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Group velocity and nonlinear dispersive wave propagation

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By the use of a Hamiltonian formulation, a basic group velocity is defined as the derivative of frequency with respect to wavenumber keeping action density constant, and is shown to represent an incremental action velocity in the general nonlinear case. The stability treatment of Whitham and Lighthill is extended to several dimensions. The water-wave analysis of Whitham (1967*a*) is extended to two space dimensions, and is shown to predict oblique-mode instabilities for $kh < 1.36$. A treatment of Lighthill's (1965) solution in the one-dimensional elliptic case resolves the problem of the energy distribution in the solution past the critical time. A note on diffraction effects on quasilinear solutions of the Whitham type is presented.

1. Introduction

The theory of the propagation of nonlinear wave trains has been developed by Whitham (1965*a*, *b*, 1967*a*, *b*, 1970) with extensions by Lighthill (1965, 1967) and others. With a few exceptions this body of theory has been developed for weakly nonlinear waves and for a propagation space comprising one distance coordinate plus time. One basic principle is the existence of an adiabatically conserved entity which has been termed wave action, characterized by a density A and a flux $\boldsymbol{B}$. Another is that the group velocity of the linear case either splits into two distinct propagation velocities for amplitude and frequency modulations (hyperbolic case) or disappears with the appearance of an unstable mode (elliptic case).

The group velocity of linear wave theory is identifiable as the velocity of action flux $\boldsymbol{B}/A$, or in some cases as the velocity of energy flux. It is identifiable as a signal velocity, the velocity at which information may be transmitted through amplitude or frequency modulations. A purpose of the present paper is to explore more fully the concept of group velocity in the nonlinear case, in particular the relations with action flux and signal velocity.

A second purpose of the present paper is to express the theory in a form wherein the strongly nonlinear case appears in essentially the same terms as does the weakly nonlinear case. Another purpose is the extension of the discussion of stability and of well- and ill-posed problems to an arbitrary number of dimensions. The quasilinear water-wave theory of Whitham is extended to a $(2+1)$ dimension propagation space. Another is an exploration of the concept of focusing in the nonlinear case.

In many respects the present paper is a continuation of two earlier papers (Hayes 1970*a*, *b*) on modal wave-action conservation and kinematic wave theory. Here we do not distinguish between a local and a modal wave system. The propagation equations for nonlinear equations are the same as those for the kinematic theory for linear wave motion, with the addition of A as a basic variable in the equation system.

2. Basic analysis

We start here with the formulation and results of Whitham. The analysis is one in a propagation space $(\boldsymbol{x}, t)$. If this is the same as the physical space the waves are termed local; if the physical space is the product of the propagation space and a cross-space the waves are termed modal. The wave motion is describable in terms of a dimensionless phase variable $\theta(\boldsymbol{x}, t)$; the quantity $\omega = -\partial\theta/\partial t$ is the frequency and $\boldsymbol{k} = \nabla\theta$ is the wavenumber. The frequency and wavenumber satisfy a consistency condition

$$\partial\boldsymbol{k}/\partial t + \nabla\omega = 0. \tag{2.1}$$

Another consistency condition is that the tensor $\nabla\boldsymbol{k}$ must be symmetric.

The wave motion is governed by a Lagrangian density through the variational principle. The average over phase (or phase shift) of the integral of this density over the cross space (if the waves are modal) is denoted $\mathscr{L}(\boldsymbol{k}, \omega, a; \boldsymbol{x}, t)$, an averaged Lagrangian density in the propagation space. The parameter a is some suitable measure of the amplitude or intensity of the wave motion. The argument space of $\mathscr{L}$ is an augmented space, and care must be taken to distinguish partial derivatives in such a space from those in the propagation space. This distinction is made partly by using subscripts for derivatives in augmented spaces and the operators $\partial/\partial t$ and ∇ in the propagation space. We shall also generally use capital letters for functions over an augmented space and lower-case letters for functions over the propagation space.

The indicated dependence of $\mathscr{L}$ on $\boldsymbol{x}$ and t reflects possible non-uniformity of the medium supporting the wave motion. In a uniform medium this explicit dependence disappears.

The analysis is an asymptotic one, valid for wave trains that approximate uniform wave trains that are planar in the propagation space. The appropriate small parameter measures the smallness of time and distance derivatives of ω, $\boldsymbol{k}$, a, and the properties defining the medium.

With the medium specified, the quantities ω, $\boldsymbol{k}$ and a must satisfy a scalar nonlinear dispersion relation in order to correspond to a possible solution. The averaged Lagrangian density $\mathscr{L}$ must be defined not only for solutions of interest but also for a neighbourhood of points in its augmented space which do not correspond to solutions. Applying the variational principle to $\mathscr{L}$ with respect to variations in a and θ yields the relations $\mathscr{L}_a = 0$ and $\partial\mathscr{L}_\omega/\partial t = \nabla\cdot\mathscr{L}_k$. The first of these serves as the needed nonlinear dispersion relation connecting ω, $\boldsymbol{k}$ and a. The second we write as

$$\partial A/\partial t + \nabla\cdot\boldsymbol{B} = 0, \tag{2.2}$$

where

$$A = \mathscr{L}_\omega \tag{2.3}$$

is interpreted as wave-action density and

$$\boldsymbol{B} = -\mathscr{L}_k \tag{2.4}$$

as wave-action flux; both quantities are to be evaluated for $\mathscr{L}_a = 0$.

It is desirable to replace $\mathscr{L}$ by a function over an augmented space with one variable eliminated, so that only solutions are represented by the space. In Lighthill's (1965) formulation the relation $\mathscr{L}_a = 0$ is used to eliminate a, to obtain a function $\mathscr{L}'(\boldsymbol{k}, \omega, \boldsymbol{x}, t)$. We eliminate instead the variable ω through a different procedure, taking advantage of the independence of the analysis to the choice of variable a measuring the intensity. It may be readily verified that the relation $\mathscr{L}_a = 0$ and the values of $\mathscr{L}_\omega$ and $\mathscr{L}_k$ are invariant with respect to a change in the definition of a. We choose to use the action density A as the measure of intensity, and write $\mathscr{L}$ as $\mathscr{L}(\boldsymbol{k}, \omega, A, \boldsymbol{x}, t)$. The nonlinear dispersion relation is then

$$\mathscr{L}_A = 0, \tag{2.5}$$

valid at a solution. This choice treats t as a distinctive variable different from the components of $\boldsymbol{x}$ in the propagation space.

The freedom available in specifying the behaviour of $\mathscr{L}$ for non-solution points permits us to keep (2.3) over the augmented space of $\mathscr{L}$. Equation (2.3) may be integrated, and yields

$$\mathscr{L} = A\omega - \mathscr{H}(\boldsymbol{k}, A, \boldsymbol{x}, t). \tag{2.6}$$

This formula will be recognized as a partial contact transformation and the function $\mathscr{H}$ which appears as a constant of integration as a Hamiltonian, one different from that obtained through a complete contact transformation (Whitham 1965*b*). In these terms the dispersion relation (2.5) takes the form

$$\omega = \mathscr{H}_A = \Omega(\boldsymbol{k}, A, \boldsymbol{x}, t), \tag{2.7}$$

which is close to the customary form for the linear case. Equation (2.4) becomes $\boldsymbol{B}(\boldsymbol{k}, A) = \mathscr{H}_k$, and $\boldsymbol{B}$ is thereby expressed naturally in a form with ω eliminated. The tensor $\boldsymbol{B}_k = \mathscr{H}_{kk}$ is seen to be symmetric.

We now define the *basic group velocity* $\boldsymbol{c}$ or $\boldsymbol{C}$ as

$$\boldsymbol{c}(\boldsymbol{x}, t) = \boldsymbol{C}(\boldsymbol{k}, A, \boldsymbol{x}, t) = \mathscr{H}_{Ak} = \Omega_k = \boldsymbol{B}_A. \tag{2.8}$$

With the boundary condition $\mathscr{H} = 0$ when $A = 0$ and with dependence on the medium dropped from the notation we can write

$$\mathscr{H}(\boldsymbol{k}, A) = \int_0^A \Omega(\boldsymbol{k}, A')\,\mathrm{d}A', \tag{2.9}$$

$$\boldsymbol{B}(\boldsymbol{k}, A) = \int_0^A \boldsymbol{C}(\boldsymbol{k}, A')\,\mathrm{d}A'. \tag{2.10}$$

These equations serve as the foundation for our basic concept of a *laminar-layer model* for the Hamiltonian and for the action flux. We conceive that the action density A has a laminar structure. In this structure each lamina of thickness $\mathrm{d}A'$ contributes an amount $\Omega(\boldsymbol{k}, A')\,\mathrm{d}A'$ to the Hamiltonian and an amount $\boldsymbol{C}(\boldsymbol{k}, A')\,\mathrm{d}A'$ to the action flux. The basic group velocity $\boldsymbol{C}(\boldsymbol{k}, A)$ is the velocity at which the wave-action density is moving in the uppermost lamina. The basic group velocities at all values of A' between 0 and A thus contribute to the total action flux.

Of the second derivatives of $\mathscr{H}$ with respect to $\boldsymbol{k}$ and A we have already considered, $\boldsymbol{C} = \mathscr{H}_{A\boldsymbol{k}}$. The derivative $\Omega_A = \mathscr{H}_{AA}$ we term the *hardness parameter*. A nonlinear spring for which the frequency of an oscillator increases with increasing amplitude is termed a hard spring. The other second derivative may be expressed

$$\mathscr{H}_{\boldsymbol{kk}} = \int_0^A \Omega_{\boldsymbol{kk}}(\boldsymbol{k}, A')\,\mathrm{d}A' \tag{2.11}$$

and is the integral over the laminar structure of the nonlinear analogue to the classical dispersion tensor.

