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Three-dimensional structure of wave-induced momentum flux in irrotational waves in combined shoaling-refraction conditions

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Abstract

In the paper, the three-dimensional structure of the wave-induced momentum flux in irrotational waves propagating over a two-dimensional, irregular bathymetry is analyzed. The expansion method developed by de Vriend and Kitou [de Vriend, H.J., Kitou, N., 1990a. Incorporation of wave effects in a 3D hydrostatic mean current model. Delft Hydraulics Report H-1295. de Vriend, H.J., Kitou, N., 1990b. Incorporation of wave effects in a 3D hydrostatic mean current model. Proc. 22nd Int. Coast. Eng. Conf. ASCE, 1005–1018.] for unidirectional waves has been extended to derive expressions for velocity components in three-dimensional waves over sloping bottom. The vertical wave-induced momentum flux resulting from this solution has been shown to be vertically-varying (contrary to the 2D-V case) and to act as a counterbalance for the vertical variability of the other wave forcing terms in the momentum equations. Thus, the total wave forcing remains depth-invariant, but—contrary to the 'traditional' solution based on the radiation stress concept—it does not depend explicitly on the direction of wave propagation and is a simple function of gradients of wave energy and water depth only. One of the most important consequences of this fact is the lack of the longshore-current-generating force in the case of non-dissipative waves approaching a shore with a bottom profile uniform in the along-shore direction. To illustrate the meaning of the new solution, the wave forcing due to waves approaching a barred beach has been analysed in detail. Also, the present solution has been shown to give the same results as the one obtained by extending of the approach by Rivero and Arcilla [Rivero, F.J., Arcilla, A. S., 1995. On the vertical distribution of $\langle \tilde{u}\tilde{w} \rangle$. Coast. Eng. 25, 137–152.] to three dimensions.

Keywords: Wave-induced momentum flux; Wave forcing; Sloping-bottom effects; Radiation stress

1. Introduction

Wave-current interaction processes at sea, resulting from simultaneous wave motion, wind-driven circulation and variations of sea surface elevation, have in recent years been subject of extensive scientific investigation. One of the important aspects of those interactions concerns wave-induced mass and momentum flux, acting as an additional driving mechanism for the large scale circulation and in some conditions significantly modifying the overall balance of forces.

Various studies concentrating on wave-driven currents have been traditionally performed by means of splitting of the total horizontal and vertical flow velocity [$\mathbf{u}$, w] into three parts, namely the slowly varying velocity components [$\mathbf{U}$, W] (named briefly as 'current'), the orbital wave motion [$\mathbf{u}_\mathrm{w}$, w_w] ('waves') and the random fluctuations [$\mathbf{u}_\mathrm{t}$, w_t] ('turbulence'):

$$\mathbf{u} = \mathbf{U} + \mathbf{u}_\mathrm{w} + \mathbf{u}_\mathrm{t},$$
$$w = W + w_\mathrm{w} + w_\mathrm{t}.$$

For a time-periodic wave motion a phase-averaging operation can be defined, which allows to formulate averaged (and relatively easy to handle) momentum equations, containing slowly-varying quantities and averaged turbulent and wave-induced momentum flux components. The basic form of those equations, together with underlying assumptions, is given in Appendix A.1. The turbulent momentum flux can be evaluated by means of one of the turbulence closure models available, which is beyond the scope of the present paper (e.g., by means of the first-order turbulence closure scheme, as

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suggested in Appendix A.1). In two-dimensional hydrodynamic models the wave-induced momentum flux has been approximated traditionally based on the linear wave theory over a horizontal bottom and the radiation stress concept of Longuet-Higgins and Stewart (1962), introduced in their analysis of vertically-integrated momentum flux in linear waves. The gradients of the radiation stress tensor have been used by many authors as a driving force in various studies dedicated to wave-driven flow, see, e.g., Mastenbroek et al. (1993), Ozer et al. (2000) or Nielsen and Apelt (2003). Recently, partly as a consequence of rapid development in fully three-dimensional hydrodynamical modelling, attempts have been made to extend the radiation stress concept to account for the vertical structure of wave-induced momentum flux and thus to enable its application in three-dimensional momentum equations. Most of those approaches reduce to evaluation of the wave-dependent terms in the momentum equations based on the linear wave theory. The only difference with respect to the original derivation procedure by Longuet-Higgins and Stewart (1962) is that the vertical integration is performed not over the entire water column, but over each layer in the model separately. This technique has been applied, e.g., by Nobuoka et al. (1998), Nobuoka and Mimura (2002) and recently by Mellor (2005).

On the other hand, it has been shown by a number of authors (e.g., de Vriend and Kitou, 1990a,b; Rivero and Arcilla, 1995) that linear wave theory over a flat bottom does not provide proper wave forcing in the case of variable water depth—so in almost any practical situation. In other words, the mild-slope approximation, based on the assumption that the vertical structure of the wave field does not change due to variations in water depth, although very useful in many applications, leads to erroneous results when the estimation of wave-induced momentum flux is concerned.

One of the 'solutions' to the problem, practiced in a number of studies, is to take for granted that the irrotational wave motion cannot give rise to phase-averaged circulation (as shown by Longuet-Higgins, 1973) and to apply in the current model wave forcing due to wave dissipation only (e.g., Deigaard and Fredsøe, 1989; Kaihatu et al., 2002, where the roller concept has been used to formulate forcing terms within the surface boundary layer). At the same time, many researchers have tried to extend the mild-slope approach to account for the sloping bottom effects and other possible sources of non-zero vertical wave-induced momentum flux, dependent on the $\overline{\mathbf{u}_w w_w}$-term. A comprehensive review of processes leading to non-zero $\overline{\mathbf{u}_w w_w}$ can be found in Rivero and Arcilla (1995).

Battjes (1968) proposed a simple wave-front curvature approach to describe modifications of the vertical wave profile due to shoaling. Rivero and Arcilla (1995) have shown that for unidirectional, irrotational waves propagating over a sloping bottom the vertical wave-induced momentum flux differs from zero, is vertically independent and leads to a reduction of the total wave forcing term in the momentum equation (depth-independent as well) by the factor of 2. de Vriend and Kitou (1990a,b) formulated a solution for unidirectional, irrotational waves propagating in water of variable depth and showed that wave forcing in this case can be separated into depth-independent and surface part and that the solution is consistent with our knowledge of the properties of irrotational wave motion. Nobuoka and Mimura (2002) have analysed, both experimentally and theoretically, vertical distribution of $\overline{u_w^2}$ (where $u_w = |\mathbf{u}_w|$), $\overline{w_w^2}$ and $\overline{\mathbf{u}_w w_w}$ in shoaling waves of different amplitudes and periods, normally incident to the bottom contours. Their theoretical model is based on the Biesel (1952) theory, valid for waves propagating over a bottom of constant slope. The authors notice that the two slightly different versions of their model result in different vertical distributions of $\overline{\mathbf{u}_w w_w}$: either the kinematic bottom boundary condition is satisfied exactly and the wave forcing is vertically non-uniform, or the kinematic bottom boundary condition is not satisfied, but the wave forcing is constant in the vertical and about two times smaller than the one predicted by linear theory. Thus, both approaches have considerable shortcomings.

Moreover, the results of all the above studies are limited to unidirectional waves. Their extension to a more complicated case of waves subject to shoaling and refraction over two-dimensional bathymetry—the main subject of this study—is not as straightforward as suggested by some of the other authors. One of the reasons is that many 2D-V models are based on an assumption that the flow at the water surface is balanced by the flow in the lower layers of a given water column, which is obviously not true in horizontally varying conditions. Another reason are the spatial changes of wave propagation direction, which should not be disregarded a priori. Also, in three-dimensional waves the horizontal momentum flux predicted by the linear theory is not depth-independent any more, which gives an indication to expect that the vertical wave induced momentum flux has in that case a more complicated vertical profile, too. As will be shown below, this is exactly the case: the horizontal and vertical momentum flux terms 'balance' each other so that the total wave forcing remains depth-independent and is a function of gradients of water depth and wave amplitude only.

