the concentration profiles. They qualitatively observe effects of stratification on velocity profiles but not in individual concentration profiles. Finally, they are not able to obtain definitive results regarding α and β. Moreover, in their analysis variability of the bottom roughness was not considered an adjustable parameter, and the difference between velocity profiles in stratified and neutral flow was solely attributed to the stratification effect, i.e., to α, β, and R_f.

[6] The effects of stratification due to suspended sediments are investigated in this paper. Exact, closed-form analytical solutions are developed for both velocity and concentration profiles in steady turbulent flows carrying sediment in suspension. From comparisons with experimental data on both velocity and suspended sediment concentration and accounting for bottom roughness variability, estimates of the values of parameters α and β appropriate for use when the stratification is caused by suspended sediments are obtained. Since a limited subset of the data used were obtained for flows over movable beds, rather than fixed beds, formulae for the reference concentration and the bed roughness which are needed for model applications are also obtained.

2. Theory

[7] We consider a gravity-driven flow of water with a free surface carrying a dilute suspension of solid particles over a plane sloping bottom. The ensemble-averaged motion is independent of streamwise position x and cross-stream position y. We neglect particle-particle interactions, develop the governing equations assuming sediment concentrations are small, and treat the fluid-sediment mixture as a continuum. The turbulent closure scheme developed in the eddy viscosity model used by *Glenn* [1983] is used here to solve the governing equations. The pressure p, velocity (u, w) and volumetric sediment concentration c can, for a uniform mean flow, be partitioned into mean and turbulent components where capital letters denote mean variables and primes denote turbulent fluctuations. The mean concentration, pressure, and velocity are treated as steady. The choice of coordinate system makes $W(z)$ = 0, the sediment particles are assumed to have the same size and density and w_s, the particle settling velocity, is treated as a constant. The sediment flux at the surface of the water column is equal to zero. Therefore expressing the turbulent flux invoking the eddy diffusivity concept ν_S, and integrating with respect to z, the governing equation for the concentration is

$$w_s C + \nu_S \frac{dC}{dz} = 0 \quad (4)$$

For a steady uniform turbulent flow the shear stress varies linearly over the depth of flow h. The governing equation for the velocity becomes

$$\nu_T \frac{dU}{dz} = u_*^2\left(1 - \frac{z}{h}\right) \quad (5)$$

where ν_T is the turbulent eddy viscosity and $u_* = \sqrt{\frac{|\tau_o|}{\rho}}$ is the shear velocity.

[8] We assume, following *Smith and McLean* [1977], that the eddy viscosity and eddy diffusivity are expressible in the following form:

$$\left.\begin{aligned}\nu_T &= \nu_{TN}\left(1-\beta R_f\right)\\ \nu_S &= \nu_{SN}\left(1-\beta R_f\right) = \frac{\nu_{TN}}{\alpha}\left(1-\beta R_f\right)\end{aligned}\right\} \tag{6}$$

where we have assumed that the stratification correction parameter β is the same for momentum and mass transfer. This simplification allows us to obtain analytical closed-form solutions without imposing the limitation of small values of βR_f relative to unity. Our solution is therefore in principle valid for values of βR_f up to unity, and (6) suggests that turbulence ceases to affect both momentum and mass transfer for the same critical value of the Richardson number. The governing equation for the concentration (4) becomes

$$\frac{dC}{dz} = \frac{-w_s\alpha}{\nu_T}C \tag{7}$$

Following *Glenn* [1983], the total density of the fluid-sediment suspension is defined as

$$\rho_T = \rho[1+(s-1)C] \tag{8}$$

where ρ is the water density and $s = \frac{\rho_s}{\rho}$ is the ratio of densities with ρ_s being the sediment density. From *Monin and Yaglom* [1971] the Flux Richardson number defined in equation (2) can be expressed as

$$R_f = \frac{-g\nu_S(s-1)\frac{dC}{dz}}{\nu_T\left(\frac{dU}{dz}\right)^2} \tag{9}$$

Replacing values of $\frac{dU}{dz}$, ν_T, ν_S, and $\frac{dC}{dz}$ given by (5), (6), and (7), we obtain after some simple algebraic manipulations

$$R_f = \frac{g(s-1)w_s C\nu_{TN}}{u_*^4\left(1-\frac{z}{h}\right)^2\left(1+\frac{\beta g(s-1)w_s\nu_{TN}}{u_*^4\left(1-\frac{z}{h}\right)^2}C\right)} \tag{10}$$

2.1. Eddy Viscosity Closure Scheme

[9] In agreement with previous studies [e.g., *Styles and Glenn*, 2000], we adopt the following model for the eddy viscosity. From z_o to a reference depth, z_r, which depends on the grain size d, we regard the sediment transport as bedload. Since the turbulent eddy scale is limited by the proximity of the bottom, a distance of several sediment diameters must be required in order to make the treatment of sediment transport as a suspension realistic. For this reason we choose $z_r = 7d$, where d is the diameter of the sediment, and regard the sediment transport for $z < z_r$ as bedload. Flow-sediment interactions in the bedload layer are complex and stratification can have opposite effects depending on the site, as shown by *Friedrichs et al.* [2000]. They illustrate the type of constant stress velocity profile which results when stratification effects, i.e., Richardson number, either decrease, increase or remain constant toward the bed, and provide real examples for each case. In particular, they show that when the Richardson number remains relatively constant above the bed, an equilibrium is reached, and the velocity profile remains log-linear. For simplicity, in our analysis, stratification is assumed to have no effect on velocity in the bedload layer. Above z_r, stratification influences both the velocity and the concentration. We assume the neutral viscosity to be parabolic and solve the governing equations (5) and (7) to develop a model for sediment concentration and velocity profiles in a stratified sediment-laden flow.

2.2. Solution for Parabolic Neutral Eddy Viscosity

[10] We consider a parabolic neutral eddy viscosity

$$\nu_{TN} = \kappa u_* z\left(1-\frac{z}{h}\right) \tag{11}$$

Integration of (5) and applying $U(z_o) = 0$ leads to the classical logarithmic velocity profile

$$U_N(z) = \frac{u_*}{\kappa}\ln\left(\frac{z}{z_o}\right) \tag{12}$$

Introducing the constant $q = \frac{w_s\alpha}{\kappa u_*}$ known as the Rouse Number, integration of (7) and use of $C(z_r) = C_r$ results in the neutral concentration distribution

$$C_N(z) = C_r\left(\frac{(h-z)z_r}{(h-z_r)z}\right)^q \tag{13}$$

From equations (10) and (11), we obtain for the Flux Richardson Number

$$R_f = \frac{azC}{1-\frac{z}{h}+\beta azC} \tag{14}$$

with $a = \frac{g(s-1)w_s\kappa}{u_*^3} = (LC)^{-1}$ where L is the Monin-Obukov length [*Styles and Glenn*, 2000]. The governing equations for the stratified concentration are valid above the reference elevation z_r. Inserting (14) in (7), the governing equation for the concentration may be written

$$\frac{dC}{dz} + \frac{w_s\alpha}{\kappa u_*}\frac{C}{z\left(1-\frac{z}{h}\right)} + \frac{w_s\alpha\beta a}{\kappa u_*}\frac{C^2}{\left(1-\frac{z}{h}\right)^2} = 0 \tag{15}$$

Introducing

$$\begin{aligned} Z_r(z) &= \frac{1}{h(q-1)}\left[\left(\frac{h}{z_r}-1\right)^{q-1} - \left(\frac{h}{z}-1\right)^{q-1}\right] && \text{for } q \neq 1\\ Z_r(z) &= \frac{1}{h}\ln\left(\frac{z(h-z_r)}{z_r(h-z)}\right) && \text{for } q = 1\end{aligned} \tag{16}$$

the solution for the stratified concentration with a parabolic neutral eddy viscosity is (see Appendix A for details)

$$C(z) = C_N(z)\frac{1}{1+\frac{h^2 q\beta a z_r^q C_r Z_r(z)}{(h-z_r)^q}} \tag{17}$$