Despite the arbitrariness possessed by Whitham's Lagrangian $\mathscr{L}$ for non-solutions it is unique for solutions. With this property an algorithm may be presented for producing $\mathscr{H}(\boldsymbol{k}, A)$ for any $\mathscr{L}(\boldsymbol{k}, \omega, a)$. The quantity $A = \mathscr{L}_\omega$ is calculated and used to eliminate a in favour of A; the result is a function $\mathscr{L}(\boldsymbol{k}, \omega, A)$ which is, in general, not of the form (2.6). The relation $\mathscr{L}_A = 0$ is used to eliminate ω from the function $\omega A - \mathscr{L}$; the resulting function is $\mathscr{H}(\boldsymbol{k}, A)$. The relations between $\mathscr{H}$ and Lighthill's function $\mathscr{L}'(\boldsymbol{k}, \omega)$ are given in appendix A. An alternative approach to our formulation using $\mathscr{H}$ is outlined in appendix B.

Whitham has included in his theory a special treatment for dependent variables which do not appear explicitly in the original (unaveraged) Lagrangian density. Such a variable is termed a potential variable, and leads to consideration of a pseudo frequency γ and a pseudo wavenumber $\boldsymbol{\beta}$ satisfying an analogue of (2.1). The presence of such variables leads to little complication in principle but some complication in detail. We consider them here in § 5. In the earlier sections we consistently assume that no potential variables are present.

A galilean transformation to a new coordinate system moving with velocity $-\boldsymbol{U}$ entails a term $A\boldsymbol{k}\cdot\boldsymbol{U}$ added to the function $\mathscr{H}$.

3. Wave propagation theory

In the theory for linear waves the equations governing propagation of the phase and its derivatives are decoupled from that governing propagation of action, amplitude, or intensity. The degenerate nature of the linear case lies in this decoupling and in the fact that the characteristic velocity for both propagations is the same. The first set of equations is purely kinematic once the dispersion relation is given; the theory is thus appropriately termed a kinematic theory. In theories whose traditions lie in the field of optics, the theory is often termed geometric.

In the nonlinear case the equations governing phase propagation and action propagation are coupled, and the theory may no longer be considered kinematic.

The theory does follow the kinematic theory for linear waves in approach, as closely as is feasible.

The basic equations are (2.1) and (2.2), both expressed in the propagation space. The derivatives of $\omega = \Omega$ and of $\boldsymbol{B}$ in the propagation space are converted to equations in the augmented space of $\mathscr{H}$, by using the symmetry of $\nabla \boldsymbol{k}$. The result is

$$\partial \boldsymbol{k}/\partial t + \boldsymbol{C}\cdot\nabla \boldsymbol{k} + \Omega_A \nabla A = -\Omega_x, \tag{3.1}$$

$$\partial A/\partial t + \boldsymbol{C}\cdot\nabla A + \mathscr{H}_{kk}:\nabla \boldsymbol{k} = -\operatorname{tr}(\mathscr{H}_{kx}). \tag{3.2}$$

The right-hand sides of these equations arise solely from non-uniformity of the medium. Equations (3.1) and (3.2) here are not restricted to the case of weak nonlinearity.

Lighthill (1967) has identified restrictions on various theories as (i) that *the modulations are small*, (ii) that *the wave amplitude is moderate*, and (iii) that *the modulations are slow*. The propagation theory of Whitham's type for which (3.1) and (3.2) are the principal equations demands (iii) alone. These equations are nonlinear in $\boldsymbol{k}$ and A, and thus give a nonlinear theory for nonlinear wave propagation. The equations of the standard kinematic wave theory are nonlinear in $\boldsymbol{k}$, and thus give a nonlinear theory for linear wave propagation. In order to determine the nature of equations (3.1) and (3.2), we impose in the next section restriction (i), to obtain a linear theory for nonlinear wave propagation, or more particularly for the propagation of modulations.

It may be noted that for an observer moving with the basic group velocity $\boldsymbol{c} = \boldsymbol{C}$, equations (3.1) and (3.2) appear in a simpler, more symmetric form. We can expect that in some sense $\boldsymbol{c}$ will represent the velocity of the centre of a wave group or of the group as a whole, and that (3.1) and (3.2) with the $\boldsymbol{c}$ terms missing will govern the development of the moving group. The trajectories defined by $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{c}$ are termed *basic rays*.

4. Linearized modulation waves

We consider equations (3.1) and (3.2) in a uniform medium with $\boldsymbol{k}$ and A slightly perturbed from fixed values $\boldsymbol{k}_0$ and A_0. The coefficients $\boldsymbol{c} = \boldsymbol{C}$, Ω_A, and $\mathscr{H}_{kk}$ are assumed to have constant values corresponding to $\boldsymbol{k}_0$ and A_0. Finally, a galilean transformation is made to a frame moving with the basic group velocity $\boldsymbol{c}$. The resultant equations

$$\partial \boldsymbol{k}/\partial t + \Omega_A \nabla A = 0, \tag{4.1}$$

$$\partial A/\partial t + \mathscr{H}_{kk}:\nabla \boldsymbol{k} = 0 \tag{4.2}$$

are linear. Conclusions on the behaviour of solutions to this pair of equations may be expected to apply locally to the more general nonlinear equations.

The phase θ is considered to be perturbed from its undisturbed value

$$\theta_0 = \boldsymbol{k}_0\cdot\boldsymbol{x} - \omega_0 t, \tag{4.3}$$

so that
$$\theta = \theta_0 + \epsilon \int F(\theta_1)\, \mathrm{d}\theta_1, \tag{4.4}$$
where
$$\theta_1 = k_1 \boldsymbol{n} \cdot \boldsymbol{x} - \omega_1 t \tag{4.5}$$
and F is an arbitrary signal which is constant on perturbation wave fronts $\theta_1 = \text{const.}$ The vector $\boldsymbol{n}$ is a unit vector normal to the perturbation wave fronts. The perturbation frequency ω_1 and wavenumber $\boldsymbol{k}_1 = k_1 \boldsymbol{n}$ may be changed by an arbitrary constant factor with a corresponding redefinition of F, and k_1 may be set equal to one if desired. The perturbation wave fronts are the wave fronts for what are termed modulation waves.

The perturbed values of $\boldsymbol{k}$, ω and A take the form
$$\boldsymbol{k} = \boldsymbol{k}_0 + \epsilon k_1 \boldsymbol{n} F(\theta_1), \tag{4.6a}$$
$$\omega = \omega_0 + \epsilon \omega_1 F(\theta_1), \tag{4.6b}$$
$$A = A_0 + \epsilon k_1 A_1 F(\theta_1). \tag{4.6c}$$
Substituting (4.6) into (4.1) and (4.2) leads to the non-dispersive dispersion relation
$$\omega_1^2 / k_1^2 = v^2 = \Omega_A \mathscr{H}_{kk} : \boldsymbol{nn} \tag{4.7}$$
for the modulation waves and to the modal relations
$$\omega_1 / k_1 = v = \Omega_A A_1 = A_1^{-1} \mathscr{H}_{kk} : \boldsymbol{nn}. \tag{4.8}$$
Here $v(\boldsymbol{n}) = \omega_1 / k_1$ is the normal phase velocity of the perturbation fronts, and $\boldsymbol{n}/v$ is the slowness vector of the fronts.

A solution exists for all values of $\boldsymbol{n}$ only if the right-hand side of (4.7) is a positive-definite quadratic form. No solution exists if the right-hand side of (4.7) is a negative-definite form. We term a solution a mode. Modes appear in pairs; with given values of $\boldsymbol{n}$, v and A another mode exists with $\boldsymbol{n}$ reversed in sign. This property is an indication of the symmetry of the equation system (4.1) and (4.2).

The standard kinematic wave theory (see Hayes (1970*b*) and references cited therein) defines a perturbation group velocity $\boldsymbol{c}_1$ for a solution, given by
$$\boldsymbol{c}_1 = \frac{\Omega_A \mathscr{H}_{kk} \cdot \boldsymbol{n}}{(\Omega_A \mathscr{H}_{kk} : \boldsymbol{nn})^{\frac{1}{2}}} = \mathscr{H}_{kk} \cdot \frac{\epsilon k_1 \boldsymbol{n} F}{\epsilon k_1 A_1 F}. \tag{4.9}$$
The total group velocity for a perturbation mode with respect to the original frame is then identifiable as
$$\boldsymbol{c} + \boldsymbol{c}_1 = \boldsymbol{B}_A + \boldsymbol{B}_k \cdot \frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}A} = \frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}A}. \tag{4.10}$$
Thus the total derivative $\mathrm{d}\boldsymbol{B}/\mathrm{d}A$ definable for a particular perturbation mode is the velocity with which the perturbation propagates. We term $\mathrm{d}\boldsymbol{B}/\mathrm{d}A$ the *total signal velocity* for a mode. The main two characteristic velocities found by Whitham to arise from a splitting of the linear group velocity are of this type. In some of his examples another characteristic velocity appears because of the presence of a pseudo frequency and wavenumber.

Equation (4.10) shows that the linear interpretation of the propagation velocity as the velocity at which action or energy flows also appears in the case of nonlinear waves, but in a particular sense. The increment of action density times the total signal velocity equals the increment in action flux, and the total signal velocity may be interpreted as the velocity with which an increment of action flows.

Boundary problems of Cauchy type

We next summarize conclusions on boundary problems for equations (4.1) and (4.2) which can be obtained from the standard theory of partial differential equations. Elimination of $\boldsymbol{k}$ between (4.1) and (4.2) leads to the equation

$$\partial^2 A/\partial t^2 - \Omega_A \mathscr{H}_{kk} : \nabla\nabla A = 0, \tag{4.11}$$

in the alternative form of a single linear second-order partial differential equation. A similar equation holds for $\boldsymbol{k}$, with an additional term appearing if the $\boldsymbol{x}$ space is curved. Initial or boundary conditions are normally specified of Cauchy type on a hypersurface in the propagation space. In this case Cauchy conditions require the specification of $\boldsymbol{k}$ and A on the hypersurface. The orientation of the hypersurface is characterized by its slowness vector $\boldsymbol{s}_{\mathrm{b}}$, defined so that the velocity $\boldsymbol{v}_b$ of any point constrained to lie on the hypersurface satisfies the condition $\boldsymbol{v}_{\mathrm{b}} \cdot \boldsymbol{s}_{\mathrm{b}} = 1$. The standard hypersurface $t =$ const. for initial conditions is characterized by $\boldsymbol{s}_{\mathrm{b}} = 0$.

