Nonlinear interactions treated by the methods of theoretical physics (with application to the generation of waves by wind)

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The 'Feynman' diagram method for analysing wave–wave interactions in random wave fields is generalized to include non-conservative interactions between wave fields and external fields. The particle interpretation is no longer applicable, but the transfer expressions can still be conveniently summarized in terms of 'transfer' diagrams, which correspond to collision diagrams in the particle picture. The method is applied to interactions between gravity waves and the turbulent atmospheric boundary layer. The complete set of lowest order transfer diagrams contains the Phillips and Miles mechanisms of wave generation and an additional set of wave-turbulence interactions, which have not been considered previously.

The closure hypothesis invoked in the derivation of the transfer expressions is discussed briefly in appendix A. It is pointed out that Benney & Saffman's recent derivation of the transfer expressions without the usual closure hypothesis contradicts the irreversibility of the transfer expressions and is valid only initially. The relevant statistical properties depend on the distinction between coarse grained and fine grained distributions. This is illustrated in appendix B by a discussion of the Gaussian property of linear, random wave fields.

1. Internal and external interactions of a wave system

We shall consider random wave fields whose statistical properties vary slowly with time and position. The processes which effect these variations can then be treated as small perturbations about the free field steady state.

An example of such a system is the set of surface gravity waves g, internal gravity waves i and seismic waves s in a nonrotating stratified ocean of finite depth (figure 1). The complete motion of the ocean consists of the wave motions and a residual horizontal shear flow h.

We shall consider also the interaction between the ocean and the atmosphere, which we assume to consist of a mean flow m and a fluctuating turbulent flow f. The mean flow is assumed to be independent of time t and the horizontal coordinate vector $\mathbf{x} = (x_1, x_2)$. Similarly, the turbulent flow is regarded as stationary and homogeneous with respect to $\mathbf{x}$.

As additional fields we may include physical inhomogeneities, such as a bottom irregularity b. These fields will also be assumed to be random and homogeneous with respect to $\mathbf{x}$.

We denote the set of wave fields, g, i, s, as the *wave system*. Interactions which affect the wave system may then be divided into two classes: *internal* (wave–wave) interactions, which involve wave components only, and *external* interactions, which involve at least one of the nonwave fields h, m, f, or b.† We shall assume that the energy and momentum of the wave system is conserved by the internal inter-

† The fields h and b can also be regarded as degenerate wave fields of zero frequency (Hasselmann 1966*a*).

actions, but not by the external interactions. Nonconservative interactions between wave components will be regarded formally as external interactions.

Wave–wave interactions were first treated in detail in Peierls's (1929) fundamental paper on the heat conduction in solids. They have since been studied extensively in solid state theory and in other fields of physics, particularly in quantum field scattering theory. The theory has been applied to plasma wave interactions by Litvak (1960). Scattering in geophysical fields has recently received interest through the work of Phillips (1960), Hasselmann (1960, 1962), Benney (1962),

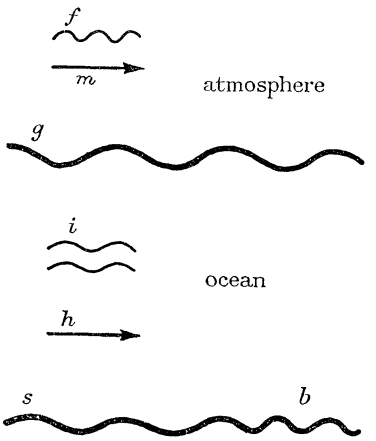

FIGURE 1. Wave fields and 'external' fields of a stratified ocean of constant depth. *Wave system*: g, surface gravity waves; i, internal gravity waves; s, seismic waves. *External fields*: m, mean boundary layer flow; f, turbulent fluctuations; h, horizontal shear flow; b, bottom irregularity.

Longuet-Higgins (1962) and others. The application of concepts developed in solid state and quantum field theory to geophysical scattering problems is described in Hasselmann (1966*a*).

In this paper we shall be concerned primarily with the generalization of these concepts to include external interactions. As an application of the generalized theory we shall consider the interactions between surface gravity waves and the atmospheric boundary layer. The complete set of lowest order interactions will be shown to contain Miles's (1957) and Phillips's (1957) theories of wave generation, and a further set of wave-turbulence interactions which have not been considered previously, but which may conceivably be the principal source of wave energy.