Equation (17) clearly demonstrates that stratification results in a decrease in concentration compared with a neutral

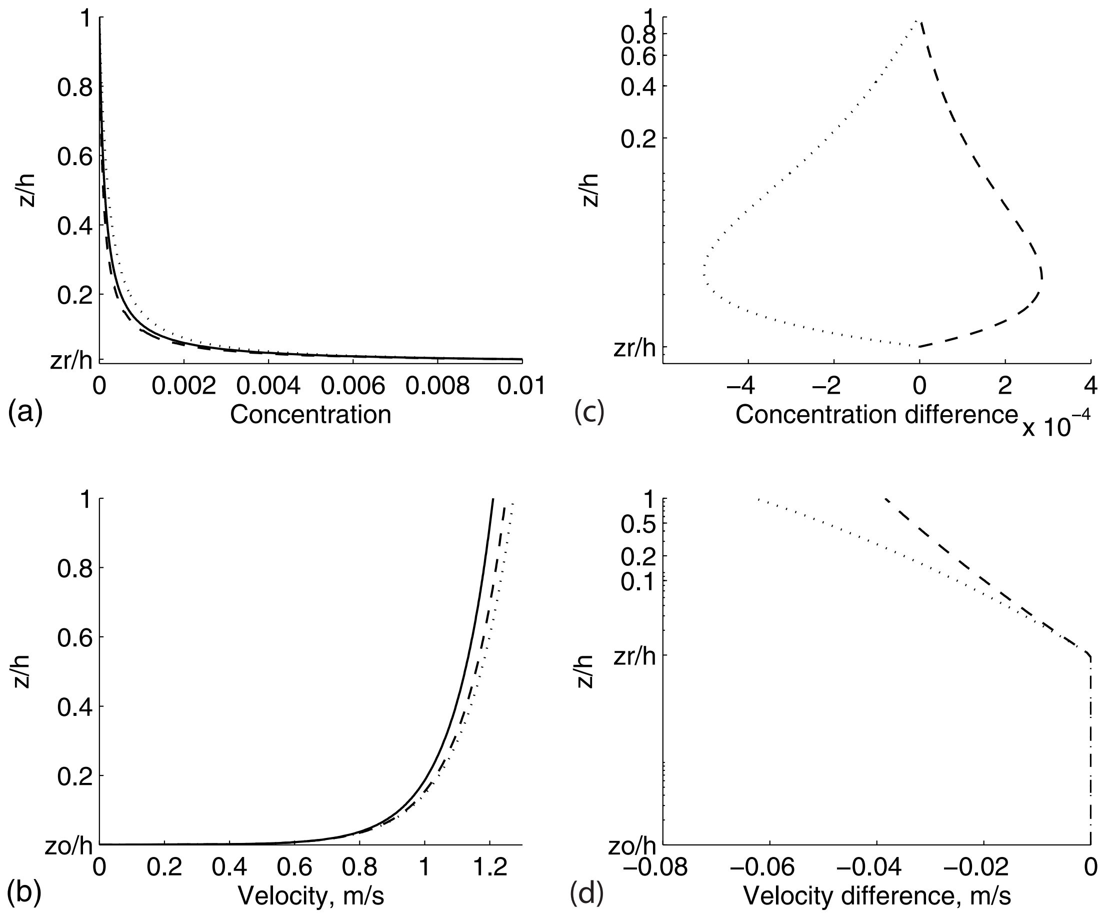

Figure 1. Concentration and velocity profiles for stratified and neutral steady uniform sediment-laden flows in an open wide channel. (a) Concentration and (b) velocity; (c) difference between neutral and stratified concentrations and (d) velocities. Solid line represents neutral with [α, β] = [1, 0]; dashed line represents stratified with [α, β] = [1, 4]; dotted line represents stratified with [α, β] = [0.8, 4].

nonstratified flow since $C(z) \leq C_N(z)$. The expression obtained for the stratified concentration distribution is valid for the portion of the water column where the mode of transport is suspended sediment transport. Immediately above the water-sediment interface the mode of sediment transport is bedload, which is not governed by (7) since sediment grains in the bedload layer are rolling, sliding, or jumping along the bottom. It is customary to relate the reference concentration for suspended load C_r to the bedload transport rate [e.g., *Einstein*, 1950], i.e., consider transport for $z < z_r$ to be bedload. Adopting this concept, (17) is valid only for $z > z_r$. For $z_o < z < z_r$ we adopt for simplicity the unstratified velocity profile given by (12) where z_o expresses the effect of bedload particles on the velocity distribution, i.e., z_o is a measure of the movable bottom roughness.

[11] Above $z = z_r$, using (6) with R_f given by (14), integrating the governing equation (5) once and using the boundary condition $U(z_o) = 0$, we obtain the stratified velocity profile

$$U(z) = U_N(z) + \frac{u_* \beta a}{\kappa} \int_{z_r}^{z} \frac{C(z')}{1 - \frac{z'}{h}} dz' \tag{18}$$

with C given by (17). As demonstrated by (18), stratification results in an increase in velocity compared to the neutral nonstratified flow provided z_o, the bottom roughness, is assumed the same. Other neutral eddy viscosity models, simpler than the parabolic model adopted here, e.g. linear-constant or entirely linear over depth, lead to simpler analytical solutions for $C(z)$ and $U(z)$ than those obtained here [see *Herrmann*, 2003].

2.3. Example Profiles

[12] To show the effect of stratification on velocity profile and sediment concentration distribution, we present a concrete example and compare neutral and stratified profiles. We use the formulae with the parabolic neutral eddy viscosity, i.e., equations (17), (12), and (18) for the stratified profiles and equations (13) and (12) for the neutral concentration and velocity profiles. We plot profiles for $U(z)$ and $C(z)$ for stratified and neutral cases, taking for this example $\beta = 4$, $\alpha = 1$, $\kappa = 0.4$, $g = 9.8$ m/s^2, $s = 2.65$, $h = 16$ cm, $u_* =$ 5 cm/s, $z_o = 0.01$ mm, $w_s = 2$ cm/s, the grain diameter $d =$ 0.3 mm and the reference volume concentration $C_r = 0.01$ at $z_r = 7d = 2$ mm. These values correspond to typical laboratory conditions. The results are shown in Figure 1.

We also show the stratified concentration distribution and velocity profile obtained when taking $\alpha = 0.8$ instead of $\alpha = 1$.

[13] Stratification decreases the eddy diffusivity ν_S compared to the neutral case. This is expressed in equation (6), where the Richardson Number R_f and the constant β, both positive, account for stratification. From (7), everything else being equal, particularly the roughness z_o and the reference concentration C_r, stratification should therefore result in an decrease of the concentration compared to the neutral case. This effect can be observed in Figure 1a (dashed line). However, if there is a difference between α for the neutral case and α for the stratified case (for example α = 0.8 for the stratified case and α = 1 for the neutral case), everything else being equal, the stratified concentration can be larger than the neutral concentration (dotted line in Figure 1a).

[14] Stratification also decreases the eddy viscosity ν_T, which is expressed in (6), due to the contribution of the Flux Richardson Number and the constant β. From (5), everything else being equal, stratification should therefore result in an increase of velocity compared to the neutral case. This effect can be observed in Figure 1b (dashed line). We obtain a larger stratified concentration for α = 0.8 than for α = 1 (Figure 1a) and hence a smaller eddy viscosity. Thus a difference between α for the neutral and for the stratified case (for example, α = 0.8 for the stratified case and α = 1 for the neutral case), everything else being equal, therefore results in an even larger difference between the stratified and the neutral velocity profiles, as seen from dotted line in Figure 1b.

[15] From Figure 1a, we can observe that most of the effect due to stratification on the concentration comes from the near-bottom region, between $z = z_r$ and $z \approx 0.4h$, where the concentration is the largest (and more accurately determined experimentally). This is not the case for the velocity for which the difference increases to a maximum at $z = h$ (see Figure 1b). However, in the upper part of the water column, other effects (e.g., wake and sidewall effects) come into play that may obscure the effect of stratification.