We consider the boundary problem to be well-posed if it is *causal* and if the hypersurface is *duly inclined* in the classic sense of Hadamard. By causal is meant that the solution at a point in the propagation space must be determinable from boundary conditions on a portion of the hypersurface which is all at earlier times than the time at the solution point. Imposing such a causal condition again treats t as a distinctive variable different in nature from the components of $\boldsymbol{x}$ in the propagation space.

The existence of well-posed Cauchy boundary problems for (4.11) or the system (4.1), (4.2) depends upon the signature of the quadratic form $v^2 - \Omega_A \mathscr{H}_{kk} : \boldsymbol{nn}$ in the vector $(\boldsymbol{n}, v)$. We define the signature as an integer triplet, giving first the number of positive eigenvalues of the matrix of the form, secondly the number of negative eigenvalues, and thirdly the number of zero eigenvalues. The sum of the three integers equals the number of dimensions of the propagation space, considered here to be 2, 3, or 4. Omitting the degenerate cases in which one or more eigenvalues are zero, there are four cases to consider:

(1) Signature $(1,1,0)$, $(1,2,0)$, or $(1,3,0)$. In this case the form $\Omega_A \mathscr{H}_{kk} : \boldsymbol{k}_1 \boldsymbol{k}_1$ is positive-definite. Duly-inclined surfaces exist, defined by the condition

$$1 - \Omega_A \mathscr{H}_{kk} : \boldsymbol{s}_{\mathrm{b}} \boldsymbol{s}_{\mathrm{b}} > 0. \tag{4.12}$$

Among the duly-inclined surfaces, causal ones exist. Thus, well-posed problems exist in this case, and include initial-value problems.

(2) Signature $(2,2,0)$. Duly-inclined surfaces do not exist.

(3) Signature $(1,1,0)$, $(2,1,0)$, or $(3,1,0)$. Duly-inclined surfaces defined by the condition $1 - \Omega_A \mathscr{H}_{kk} : \boldsymbol{s}_{\mathrm{b}} \boldsymbol{s}_{\mathrm{b}} < 0$ exist, but none are causal.

(4) Signature (2, 0, 0,), (3, 0, 0), or (4, 0, 0). The basic system is elliptic, and no surface is duly-inclined. In this case the form $\Omega_A \mathscr{H}_{kk}:\boldsymbol{nn}$ is negative-definite.

Only in case 1, with the tensor $\Omega_A \mathscr{H}_{kk}$ a positive-definite one, do well-posed problems exist in the sense demanded. In this sense the shape of the duly inclined boundary and the values of $\boldsymbol{k}$ and A on the boundary may be quite arbitrary and, in particular, may be described by non-analytic functions.

In another sense, properly posed Cauchy problems exist for elliptic systems even though the problems are what are termed ill-posed. In our case (4), if the boundary hypersurface is smooth in the sense that it is described by analytic functions, and if the values of $\boldsymbol{k}$ and A on the hypersurface are given by analytic functions, the Cauchy–Kowalewski theorem ensures that the problem is properly posed in some neighbourhood of the boundary. As long as some portion of the boundary is at earlier time from that of a solution point, the principle of analytic continuation ensures that the causal requirement is met.

If the propagation space has more than two dimensions, cases (2) and (3) can appear. These cases are intermediate between the well-posed case (1) and the elliptic case (4). The boundary specifications may be expected to permit a restricted type of non-analyticity but to require analyticity in a restricted sense. If the boundary is a non-duly-inclined hyperplane the linear problem can be resolved through Fourier expansion of $\boldsymbol{k}$ and A in the hyperplane. Further exploration of these cases would be of interest.

Where physical considerations dictate Cauchy boundary conditions that are ill-posed, the appropriate physical interpretation is that the solution is unstable. For a direction $\boldsymbol{n}$ for which (4.7) has no real solution, an F which is sinusoidal in θ_1, and a given k_1, equation (4.7) gives an imaginary ω_1 whose magnitude is the exponential time rate of growth of the perturbation. A corresponding sinusoidal distribution of perturbation boundary values leads in a linear problem to an unstable perturbation solution. Thus, 'ill-posed' is in this sense to be considered equivalent to 'unstable'. Note that the exponential growth is determined relative to an observer moving with the basic group velocity.

In physical wave problems, two types of boundary conditions are most frequently considered and can be treated as standard. One is that of initial conditions on a boundary $t = \text{const.}$, with $\boldsymbol{s}_\text{b} = 0$. The other type is that of a wave-maker fixed in one galilean frame and located on a surface in the $\boldsymbol{x}$ space with orientation determined by a normal unit vector $\boldsymbol{n}_\text{b}$. In this galilean frame, the basic group velocity $\boldsymbol{c}$ is the velocity of the frame in which (4.1) and (4.2) are valid. The slowness vector of the boundary in the second frame is $\boldsymbol{s}_\text{b} = \boldsymbol{n}_\text{b}/\boldsymbol{n}_\text{b}\cdot\boldsymbol{c}$, and condition (4.12) for case (1) may be expressed

$$(\boldsymbol{n}_\text{b}\cdot\boldsymbol{c})^2 > \Omega_A \mathscr{H}_{kk}:\boldsymbol{n}_\text{b}\boldsymbol{n}_\text{b}. \tag{4.13}$$

This condition must be met by the wave-maker in case (1) in order for the boundary problem to be well-posed.

The inclusion of the inhomogeneous right-hand terms arising from non-uniformity of the medium do not change the general conclusions of this section. In the cases requiring analyticity of certain functions in order for a boundary problem to be properly posed a requirement of analyticity may be expected of the inhomogeneous terms.

Isotropic media

An important special case is that in which the medium is isotropic, with dependence of $\mathscr{H}$ or Ω on $\boldsymbol{k}$ replaced by dependence upon $k = (\boldsymbol{k}\cdot\boldsymbol{k})^{\frac{1}{2}}$. Convenient in this case is the concept of the dispersion exponent.

In the kinematic theory of linear wave propagation the dispersion exponent is $\delta = \boldsymbol{k}\cdot\Omega_k/\Omega$ (termed β in Hayes 1970*b*), and measures the ratio of the component of the group velocity normal to wave fronts to the phase velocity. The appropriate analogue for nonlinear waves is

$$\delta = \frac{\boldsymbol{k}\cdot\mathscr{H}_k}{\mathscr{H}}. \tag{4.14}$$

For isotropic media $\mathscr{H}(\boldsymbol{k}) = \mathscr{H}'(k)$, and

$$\mathscr{H}_k = \frac{\boldsymbol{k}\mathscr{H}'_k}{k} = \frac{\delta(k)\,\boldsymbol{k}\mathscr{H}'}{k^2}, \tag{4.15}$$

with the dependence upon A understood.

The second derivative may be expressed

$$\mathscr{H}_{kk} = \frac{\mathscr{H}}{k^2}\left(\delta\boldsymbol{I} + (\alpha-\delta)\frac{\boldsymbol{kk}}{k^2}\right), \tag{4.16}$$

where
$$\alpha = k\delta_k + \delta^2 - \delta = \frac{k^2\mathscr{H}'_{kk}}{\mathscr{H}'} \tag{4.17}$$

and $\boldsymbol{I}$ is the idemfactor or unit tensor.

The eigenvalues of the tensor $\Omega_A\mathscr{H}_{kk}$ are then $\Omega_A\mathscr{H}k^{-2}(\alpha,\delta,\delta)$ in a propagation space of $3+1$ dimensions. With the exception of a few anomalous cases, the action flow for an isotropic medium is in the same direction as that of the phase velocity, and the parameter $\delta > 0$. The conditions that a problem be well-posed is then $\Omega_A > 0$, $\alpha > 0$.

5. Potential variables

With a single potential variable present there can be a variable $\gamma(\boldsymbol{x},t)$ termed a pseudo frequency and a variable $\boldsymbol{\beta}(\boldsymbol{x},t)$ termed a pseudo wavenumber which survive the averaging over phase and appear in the augmented argument space of $\mathscr{L}$. These variables obey a consistency relation

$$\partial\boldsymbol{\beta}/\partial t + \nabla\gamma = 0, \tag{5.1}$$

analogous to (2.1). Also, the tensor $\nabla\boldsymbol{\beta}$ is symmetric. (Here γ has its sign reversed from that in Hayes (1970*a*).)

A variable b appears as a constant of integration and appears in the argument of $\mathscr{L}$ as a pseudo amplitude. As with the amplitude a the relation $\mathscr{L}_b = 0$ holds as a pseudo dispersion relation. The analogy is completed in Whitham's theory with a law

$$\partial P/\partial t + \nabla \cdot \boldsymbol{Q} = 0 \tag{5.2}$$

for conservation of a pseudo action, where

$$P = \mathscr{L}_\gamma, \tag{5.3}$$

$$\boldsymbol{Q} = -\mathscr{L}_{\boldsymbol{\beta}}, \tag{5.4}$$

are pseudo action density and flux.