The content of the paper is as follows: the next Section presents a derivation of the formulae for the horizontal and vertical velocity components in waves propagating over two-dimensional bathymetry, which is an extension of the method developed for 2D-V waves by de Vriend and Kitou (1990a). In Section 3, the properties of the $\overline{\mathbf{u}_w w_w}$-term resulting from this theory are discussed. The form of the total wave forcing terms is presented in Section 4, together with an analysis of the differences between the formulae presented herein and other available formulations and with discussion of their consequences for the calculation of wave-induced currents. A simple case of waves approaching a barred beach with uniform depth profile in the along-shore direction is analysed in detail to better visualize the main properties of the corrected solution. It is also shown that a simple technique of $\overline{\mathbf{u}_w w_w}$-evaluation proposed by Rivero and Arcilla (1995), when extended to waves propagating in arbitrary direction, gives exactly the same values of wave forcing term as those obtained by the present theory. Finally, Section 5 summarizes and discusses the results.

2. Velocity field in irrotational waves over a sloping bottom

2.1. Assumptions

As has been stated in the introduction, the method of the derivation of the equations formulated in this paper is almost identical to the one used by de Vriend and Kitou (1990a,b), with the most important differences concerning the treatment of the combined wave shoaling and refraction, for obvious reasons not present in the one dimensional case.

Let ξ denote the free surface disturbance related to the short wave motion, so that the total free surface elevation in the Cartesian coordinate system ($\mathbf{x}$, z) can be written as:

$$z = \eta(\mathbf{x}, t) + \xi(\mathbf{x}, t),$$

where η denotes the slowly-varying free surface disturbance resulting from other processes (e.g., large-scale circulation). In the following, the time scale of the variability of η is assumed to be much larger than the characteristic short wave period, so that the wave motion remains unaffected by temporal changes of η.

The wave motion, described by the elevation ξ and the horizontal ($\mathbf{u}_{\mathrm{w}}$) and vertical (w_{w}) velocity components, is assumed to be irrotational and—following de Vriend and Kitou (1990a)—can be expressed as a sum of the 'classical' solution for linear waves over a flat bottom and some correction terms that enable to account for nonuniformities in water depth and wave field itself:

$$\xi = a\cos\varphi, \tag{1.1}$$

$$\mathbf{u}_{\mathrm{w}} = \frac{agk}{\omega} f_{\mathrm{c}} \frac{\mathbf{k}}{k} \cos\varphi + \tilde{\mathbf{u}}_{\mathrm{w}} \sin\varphi = \nabla \Phi, \tag{1.2}$$

$$w_{\mathrm{w}} = \frac{agk}{\omega} f_{\mathrm{s}} \sin\varphi + \tilde{w}_{\mathrm{w}} \cos\varphi = \frac{\partial \Phi}{\partial z}, \tag{1.3}$$

where φ denotes phase, $\omega = -\frac{\partial \varphi}{\partial t}$ is the wave frequency, $\mathbf{k} = \nabla\varphi$, $k = |\mathbf{k}|$ denotes the wave number, α—wave amplitude, g—acceleration due to gravity and Φ—velocity potential. The functions f_{c} and f_{s} describe the vertical structure of the wave field and are given by:

$$f_{\mathrm{c}} = \frac{\cosh[k(z+h)]}{\cosh[kd]} \quad \text{and} \quad f_{\mathrm{s}} = \frac{\sinh[k(z+h)]}{\cosh[kd]},$$

where d is the total water depth, $d = h + \eta$, and $z = -h(\mathbf{x})$ is the bottom level.

A detailed form of the functions denoted above with $\tilde{\mathbf{u}}_{\mathrm{w}}$ and $\tilde{w}_{\mathrm{w}}$ will be found below. At this point we will only assume that they are of order (∇a, ∇h, $\nabla \eta$). One of the most important consequences following from this assumption is that all phase-averaged wave dependent quantities occurring in the wave forcing terms of the momentum equations (see Section 4 for a detailed discussion and Appendix A.1 for the definition of averaging operation)—except $\overline{\mathbf{u}_{\mathrm{w}} w_{\mathrm{w}}}$—are equal to those resulting from the linear wave theory, with an error of order ($\nabla^2 a$, $\nabla^2 h$, $\nabla^2 \eta$).

Additional restrictions concerning the form of $\tilde{\mathbf{u}}_{\mathrm{w}}$ and $\tilde{w}_{\mathrm{w}}$ follow from the kinematic bottom and surface (linearized) boundary conditions:

$$w_{\mathrm{w}} = -\mathbf{u}_{\mathrm{w}} \cdot \nabla h \quad \text{at} \quad z = -h,$$

$$w_{\mathrm{w}} = \mathbf{u}_{\mathrm{w}} \cdot \nabla \eta \quad \text{at} \quad z = \eta,$$

which leads to:

$$\tilde{w}_{\mathrm{w}} = -\frac{agk}{\omega} \frac{1}{\cosh[kd]} \frac{\mathbf{k}}{k} \cdot \nabla h \quad \text{at} \quad z = -h, \tag{2.1}$$

$$\tilde{w}_{\mathrm{w}} = \frac{agk}{\omega} \frac{\mathbf{k}}{k} \cdot \nabla \eta \quad \text{at}d \quad z = \eta \tag{2.2}$$

up to the first order accuracy in gradients of a, h and η (and under an assumption made earlier, that temporal changes of η are so slow, that they do not influence the wave behaviour).

2.2. Form of the functions $\tilde{\mathbf{u}}_w$ and $\tilde{w}_w$

From (1.2) and (1.3) it follows that:

$$\Phi = \frac{ag}{\omega} f_{\mathrm{c}} \sin\varphi + \int \tilde{w}_{\mathrm{w}} dz \cos\varphi$$

and

$$\mathbf{u}_{\mathrm{w}} = \frac{g}{\omega} f_{\mathrm{c}} \nabla a + \frac{agk}{\omega} f_{\mathrm{s}} \nabla h - \frac{agk}{\omega} \frac{G}{G+1} \times \left[\frac{z+h}{d} f_{\mathrm{s}} + \frac{\tanh[kd]}{G} f_{\mathrm{c}} \right] \nabla d - \mathbf{k} \int \tilde{w}_{\mathrm{w}} dz, \tag{3}$$

where:

$$G = \frac{2kd}{\sinh[2kd]}$$

and the wave number gradient has been evaluated on the basis of the differentiated linear dispersion relation ($\omega^2 = gk \tanh[kd]$):

$$\nabla k = -\frac{G}{G+1} \frac{k}{d} \nabla d. \tag{4}$$

Substitution of expressions (3) and (4), together with (1.2)–(1.3), into the continuity equation:

$$\nabla \cdot \mathbf{u}_{\mathrm{w}} + \frac{\partial w_{\mathrm{w}}}{\partial z} = 0$$

leads to:

$$\frac{\partial \tilde{w}_{\mathrm{w}}}{\partial z} - k^2 \int \tilde{w}_{\mathrm{w}} dz = -2 \frac{gk}{\omega} f_{\mathrm{c}} \frac{\mathbf{k}}{k} \cdot \nabla a - 2 \frac{agk^2}{\omega} f_{\mathrm{s}} \frac{\mathbf{k}}{k} \cdot \nabla h + 2 \frac{agk^2}{\omega} \frac{G}{G+1} \times \left[\frac{z+h}{d} f_{\mathrm{s}} + \frac{1}{kd} \left(\sinh^2[kd] + \frac{1}{2} \right) f_{\mathrm{c}} \right] \frac{\mathbf{k}}{k} \cdot \nabla d - \frac{agk}{\omega} f_{\mathrm{c}} \nabla \cdot \left(\frac{\mathbf{k}}{k} \right).$$

The last term on the right-hand side of the above equation describes the effects of changes of wave propagation direction. For non-dissipative waves it can be expressed as a function of ∇a and ∇d based on the energy conservation equation:

$$\nabla \cdot \left[c_{\mathrm{g}} E \frac{\mathbf{k}}{k} \right] = 0, \tag{5}$$

where $E = \frac{1}{2}\rho g a^2$, is wave energy, $c_{\mathrm{g}} = \frac{\omega}{2k}(G+1)$ is the group speed and ρ denotes water density.