3. Determination of Model Parameters

[16] We perform an analysis of experimental data, based on our theory, to obtain the best choice of values for the model parameters α and β. To calibrate our model, we use experimental data sets reported by *Vanoni* [1946], *Brooks* [1954], *Einstein and Chien* [1955], *Barton and Lin* [1955], *Vanoni and Nomicos* [1960], *Coleman* [1981], *Coleman* [1986], and *Lyn* [1986]. We use the same experiment numbering system as used by *Villaret and Trowbridge* [1991]. In these experiments, velocity and concentration profiles were measured in laboratory channels over plane beds. These measurements were reported, as well as the water depth h and temperature, the settling velocity w_s, the shear velocity u_*, the grain diameter d, in data sets which were generously provided by John Trowbridge (of *Villaret and Trowbridge* [1991]). Only the experiments in which the bed remains flat are considered. There are two types of experiments: "starved bed" experiments and "equilibrium bed" experiments. In "starved bed" experiments, no sand bed is present, the water flows either over a smooth bottom or over a bottom to which sand particles have been previously glued. The experimenter introduces sediments in the flow and stops the experiments when sand bed formation is observed. In "equilibrium bed" experiments, sediments are provided by the sand bed to the overlying water column and the experimenter takes measurements once equilibrium between flow and sand bed has been established.

[17] To measure or compute the sediment settling velocity w_s, the experimenters used different methods. To be consistent, we need to use a settling velocity obtained by the same method for every experiment. Therefore we use the recent formula for naturally worn sands by *Jiménez and Madsen* [2003] to obtain settling velocities that are in agreement with those provided by the experimenters: $w_s = (0.92 \pm 0.10) w_{s,s}$ where $w_{s,s}$ is the settling velocity specified by the investigator. In the following analysis, we also use $\kappa = 0.4$, $g = 9.8$ m/s^2 and $s = 2.65$ (all sediments are quartz and the fluid is water).

3.1. Data Analysis

[18] Stratification effects are expected to be pronounced only in some regions of the water column. Indeed, in the higher region of the water column, the sediment concentration is too small to result in any significant density difference. In the lower region, the momentum transporting eddies are very small, therefore the density difference over the eddy length scale is small. We therefore expect stratification effects to be most pronounced in an intermediate layer. Moreover, we expect the velocity data in the upper layer to be influenced by wake and sidewall effects. We therefore consider only data obtained in the lower portion of the water column corresponding to $z \leq 0.4h$.

[19] The most convenient and consistent choice of reference elevation z_r for the reference concentration C_r would, as previously mentioned, be one that separates the water column into two regions so that sediment transport may be regarded as bed load below z_r and as suspended load above z_r. This suggests z_r to be related to the thickness of the bed load layer, which may be approximated as a multiple of the grain diameter [*Madsen*, 1991]. For this reason, the reference elevation $z_r = 7d = 7d_n$, where $d_n = d_{sieve}/0.9$ is the nominal sediment diameter for naturally worn sands [*Jiménez and Madsen*, 2003], is adopted here.

[20] For a given choice of α and β, the reference concentration C_r remains the only unknown in equations (13) and (17). In order to determine it, we vary C_r in order to minimize the variance

$$\epsilon_c(\alpha, \beta) = \frac{1}{N} \sum_{1}^{N} \left(\frac{C_p - C_m}{C_p} \right)^2 \tag{19}$$

where C_m is the measured concentration reported in the data set, C_p corresponds to the concentration predicted by our model at every measurement point and N is the number of data points considered in a particular experiment ($z < 0.4h$). For our given choice of α and β, once C_r has been determined, the concentration can be calculated over the entire depth using (13) or (17).

[21] With the best value of C_r, the stratified concentration can also be integrated in order to compute the velocity. However, in the velocity profile equations (12) or (18), z_o is still unknown. Keeping the same α and β, we use the same

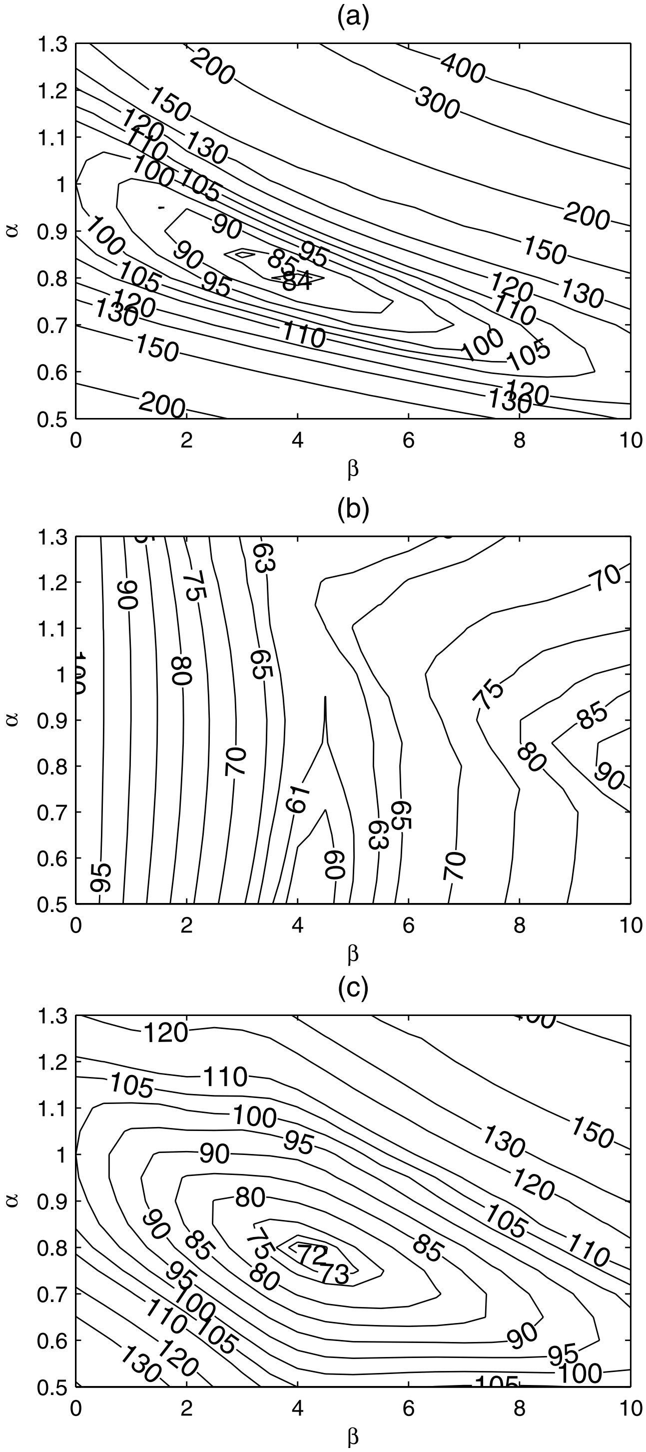

Figure 2. Contours of the relative error (in percent) in α, β-plane. (a) Concentration $\overline{\epsilon_{c,rel}}(\alpha, \beta)$ as given by (22), (b) velocity $\overline{\epsilon_{u,rel}}(\alpha, \beta)$ as given by (22) and (c) combined concentration and velocity $\overline{\epsilon_{rel}}(\alpha, \beta)$ as given by (23).

method as previously outlined for C_r and obtain the best z_o by minimizing the variance

$$\epsilon_u(\alpha, \beta) = \frac{1}{N} \sum_1^N \left(\frac{U_p - U_m}{U_p} \right)^2 \tag{20}$$

Introducing this value of z_o in equations (12) or (18), we are then able to compute the velocity over the entire depth.