Our procedure again follows by analogy. The variable P is used as pseudo amplitude as a choice of b, and (5.3) is taken over the entire augmented space. We set

$$\mathscr{L}(\boldsymbol{k}, \omega, A; \boldsymbol{\beta}, \gamma, P) = A\omega + P\gamma - \mathscr{H}(\boldsymbol{k}, A, \boldsymbol{\beta}, P) \tag{5.5}$$

in analogy with (2.6). Dependence upon $\boldsymbol{x}, t$ from non-uniformity of the medium is dropped from the notation. The pseudo dispersion relation takes the form

$$\gamma(\boldsymbol{x}, t) = \Gamma(\boldsymbol{k}, A, \boldsymbol{\beta}, P) = \mathscr{H}_P, \tag{5.6}$$

and the pseudo action flux the form

$$\boldsymbol{Q} = \mathscr{H}_{\boldsymbol{\beta}}. \tag{5.7}$$

The equations of § 2 remain valid in this case. The second derivatives of $\mathscr{H}$ include a pseudo group velocity $\Gamma_{\boldsymbol{\beta}} = \mathscr{H}_{P\boldsymbol{\beta}}$. Identities based upon other second derivatives include $\Gamma_A = \Omega_P$, $\boldsymbol{B}_P = \Gamma_{\boldsymbol{k}}$, and $\Omega_{\boldsymbol{\beta}} = \boldsymbol{Q}_A$. Again, the algorithm described in § 2 for obtaining $\mathscr{H}$ from an $\mathscr{L}$ not of form (2.6) applies here with an obvious extension. A galilean transformation in this case brings in the additional term $P\boldsymbol{\beta}\cdot \boldsymbol{U}$ in $\mathscr{H}$.

Unless it can be determined in a particular case that $P = 0$ characterizes a natural rest state of the system when $A = 0$, the extension of the concept of laminar structure to encompass a structure in P would not seem to be fruitful. The concept of a laminar structure in A applies here as before, with $\boldsymbol{\beta}$ and P considered as parameters. The condition $\mathscr{H} = 0$ at $A = 0$ is not necessary.

Propagation theory

The propagation theory in this case comes from the set of equations obtained by writing (2.1), (2.2), (5.1) and (5.2) in the augmented space. The result is

$$\frac{\partial}{\partial t}\begin{pmatrix} \boldsymbol{k} \\ A \\ \boldsymbol{\beta} \\ P \end{pmatrix} + \begin{pmatrix} \boldsymbol{C}\cdot & \Omega_A & \Omega_{\boldsymbol{\beta}}\cdot & \Omega_P \\ \mathscr{H}_{\boldsymbol{kk}}: & \boldsymbol{C}\cdot & \mathscr{H}_{\boldsymbol{\beta k}}: & \Gamma_{\boldsymbol{k}}\cdot \\ \Gamma_{\boldsymbol{k}}\cdot & \Omega_P & \Gamma_{\boldsymbol{\beta}}\cdot & \Gamma_P \\ \mathscr{H}_{\boldsymbol{\beta k}}: & \Omega_{\boldsymbol{\beta}}\cdot & \mathscr{H}_{\boldsymbol{\beta\beta}}: & \Gamma_{\boldsymbol{\beta}}\cdot \end{pmatrix} \begin{pmatrix} \nabla \boldsymbol{k} \\ \nabla A \\ \nabla \boldsymbol{\beta} \\ \nabla P \end{pmatrix} = -\begin{pmatrix} \Omega_{\boldsymbol{x}} \\ \operatorname{tr} \mathscr{H}_{\boldsymbol{kx}} \\ \Gamma_{\boldsymbol{x}} \\ \operatorname{tr} \mathscr{H}_{\boldsymbol{\beta x}} \end{pmatrix}, \tag{5.8}$$

the right-hand side appearing from the influence of non-uniformity of the medium. The pseudo solution is uncoupled only if $\mathscr{H}_{\boldsymbol{\beta k}}$, $\Omega_{\boldsymbol{\beta}}$, $\Gamma_{\boldsymbol{k}}$ and Ω_P are all zero.

For the theory for linearized modulation waves we consider (5.8) with the right-hand side null and with the coefficient matrix constant, without making the galilean transformation of (4.1) and (4.2). In addition to (4.6) we assume perturbation of the form

$$\boldsymbol{\beta} = \boldsymbol{\beta}_0 + \epsilon k_1 \beta_1 \boldsymbol{n} F(\theta_1), \tag{5.9a}$$

$$\gamma = \gamma_0 + \epsilon k_1 \gamma_1 F(\theta_1), \tag{5.9b}$$

$$P = P_0 + \epsilon k_1 P_1 F(\theta_1). \tag{5.9c}$$

The fact that $\nabla\boldsymbol{\beta}$ must be a symmetric tensor is responsible for the requirement that the perturbation of $\boldsymbol{\beta}$ be in the direction $\boldsymbol{n}$.

Substitution of (4.6) and (5.9) into the linear homogeneous form of (5.8) leads to the eigenvalue problem

$$\begin{pmatrix} \boldsymbol{C}\cdot\boldsymbol{n}-v & \Omega_A & \Omega_\beta\cdot\boldsymbol{n} & \Omega_P \\ \mathscr{H}_{kk}:\boldsymbol{nn} & \boldsymbol{C}\cdot\boldsymbol{n}-v & \mathscr{H}_{\beta k}:\boldsymbol{nn} & \Gamma_k\cdot\boldsymbol{n} \\ \Gamma_k\cdot\boldsymbol{n} & \Omega_P & \Gamma_\beta\cdot\boldsymbol{n}-v & \Gamma_P \\ \mathscr{H}_{\beta k}:\boldsymbol{nn} & \Omega_\beta\cdot\boldsymbol{n} & \mathscr{H}_{\beta\beta}:\boldsymbol{nn} & \Gamma_\beta\cdot\boldsymbol{n}-v \end{pmatrix} \begin{pmatrix} 1 \\ A_1 \\ \beta_1 \\ P_1 \end{pmatrix} = 0. \tag{5.10}$$

The determinant of the square matrix in (5.10) is a form $\Delta(\boldsymbol{n}, v)$ which is quartic in $\boldsymbol{n}$ and v. For a given $\boldsymbol{n}$, eigenvalues of v are solutions to the quartic equation $\Delta = 0$ and the corresponding modes are the eigenvectors given by (5.10). The perturbation γ_1 is given by

$$\gamma_1 = (\Gamma_k\cdot\boldsymbol{n}\,\Gamma_A\;\Gamma_\beta\cdot\boldsymbol{n}\,\Gamma_P) \begin{pmatrix} 1 \\ A_1 \\ \beta_1 \\ P_1 \end{pmatrix}. \tag{5.11}$$

Real eigenvalues correspond to stable modes for propagating modulation waves, and pairs of complex eigenvalues correspond to unstable modes (with their damped conjugate modes).

For a Cauchy boundary problem to be well-posed in the sense of § 4 it is necessary that no unstable modes exist, that $\Delta = 0$ has four real roots for v each choice of $\boldsymbol{n}$. Cauchy boundary conditions consist of the specification of $\{\boldsymbol{k}, A, \boldsymbol{\beta}, P\}$ on a boundary surface. The boundary surface must be duly-inclined, which requires in this case that $\Delta(s_b, 1) > 0$; $\Delta(0,1) = 1$, and initial-value boundaries are duly-inclined. Again, ill-posed problems may be properly posed with appropriate restrictions on analyticity imposed on the boundary problem.

Isotropic media

With the medium isotropic no great simplification appears. The Hamiltonian can be expressed in the form $\mathscr{H}(\boldsymbol{k}, \boldsymbol{\beta}) = \mathscr{H}'(k, \beta, \Lambda)$, where $\Lambda = \boldsymbol{k}\cdot\boldsymbol{\beta} = k\beta\lambda$ and λ

is the direction cosine between $\boldsymbol{k}$ and $\boldsymbol{\beta}$. To free the determinant from dependence upon the directions of $\boldsymbol{k}$ and $\boldsymbol{\beta}$ we can use the two direction cosines

$$\mu = \boldsymbol{n}\cdot\boldsymbol{k}/k = \cos\psi, \tag{5.12a}$$

$$\nu = \boldsymbol{n}\cdot\boldsymbol{\beta}/\beta. \tag{5.12b}$$

Then we can express

$$\boldsymbol{C}\cdot\boldsymbol{n} = \mu\Omega'_k + \nu\beta\Omega'_A, \tag{5.13a}$$

$$\Omega_{\boldsymbol{\beta}}\cdot\boldsymbol{n} = \nu\Omega'_\beta + \mu k\Omega'_A, \tag{5.13b}$$

$$\Gamma_{\boldsymbol{k}}\cdot\boldsymbol{n} = \mu\Gamma'_k + \nu\beta\Gamma'_A, \tag{5.13c}$$

$$\Gamma_{\boldsymbol{\beta}}\cdot\boldsymbol{n} = \nu\Gamma'_\beta + \mu k\Gamma'_A, \tag{5.13d}$$

$$\mathscr{H}_{\boldsymbol{kk}}:\boldsymbol{nn} = (1-\mu^2)\,k^{-1}\mathscr{H}'_k + \mu^2\mathscr{H}'_{kk} + 2\mu\nu\beta\mathscr{H}'_{kA} + \nu^2\beta^2\mathscr{H}'_{AA}, \tag{5.13e}$$

$$\mathscr{H}_{\boldsymbol{k\beta}}:\boldsymbol{nn} = \mu\nu(\mathscr{H}'_{k\beta} + k\beta\mathscr{H}'_{AA}) + \mathscr{H}'_A + \mu^2 k\mathscr{H}'_{kA} + \nu^2\beta\mathscr{H}'_{\beta A}, \tag{5.13f}$$

$$\mathscr{H}_{\boldsymbol{\beta\beta}}:\boldsymbol{nn} = (1-\nu^2)\,\beta^{-1}\mathscr{H}'_\beta + \nu^2\mathscr{H}'_{\beta\beta} + 2\mu\nu k\mathscr{H}'_{\beta A} + \mu^2k^2\mathscr{H}'_{AA}. \tag{5.13g}$$

Using (5.13) the equation $\Delta(\boldsymbol{n}, v) - 0$ is a quartic in v with coefficients whose dependence upon $\boldsymbol{k}$, $\boldsymbol{\beta}$ and $\boldsymbol{n}$ is replaced by a dependence upon k, β, A, μ and ν.