We have:

$$a\nabla \cdot \left(\frac{\mathbf{k}}{k} \right) = -2\frac{\mathbf{k}}{k} \cdot \nabla a + 2\frac{G}{(G+1)^2}(kd\tanh[kd]-1)\frac{a}{d}\frac{\mathbf{k}}{k} \cdot \nabla d$$

and finally:

$$\begin{aligned}\frac{\partial \widetilde{w}_{\mathrm{w}}}{\partial z} - k^2 \int \widetilde{w}_{\mathrm{w}} dz &= 2(\delta_{\mathrm{r}}-1)\frac{gk}{\omega} f_{\mathrm{c}} \frac{\mathbf{k}}{k} \cdot \nabla a - 2\frac{agk^2}{\omega} f_{\mathrm{s}} \frac{\mathbf{k}}{k} \cdot \nabla h + 2\frac{agk^2}{\omega}\frac{G}{G+1} \\ &\times \left[\frac{z+h}{d} f_{\mathrm{s}} + \left(\frac{\cosh^2[kd] + \frac{1}{2}}{kd} - \delta_{\mathrm{r}} \frac{1}{G+1}\left(\tanh[kd] - \frac{1}{kd} \right) \right) f_{\mathrm{c}} \right] \\ &\times \frac{\mathbf{k}}{k} \cdot \nabla d.\end{aligned}$$

For waves satisfying Eq. (5), the value of the δ_{r} parameter is 1, but for the time being it is useful to keep it in its symbolic form—it will serve as kind of a marker throughout the subsequent transformations of the equations, as this is what constitutes the main qualitative difference between the 1D solution of de Vriend and Kitou (1990a) and the 2D solution presented herewith. As can be seen, for waves satisfying Eq. (5) the amplitude-gradient term in the equation for $\tilde{w}_{\mathrm{w}}$ disappears and $\tilde{w}_{\mathrm{w}}$ is determined only by the geometry of the bottom and the sea surface.

Taking z-derivative of the equation for $\tilde{w}_{\mathrm{w}}$ results in:

$$\begin{aligned}\frac{\partial^2 \widetilde{w}_{\mathrm{w}}}{\partial z^2} - k^2 \widetilde{w}_{\mathrm{w}} &= 2(\delta_{\mathrm{r}}-1)\frac{gk^2}{\omega} f_{\mathrm{s}} \frac{\mathbf{k}}{k} \cdot \nabla a - 2\frac{agk^3}{\omega} f_{\mathrm{c}} \frac{\mathbf{k}}{k} \cdot \nabla h + 2\frac{agk^3}{\omega}\frac{G}{G+1} \\ &\times \left[\frac{z+h}{d} f_{\mathrm{c}} + \left(\frac{\cosh^2[kd] + \frac{3}{2}}{kd} - \delta_{\mathrm{r}} \frac{1}{G+1}\left(\tanh[kd] - \frac{1}{kd} \right) \right) f_{\mathrm{s}} \right] \\ &\times \frac{\mathbf{k}}{k} \cdot \nabla d.\end{aligned}$$

A solution to this equation, satisfying boundary conditions (2.1) and (2.2), can be found easily, although it requires a lot of lengthy algebra (see Appendix A.2 for the form of the general solution to a differential equation of the above type):

$$\widetilde{w}_{\mathrm{w}} = \widetilde{w}_{\mathrm{w},a} + \widetilde{w}_{\mathrm{w},h} + \widetilde{w}_{\mathrm{w},d}, \tag{6.1}$$

where:

$$\widetilde{w}_{\mathrm{w},a} = (\delta_{\mathrm{r}}-1)\frac{gk}{\omega} d \left[\frac{z+h}{d} f_{\mathrm{c}} - \frac{1}{\tanh[\mathrm{kd}]} f_{\mathrm{s}} \right] \frac{\mathbf{k}}{k} \cdot \nabla a, \tag{6.2}$$

$$\widetilde{w}_{\mathrm{w},h} = -\frac{agk}{\omega}[k(z-\eta)f_{\mathrm{s}} + f_{\mathrm{c}}]\frac{\mathbf{k}}{k} \cdot \nabla h, \tag{6.3}$$

$$\begin{aligned}\widetilde{w}_{\mathrm{w},d} = \frac{agk}{\omega} &\left\{ \frac{G}{G+1}\left[\cosh^2[kd] - \delta_{\mathrm{r}}\frac{1}{G+1}(kd\tanh[kd]-1) \right] \right. \\ &\times \left[\frac{z+h}{d} f_{\mathrm{c}} - \frac{1}{\tanh[kd]} f_{\mathrm{s}} \right] + \frac{1}{2}\frac{G}{G+1} k(z-\eta)\frac{z+h+d}{d} f_{\mathrm{s}} \\ &\left. + \frac{1}{\tanh[kd]} f_{\mathrm{s}} \right\} \frac{\mathbf{k}}{k} \cdot \nabla d.\end{aligned} \tag{6.4}$$

The corresponding $\tilde{\mathbf{u}}_{\mathrm{w}}$ can be found from expression (3):

$$\tilde{\mathbf{u}}_{\mathrm{w}} = \tilde{\mathbf{u}}_{\mathrm{w},a} + \tilde{\mathbf{u}}_{\mathrm{w},h} + \tilde{\mathbf{u}}_{\mathrm{w},d}, \tag{7.1}$$

where:

$$\begin{aligned}\tilde{\mathbf{u}}_{\mathrm{w},a} &= \frac{g}{\omega} f_{\mathrm{c}} \nabla a - \frac{\mathbf{k}}{k}(\delta_{\mathrm{r}}-1)\frac{gk}{\omega} d \left[\frac{z+h}{d} f_{\mathrm{s}} - \frac{1}{kd} f_{\mathrm{c}} - \frac{1}{\tanh[kd]} f_{\mathrm{c}} \right] \\ &\times \frac{\mathbf{k}}{k} \cdot \nabla a,\end{aligned} \tag{7.2}$$

$$\tilde{\mathbf{u}}_{\mathrm{w},h} = \frac{agk}{\omega} f_{\mathrm{s}} \nabla h + \frac{\mathbf{k}}{k}\frac{agk}{\omega} k(z-\eta) f_{\mathrm{c}} \frac{\mathbf{k}}{k} \cdot \nabla h, \tag{7.3}$$

$$\begin{aligned}\tilde{\mathbf{u}}_{\mathrm{w},d} = &-\frac{agk}{\omega}\frac{G}{G+1}\left[\frac{z+h}{d} f_{\mathrm{s}} + \frac{\tanh[kd]}{G} f_{\mathrm{c}} \right] \nabla d \\ &- \frac{\mathbf{k}}{k}\frac{agk}{\omega}\left\{ \frac{G}{G+1}\left[\cosh^2[kd] - \delta_{\mathrm{r}}\frac{1}{G+1}(kd\tanh[kd]-1) \right] \right. \\ &\times \left[\frac{z+h}{d} f_{\mathrm{s}} - \frac{1}{kd} f_{\mathrm{c}} - \frac{1}{\tanh[kd]} f_{\mathrm{c}} \right] \\ &+ \frac{G}{G+1}\left[\frac{1}{kd} + \frac{1}{2}k(z-\eta)\frac{z+h+d}{d} \right] f_{\mathrm{c}} - \frac{G}{G+1}\frac{z+h}{d} f_{\mathrm{s}} \\ &\left. + \frac{1}{\tanh[kd]} f_{\mathrm{c}} \right\} \frac{\mathbf{k}}{k} \cdot \nabla d.\end{aligned} \tag{7.4}$$

Expressions (6) and (7), inserted into (1.2),(1.3), provide a solution to the Laplace equation with the boundary conditions (2.1),(2.2), which is first order accurate in the gradients of a, h and η.