3.2. Determination of α and β

[22] For each data set we define relative errors using the neutral case, where $\alpha = 1$ and $\beta = 0$, as the reference, i.e.,

$$\epsilon_{c,rel}(\alpha, \beta) = \frac{\epsilon_c(\alpha, \beta)}{\epsilon_c(1, 0)} \quad \text{and} \quad \epsilon_{u,rel}(\alpha, \beta) = \frac{\epsilon_u(\alpha, \beta)}{\epsilon_u(1, 0)} \tag{21}$$

For each couple $[\alpha, \beta]$, we define the average relative errors

$$\left. \begin{aligned} \overline{\epsilon_{c,rel}}(\alpha, \beta) &= \sqrt[N_e]{\textstyle\prod_{N_e} \epsilon_{c,rel}(\alpha, \beta)} \\ \overline{\epsilon_{u,rel}}(\alpha, \beta) &= \sqrt[N_e]{\textstyle\prod_{N_e} \epsilon_{u,rel}(\alpha, \beta)} \end{aligned} \right\} \tag{22}$$

where N_e is the total number of experiments considered in our analysis. Finally, we define a "combined" average relative error

$$\overline{\epsilon_{rel}}(\alpha, \beta) = \sqrt[N_e]{\prod_{N_e} \sqrt{\epsilon_{u,rel}(\alpha, \beta)\epsilon_{c,rel}(\alpha, \beta)}} \tag{23}$$

[23] Optimal values of the fitting parameters C_r and z_o are computed for every couple of $[\alpha, \beta]$ for $0.5 \leq \alpha \leq 1.3$ with 0.05 intervals and $0 < \beta \leq 10$ with 0.5 intervals for stratified cases and $\beta = 0$ for neutral cases. From these computations, we first obtain for each experiment a best couple $[\alpha_c, \beta_c]$ which minimizes the error ϵ_c, a best couple $[\alpha_u, \beta_u]$ which minimizes the error ϵ_u, and a best couple $[\alpha, \beta]$ which minimizes the product of errors $\epsilon_u \epsilon_c$. The reference case is defined as the results obtained for the neutral case $[\alpha, \beta] = [1, 0]$. All the cases where values of C_r obtained for the neutral reference case are not physically consistent are eliminated from our study. Our consistency criteria is $C_r < C_{r,cr} = 0.1$. Enforcing this consistency criteria results in our elimination of 24 experiments (*Einstein* [1950] runs 5, 12, 13, 14, 15, 16 and all *Coleman* [1981] coarse sand experiments) out of 75 available.

[24] From these computations, we obtain the values of $\overline{\epsilon_{c,rel}}(\alpha, \beta)$, $\overline{\epsilon_{u,rel}}(\alpha, \beta)$, and $\overline{\epsilon_{rel}}(\alpha, \beta)$ for each $[\alpha, \beta]$ couple and plot these in the α, β plane from which contour plots of relative error may be obtained (Figure 2). The average relative concentration error $\overline{\epsilon_{c,rel}}(\alpha, \beta)$ (which is a ratio of variances) is minimum and equal to 84% for $[\alpha, \beta] = [0.8, 4]$ as seen from Figure 2a. This relative error corresponds to a 9% (= $1 - \sqrt{0.84}$) improvement of the concentration prediction compared to the neutral case. For the velocity, we observe in Figure 2b a minimum valley of $\overline{\epsilon_{u,rel}}(\alpha, \beta)$ for $\beta = 4$, and the error decreases very slowly as α decreases. The influence of β on the velocity is much more pronounced than the influence of α. The average relative error (61%) for the velocity, $\overline{\epsilon_{u,rel}}(\alpha, \beta)$, corresponds to a 22% improvement of the velocity prediction compared to the neutral case.

[25] The minimum of the combined average relative error $\overline{\epsilon_{rel}}(\alpha, \beta)$ is obtained for $[\alpha, \beta] = [0.8, 4]$, as seen from Figure 2c, and corresponds to an average error of approximately 70%, i.e., an improvement of roughly 16% of the overall predictive capability of the stratification model with $[\alpha, \beta] = [0.8, 4]$ compared to the neutral case. For the neutral case, which corresponds to $\beta = 0$ in Figure 2, the

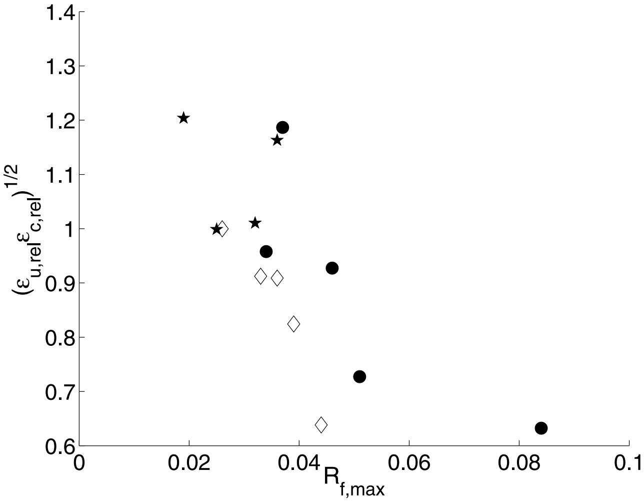

Figure 3. Relative error $\sqrt{\epsilon_{c,rel}\epsilon_{u,rel}}$ dependency on the maximum Flux Richardson number $R_{f,\max}$ for "equilibrium bed" experiments. Solid circle represents *Barton and Lin* [1955], diamond represents *Brooks* [1954], and star represents *Lyn* [1986].

minimum for the average relative concentration and combined errors occurs at $\alpha = 1$ (Figures 2a and 2c), whereas no clear minimum is present in the velocity error (Figure 2b). *Businger et al.* [1971] obtained $\alpha = 1.35$ from their thermal data, corresponding to neutral conditions.

[26] Since our interest is in the prediction of stratification effects on natural river flows the subset of experiments corresponding to equilibrium bed experiments are particularly relevant to our analysis. For this reason we present in Figure 3 the relative error $\sqrt{\epsilon_{c,rel}\epsilon_{u,rel}}$ vs. the maximum Flux Richardson number over the water column, $R_{f,\max}$, for each of the 14 equilibrium bed experiments, using $[\alpha, \beta] = [0.8, 4]$. The error decreases as the maximum Flux Richardson number, i.e., the stratification effects, increases. Including stratification effects in our turbulence model to study sediment transport in natural flows therefore represents an improvement that is all the more important when these effects are stronger.

3.3. Starved Experiments

[27] Coleman conducted 20 fine sand starved bed experiments, keeping the shear stress constant and increasing the sediment load between each run [*Coleman*, 1981]. The parameters $[\alpha, \beta]$ are now fixed equal to $[\alpha, \beta] = [0.8, 4]$ when stratification is considered and $[\alpha, \beta] = [1, 0]$ when it is not. Using the same optimization method, we obtain the optimal values of the reference concentration C_r and the bed roughness z_o that minimize the absolute errors for the concentration ϵ_c and the velocity ϵ_u for each starved experiment. For both stratified and neutral cases, it can be observed that the roughness z_o increases with the reference concentration C_r (see Figure 4). This can be physically interpreted. The larger C_r is, the larger the near-bottom concentration is. Near the bottom, each sand particle that hits the plexiglas bottom does not bounce elastically but drags along the bottom a little before returning to the flow. If the bottom concentration is larger, the number of particles that hit the bottom is also larger, and the particle-induced drag on the near-bottom fluid is therefore larger. In practice, this results in a larger effective flow resistance as reflected by the increase of z_o as C_r increases. *Villaret and Trowbridge* [1991] attributed the shift between the neutral velocity profile and the stratified velocity profile solely to the stratification effect. Actually, there are two effects: the stratification effect and the bed roughness effect as explained above.