6. Water waves

Whitham (1967*a*) has given a detailed theory for water waves with weak non-linearity. The theory applies for all values of the depth parameter kh except for the small values $kh \ll 1$ of shallow-water waves. He expresses an averaged Lagrangian $\mathscr{L}$ in his equation (25). Here we change his term $\frac{1}{2}gb^2$ to $\frac{1}{2}gh^2$ (to simplify notation), change the sign of $\mathscr{L}$ (to make A positive), extend the expression to a propagation space of $2+1$ dimensions by replacing βk by $\boldsymbol{\beta}\cdot\boldsymbol{k}$, and apply the algorithm for finding $\mathscr{H}$. We note that $P = h$ in this case, and so use h for P. We drop the parameter h_0 and use h consistently, even though h is not a constant.

The result for the Hamiltonian is

$$\mathscr{H}(\boldsymbol{k}, A, \boldsymbol{\beta}, h) = \tfrac{1}{2}\beta^2 h + \tfrac{1}{2}gh^2 + A\boldsymbol{\beta}\cdot\boldsymbol{k} + A\omega_0(k, h) + \tfrac{1}{2}k^3DA^2 + O(A^3), \tag{6.1}$$

where

$$\omega_0 = (gk\tanh kh)^{\frac{1}{2}}, \tag{6.2}$$

$$D = (9T^4 - 10T^2 + 9)/8T^3, \tag{6.3}$$

and $T = \tanh kh$. We calculate

$$\Omega = \boldsymbol{\beta}\cdot\boldsymbol{k} + \omega_0(k, h) + k^3DA + O(A^2), \tag{6.4a}$$

$$\Gamma = \tfrac{1}{2}\beta^2 + gh + kBA/h + O(A^2), \tag{6.4b}$$

$$\boldsymbol{C} = \boldsymbol{\beta} + \boldsymbol{C}_0 + (k^3D)_k\,k^{-1}\boldsymbol{k}A + O(A^2), \tag{6.4c}$$

$$\Omega_A = k^3D + O(A), \tag{6.4d}$$

$$\Omega_h = kB/h + O(A), \tag{6.4e}$$

$$\Omega_{\boldsymbol{\beta}} = \boldsymbol{k} + O(A^2), \tag{6.4f}$$

$$\Gamma_h = g + O(A), \tag{6.4g}$$

$$\Gamma_\beta = \boldsymbol{\beta} + O(A^2), \tag{6.4h}$$

$$\mathscr{H}_{\beta\beta} = h\boldsymbol{I} + O(A^3), \tag{6.4i}$$

$$\mathscr{H}_{\beta k} = A\boldsymbol{I} + O(A^3), \tag{6.4j}$$

$$\mathscr{H}_{kk} = -[\tfrac{3}{4}k^{-2}\omega_0 + \tfrac{1}{2}h(T^{-1}+3T)B]\,k^{-2}\boldsymbol{kk}A + (\tfrac{1}{2}k^{-2}\omega_0 + k^{-1}B)\,\boldsymbol{I}A + O(A^2), \tag{6.4k}$$

$$\Gamma_k = O(A), \tag{6.4l}$$

where
$$B = \frac{ghk}{2\omega_0\cosh^2 kh} = \tfrac{1}{2}hg^{\frac{1}{2}}k^{\frac{1}{2}}(T^{-\frac{1}{2}} - T^{\frac{3}{2}}), \tag{6.5}$$

$$\boldsymbol{C}_0 = (\tfrac{1}{2}k^{-1}\omega_0 + B)\,k^{-1}\boldsymbol{k}. \tag{6.6}$$

The determinant $\Delta(\boldsymbol{n}, v)$ is to be calculated to the lowest effective order in A. We set $\mu = \boldsymbol{n}\cdot\boldsymbol{k}/k = \cos\psi$ for the principal perturbation cosine, and let $\boldsymbol{C}_0\cdot\boldsymbol{n} = \mu C_0$ and $\boldsymbol{k}\cdot\boldsymbol{n} = \mu k$. The determinantal equation takes the form

$$\Delta = \begin{vmatrix} \mu C_0 + \boldsymbol{\beta}\cdot\boldsymbol{n} - v & \Omega_A & \mu k & \Omega_h \\ \mathscr{H}_{kk}:\boldsymbol{nn} & \mu C_0 + \boldsymbol{\beta}\cdot\boldsymbol{n} - v & A & O(A) \\ O(A) & \Omega_h & \boldsymbol{\beta}\cdot\boldsymbol{n} - v & g \\ A & \mu k & h & \boldsymbol{\beta}\cdot\boldsymbol{n} - v \end{vmatrix} = 0. \tag{6.7}$$

Two of the roots are close to $\pm(gh)^{\frac{1}{2}}$ and correspond to long-period waves affecting the mean depth h. These are discarded as corresponding to a perturbation mode which is not of interest.

For the other two roots we replace each of the last two diagonal terms by $-\mu C_0$ and drop the terms in A and $O(A)$. The equation can then be directly solved, and

$$(v - \boldsymbol{\beta}\cdot\boldsymbol{n} - \mu C_0)^2 = \Omega'_A \mathscr{H}_{kk}:\boldsymbol{nn}, \tag{6.8}$$

where Ω'_A is an effective hardness parameter given by

$$\Omega'_A = \Omega_A - \frac{\mu^2k^2g + 2\mu^2kC_0\Omega_h + h\Omega_h^2}{gh - \mu^2C_0^2}. \tag{6.9}$$

When we use the value for Ω_h calculated in (6.4*e*) this becomes

$$\Omega'_A = \Omega_A - \left(\frac{k^2}{h}\right)\frac{\mu^2(gh + 2C_0B) + B^2}{gh - \mu^2C_0^2}, \tag{6.10}$$

in a form in which the correspondence at $\mu = 1$ with Whitham's equation (57) is evident.

The other factor on the right-hand side of (6.8) is

$$\mathscr{H}_{kk}:\boldsymbol{nn} = [-\tfrac{1}{4}k^{-2}\omega_0 + \tfrac{1}{2}(2k^{-1} - hT^{-1} - 3hT)\,B]\,\mu^2A + [\tfrac{1}{2}k^{-2}\omega_0 + k^{-1}B]\,(1-\mu^2)\,A. \tag{6.11}$$

The coefficient of $(1-\mu^2)A$, equivalent to $k^{-2}\omega_0$ times the dispersion parameter δ of (4.14), is always positive. The coefficient of μ^2A in (6.11), equivalent to $k^{-2}\omega_0$ times the parameter α of (4.17), is always negative but approaches zero as kh approaches 0. The form (6.11) is always indefinite in μ.

Whether the mode corresponding to a direction $\boldsymbol{n}$ is stable or not depends upon the sign of the product of two quadratic forms in μ. One of these is the dispersion form $\mathscr{H}_{kk}:\boldsymbol{nn}$, and the other is $(gh-\mu^2C_0^2)\,\Omega'_A$. Calculation yields a stability plot of $\cos^{-1}\mu = \psi(kh)$ as two curves for neutral stability separating stable and unstable regions (figure 1). The two curves cross at $kh = 0.380$. They are extremely close together from $kh = 0$ to about 0.5, and the theoretically predicted instability (except at the crossing-point) in this range is certainly non-existent practically.

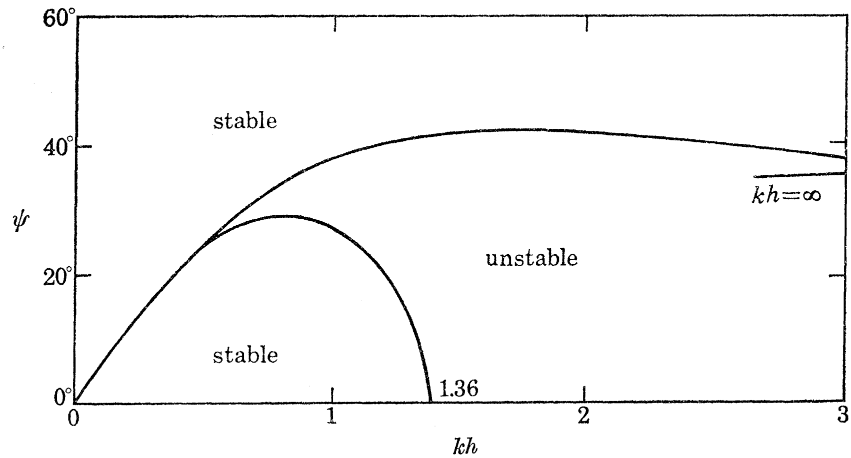

FIGURE 1. Stability plot for water waves.

The hardness neutral curve $\Omega'_A = 0$ intersects the line $\psi = 0$ at $kh = 1.363$, in agreement with the corresponding result of Whitham. The dispersion neutral curve is asymptotic to $\sin^{-1}(1/3^{\frac{1}{2}})$ as $kh \to \infty$.

A separate investigation indicates that the effect of the dropped terms in A and $O(A)$ is stabilizing. If the domain is finite in extent diffraction effects enter, and these are stabilizing also. These observations reinforce the conclusion that the predicted instability for $kh < 0.5$ is non-existent practically.

7. One-dimensional propagation, hyperbolic case

With one-dimensional propagation the propagation space (x, t) is two-dimensional and the wavenumber has but a single component k. We assume there are no potential variables and that $\Omega_A = \mathscr{H}_{AA}$ and $\mathscr{H}_{kk}$ are non-zero. Two cases appear, the hyperbolic case $\Omega_A\mathscr{H}_{kk} > 0$ and the elliptical case $\Omega_A\mathscr{H}_{kk} < 0$. The hyperbolic case has been extensively studied, primarily in the papers cited of Whitham and Lighthill. Here the theory will be briefly summarized, with particular attention to the strongly nonlinear case in our Hamiltonian formulation. For most purposes we restrict ourselves to the case of a uniform medium, in which $\mathscr{H}$ has no explicit dependence upon x or t.