3. Vertical wave-induced momentum flux

3.1. General form

The expressions derived in the previous section allow for a direct evaluation of the $\overline{\mathbf{u}_{\mathrm{w}} w_{\mathrm{w}}}$ parameter, which—contrary to the corresponding quantity resulting from the mild-slope approximation—is now different from zero and depends on the spatial distribution of wave energy and water depth through the functions $\tilde{\mathbf{u}}_\mathrm{w}$ and $\tilde{w}_\mathrm{w}$:

$$\overline{\mathbf{u}_\mathrm{w} w_\mathrm{w}} = \frac{1}{2}\frac{agk}{\omega}\left(f_\mathrm{c}\tilde{w}_\mathrm{w}\frac{\mathbf{k}}{k} + f_\mathrm{s}\tilde{u}_\mathrm{w}\right). \tag{8}$$

Similarly to (6) and (7), it is convenient to write:

$$\overline{\mathbf{u}_\mathrm{w} w_\mathrm{w}} = \overline{\mathbf{u}_\mathrm{w} w_{\mathrm{w},a}} + \overline{\mathbf{u}_\mathrm{w} w_{\mathrm{w},h}} + \overline{\mathbf{u}_\mathrm{w} w_{\mathrm{w},d}}. \tag{9.1}$$

Then from (8) and from the identity:

$$\frac{1}{2}\left(\frac{agk}{\omega}\right)^2 = \frac{E}{\rho d}\frac{kd}{\tanh[kd]}$$

it follows that:

$$\begin{aligned}\overline{\mathbf{u}_\mathrm{w} w_{\mathrm{w},a}} = {} & (\delta_\mathrm{r}-1)\frac{1}{2}\left[G\frac{z+h}{d} + \frac{1}{\tanh[kd]}f_\mathrm{s}f_\mathrm{c}\right]\left(\frac{\mathbf{k}}{k}\cdot\nabla\left(\frac{E}{\rho}\right)\right)\frac{\mathbf{k}}{k} \\ & + \frac{1}{2}\frac{1}{\tanh[kd]}f_\mathrm{s}f_\mathrm{c}\nabla\left(\frac{E}{\rho}\right),\end{aligned} \tag{9.2}$$

$$\overline{\mathbf{u}_\mathrm{w} w_{\mathrm{w},h}} = -\frac{E}{\rho d}\frac{kd}{\tanh[kd]}\left[f_\mathrm{c}^2\left(\frac{\mathbf{k}}{k}\cdot\nabla h\right)\frac{\mathbf{k}}{k} - f_\mathrm{s}^2\nabla h\right], \tag{9.3}$$

$$\begin{aligned}\overline{\mathbf{u}_\mathrm{w} w_{\mathrm{w},d}} = {} & \frac{E}{\rho d}\alpha_1(kd)\left[\alpha_2(kd)\left(\frac{1}{\cosh^2[kd]}\frac{z+h}{d} + \frac{1}{kd}f_\mathrm{s}f_\mathrm{c}\right)\right. \\ & \left. + \frac{z+h}{d}f_\mathrm{c}^2\right]\left(\frac{\mathbf{k}}{k}\cdot\nabla d\right)\frac{\mathbf{k}}{k} \\ & - \frac{E}{\rho d}\alpha_1(kd)\left[\frac{z+h}{d}f_\mathrm{s}^2 + \frac{\sinh^2[kd]}{kd}f_\mathrm{s}f_\mathrm{c}\right]\nabla d,\end{aligned} \tag{9.4}$$

where for brevity parameters:

$$\alpha_1(kd) = \frac{kd}{\tanh[kd]}\frac{G}{G+1} = \frac{G^2}{G+1}\cosh^2[kd],$$

$$\alpha_2(kd) = \sinh^2[kd] - \delta_\mathrm{r}\frac{1}{G+1}(kd\tanh[kd]-1)$$

have been introduced.

As can be seen, in a general case of waves propagating in an arbitrary direction relative to gradients of the bottom and/or mean water level, $\overline{\mathbf{u}_\mathrm{w} w_\mathrm{w}}$ has a very complicated form. Below, properties of the formulae (9.1)–(9.4) will be analysed in more detail. For the purpose of this analysis it is convenient to rewrite the expressions (9.2)–(9.4) in the following, compact form:

$$\begin{aligned}\overline{\mathbf{u}_\mathrm{w} w_{\mathrm{w},a}} &= \alpha_{a,1}\left(\frac{\mathbf{k}}{k}\cdot\nabla\left(\frac{E}{\rho}\right)\right)\frac{\mathbf{k}}{k} + \alpha_{a,2}\nabla\left(\frac{E}{\rho}\right), \\ \overline{\mathbf{u}_\mathrm{w} w_{\mathrm{w},h}} &= \frac{E}{\rho d}\left[\alpha_{h,1}\left(\frac{\mathbf{k}}{k}\cdot\nabla h\right)\frac{\mathbf{k}}{k} + \alpha_{h,2}\nabla h\right], \\ \overline{\mathbf{u}_\mathrm{w} w_{\mathrm{w},d}} &= \frac{E}{\rho d}\left[\alpha_{d,1}\left(\frac{\mathbf{k}}{k}\cdot\nabla d\right)\frac{\mathbf{k}}{k} + \alpha_{d,2}\nabla d\right],\end{aligned}$$

where the form of the coefficients $\alpha_{a,1}$, $\alpha_{a,2}$, $\alpha_{h,1}$, $\alpha_{h,2}$, $\alpha_{d,1}$and $\alpha_{d,2}$ follows directly from (9.2)–(9.4).

3.2. Unidirectional waves

In a special case when both the wave energy and water depth vary only in wave direction (2DV-waves), formulae (9.2)–(9.4) reduce to:

$$\overline{\mathbf{u}_\mathrm{w} w_{\mathrm{w},a}} = -\frac{1}{2}G\frac{z+h}{d}\nabla\left(\frac{E}{\rho}\right), \tag{10.1}$$

$$\overline{\mathbf{u}_\mathrm{w} w_{\mathrm{w},h}} = -\frac{E}{\rho d}G\nabla h, \tag{10.2}$$

$$\overline{\mathbf{u}_\mathrm{w} w_{\mathrm{w},d}} = \frac{E}{\rho d}\alpha_1(kd)\frac{z+h}{d}\nabla d, \tag{10.3}$$

which is—not surprisingly—identical to the solution obtained by de Vriend and Kitou (1990a).

Thus, in waves subject to shoaling, but no refraction (e.g., coming towards a shore perpendicularly to the bottom contours), $\overline{\mathbf{u}_\mathrm{w} w_\mathrm{w}}$ varies linearly with depth, in a way shown in Fig. 1 for kd values ranging from 0.5 to 4. In the plots, the scaled vertical coordinate σ has been used, defined as:

$$\sigma = \frac{z-\eta}{d}. \tag{11}$$

If the waves are non-dissipative and the energy conservation equation holds true, ∇E can be expressed as a function of ∇d:

$$\nabla\left(\frac{E}{\rho}\right) = 2\frac{E}{\rho d}\frac{G}{(G+1)^2}[G\sinh^2[kd]-1]\nabla d,$$

which leads to:

$$\overline{\mathbf{u}_\mathrm{w} w_\mathrm{w}} = \frac{E}{\rho d}G\left[\frac{G}{G+1}\left(1+\frac{\cosh^2[kd]}{G+1}\right)\frac{z+h}{d}\nabla d - \nabla h\right].$$

3.3. Waves subject to shoaling and refraction

In their paper presenting 2D-V wave-driven circulation model, de Vriend and Kitou (1990b) suggest, that their one-dimensional solution to the Laplace's equation can be generalized to a fully two-dimensional one by simple 'rotation' of this solution toward the direction of wave propagation. Even without having the results of the above presented analysis in mind, this statement seems rather unfounded and the approach oversimplified. In Figs. 2, 3 and 4 an example of the vertical structure of the components of $\overline{\mathbf{u}_\mathrm{w} w_\mathrm{w}}$-term over a sloping bottom is shown for an angle between the direction of wave propagation and gradient of water depth equal to 45°. As can be seen, the vertical variability of $\overline{\mathbf{u}_\mathrm{w} w_\mathrm{w}}$ is very different from the one shown in Fig. 1. For δ_r=1 the influence of the bottom topography on