[28] Coleman's run 20 is the run with the largest amount of sediment in suspension and consequently with the highest C_r. We show *Coleman* [1981] run 20 concentration data in Figure 5a and clear water run and run 20 velocity data in Figure 5b. We first fit the clear water velocity data between the bed and $0.4h$ using the neutral model. We obtain a reasonably good fit (see Figure 5b, dotted line) with $z_o = 5.7 \cdot 10^{-6}$m. When the concentration increases, we would expect the velocity to increase, since the eddy viscosity decreases. However, we can observe in Figure 5b that this is

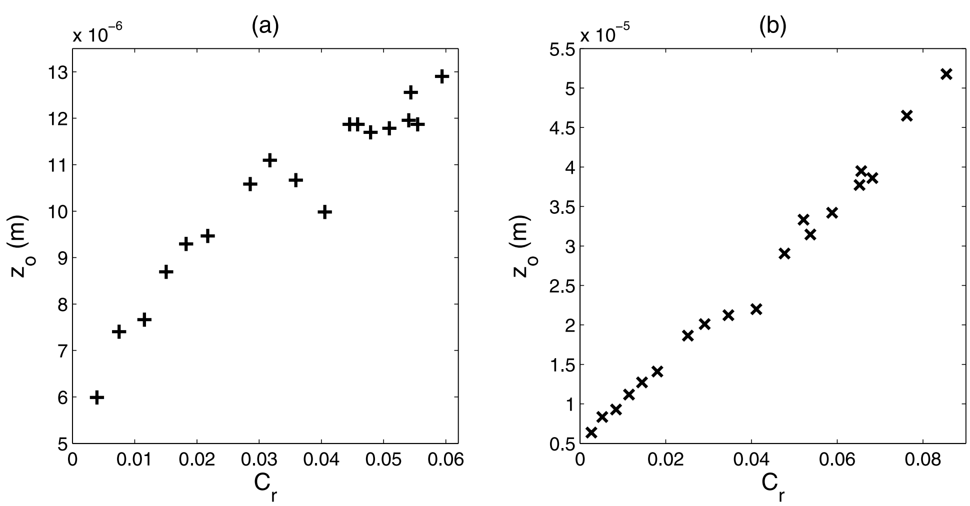

Figure 4. Movable bed roughness, $z_o = k_N/30$, dependency on reference concentration, C_r, for *Coleman* [1981] fine sand starved-bed experiments, $d = 0.11$ mm, (a) neutral case: $\alpha = 1$, $\beta = 0$, (b) stratified case: $\alpha = 0.8$, $\beta = 4$.

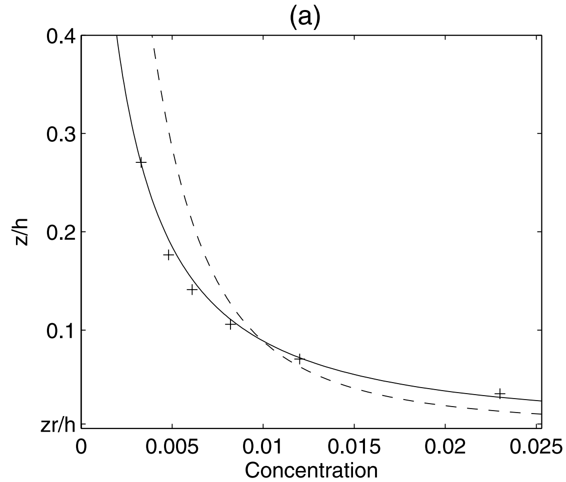

Figure 5. Predicted and measured velocity and concentration profiles for *Coleman* [1981] starved-bed experiment run 20 and clear water experiment. Solid line represents best-fit for the stratified model ($z_{o,S} = 5.2 \cdot 10^{-5}$m, $C_{r,S} = 0.0855$). Dashed line represents best-fit for the neutral model ($z_{o,N} = 1.3 \cdot 10^{-5}$m, $C_{r,N} = 0.0448$). Dotted line represents best-fit for Coleman clear water run using the neutral model ($z_o = 5.7 \cdot 10^{-6}$m). Cross represents Coleman run 20 data. Open circle represents Coleman clear water velocity data. (a) Concentration profiles. (b) Velocity profiles.

not the case. On the contrary, the velocity decreases. If we fit run 20 data using the neutral model $[\alpha, \beta] = [1, 0]$, we obtain $z_{o,N} = 1.3 \cdot 10^{-5}$m and the velocity shifts toward lower values. As explained above, increasing the sediment concentration may increase the bottom roughness significantly and this can result in a decrease of the velocity. However, both the concentration and velocity profiles obtained with the neutral model (dashed line in Figures 5a and 5b) fit very poorly run 20 data. Using the stratified model $[\alpha, \beta] = [0.8, 4]$, we obtain $z_{o,S} = 5.2 \cdot 10^{-5}$m, and the predicted concentration and velocity (full lines in Figure 5) provide excellent fits to run 20. To quantify the fits presented in Figure 5a, the absolute concentration error defined as $\sqrt{\epsilon_c}$, with ϵ_c given by (19), is 0.060 and 0.280 for the stratified (full line) and neutral (dashed line) fits to run 20, respectively. To quantify the fits presented in Figure 5b, the absolute velocity error defined as $\sqrt{\epsilon_u}$, with ϵ_u given by (20), is 0.020 and 0.070 for the stratified (full line) and neutral (dashed line) fits to run 20, respectively.

[29] In Figure 6 we show the neutral and stratified eddy viscosity profiles for Coleman's run 20. For $z \geq z_r$, Figure 6a, we multiply the neutral viscosity by $(1 - \beta R_f)$ to obtain the stratified eddy viscosity, whereas these are assumed equal for $z < z_r$, Figure 6b. Since the stratification correction $(1 - \beta R_f)$ is smaller than unity at $z = z_r$, the stratified eddy viscosity is discontinuous at $z = z_r$, where it jumps to a lower value. The Flux Richardson Number for Coleman's run 20 is nearly constant over most of the depth and therefore so is the stratification term $(1 - \beta R_f)$, as seen in Figure 7a. Thus at first sight, it appears possible to account for stratification by simply scaling the neutral eddy viscosity by a constant factor, say λ_s = average of $(1 - \beta R_f)$ over depth. This approximation, which corresponds to the method originally proposed by *Vanoni* [1946], does indeed appear to provide a reasonable representation of the stratified eddy viscosity for $z > z_r$ as seen in Figure 6a, where $\lambda_s \approx 0.54$ is obtained from Figure 7a for Coleman's run 20. However, when we zoom in on the region immediately above the bottom as done in Figure 6b we see that there is a significant difference between the stratified and the scaled neutral eddy viscosities in this region. For $z \ll h$ the governing equation (5) for the velocity becomes

$$\frac{dU}{dz} = \frac{u_*}{\kappa z\left(1 - \beta R_f\right)} \tag{24}$$

and, assuming $(1 - \beta R_f) \approx$ constant, we obtain for $z \ll h$

$$U(z) = \frac{u_*}{\kappa z\left(1 - \beta R_f\right)} \ln\left(\frac{z}{z_0}\right) \tag{25}$$

which shows that the velocity immediately above the bottom is approximately inversely proportional to the stratification correction factor $(1 - \beta R_f)$.

[30] For Coleman's run 20, which corresponds to a Rouse Number = $q = \alpha w_s/\kappa u_* = 0.49 < 1$, we have $\lambda_s \approx 0.54$, whereas Figure 7a shows that $(1 - \beta R_f) \approx 0.8$ very near the bottom. Thus replacing $(1 - \beta R_f) \approx 0.8$ by $\lambda_s = 0.54$ in (25) can lead to significant errors in the velocity predictions, with the scaled neutral eddy viscosity model yielding much higher values of U than the stratified model. For Coleman's run 20 this overprediction amounts to roughly 0.35 m/s (or $\approx$ 40%) over most of the depth.