2. Equations of motion

(*a*) *Internal interactions*

We assume that the state of the wave system can be described by a set of coordinates $q^{\nu}_{\mathbf{k}}$ which in the linear approximation represent the amplitudes of the normal mode eigenfunctions $\phi^{\nu}_{\mathbf{k}}(z)\,e^{i\mathbf{k}\cdot\mathbf{x}}$, where z is the vertical coordinate and $\mathbf{k}$ the horizontal wavenumber.

Let the evolution of the wave system itself, excluding the external interactions, be governed by a Lagrangian

$$\left.\begin{aligned} L(q^\nu_{\mathbf{k}}, \dot{q}^\nu_{\mathbf{k}}) &= L_2 + L_3 + L_4 + \ldots, \\ L_2 &= \sum_{\mathbf{k},\nu} \tfrac{1}{2}(\dot{q}^\nu_{\mathbf{k}} \dot{q}^\nu_{-\mathbf{k}} - (\omega^\nu_{\mathbf{k}})^2 q^\nu_{\mathbf{k}} q^\nu_{-\mathbf{k}}) \end{aligned}\right\} \quad (2{\cdot}1)$$

where

is the Lagrangian of the linear system and L_3, L_4, ... are homogeneous interaction Lagrangians of third, fourth, ... orders. The Lagrangian L_2 represents a system of uncoupled oscillators. It is determined uniquely (except for normalization factors) by the invariance of the system under horizontal translations and reflexions. For convenience of notation $\mathbf{k}$ is treated in (2·1) as a discrete variable, but we shall consider later the limiting case of a continuous spectrum. The interaction Lagrangians describe the nonlinear coupling. Examples for a number of scattering processes in the oceanic waveguide are given in Hasselmann (1966*a*).

The equations of motion can be rewritten in a more convenient form by introducing the Hamilton function

$$H = \Sigma p^\nu_{\mathbf{k}} \dot{q}^\nu_{\mathbf{k}} - L, \quad \text{with} \quad p^\nu_{\mathbf{k}} = \partial L / \partial \dot{q}^\nu_{\mathbf{k}}$$

and then transforming from the canonical variables $q^\nu_{\mathbf{k}}$, $p^\nu_{\mathbf{k}}$ to the normal variables

$$\left.\begin{aligned} a^\nu_{\mathbf{k}} &= \tfrac{1}{\sqrt{2}}(p^\nu_{-\mathbf{k}} - \mathrm{i}\omega^\nu_{\mathbf{k}} q^\nu_{\mathbf{k}}), \\ a^{-\nu}_{\mathbf{k}} &= \tfrac{1}{\sqrt{2}}(p^\nu_{-\mathbf{k}} + i\omega^\nu_{\mathbf{k}} q^\nu_{\mathbf{k}}). \end{aligned}\right\} \quad (2{\cdot}2)$$

The equations of motion then become

$$\dot{a}^\nu_{\mathbf{k}} = -\mathrm{i}\omega^\nu_{\mathbf{k}} \frac{\partial H}{\partial a^{-\nu}_{-\mathbf{k}}} \quad (\nu \gtrless 0). \quad (2{\cdot}3)$$

Equation (2·3) is valid for both signs of the index ν, if for negative indices we define

$$\omega^{-\nu}_{\mathbf{k}} = -\omega^\nu_{\mathbf{k}}.$$

The Hamiltonian then becomes

$$H = H_2 + H_3 + H_4 + \ldots,$$

with

$$H_2 = \Sigma \tfrac{1}{2}(a^\nu_{\mathbf{k}} a^{-\nu}_{-\mathbf{k}}) \quad (2{\cdot}4)$$

and for $n \geqslant 3$

$$H_n = \Sigma D^{\nu_1 \cdots \nu_n}_{\mathbf{k}_1 \ldots \mathbf{k}_n} a^{\nu_1 \cdots}_{\mathbf{k}_1 \ldots} a^{\nu_n}_{\mathbf{k}_n}, \quad (2{\cdot}5)$$

where $D^{\nu_1 \cdots \nu_n}_{\mathbf{k}_1 \ldots \mathbf{k}_n}$ are constant, symmetrical interaction coefficients.