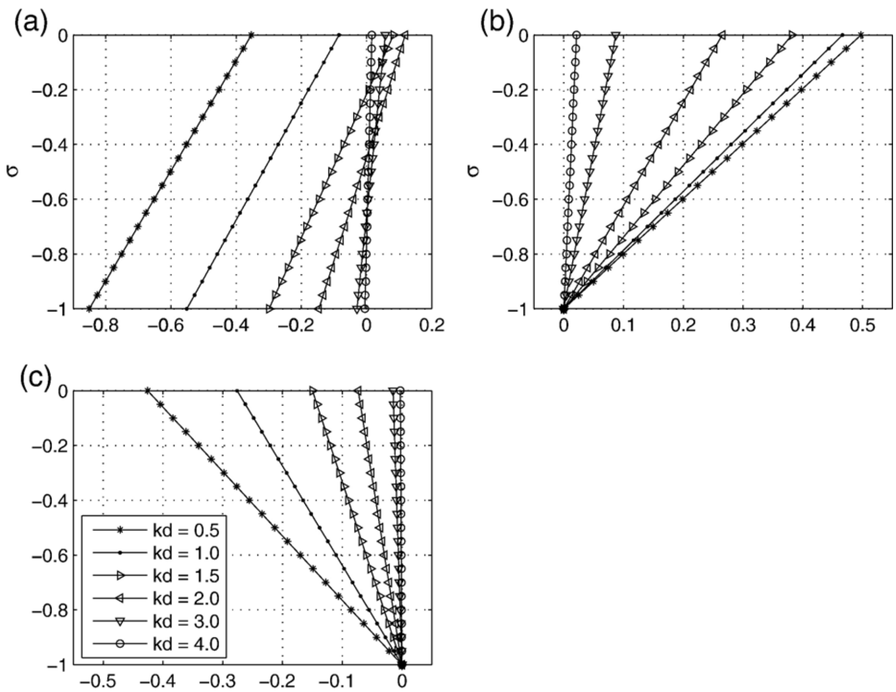

Fig. 1. Vertical variability of $\overline{\mathbf{u}_\mathrm{w} w_\mathrm{w}}$ in unidirectional waves over a sloping bottom for different kd values, due to gradient of bottom level ($\alpha_{h,1}+\alpha_{h,2}+\alpha_{d,1}+\alpha_{d,2}$; a), water surface elevation ($\alpha_{d,1}+\alpha_{d,2}$; b) and wave amplitude ($\alpha_{a,1}+\alpha_{a,2}$; c).

$\overline{\mathbf{u}_\mathrm{w} w_\mathrm{w}}$ decreases to zero at the water surface. Refraction has also a very pronounced influence on the $\overline{\mathbf{u}_\mathrm{w} w_{\mathrm{w},a}}$-profile—in the direction of ∇E its values vary between zero at the bottom and 1/2 at the surface; the component perpendicular to ∇E vanishes.

4. Wave forcing

The results presented so far make it possible to evaluate the wave forcing term, formulated in Appendix A.1 (formula A.1.1). After a lengthy algebra the following—surprisingly simple—expression is obtained:

$$\mathbf{F}_\mathrm{w} = \frac{E}{\rho d}\left[\frac{kd}{\tanh[kd]}\frac{G}{G+1}\frac{\nabla d}{d} - \frac{G}{2}\frac{\nabla E}{E}\right] \tag{12}$$

or even more simply, but with gradients of wave number k included explicitly (through G):

$$\mathbf{F}_\mathrm{w} = -\frac{1}{2}\nabla\left(\frac{E}{\rho d}G\right). \tag{13}$$

Thus, the wave forcing is—as it should be, considering the assumptions of the underlying wave theory—constant over depth. The vertical variability of the $\overline{\mathbf{u}_\mathrm{w} w_\mathrm{w}}$-term, analysed in the previous section, acts as a counterbalance for the corresponding variability of the horizontal wave-induced momentum flux. What's more, the form of (12) does not depend on the δ_r parameter.

Compared to the 'traditional' formulation of the wave forcing in the vertically integrated momentum equations, based on gradients of the radiation stress tensor components,

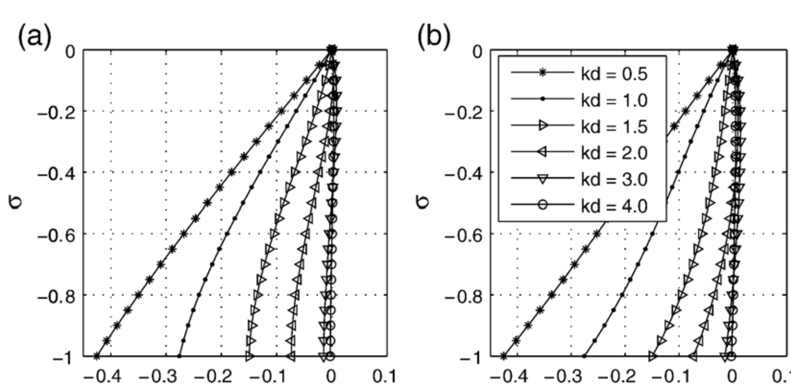

Fig. 2. Vertical variability of $\overline{\mathbf{u}_\mathrm{w} w_\mathrm{w}}$ in waves propagating over a sloping bottom at an angle $\theta=45°$ relative to the direction of the bottom level gradient ∇h, for $\delta_\mathrm{r}=1$. The plots (a) and (b) show the components of $\overline{\mathbf{u}_\mathrm{w} w_{\mathrm{w},h}} + \overline{\mathbf{u}_\mathrm{w} w_{\mathrm{w},d}}$ parallel and perpendicular to the direction of ∇h: $(\alpha_{h,1}+\alpha_{d,1})\cos^2\theta+\alpha_{h,2}+\alpha_{d,2}$ and $(\alpha_{h,1}+\alpha_{d,1})\sin\theta\cos\theta$, respectively.

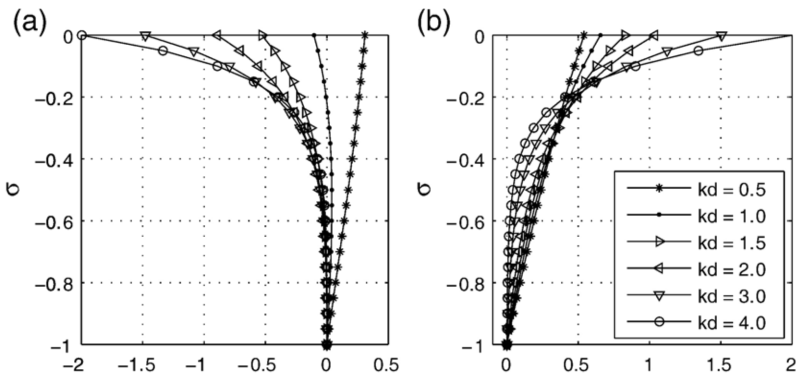

Fig. 3. Vertical variability of $\overline{\mathbf{u}_w w_w}$ in waves propagating over a sloping sea surface at an angle $\theta=45°$ relative to the direction of the sea level gradient $\nabla\eta$, for $\delta_r=1$. The plots (a) and (b) show the components of $\overline{\mathbf{u}_w w_{w,d}}$ parallel and perpendicular to the direction of $\nabla\eta$: $\alpha_{d,1}\cos^2\theta+\alpha_{a,2}$ and $\alpha_{d,1}\sin\theta\cos\theta$, respectively.

expressions (12) and (13) show fundamental differences, which will be discussed in the following sections.