[31] For cases when the Rouse Number $q > 1$, suspended sediments are present only in the lower portion of the water column. As a consequence, stratification effects are limited in vertical extent and absent higher up in the water column. This is illustrated by the variation of the stratification correction $(1 - \beta R_f)$ shown in Figure 7b for run 4 of *Einstein and Chien* [1955], which corresponds to a Rouse Number = $q = 2.04 > 1$. In contrast to Coleman's run 20 ($q = 0.49 < 1$) shown in Figure 7a, it is seen that $(1 - \beta R_f)$ decreases as the bottom is approached from above. Thus when $q > 1$ the value of λ_s would be larger than the near-bottom value of $(1 - \beta\, R_f)$, and (25) therefore suggests that the scaled neutral eddy viscosity model would underpredict

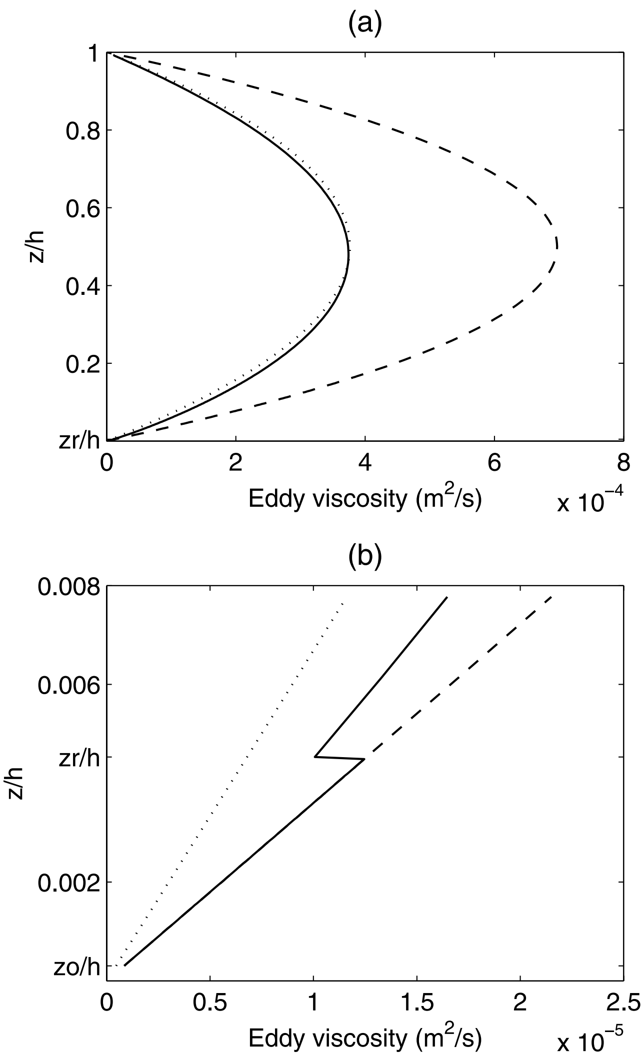

Figure 6. Coleman run 20 eddy viscosity. Dashed line represents parabolic neutral eddy viscosity $\nu_{TN} = \kappa u_* z(1 - \frac{z}{h})$. Solid line represents stratified eddy viscosity $\nu_T = \nu_{TN}(1 - \beta R_f)$ with R_f given by (14). Dotted line represents neutral eddy viscosity ν_{TN} multiplied by 0.54. (a) Eddy viscosity profiles for $z > z_r$. (b) Near-bottom eddy viscosity profiles for $z \ll h$.

the velocity obtained from the stratified model, i.e., the opposite effect of the one noted when $q < 1$. For the Einstein and Chien run 4, the underprediction amounts to ≈0.13 m/s (≈6%) over most of the depth.

[32] The behavior of the stratification correction factor $(1 - \beta R_f)$ shown in Figure 7 is typical for cases when the Rouse Number = $q < 1$, Figure 7a, and $q > 1$, Figure 7b, and we therefore conclude that the simple scaling of the neutral eddy viscosity proposed by *Vanoni* [1946] will in general not lead to acceptable predictions of the velocity in a sediment-laden flow.

4. Determination of Movable Bed Roughness and Reference Concentration

[33] Having established our theoretical model for turbulent flows stratified by suspended sediment, $[\alpha, \beta]$ are now fixed equal to [0.8, 4] when stratification is considered and [1, 0] when it is not. Adopting these values for $[\alpha, \beta]$ and using the same optimization method we obtain the optimal values of the bed roughness z_o and the reference concentration C_r that minimize the absolute errors for the concentration ϵ_c and the velocity ϵ_u for each experiment. It is then possible to determine correlations between z_o and C_r and the parameters of the flow. From a practical point of view, we want to be able to predict velocity profiles and sediment concentration distributions in natural rivers knowing the parameters of the flow and the sediment. Therefore we will use only the "equilibrium bed" experiments in our analysis. Note that the experiments which were physically unrealistic (i.e., $C_r \geq 0.1$) are not considered here (i.e., *Einstein* [1950] runs 5, 12, 13, 14, 15, 16 and all *Coleman* [1981] coarse sand experiments). Moreover, the values of z_o/d_n for *Barton and Lin* [1955] run 26 and 29 are one order of magnitude larger than the values obtained for the 12 other bed experiments for both neutral and stratified conditions. Therefore these values are not taken into account when we perform our analysis to determine z_o. In order to be consistent, these two experiments are not taken into account when performing the analysis for the reference concentration C_r, even though these are within the same range as the values obtained from accepted equilibrium experiments.

4.1. Movable Bed Roughness z_o

[34] The Shields Parameter is the ratio of mobilizing (drag) and stabilizing (submerged weight) forces acting on a surficial sand grain

$$\Psi = \frac{\tau_o}{(\rho_s - \rho)gd} = \frac{u_*^2}{(s-1)gd} \tag{26}$$

The critical Shields Parameter Ψ_{cr} expresses the conditions of neutral stability of a sediment grain on the fluid-sediment interface and is the Shields Parameter at which sediment motion starts. The critical Shields Parameter Ψ_{cr} is a function of the fluid-sediment parameter $S_* = \frac{d\sqrt{(s-1)gd}}{4\nu}$ [*Madsen and Grant*, 1976]. In our analysis, we consider the nominal diameter $d_n = d_{sieve}/0.9$ when computing the value of S_* and Ψ. A formula for Ψ_{cr}, given by *Soulsby* [1997], may be written as

$$\Psi_{cr} = 0.095 S_*^{-2/3} + 0.056\left(1 - \exp\left(\frac{-S_*^{3/4}}{20}\right)\right) \tag{27}$$

If the flow corresponds to $Re_* = \frac{30 z_o u_*}{\nu} > 3.3$ the flow is considered rough turbulent. This is the case here for all experiments. From the extensive experiments by Nikuradse, summarized in most standard fluid mechanics texts [e.g., *Schlichting*, 1968], the equivalent Nikuradse sand grain roughness is then defined as $k_N = 30z_0$. When the sediment is not moving ($\Psi < \Psi_{cr}$) $k_N = d$, and when it is moving ($\Psi > \Psi_{cr}$) the equivalent Nikuradse sand grain roughness is $k_N \geq d$. This suggests that the movable bed roughness can be expected to behave as

$$k_N = 30z_o = [a + b(\Psi - \Psi_{cr})]d \quad \text{for} \quad \Psi > \Psi_{cr} \tag{28}$$

Similarly, *Smith and McLean* [1977] suggested a general expression for $z_o = \alpha_o d(\Psi - \Psi_{cr}) + z_n$ where z_n is the z_o

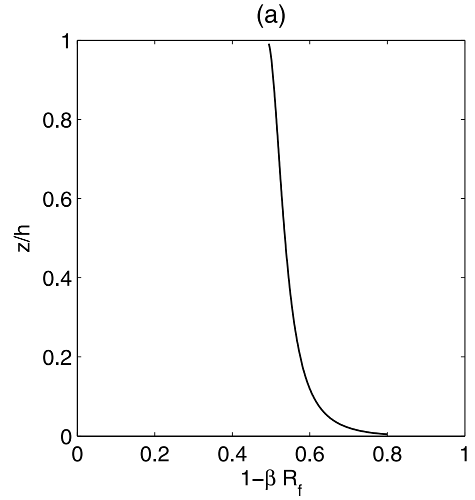

Figure 7. Stratification correction term. $(1 - \beta R_f)$ between z_r and h in (a) *Coleman* [1981] run 20 (Rouse Number = q = 0.49 and (b) *Einstein and Chien* [1955] (Rouse Number = q = 2.00). R_f given by (14).

given by Nikuradse's work and α_o is a constant. Values of α_o vary among the different studies that have been published. *Owen* [1964] and *Smith and McLean* [1977] obtained α_o = 22.8 and 26.3, respectively. *Wilson* [1989] also suggested that when sediment motion occurs k_N should be of the form $k_N = 5\Psi d$ for steady sheet-flow in closed conduits. Wilson did not suggest $k_N = 5(\Psi - \Psi_{cr})d$, however, in his analysis, $\Psi \gg \Psi_{cr}$ so that $5(\Psi - \Psi_{cr})d \approx 5\Psi d$.