With one-dimensional propagation the propagation equations (3.1) and (3.2) become

$$k_t + \Omega_k k_x + \Omega_A A_x = -\Omega_x, \tag{7.1a}$$

$$A_t + \Omega_k A_x + \mathscr{H}_{kk} k_x = -\mathscr{H}_{kx}. \tag{7.1b}$$

In a uniform medium the right-hand sides vanish. In this case the equations for $x(k, A)$ and $t(k, A)$ obtained by a transformation of hodograph type are linear, and may be expressed

$$(x - \Omega_k t)_A + (\Omega_A t)_k = 0, \tag{7.2a}$$

$$(x - \Omega_k t)_k + (\mathscr{H}_{kk} t)_A = 0. \tag{7.2b}$$

A 'potential' $\phi(k, A)$ and a 'streamfunction' $\Phi(k, A)$ may be defined by

$$\phi_k = x - \Omega_k t, \quad \phi_A = -\Omega_A t, \tag{7.3}$$

$$\Phi_A = x - \Omega_k t, \quad \Phi_k = -\mathscr{H}_{kk} t. \tag{7.4}$$

These functions satisfy the equations

$$\phi_{kk} - \left(\frac{\mathscr{H}_{kk}}{\Omega_A} \phi_A\right)_A = 0, \tag{7.5}$$

$$\Phi_{AA} - \left(\frac{\Omega_A}{\mathscr{H}_{kk}} \Phi_k\right)_k = 0. \tag{7.6}$$

The superposed solution $\phi = -\Omega t_0$, $\Phi = -\mathscr{H}_k t_0$ corresponds to a time translation by t_0. The superposed solution $\phi = k x_0$, $\Phi = A x_0$ corresponds to a distance translation by x_0. A galilean transformation to a frame with velocity $-U$ is obtained by adding AUk to $\mathscr{H}$.

In the hyperbolic case, with Ω_A and $\mathscr{H}_{kk}$ of the same sign, the method of characteristics may be applied to the equations above. The characteristics for (7.1) are given by

$$\mathrm{d}x/\mathrm{d}t = \Omega_k \pm (\Omega_A \mathscr{H}_{kk})^{\frac{1}{2}} \tag{7.7}$$

in the (x, t) space, with

$$\mathrm{d}A \pm \left(\frac{\mathscr{H}_{kk}}{\Omega_A}\right)^{\frac{1}{2}} \mathrm{d}k = -\left[\mathscr{H}_{kx} \pm \left(\frac{\mathscr{H}_{kk}}{\Omega_A}\right)^{\frac{1}{2}} \Omega_x\right] \mathrm{d}t, \tag{7.8}$$

as the corresponding characteristic relations. For (7.2) in a uniform medium the roles are reversed; the characteristics in the (k, A) space are given by (7.8) with right-hand side zero, while (7.7) gives the characteristic relations for the dependent variables (x, t). Integrals of (7.8) with right-hand side zero give analogues of the classical Riemann invariants.

If the ratio $\mathscr{H}_{kk}/\Omega_A$ may be considered accurately enough represented by a product of functions of k and of A, equations (7.5) and (7.6) are subject to the method of separation of variables; in the hyperbolic case the Riemann invariants are naturally defined through quadratures.

Another convenient approximation may be expressed, for multidimensional propagation, that $\mathscr{H}_{AAk} = \Omega_{Ak} = 0$. This approximation puts $\mathscr{H}$ in the functional form

$$\mathscr{H} = \mathscr{H}_0(\boldsymbol{k}) + A\Omega_0(\boldsymbol{k}) + F(A), \tag{7.9}$$

with $\Omega = \Omega_0 + F'$, $\Omega_k = \Omega_{0,k}$, $\Omega_A = F''$ and $\mathscr{H}_{kk} = \mathscr{H}_{0,kk} + A\Omega_{0,kk}$. If the approximation encompasses the linear limit $A = 0$ the boundary condition $\mathscr{H}(\boldsymbol{k}, 0) = 0$ yields $\mathscr{H}_0 = 0$, $F(0) = 0$ and $F'(0) = 0$. Then the ratio $\mathscr{H}_{kk}/\Omega_A = \Omega_{0,kk}(A/F'')$ and is in the product form in the one-dimensional case. An essential property of this approximation is that the action flux $\boldsymbol{B} = \mathscr{H}_k = A\Omega_k$ is equal to the action density times the basic group velocity. The basic group velocity is thus the action flow velocity in this approximation, with the action following basic rays.

Additional approximating assumptions involve taking finite numbers of terms in Taylor expansions of $\mathscr{H}$. A standard approximation is the *quasi-linear approximation,* in which Ω_A and $\Omega_{0,kk}$ are taken to be constant, and $F = \frac{1}{2}\Omega_A A^2$. This approximation is basic to Lighthill's transformation using an amplitude variable r or s proportional to $A^{\frac{1}{2}}$, in which an axisymmetric wave or Laplace equation is obtained. The elliptic case is discussed in the following section. With this quasilinear approximation the Riemann invariants from (7.8) may be expressed as

$$2A^{\frac{1}{2}} \pm (\Omega_{0,kk}/\Omega_A)^{\frac{1}{2}}\, k.$$

The approximation $\mathscr{H}_{kk} = \text{const.}$, $\Omega_A = \text{const.}$ cannot encompass the linear limit $A = 0$. This approximation may be used if the waves are never weak, and is the approximation underlying § 4 above. In a uniform medium this leads to simple wave equations for either (k, A) as functions of (x, t) or (x, t, ϕ, Φ) as functions of (k, A) in the hyperbolic case. In the elliptic case Laplace equations are obtained. (In the special case $\Omega_A = A\Omega_{AA}$, $\mathscr{H}_{kk} = A\Omega_{kk}$, wave or Laplace equations are also obtained.)

One of Whitham's interesting results was that the characteristics (7.7) of either family may converge to an envelope. Near such points of convergence, of course, the approximations underlying the basic Whitham theory must break down. Speculations have been advanced on the possible existence of an entity analogous to a shock wave, created near such an envelope. This writer's opinion is that the region near such an envelope is a diffraction region, and that the multivalued solutions behind the envelope should be interpreted as superpositions of separate quasilinear wave systems.

The convergence of characteristics of type (7.7) would appear superficially to represent a type of focusing. It is clear that this convergence is completely different in nature to the focusing of the kinematic (or geometric) theory for linear wave propagation. Focusing in the kinematic theory comes from a convergence of the rays representing action flow, and leads within the theory to infinite action density. With nonlinear wave propagation focusing must correspond to convergence of the vector $\boldsymbol{B} = \mathscr{H}_k$, or within the quasilinear approximation to convergence of the basic rays. Convergence of characteristics in the hyperbolic case is not accompanied by convergence of the basic rays.

Consideration of a number of examples indicates that a principal nonlinear effect in the hyperbolic case is the prevention of focusing. Two basic rays which are converging and which would focus in the linear kinematic theory are forced apart by the nonlinear terms. This observation is in accord with the interpretation of nonlinear effects in the hyperbolic case as being stabilizing.

8. One-dimensional propagation, elliptic case

The elliptic case of one-dimensional propagation has been studied extensively by Lighthill. Some unresolved questions remain, and our main purpose in this Section is to discuss some of these. With the exception of the characteristic equations (7.7) and (7.8) the equations of the last section apply equally to the elliptic case.

The basic group velocity in the quasilinear approximation is $\Omega_k = c_0 + \Omega_{kk}(k-k_0)$, while $\mathscr{H}_{kk} = A\Omega_{kk}$. A galilean transformation to a frame moving with velocity c_0 removes the term in c_0 from the equations. Equations (7.1) then become, for a uniform medium,

$$k_t + \Omega_{kk}(k-k_0)\,k_x + \Omega_A A_x = 0, \tag{8.1a}$$

$$A_t + \Omega_{kk}(k-k_0)\,A_x + \Omega_{kk} A k_x = 0. \tag{8.1b}$$

The transformation

$$\tau = -\tfrac{1}{2}\Omega_{kk} t, \quad A = -\frac{\Omega_{kk}}{4\Omega_A} s^2 \tag{8.2}$$

is made, and gives

$$k_\tau - 2(k-k_0)\,k_x + s s_x = 0, \tag{8.3a}$$

$$s_\tau - 2(k-k_0)\,s_x - s k_x = 0. \tag{8.3b}$$

If $\Omega_{kk} < 0$ and $\Omega_A > 0$ the variable τ is a reduced time. If $\Omega_{kk} > 0$ and $\Omega_A < 0$ the variable τ is a reduced negative time. Equations (8.3) are invariant under the transformation

$$(k, s, \tau, x)^{\mathrm{T}} = (fk, \pm fs, gf^{-2}\tau, gf^{-1}x), \tag{8.4}$$

where f and g are arbitrary scale parameters.

An inversion of hodograph type gives the equations

$$x_s + 2(k-k_0)\,\tau_s + s\tau_k = 0, \tag{8.5a}$$

$$x_k + 2(k-k_0)\,\tau_k - s\tau_s = 0. \tag{8.5b}$$

The 'potential' ϕ and 'streamfunction' ψ may be defined by

$$\phi_k = s^{-1}\psi_s = x + 2(k-k_0)\,\tau, \tag{8.6a}$$

$$-s^{-1}\phi_s = s^{-2}\psi_k = \tau. \tag{8.6b}$$

The potential ϕ is the same as the ϕ of (7.3) and (7.5). The streamfunction Φ of (7.4) and (7.6) equals $(-\Omega_{kk}/2\Omega_A)\,\psi$. These functions satisfy

$$\phi_{kk} + \phi_{ss} + s^{-1}\phi_s = 0, \tag{8.7}$$

the axisymmetric Laplace equation, and

$$\psi_{kk} + \psi_{ss} - s^{-1}\psi_s = 0. \tag{8.8}$$

Lighthill (1965) has shown how Garabedian's method of imaginary rays may be used with the Riemann method and the Riemann function for the axisymmetric wave equation to give solutions to (8.7) with Cauchy-type boundary conditions. In the class of problems considered $k = k_0 =$ const. and a (analytic) function $s(x)$ are specified at $\tau = 0$. The function $s(x)$ is inverted to give a function $\phi_k = x(s)$. The condition $\phi_s = 0$ is integrated to yield $\phi = 0$. The initial line $k = k_0$ in the two-dimensional real space (k, s) is considered to be analytically continued to give an initial surface in the four-dimensional complex space $(k + \mathrm{i}\kappa, s + \mathrm{i}\sigma)$ given by $k = k_0$, $\kappa = 0$, and the initial values $\phi_k = x(s)$ and $\phi = 0$ are analytically continued over this initial surface. The intersections of imaginary characteristics from arbitrary points in the real (k, s) space with the initial surface are determined, and the value of ϕ there expressed as a Riemann integral between these intersection points on a path in the initial surface.