4.1. Wave refraction on a barred beach

To illustrate those differences clearly, it is useful to consider a simple case of waves approaching a straight, infinitely long beach with parallel bottom contours. For the purpose of this example it is useful to introduce a Cartesian coordinate system $\mathbf{x}=[x_1, x_2]$ such that the bottom contours are parallel to the x_2-axis. If linear wave theory over a flat bottom is applied, then $\overline{\mathbf{u}_w w_w}=0$ and the two components of the wave forcing are (see Eq. (A.1.2) in Appendix A.1 for the general expression):

$$\widetilde{F}_{w,1}=-\frac{\partial\left(\overline{u_{w,1}^2}-\overline{w_w^2}\right)}{\partial x_1}=-\frac{\partial}{\partial x_1}\left[\frac{E}{\rho d}G\left(1-\sin^2\theta\cosh^2[k(z+h)]\right)\right],$$
$$\widetilde{F}_{w,2}=-\frac{\partial\left(\overline{u_{w,1}u_{w,2}}\right)}{\partial x_1}=-\frac{\partial}{\partial x_1}\left[\frac{E}{\rho d}G\sin\theta\cos\theta\cosh^2[k(z+h)]\right],$$

where the angle θ between the wave propagation direction and the x_1-axis has been introduced. The vertically integrated $\tilde{F}_{w,1}$ and $\tilde{F}_{w,2}$ terms correspond with the radiation stress terms in the momentum equations (see, e.g., Longuet-Higgins and Stewart, 1962 or Dingemans et al., 1987). Although the wave field is uniform in the direction along the beach, for $\theta\neq 0$ both components of the wave forcing are different from zero. Moreover, both are depth-varying. On the contrary, expression (13) reduces in this case to:

$$F_{w,1}=-\frac{1}{2}\frac{\partial}{\partial x_1}\left(\frac{E}{\rho d}G\right),$$
$$F_{w,2}\equiv 0.$$

There is no wave-induced force in the direction along the beach, no matter what is the value of θ. Hence, for non-dissipative waves outside the surf zone there is no mechanism driving the longshore current—a result not only quantitatively different, but also qualitatively opposing to the ‘classical’ conviction that the vertically integrated $\tilde{F}_{w,2}$ term is the main driving force in the along-shore momentum equation.

The discussion of the consequences of this result for the balance of forces in the nearshore zone will be given in the summary in Section 5. Here let us analyze in more detail an example case of waves approaching a barred beach of a profile shown in Figs. 5 and 6. Let us assume that in the constant depth area seaward from the bar the waves propagate at the angle $\theta_0=45°$ relative to the bottom contours, $(kd)_0=1.5$ and $a_0=1$ m there and $\eta=0$ (which is of course not realistic, but does not

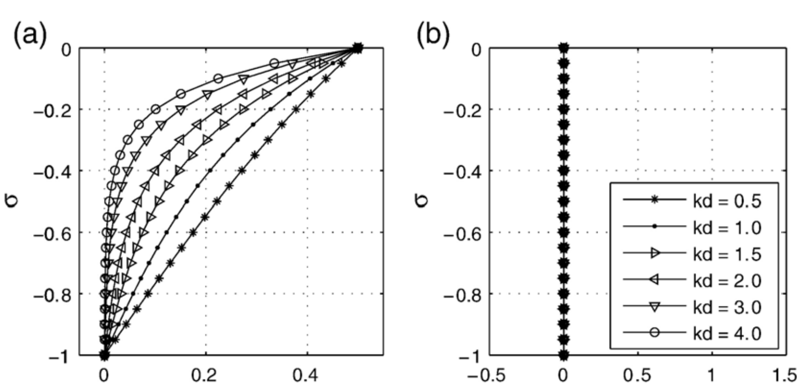

Fig. 4. Vertical variability of $\overline{\mathbf{u}_w w_w}$ in waves with spatially varying energy, propagating at an angle $\theta=45°$ relative to the direction of the energy gradient ∇E, for $\delta_r=1$. The plots (a) and (b) show the components of $\overline{\mathbf{u}_w w_{w,a}}$ parallel and perpendicular to the direction of ∇E: $\alpha_{a,1}\cos^2\theta+\alpha_{a,2}$ and $\alpha_{a,1}\sin\theta\cos\theta$, respectively.

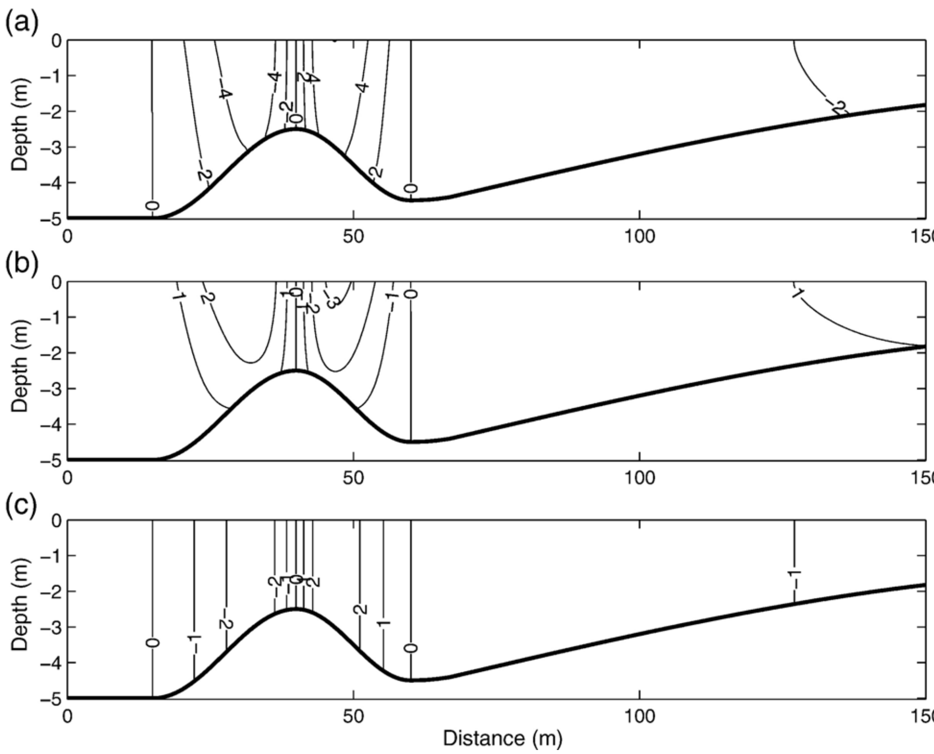

Fig. 5. Vertical structure of the cross-shore component of wave forcing (in 10^{-2} m/s^2) in waves approaching a barred beach: horizontal wave-induced momentum flux $\tilde{F}_{x,1}$ (a), vertical wave-induced momentum flux $\partial\overline{u_{\mathrm{w},1}w_{\mathrm{w}}}/\partial z$ (b) and the sum of these two terms $F_{x,1}$ (c).

introduce any error relevant from the point of view of the present discussion). The distribution of the cross-shore ($\tilde{F}_{\mathrm{w},1}$, the $\overline{u_{\mathrm{w},1}w_{\mathrm{w}}}$-dependent term and $\tilde{F}_{\mathrm{w},1}$) and along-shore ($\tilde{F}_{\mathrm{w},2}$, the $\overline{u_{\mathrm{w},2}w_{\mathrm{w}}}$-dependent term and $F_{\mathrm{w},1}$) components of the wave forcing in this simple case is shown in Figs. 5 and 6, respectively.