[35] For stratified conditions we perform a linear regression analysis between $y = 30z_o = k_N$ and $x = (\Psi - \Psi_{cr})$. The same analysis is performed for neutral conditions. A measure of the error is given by the average $\overline{x_{z_o}}$ and the standard deviation σ_{z_o} of x_{z_o}, the ratio of the best z_o value obtained from the measured velocity, $z_{o,m}$, and the value predicted by the regression formula obtained here, $z_{o,p}$, for each experiment, i.e., $x_{z_o} = \frac{z_{o,m}}{z_{o,p}}$. The results are presented graphically in Figure 8. A linear fit of the form $y = ax^b$ is performed in order to check the extent to which the linear assumption implied by (28) is correct. We obtain b = 0.717 for the neutral case and b = 0.843 for the stratified case and therefore accept the assumption of a linear relationship between k_N and $\Psi - \Psi_{cr}$. From our analysis, the formulae for the bottom roughness as function of the Shields Parameter is for application in conjunction with our neutral and stratified models

$$\left.\begin{aligned} \{k_N\}_N &= 30z_{o,N} = \{4.5(\Psi - \Psi_{cr}) + 1.7\}d_n(1 \pm 0.36) \\ \{k_N\}_S &= 30z_{o,S} = \{7.4(\Psi - \Psi_{cr}) + 1.6\}d_n(1 \pm 0.29) \end{aligned}\right\} \quad (29)$$

We obtain multiplying factors very close to *Wilson* [1989] (4.5 and 7.4 versus 5) but much smaller than those of *Smith and McLean* [1977] and *Owen* [1964].

4.2. Reference Concentration C_r

[36] When there is no sediment motion, i.e., $\Psi < \Psi_{cr}$, there is no suspended sediment and the reference concentration C_r at $z_r = 7d_n$ is equal to 0. Performing a Taylor expansion, we therefore have $C_r \cong \lambda(\Psi - \Psi_c)$ which suggests the reference concentration C_r to be proportional to the difference $\Psi - \Psi_{cr}$ and therefore to the excess shear stress $\tau_o - \tau_{cr}$. *Smith* [1977] and *Smith and McLean* [1977] suggested that the near-bottom reference concentration C_r at their reference elevation $z_r = z_o$ should be proportional to the excess shear stress $\tau_o - \tau_{cr}$ normalized by the critical shear stress τ_{cr}: they suggested $C_r = \gamma_1(\frac{\tau_o}{\tau_{cr}} - 1)$, with $\gamma_1 = 1.6 \cdot 10^{-3}$. Following *Smith and McLean* [1977], we may therefore expect the relationship for the reference concentration C_r to be of the form

$$C_r = \lambda_r\left(\frac{\Psi}{\Psi_{cr}} - 1\right) \text{ at } z_r = 7d \quad (30)$$

We perform a linear regression analysis between $y = C_r$ and $x = (\frac{\Psi}{\Psi_{cr}} - 1)$. For each analysis, a measure of the average error is given by the average $\overline{x_{C_r}}$ and standard deviation σ_{C_r} of x_{C_r}, the ratio of the best C_r value obtained from the

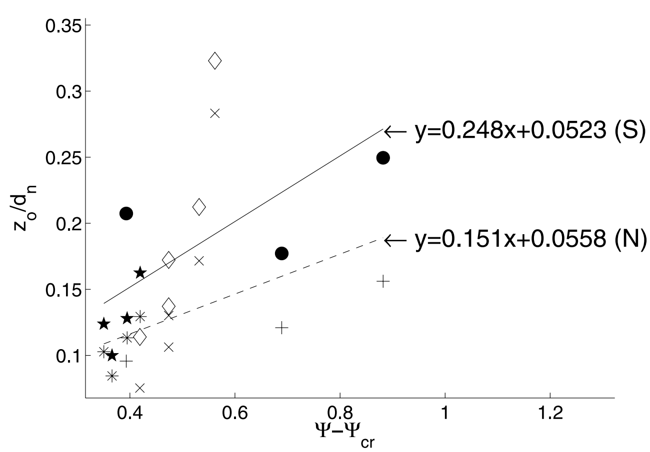

Figure 8. Movable bottom roughness relationship, z_o/d_n versus $\Psi - \Psi_{cr}$. Dashed line represents neutral. Solid line represents stratified. Analysis of data with the neutral model (N): plus represents *Barton and Lin* [1955], cross represents *Brooks* [1954], star represents *Lyn* [1986]. Analysis of data with the stratified model (S): solid circle represents *Barton and Lin* [1955], diamond represents *Brooks* [1954], star represents *Lyn* [1986].

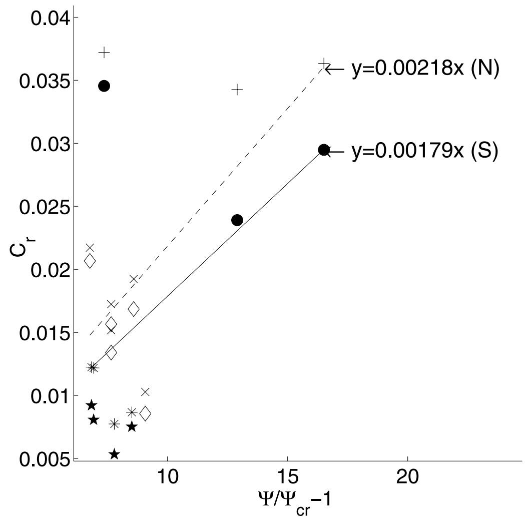

Figure 9. Reference concentration relationship. C_r at $z = z_r = 7d_n$ versus $\Psi - \Psi_{cr}$. Dashed line represents neutral. Full line represents stratified. Analysis of data with the neutral model (N): plus represents *Barton and Lin* [1955], cross represents *Brooks* [1954], star represents *Lyn* [1986]. Analysis of data with the stratified model (S): solid circle represents *Barton and Lin* [1955], diamond represents *Brooks* [1954], star represents *Lyn* [1986].

measured concentration, $C_{r,m}$, and the value predicted by the regression formula presented here, $C_{r,p}$, for each experiment, i.e., $x_{C_r} = \frac{C_{r,m}}{C_{r,p}}$. This analysis is performed for neutral and stratified conditions. The results are shown graphically in Figure 9. Again, we check the linearity implied by (30) by fitting $y = ax^b$, and obtain $b = 1.01$ and 0.926 for the neutral and stratified cases, respectively. These values are very close to unity and therefore justify the assumed linear relationship between C_r and $\frac{\Psi}{\Psi_{cr}} - 1$. We obtain the following predictive formulae for the reference concentration to be used in conjunction with our neutral and stratified model

$$\left.\begin{aligned} \{C_r\}_N &= \left\{0.0022\left(\frac{\Psi}{\Psi_{cr}} - 1\right)\right\}(1 \pm 0.51) \\ \{C_r\}_S &= \left\{0.0018\left(\frac{\Psi}{\Psi_{cr}} - 1\right)\right\}(1 \pm 0.62) \end{aligned}\right\} \text{at } z_r = 7d \qquad (31)$$

The proportionality factors are very close to that of *Smith and McLean* [1977] who obtained 0.0016 using the reference elevation $z_r = z_o$, with $z_o = \alpha_o(\Psi - \Psi_{cr})d + z_n$ and $\alpha_o = 26.3$. For *Smith and McLean* [1977] experiments with shear velocities and sediment diameters comparable to the bed experiments analyzed in our study, thus with comparable Shields parameters, their values for z_r/d are within a factor of 1.5 the same as our value, and our values for proportionality factors are therefore comparable and nearly the same, 0.0016 versus 0.0022. *Zyserman and Fredsoe* [1994] also related the reference concentration to Ψ and proposed a formula: $C_r = \frac{0.331(\Psi - 0.045)^{1.75}}{1 + 0.720(\Psi - 0.045)^{1.75}}$ for $z_r = 2d$. In our range of Shields Parameters, this formula provides values for the reference concentration that are almost one order of magnitude larger than the values found in our data analysis. Since *Zyserman and Fredsoe* [1994] chose $z_r = 2d$ whereas we chose $z_r = 7d$, this difference is explained by the reference concentration being much larger at $z_r = 2d$ than at $z_r = 7d$.