In case k is not equal to a constant at $t = 0$ the procedure of Lighthill may be carried out, but is considerably more involved. The following additional non-trivial calculations are required: (*a*) the values of $\phi = \int \phi_k \,\mathrm{d}k$ on the real initial line $k(s) = k(x(s))$ must be obtained; (*b*) the initial line $k(s)$ must be analytically continued to find the now non-trivial initial surface; (*c*) the functions ϕ as well as ϕ_k must be analytically continued over the initial surface; (*d*) the now non-trivial intersection points of the imaginary characteristics with the initial surface must be found; and (*e*) the Riemann integral now includes an integral using derivatives of the Riemann function as well as terms from the intersection points (since ϕ is now non-zero). Thus having k non-constant at $t = 0$ involves no difference in principle but a tremendous difference in practice.

Equation (8.7) may be investigated by standard means, of which one is that of harmonic Legendre solutions. These solutions are of the form

$$\phi = R^n[C_n P_n(\mu) + D_n Q_n(\mu)] + R^{-n-1}[C_{-n-1} P_n(\mu) + D_{-n-1} Q_n(\mu)], \tag{8.9}$$

$$\frac{\psi}{1-\mu^2} = \frac{R^{n+1}}{n+1}[C_n P_n' + D_n Q_n'] - \frac{R^{-n}}{n}[C_{-n-1} P_n' + D_{-n-1} Q_n'], \tag{8.10}$$

where
$$R = [s^2 + (k-k_0)^2]^{\frac{1}{2}}, \quad \mu = k/R. \tag{8.11}$$

A superposed solution for ϕ proportional to RP_1 corresponds to a distance translation; a superposed solution proportional to R^2P_2 corresponds to a time translation.

A simple solution representing a focusing is given by

$$\phi = -CR^{-1}P_0 = -C[s^2 + (k-k_0)^2]^{-\frac{1}{2}}.$$

In this solution $-\tau = C[s^2 + (k-k_0)^2]^{-\frac{3}{2}}$, and, within the range $x^2 \leqslant C^2(-\tau/A)^{\frac{4}{3}}$, is given by

$$k - k_0 = \frac{x}{-3\tau}, \tag{8.12a}$$

$$s = \left[\left(\frac{C}{-\tau}\right)^{\frac{2}{3}} - \left(\frac{x}{-3\tau}\right)^2\right]^{\frac{1}{2}}. \tag{8.12b}$$

This solution is expressed for negative τ; the corresponding solution for positive τ is obtained by changing the sign of A. In this case ψ is not given by (8.10).

In this simple example, the initial conditions in terms of k and s as functions of x at some negative time would lead to focusing in the kinematic theory, with straight rays in (x, t) space and with no nonlinear effects. In this elliptic quasilinear solution the basic rays are not straight, and focusing occurs in two-thirds of the time which would be needed without nonlinear effects. The effect of the nonlinear terms is to pull the basic rays together to give earlier focusing.

Lighthill (1965) has given an interesting example of the history of a wave packet starting with an amplitude modulation but with zero wavenumber modulation. In his solution a singular point is reached at $k = k_0$, $s = s_1$, $x = 0$, $t = t_{\max}$, and difficulties appear in continuing the solution for $t > t_{\max}$. This singularity is not a focus; the basic rays remain separate and do not converge and, correspondingly, the intensity does not become infinite. We suggest a singularity of the type of that in Lighthill's solution be termed a *pinch*.

To investigate singularities of the pinch type on a simple basis, we drop the $s^{-1}\phi_s$ term in (8.7) and consider the time scale shifted so that $t_{\max} = 0$. A solution is assumed of the form (locally, of course)

$$\phi = -\{B/(m+1)\}\,\mathrm{Re}\,[(s_1 - s) + i(k - k_0)]^{m+1}. \tag{8.13}$$

We obtain

$$\phi_s + i\phi_k = -s_1\tau + i[x + 2(k - k_0)\tau] = B[(s_1 - s) + i(k - k_0)]^m. \tag{8.14}$$

Pinch-type solutions are obtained for $m > 1$, with Lighthill's case appearing with $m = 2$. In the $m = 2$ case we have

$$-\tau = (B/s_1)\,[(s_1 - s)^2 - (k - k_0)^2], \tag{8.15}$$

$$x + 2(k - k_0)\,\tau = 2B(k - k_0)\,(s_1 - s). \tag{8.16}$$

The solutions for s and k as functions of x and t or τ are multi-valued, double-valued if $m = 2$, triply-valued if $m = 3$, and infinitely-valued if m is irrational. Besides the solution of physical interest there are thus other mathematical solutions which do not exist physically, which are *virtual solutions*. The pinch singularity is a branch point in the (x, t) space, and each physical or virtual solution may be considered to lie in one Riemann sheet.

The nature of the continuation of a solution past a pinch can be discussed in these terms. Each solution, physical or virtual, is split into right-hand and left-hand parts by the pinch singularity. For $\tau > 0$ or $t > t_{\max}$ the right-hand side of the physical solution is joined with the left-hand side of the appropriate virtual solution, and conversely. The total solution consists of two superposed solutions. Each of the two component solutions is analytically incomplete, being part physical and part virtual. The zone covered by the virtual part of each component solution is a shadow zone, whose boundary is a basic ray passing through the singular point.

The basic ray which enters the singular point bifurcates there into two. For

$t > t_{\max}$ that corresponding to the right-hand physical solution is deflected to the left, and conversely. The cusp-shaped (in x, t space) region between the two branches of this 'central' basic ray is covered by the physical parts of both component solutions. Action or energy calculated taking into account the double coverage by superposed solutions in this region is conserved. This observation resolves the problem posed by Lighthill as to the apparent disappearance of part of the energy in his solution.

To calculate the central basic ray associated with the right-hand physical solution for $\tau > 0$, $t > t_{\max}$, to lowest order, the approximation $s = s_1$ in (8.15) is adequate. This gives $k - k_0 = (s_1/B)^{\frac{1}{2}}\tau^{\frac{1}{2}}$. The differential equation for the basic ray is $\mathrm{d}x/\mathrm{d}\tau = -2(k-k_0)$ and integrates to

$$x = -\tfrac{4}{3}(s_1/B)^{\frac{1}{2}}\tau^{\frac{3}{2}} = -\tfrac{4}{3}(k-k_0)\,\tau. \tag{8.17}$$

The central basic ray associated with the left-hand physical solution is the same with sign reversed. For other values of x we can expect analogous formulas with x proportional to $\tau^{(m+1)/m}$.

The solution hereby obtained cannot be valid near the pinch or near the branched central basic rays past the pinch. Near the pinch singularity diffraction effects predominate, while each central basic ray should bear a zone of Fresnel-type diffraction. There should be nonlinear interaction between the component solutions near the pinch where they are nearly in phase. The correct solution near the pinch and near the central basic rays can only be obtained through a consistent quasilinear theory encompassing diffraction. However, the picture of superposed component solutions in the cusped overlap region far from its boundaries may be expected to be sound.

The solution (8.12) of type $R^{-1}P_0$ and Lighthill's solution show that both focuses and pinches can occur in elliptic problems. In general, a principal nonlinear effect in the elliptic case is a tendency to cause earlier focusing. Two basic rays which are converging and which would focus in the linear kinematic theory are usually pulled together by the nonlinear terms. This observation is in accord with the interpretation of nonlinear effects in the elliptic case as being destabilizing. Whether a pinch occurs and its nature if it does depends upon the subtleties of the analytic continuation of initial conditions. Lighthill's example is one in which there would be no focusing in the linear kinematic theory.

9. Note on diffraction effects

A treatment of diffraction effects is outside the scope of this paper. Here we present some remarks on the influence such effects may have on the conclusions of this paper. In this context diffraction effects may be interpreted as arising from the necessity of including derivatives of A and perhaps also of $\boldsymbol{k}$ in the argument space of the functions Ω and $\boldsymbol{B}$ of (2.7) and (2.10), with no alteration in the basic relations (2.1) and (2.2). Within the quasilinear approximation this additional dependence may be made specific.

A number of investigators, primarily in plasma physics or fluid mechanics, have arrived at a governing equation termed usually a nonlinear Schroedinger equation. Interpreted in the notation of this paper this equation is one for a complex variable $\psi = a\,e^{i\theta}$ for wave propagation in a uniform medium, with amplitude $|\psi| = a = A^{\frac{1}{2}}$ and with θ a perturbation phase. The equation is

$$i(\psi_t + \boldsymbol{c}_0 \cdot \nabla\psi) + \tfrac{1}{2}\Omega_{kk} : \nabla\nabla\psi = \Omega_A |\psi|^2 \psi. \tag{9.1}$$

This equation yields two equations for a and θ. One may be put in the form

$$(a^2)_t + \nabla \cdot [(\boldsymbol{c}_0 + \Omega_{kk} \cdot \nabla\theta)\, a^2] = 0, \tag{9.2}$$

which may be recognized as (2.2) with $a^2 = A$ and $\boldsymbol{B} = A\boldsymbol{c}$. The other equation, with $-\theta_t$ identified as a *local* perturbation frequency $\omega - \omega_0$ and $\nabla\theta$ as a *local* perturbation wavenumber $(\boldsymbol{k} - \boldsymbol{k}_0)$, may be expressed

$$\omega - \omega_0 = \boldsymbol{c}_0 \cdot (\boldsymbol{k} - \boldsymbol{k}_0) + \tfrac{1}{2}\Omega_{kk} : (\boldsymbol{k} - \boldsymbol{k}_0)^2 + \Omega_A a^2 - \tfrac{1}{2} a^{-1} \Omega_{kk} : \nabla\nabla a. \tag{9.3}$$

The last term on the right-hand side of (9.3) appears in addition to the terms expected from a Taylor expansion of the dispersion relation (2.7). Equations (9.2) and (9.3) without this term correspond to Whitham's theory. This term is sometimes considered to represent a dispersive term in the modulation equations. The perturbation equations of § 4 are of non-dispersive form, and this property is inherent in the invariance of the propagation equations (without the term) to a scale transformation in $(\boldsymbol{x}, t)$. We prefer to consider the last term in (9.3) as a diffraction term.