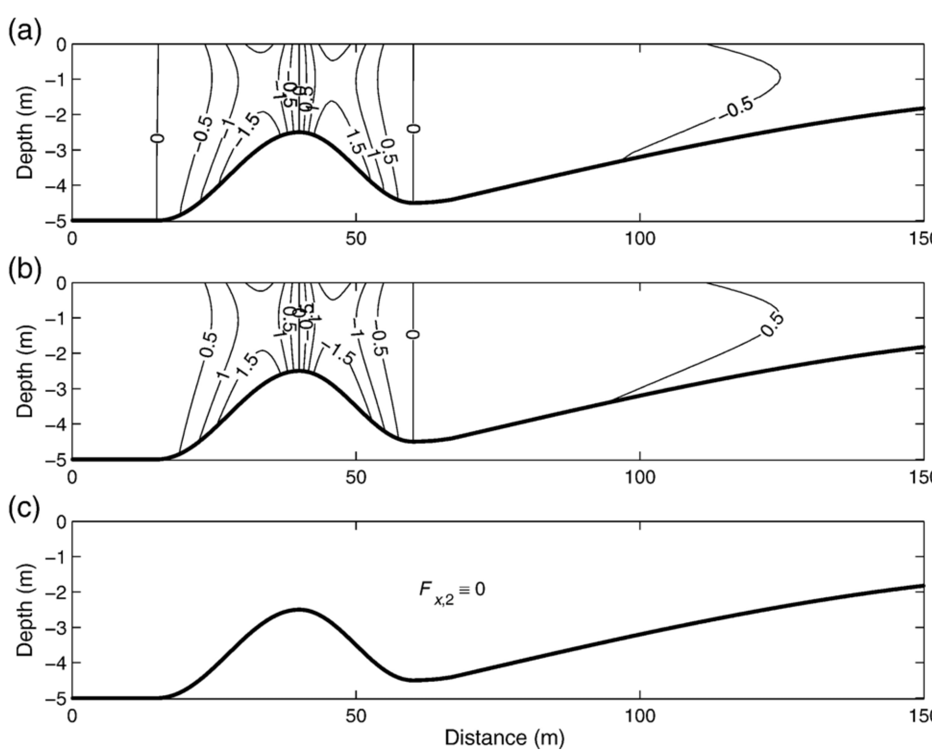

Fig. 6. Vertical structure of the along-shore component of wave forcing (in 10^{-2} m/s^2) in waves approaching a barred beach: horizontal wave-induced momentum flux $\tilde{F}_{x,2}$ (a), vertical wave-induced momentum flux $\partial\overline{u_{\mathrm{w},1}w_{\mathrm{w}}}/\partial z$ (b) and the sum of these two terms $F_{x,2}$ (c).

In the cross-shore direction, the sum of the depth-varying horizontal wave-induced momentum flux (negative for decreasing depth and positive for increasing depth; Fig. 5a) and the depth-varying vertical wave-induced momentum flux (positive for decreasing depth and negative for increasing depth; Fig. 5b) is a depth-independent function of the gradient of wave energy E and water depth d, exactly as described by expression (12). In the along-shore direction, the vertical and horizontal wave-induced momentum flux have opposite values and their sum equals zero everywhere in the domain.

4.2. Comparison with the solution by Rivero and Arcilla (1995)

Rivero and Arcilla (1995) formulated a simple expression relating vorticity with horizontal and vertical momentum flux in unidirectional water waves. This approach can be easily extended to the fully three-dimensional case, as shown below.

Let $\mathbf{v}_{\mathrm{w}}=[\mathbf{v}_{\mathrm{w},h},\ v_{\mathrm{w},z}]$ denote the vorticity of the wave field. From irrotationality of the wave number vector $\mathbf{k}$ it follows that $v_{w,z}=0$. For $\mathbf{v}_{\mathrm{w},h}$ we have:

$$\mathbf{v}_{\mathrm{w},h}=\frac{\partial u_{\mathrm{w}}}{\partial z}\frac{\mathbf{k}}{k}-\nabla w.$$

Thus:

$$\begin{aligned} w_{\mathrm{w}}\mathbf{v}_{\mathrm{w},h} &= w_{\mathrm{w}}\frac{\partial u_{\mathrm{w}}}{\partial z}\frac{\mathbf{k}}{k}-w_{\mathrm{w}}\nabla w_{\mathrm{w}} \\ &= \frac{\partial(u_{\mathrm{w}}w_{\mathrm{w}})}{\partial z}\frac{\mathbf{k}}{k}-u_{\mathrm{w}}\frac{\partial w_{\mathrm{w}}}{\partial z}\frac{\mathbf{k}}{k}-\frac{1}{2}\nabla(w_{\mathrm{w}}^2) \end{aligned}$$

and from the continuity equation:

$$\begin{aligned} w_{\mathrm{w}}\mathbf{v}_{\mathrm{w},h} &= \frac{\partial(u_{\mathrm{w}}w_{\mathrm{w}})}{\partial z}\frac{\mathbf{k}}{k}-\frac{1}{2}\nabla(w_{\mathrm{w}}^2)+u_{\mathrm{w}}\left(\nabla\cdot\left(u_{\mathrm{w}}\frac{\mathbf{k}}{k}\right)\right)\frac{\mathbf{k}}{k} \\ &= \frac{\partial(u_{\mathrm{w}}w_{\mathrm{w}})}{\partial z}\frac{\mathbf{k}}{k}-\frac{1}{2}\nabla(w_{\mathrm{w}}^2)+\left[\frac{1}{2}\nabla(u_{\mathrm{w}}^2)\cdot\frac{\mathbf{k}}{k}+u_{\mathrm{w}}^2\nabla\cdot\frac{\mathbf{k}}{k}\right]\frac{\mathbf{k}}{k}. \end{aligned}$$

For irrotational waves analysed in this work the above equation enables to evaluate the vertical wave-induced momentum flux as a function of other wave parameters (known, e.g., from linear wave theory), exactly as suggested by Rivero and Arcilla (1995). The resulting wave forcing term, given by equation (A.1.1) in Appendix A.1, is then:

$$\mathbf{F}_{\mathrm{w}}=\left[-\frac{1}{2}\frac{\mathbf{k}}{k}\cdot\nabla(\overline{u_{\mathrm{w}}^2})+u_{\mathrm{w}}^2\frac{\mathbf{k}}{k}\cdot\frac{\nabla k}{k}\right]\frac{\mathbf{k}}{k}-\overline{u_{\mathrm{w}}^2}\frac{\nabla k}{k}+\frac{1}{2}\nabla(w_{\mathrm{w}}^2).$$

It is easy to show that evaluation of the terms in the above equation leads exactly to the formula (12). In other words, the simple approach of Rivero and Arcilla (1995) leads to the same wave forcing as the approach of de Vriend and Kitou (1990a,b). As noticed by Rivero and Arcilla (1995) and as can be seen from the above expression, for unidirectional waves the inclusion of the vertical wave-induced momentum flux leads to reduction of the wave forcing term by the factor of 2.

5. Summary and conclusions

In the paper, the expansion method of de Vriend and Kitou (1990a,b) has been extended to waves propagating over two-dimensional bathymetry. The resulting solution, which is first-order accurate in the gradients of water depth and wave energy, has been used to formulate corrected expressions for the vertical component of the wave-induced momentum flux in waves subject to shoaling and refraction. It has been shown that the proposed solution gives consistent results in terms of the formulation of wave forcing terms in the momentum equations and that this solution cannot be considered as a straightforward extension of the one-dimensional problem. The vertical variability of the vertical wave-induced momentum flux acts as a counterbalance for the corresponding variability of the horizontal momentum flux—the total wave forcing term is thus a depth-independent function of gradients of wave energy and water depth and does not depend explicitly on wave direction. This implies, among other things, that in the absence of the along-shore variability in the wave field the along-shore component of wave forcing equals zero.

Finally, it is worth discussing some consequences of the results obtained in this work for modelling of wave-induced nearshore currents. If one accepts correctness of these results (outside the surf zone), then, considering their considerable differences with respect to the forcing elaborated on the basis of mild-slope linear wave theory, an unavoidable question is why the classical method gives so satisfactory results in many practical applications.

One of the reasons is—as has been already suggested by Rivero and Arcilla (1995)—that the models usually contain certain number of coefficients and parameters (sometimes without any physical meaning) that can be tuned so that the errors in specifying some terms in the model equations are compensated by errors in other terms. For example, one of the most important factors in longshore currents modelling is the bottom shear stress. However, as Garcez Faria et al. (1998) have shown in their paper describing the results of the DUCK’94 Experiment, the bottom shear stress coefficient can vary by an order of magnitude in the nearshore zone. It seems reasonable to assume that in many studies, especially those dealing with complicated field conditions, the bottom shear stress is not estimated with an accuracy sufficient to discover inconsistencies in the formulation of wave forcing terms. The more so, if one considers that some of those effect are usually not very pronounced and difficult to measure. As has been noticed earlier, the deviation of the $\overline{u_{\mathrm{w}}w_{\mathrm{w}}}$-profile from a linear one increases with increasing kd and with increasing angle between the wave propagation direction and water depth gradient. Both conditions are more likely satisfied relatively far from the shore, where other processes dominate the momentum balance and the influence of wave forcing is not so pronounced (which again makes it easier to compensate for possible errors in its formulation). Close to the shore kd decreases and due to refraction the propagation direction changes towards shore-perpendicular, so that the vertical profile of the wave forcing can be satisfactorily approximated by a linear function, resulting from studies of unidirectional waves.