[37] For given Ψ and Ψ_{cr}, (31) shows the neutral reference concentration to be larger than the stratified reference concentration. Equation (17) relates the stratified and neutral concentration. From this equation, for given q and C_r, $C(z) < C_N(z)$. To fit a same data set, we would therefore expect that $C_{r,N} < C_{r,S}$. This is not the case in (31), due to the fact that $\alpha = 1$ for the neutral case and $\alpha = 0.8$ for the stratified case, thus $q = \frac{w_s \alpha}{\kappa u_*}$ is not the same for the neutral case and the stratified case. We observed in Figure 1a that if α is different for the neutral case and the stratified case, for example $\alpha = 0.8$ for the stratified case and $\alpha = 1$ for the neutral case, the concentration can be larger for the stratified case than for the neutral case. To fit a same data set, the stratified reference concentration should then be smaller than the neutral reference concentration.

5. Conclusion

[38] The early work concerning sediment stratification relates stratification effects to the von Karman constant's variability [*Vanoni*, 1946]. Subsequent attempts to account for sediment-induced stratification in turbulent flows were based on the stratified flow analogy [*Smith and McLean*, 1977; *Glenn and Grant*, 1987; *Styles and Glenn*, 2000]. These investigators assumed that similarly to thermal stratification, sediment stratification can be expressed through a modification to neutral eddy viscosity and eddy diffusivity. They introduced the parameters α and β, whose values were assumed to be the same as those obtained for thermally stratified atmospheric boundary layers [*Businger et al.*, 1971]. However, the correctness of this assumption was not demonstrated.

[39] *Villaret and Trowbridge* [1991] were the first investigators to attempt to account for stratification effects using suspended sediment data. Following these investigators, we also assume that stratification effects are expressed through the parameters α and β. We solve the governing equations for velocity and sediment concentration for a sediment-induced stratified steady unidirectional flow in an open rectangular channel for a parabolic neutral eddy viscosity model. Analytical closed-form solutions are obtained for both velocity profile and sediment concentration distribution. This is in contrast to other works [e.g., *Styles and Glenn*, 2000], where the two equations are solved iteratively. The velocity and concentration formulae that we established are intended to predict velocity and concentration between the bed and $0.4h$, where stratification effects are expected to be most pronounced. Above $0.4h$, other processes affect the flow and stratification is not expected to be significant.

[40] We use our model with the extensive data set used by *Villaret and Trowbridge* [1991]. Since the information necessary to evaluate the sand grading effect were not provided by the experimenters, we do not, as elegantly done by *Villaret and Trowbridge* [1991], include the effects of a slight nonuniformity of the suspended sediment, but treat it as uniform with a constant settling velocity, w_s. In contrast to them, we vary not only α and β but also the movable bed roughness z_o, whose variability was not considered by *Villaret and Trowbridge* [1991], to obtain

the optimal set [α, β] for the suspended sediment experiments. For neutral conditions, i.e., when stratification effects are not accounted for, $\nu_{TN} = \alpha\nu_{SN}$ and $\beta = 0$ by definition, the optimal value $\alpha = 1$ is obtained. If we want to predict velocity and sediment concentration by accounting for stratification, the optimal values $\alpha = 0.8$ and $\beta = 4.0$ are obtained. For a Flux Richardson Number equal to 0.25, stratification effects would thus completely annihilate turbulent mixing. This result is in agreement with the Miles theorem which predicts a critical Flux Richardson Number $R_{f,cr} = 0.25$ above which turbulence production is eliminated. Accounting for stratification improves the prediction of velocity and concentration in comparison with the case where we do not account for stratification. Indeed, when comparing our model to the data set used in its establishment, we observe a 16% improvement of the overall predictive capability of the stratified model compared to the neutral model, a 9% improvement of the concentration prediction, and a 22% improvement of the velocity prediction.

[41] For predictive purposes, we need to know the movable bed roughness z_o and the reference concentration C_r at $z_r = 7d$. We analyze a subset of equilibrium bed experiments, which correspond to natural river conditions, in order to establish relationships between these and the sediment and flow parameters, namely the density ratio s, the sediment diameter d_n, the shear velocity u_*, and the fluid's kinematic viscosity ν and density ρ. From a linear regression analyses, we establish relationships between the ratio of the movable bed roughness z_o to the nominal sediment diameter d_n and the difference between the Shields Parameter Ψ and the critical Shields Parameter Ψ_{cr} for neutral and stratified model applications. Knowing the flow and sediment parameters, we are then able to predict the movable bed roughness z_o under neutral conditions with a 36% uncertainty, and with a 29% uncertainty if we account for sediment-induced stratification effects.

[42] We also establish linear relationships between the reference concentration C_r at $z = z_r = 7d_n$ and the relative difference between the Shields Parameter Ψ and the critical Shields Parameter Ψ_{cr} for neutral and stratified model applications. Knowing the flow and sediment parameters, we are then able to predict the reference concentration C_r under neutral conditions with a 51% uncertainty, and with a 62% uncertainty if we account for stratification effects.

[43] The formulae established for the reference concentration and the movable bed roughness only provide rough estimates of these parameters. An examination of the effects of the uncertainty in the values for z_o and C_r on the velocity, concentration and transport rate prediction capability of our model reveals that this uncertainty overshadows the improvement resulting from accounting for stratification [*Herrmann*, 2003]. The scatter in the relationships between C_r and z_0 and the Shields Parameter could be due to the oversimplification of the viscosity model, which does not consider particle-particle and particle-turbulent flow interactions. More sophisticated models, e.g., two-phase flow models as proposed by *Villaret et al.* [2000], may have the potential to overcome this problem in the future.

[44] With our model it is feasible to account for stratification in the governing equations for a two-dimensional steady uniform flow carrying sediment in suspension. However, this simple model accounting for stratification awaits better determination of the reference concentration C_r and the movable bed roughness z_o to become of practical importance.

Appendix A: Solution of Equation (15)

[45] Here details are given for the solution (17) for equation (15). Equation (15) may formally be written

$$C' + PC + QC^2 = 0 \tag{A1}$$

with $P(z) = \frac{q}{z\left(1-\frac{z}{h}\right)}$ and $Q(z) = \frac{\beta a q}{\left(1-\frac{z}{h}\right)^2}$. In (A1) we recognize Ricatti's equation, which can be solved [e.g., *Hildebrand*, 1976] by introducing

$$C(z) = \frac{f'(z)}{Q(z)f(z)} \tag{A2}$$

Upon substitution of (A2) in (A1), we have for f

$$f'' + \left(P - \frac{Q'}{Q}\right)f' = 0$$

Integrating this equation twice, we obtain

$$f(z) = A_2 + A_1 Z_r(z)$$

with A_1 and A_2 being constants and $Z_r(z)$ given by (16). Substituting f into (A2), using the reference elevation and making use of the expression for the neutral concentration distribution given by (13), the stratified concentration with a parabolic neutral eddy viscosity is given by (17).

[46] **Acknowledgments.** This work was funded by a Fellowship from the Department of Civil and Environmental Engineering of the Massachusetts Institute of Technology and by the Massachusetts Institute of Technology Sea Grant Program. We thank an anonymous reviewer for helpful suggestions.

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