A perturbation analysis of the type given in § 4 with modulations sinusoidal in θ_1 leads to the perturbation dispersion relation

$$\omega_1^2 = a^2 \Omega_A \Omega_{kk} : \boldsymbol{k}_1 \boldsymbol{k}_1 + \tfrac{1}{4}(\Omega_{kk} : \boldsymbol{k}_1 \boldsymbol{k}_1)^2 \tag{9.4}$$

in place of (4.7), with $\boldsymbol{k}_1 = k_1 \boldsymbol{n}$ and $a^2 = A$. The effect of the diffractive term here is always stabilizing. Where $\Omega_A \Omega_{kk} : \boldsymbol{k}_1 \boldsymbol{k}_1$ is not positive definite an upper limit is placed on the exponential unstable growth rate $i\omega_1$, and given by $\omega_1^2 = -2\Omega_A^2 A^2$. If a lower limit is imposed upon the magnitude of $\boldsymbol{k}_1$ by the geometry of the system or the size of a wave packet, a proportional lower limit appears on the amplitude a required for instability.

All the propagation equations of this paper for a uniform medium, for example (3.1) and (3.2) or (8.3) are invariant under a scale transformation $(t, x)^{\mathrm{T}} = (gt, gx)$. With diffraction effects this invariance disappears. Equation (8.3a), for example, becomes

$$k_\tau - 2(k - k_0)\, k_x + s s_x + (s^{-1} s_{xx})_x = 0, \tag{9.5}$$

and the invariant transformation (8.4) is reduced by the requirement that $g = 1$. The implication of this observation is that diffractive terms may be made as small as desired through a suitable scale transformation.

It has been suggested that the propagation equations of the Whitham type are faulty and that diffractive terms need always be taken into account. This is clearly untrue globally, and is true only in the restricted sense that it has always been true in kinematic (geometric) linear wave theory: In the neighbourhood of a focus or other singular point diffraction is important. Lighthill's solution in the elliptic case is certainly valid within the assumptions of the Whitham theory, except in the immediate neighbourhood of the pinch singularity or of the central basic rays following the pinch.

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Appendix A. Lighthill's function $\mathscr{L}'(\boldsymbol{k},\omega)$

The relation $\mathscr{L}'_\omega = A$ defines the action density in Lighthill's formulation. This relation may be used to eliminate ω in favour of A from $\omega\mathscr{L}'_\omega(\boldsymbol{k},\omega)-\mathscr{L}'(\boldsymbol{k},\omega)$. The result is the function $\mathscr{H}(\boldsymbol{k},A)$ of this paper. The action flux is $\boldsymbol{B} = \mathscr{H}_k = -\mathscr{L}'_k$.

The second derivatives of $\mathscr{L}'$ are related to second derivatives of $\mathscr{H}$ through

$$\mathscr{L}'_{\omega\omega} = \frac{1}{\Omega_A} = \frac{1}{\mathscr{H}_{AA}}, \tag{A 1}$$

$$\mathscr{L}'_{\omega k} = -\frac{\boldsymbol{C}}{\Omega_A} = -\frac{\mathscr{H}_{Ak}}{\mathscr{H}_{AA}}, \tag{A 2}$$

$$\mathscr{L}'_{kk} = -\mathscr{H}_{kk}+\frac{\boldsymbol{CC}}{\mathscr{H}_{AA}}. \tag{A 3}$$

The discriminant tensor whose scalar analogue appears in Lighthill's analysis is

$$\mathscr{L}'_{\omega\omega}\mathscr{L}'_{kk}-\mathscr{L}'_{\omega k}\mathscr{L}'_{\omega k} = -\frac{1}{\Omega_A^2}(\Omega_A\mathscr{H}_{kk}). \tag{A 4}$$

Appendix B. Direct Hamiltonian formulation

The following is an outline of a development based on the approach of Hayes (1970*a*), but using partially Hamiltonian equations and leading directly to the average Hamiltonian density $\mathscr{H}$. Those who do not wish to accede to the treatment of § 2 based upon the arbitrariness inherent in Whitham's function $\mathscr{L}(\omega,\boldsymbol{k},a)$ may find this development preferable.

With the notations and definitions of Hayes (1970*a*) a Lagrangian density $L(\phi,\dot\phi,\nabla\phi;\boldsymbol{x},t)$ is assumed given. The Euler equations are then $\dot p+\nabla\cdot L_{\nabla\phi}-L_\phi = 0$, where $p = L_{\dot\phi}$. The Hamiltonian density is defined by

$$H(\phi,p,\nabla\phi;\boldsymbol{x},t) = p\dot\phi-L, \tag{B 1}$$

with $\dot\phi$ eliminated in favour of p. Computation gives $H_\phi = -L_\phi$, $H_{\nabla\phi} = -L_{\nabla\phi}$, and $H_t = -L_t$. The basic governing equations which replace the Euler equations are

$$H_p = \dot\phi, \tag{B 2a}$$

$$H_\phi = -\dot p+\nabla\cdot H_{\nabla\phi}. \tag{B 2b}$$

Next, we consider a one-parameter family of solutions to (B 2), periodic of period 2π in the parameter θ. We define action density A and flux $\boldsymbol{B}$ by

$$A = (2\pi)^{-1}\oint p\phi_\theta\,\mathrm{d}\theta, \quad \boldsymbol{B} = -(2\pi)^{-1}\oint H_{\nabla\phi}\,\phi_\theta\,\mathrm{d}\theta.$$

As with the Lagrangian formulation the result

$$\partial A/\partial t + \nabla\cdot\boldsymbol{B} = 0 \tag{B 3}$$

follows.

We next consider the space to be decomposed into a cross space $\boldsymbol{x}_\perp$ and a propagation space $(\boldsymbol{x}_\parallel, t)$, with the medium uniform in the latter. We consider solutions

$$\phi = \Phi(\vartheta, \boldsymbol{x}_\perp), \quad p = P(\vartheta, \boldsymbol{x}_\perp)$$

which are strictly periodic of period 2π in the phase $\varphi = \boldsymbol{k}\cdot\boldsymbol{x}_\parallel - \omega t - \theta$. The operator relations $\partial/\partial t = -\omega\partial/\partial\vartheta = \omega\partial/\partial\theta$, $\nabla_\parallel = -\boldsymbol{k}\,\partial/\partial\theta$ hold. The quantities A, $\boldsymbol{B}_\parallel$, and $\boldsymbol{B}_\perp$ are functions of $\boldsymbol{x}_\perp$ alone, and $\nabla_\perp\cdot\boldsymbol{B}_\perp = 0$. The Hamiltonian density may be expressed as $H(\Phi, P, -\boldsymbol{k}\Phi_\theta, \nabla_\perp\Phi; x_\perp)$ and the equations which Φ and P must satisfy are $H_p = \omega\Phi_\theta$, $H_\phi = -\omega P_\theta - \boldsymbol{k}\cdot(H_{\nabla_\parallel\phi})_\theta + \nabla_\perp\cdot H_{\nabla_\perp\phi}$. If $\boldsymbol{k}$ and an amplitude measure are given, this gives an eigenvalue problem for ω.

We define

$$\bar{H} = (2\pi)^{-1}\oint H\,\mathrm{d}\theta$$

and consider a variation of the solution Φ, P. We obtain

$$\delta\bar{H} = \omega\delta A + \boldsymbol{B}.\delta\boldsymbol{k} + \nabla_\perp\cdot[(2\pi)^{-1}\oint H_{\nabla_\perp\phi}\,\delta\Phi\,\mathrm{d}\theta]. \tag{B 4}$$

The integrals $\mathscr{H} = \int\bar{H}\,\mathrm{d}\boldsymbol{x}_\perp$, $\mathscr{A} = \int A\,\mathrm{d}\boldsymbol{x}_\perp$, $\mathscr{B} = \int\boldsymbol{B}_\parallel\,\mathrm{d}\boldsymbol{x}_\perp$ are defined. The integral of (B 4) over the cross space gives, with appropriate boundary conditions,

$$\delta\mathscr{H} = \omega\delta\mathscr{A} + \mathscr{B}\cdot\delta\boldsymbol{k}. \tag{B 5}$$

This leads, for the family of solutions covered by the variation, to $\mathscr{H} = \mathscr{H}(\boldsymbol{k},\mathscr{A})$ and $\omega = \Omega(\boldsymbol{k},\mathscr{A}) = \mathscr{H}_\mathscr{A}$, $\mathscr{B}(\boldsymbol{k},\mathscr{A}) = \mathscr{H}_k$.

Finally, we consider a medium which may have slow variations in the propagation space, and to a wave solution which locally at each point in the propagation space is closely approximated by a periodic solution of the family considered above. The phase ϑ is now a smooth function of $(\boldsymbol{x}_\parallel, t)$, and we define $\omega = -\partial\vartheta/\partial t$, $\boldsymbol{k} = \nabla_\parallel\vartheta$. This definition gives (2.1) immediately, $\partial\boldsymbol{k}/\partial t + \nabla_\parallel\omega = 0$, while the integral of (B 3) over the cross-space with appropriate boundary conditions gives (2.2) in the form $\partial\mathscr{A}/\partial t + \nabla_\parallel\cdot\mathscr{B} = 0$.

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