As for the along-shore component of wave forcing, a study by Lentz et al. (1999) shows that outside the surf zone, even in very shallow water, the alongshore momentum balance is dominated by surface and bottom stresses, which are approximately an order of magnitude larger as the wave forcing. What's more, the authors show that the sum of the two response terms (acceleration and bottom shear stress) remain up to the boundary of the surf zone balanced by the two forcing terms (surface stress and pressure gradient)—without any significant role of radiation stress forcing. Those relationships reverse of course within the surf zone, but there the assumptions and results of the linear theory do not apply any more.

Appendix A. Wave forcing terms in the momentum equations

Let us consider the set of continuity and momentum equations in the following form:

$$\nabla\cdot\mathbf{u}+\frac{\partial w}{\partial z}=0,$$
$$\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}+w\frac{\partial\mathbf{u}}{\partial z}=-\frac{1}{\rho}\nabla p+\mathbf{F}_h,$$
$$\frac{\partial w}{\partial t}+(\mathbf{u}\cdot\nabla)w+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+F_z-g,$$

where p denotes pressure and $\mathbf{F}_h$ and F_z are additional terms that are irrelevant for the present study (like, e.g., Coriolis acceleration or wind forcing). The meaning of the remaining symbols has been given in the main text.

Let us also make the following assumptions concerning the flow characteristics:

1. The flow velocity can be expressed as a sum of mean [$\mathbf{U}$, W], wave [$\mathbf{u}_\mathrm{w}$, w_w] and turbulent [$\mathbf{u}_\mathrm{t}$, w_t] components:
$$\mathbf{u}=\mathbf{U}+\mathbf{u}_\mathrm{w}+\mathbf{u}_\mathrm{t},w=W+w_\mathrm{w}+w_\mathrm{t}.$$
2. The wave motion is time-periodic, so that the phase-averaging operation can be defined:
$$\bar{\phi}\equiv\frac{1}{2\pi}\int_0^{2\pi}\varphi(\varphi)d\varphi$$
for a given dependent variable ϕ (φ denotes phase). Averaged wave quantities $\bar{\mathbf{u}}_\mathrm{w}$, $\overline{w}_\mathrm{w}$ and $\bar{\xi}$ equal (by definition) zero.
3. The averaging operation defined above is sufficient for averaging the turbulent velocities as well (equivalently: the characteristic time scale of the turbulent motion is shorter than the characteristic time scale of the wave motion).
4. The turbulent motion and the wave motion are uncorrelated, that is:
$$\overline{\mathbf{u}_\mathrm{w}\mathbf{u}_\mathrm{t}}=\overline{\mathbf{u}_\mathrm{w}w_\mathrm{t}}=\overline{w_\mathrm{w}\mathbf{u}_\mathrm{t}}=\overline{w_\mathrm{w}w_\mathrm{t}}=0.$$
5. The slowly-varying accelerations in the vertical are negligible. Hence, after separation of the wave-induced pressure p_w from the total pressure p, the remaining pressure p_h is hydrostatic.
6. The horizontal turbulent momentum flux is negligible in comparison to the horizontal wave-induced momentum flux.
7. The vertical turbulent momentum flux can be accounted for by means of an appropriate turbulence closure scheme, e.g., the first-order turbulence closure:
$$\overline{\mathbf{u}_\mathrm{t}w_\mathrm{t}}=-\mu_\mathrm{t}\frac{\partial\mathbf{U}}{\partial z}.$$

A detailed derivation of the final equations based on the above assumptions can be found in a number of other studies (see, e.g., Mellor, 2003) and will not be repeated here. The final form of the phase-averaged horizontal momentum equation, crucial for the work presented in this paper, is:

$$\frac{\partial\mathbf{U}}{\partial t}+(\mathbf{U}\cdot\nabla)\mathbf{U}+W\frac{\partial\mathbf{U}}{\partial z}=\frac{\partial}{\partial z}\left(\mu_\mathrm{t}\frac{\partial\mathbf{U}}{\partial z}\right)-\frac{1}{\rho}\nabla p_h+\mathbf{F}_h+\mathbf{F}_\mathrm{w}+\mathbf{F}_\mathrm{w,surf},$$

where $\mathbf{F}_\mathrm{w}$ denotes the depth-dependent part of the wave forcing term and $\mathbf{F}_\mathrm{w,surf}$ is its part acting on the mean ($z=\eta$) water surface and proportional to $\overline{\xi^2}$. In numerical circulation models this term is usually applied to the uppermost layer of the model.

For the purpose of consistency with the notation used in the main text, it is convenient to write the horizontal wave velocity as:

$$\mathbf{u}_\mathrm{w}=u_\mathrm{w}\frac{\boldsymbol{k}}{k}.$$

$\mathbf{F}_\mathrm{w}$ has then the following form:

$$\mathbf{F}_\mathrm{w}=\left[-\frac{\mathbf{k}}{k}\cdot\nabla\left(\overline{u_\mathrm{w}^2}\right)+\overline{u_\mathrm{w}^2}\frac{\mathbf{k}}{k}\cdot\frac{\nabla k}{k}-\overline{u_\mathrm{w}^2}\nabla\cdot\left(\frac{\mathbf{k}}{k}\right)\right]\frac{\mathbf{k}}{k}-\overline{u_\mathrm{w}^2}\frac{\nabla k}{k}+\nabla\left(\overline{w_\mathrm{w}^2}\right)-\frac{\partial\left(\overline{u_\mathrm{w}w_\mathrm{w}}\right)}{\partial z}\frac{\mathbf{k}}{k}. \tag{A.1.1}$$

The expression above is very general—no specific wave theory has been applied to formulate it; the only underlying assumption is that of irrotationality of the wave vector $\mathbf{k}$.

In the most common notation, making use of the horizontal coordinate system $\mathbf{x}=[x_1, x_2]$ and horizontal velocity components $\mathbf{u}_\mathrm{w}=[u_{\mathrm{w},1}, u_{\mathrm{w},2}]$, expression (A.1.1) can be rewritten as:

$$F_{\mathrm{w},i}=-\sum_{j=1}^{2}\frac{\partial\left(\overline{u_{\mathrm{w},i}u_{\mathrm{w},j}}\right)}{\partial x_j}+\frac{\partial\left(\overline{w_\mathrm{w}^2}\right)}{\partial x_i}-\frac{\partial\left(\overline{u_{\mathrm{w},i}w_\mathrm{w}}\right)}{\partial z}=\widetilde{F}_{\mathrm{w},i}-\frac{\partial\left(\overline{u_{\mathrm{w},i}w_\mathrm{w}}\right)}{\partial z},\quad i=1,2. \tag{A.1.2}$$

The surface part of wave forcing is given by:

$$\mathbf{F}_\mathrm{w,surf}=-\frac{1}{2}\nabla\left(\frac{E}{\rho}G\right). \tag{A.1.3}$$

Upon vertical integration over the entire water column, the sum:

$$\widetilde{F}_{\mathrm{w},i}+F_{\mathrm{w,surf},i},\quad i=1,2$$

gives the same wave forcing components as those obtained by Longuet-Higgins and Stewart (1962).

Appendix B. General form of the solution to the ordinary differential equations used in the paper

- $y''(z) - k^2 y(z) = f_c$:

$$y = \frac{2kzf_s - f_c}{4k^2} + C_1 e^{-kz} + C_2 e^{kz},$$

- $y''(z) - k^2 y(z) = f_s$:

$$y = \frac{2kzf_c - f_s}{4k^2} + C_1 e^{-kz} + C_2 e^{kz},$$

- $y''(z) - k^2 y(z) = k(z+h) f_c$:

$$y = -\frac{(1 + 2k^2 z(z+2h)) f_s + 2k(z+h) f_c}{8k^2} + C_1 e^{-kz} + C_2 e^{kz},$$

where C_1 and C_2 are the integration constants.

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