The reality of H yields the conditions

$$a^{-\nu}_{-\mathbf{k}} = (a^\nu_{\mathbf{k}})^* \quad (2{\cdot}6)$$

and

$$D^{-\nu_1 \ldots -\nu_n}_{-\mathbf{k}_1 \ldots -\mathbf{k}_n} = (D^{\nu_1 \cdots \nu_n}_{\mathbf{k}_1 \ldots \mathbf{k}_n})^*. \quad (2{\cdot}7)$$

The homogeneity of the field yields the further condition

$$D^{\nu_1 \cdots \nu_n}_{\mathbf{k}_1 \ldots \mathbf{k}_n} = 0 \quad \text{for} \quad \mathbf{k}_1 + \ldots + \mathbf{k}_n \neq 0. \quad (2{\cdot}8)$$

The equations of motion (2·3) are given explicitly by

$$\dot{a}^\nu_{\mathbf{k}} + \mathrm{i}\omega^\nu_{\mathbf{k}} a^\nu_{\mathbf{k}} = -3\mathrm{i}\omega^\nu_{\mathbf{k}} \Sigma D^{-\nu\nu_1\nu_2}_{-\mathbf{k}\mathbf{k}_1\mathbf{k}_2} a^{\nu_1}_{\mathbf{k}_1} a^{\nu_2}_{\mathbf{k}_2} - \ldots - (p+1)\,\mathrm{i}\omega^\nu_{\mathbf{k}} \Sigma D^{-\nu\nu_1\cdots\nu_p}_{-\mathbf{k}\mathbf{k}_1\ldots\mathbf{k}_p} a^{\nu_1}_{\mathbf{k}_1} \ldots a^{\nu_p}_{\mathbf{k}_p}. \quad (2{\cdot}9)$$

The linear solution of equation (2·9), without the interaction terms, is

$${}_1a^\nu_{\mathbf{k}} = A^\nu_{\mathbf{k}} \exp\{-\mathrm{i}\omega^\nu_{\mathbf{k}} t\}, \quad A^\nu_{\mathbf{k}} \text{ const.} \quad (2{\cdot}10)$$

(*b*) *External interactions*

Interactions with external fields can be readily included in the above formalism. We assume that the external fields can be described by a set of variables $b^{\mu}_{\mathbf{k}}(t)$, where in analogy with the linear wave solution (2·10)

$$b^{\mu}_{\mathbf{k}}(t) = B^{\mu}_{\mathbf{k}} \exp\{-\mathrm{i}\omega^{\mu}_{\mathbf{k}} t\}, \quad B^{\mu}_{\mathbf{k}} \text{ const.} \tag{2·11}$$

Here μ represents a generalized index which may be either discrete, continuous or a combination of both.

For example, if the external field is a homogeneous, random irregularity δH of the ocean depth H, we may write

$$\delta H(\mathbf{x}) = \Sigma B_{\mathbf{k}} \exp\{\mathrm{i}\mathbf{k}.\mathbf{x}\}, \quad B \text{ const.},$$

so that $\mu = \text{const.}$, $\omega^{\mu}_{\mathbf{k}} = 0$.

If the external field is a stationary, horizontally homogeneous turbulent velocity field $u_i(\mathbf{x}, z, t)$, we may write

$$u_i = \sum_{\mathbf{k},\,\omega} U_i(k, \omega, z) \exp\{\mathrm{i}(\mathbf{k}.\mathbf{x} + \omega t)\} \tag{2·12}$$

so that $\mu \equiv (i, z, \omega)$ and $\omega^{\mu}_{\mathbf{k}} \equiv -\omega$.

The reality condition corresponding to equation (2·6) becomes

$$b^{\overline{\mu}}_{\mathbf{k}} = (b^{\mu}_{\mathbf{k}})^*, \tag{2·13}$$

where $\overline{\mu}$ is the *conjugate* index of μ, for which

$$\omega^{\overline{\mu}}_{\mathbf{k}} = -\omega^{\mu}_{\mathbf{k}}.$$

In the case of the turbulent field, for example,

$$\mu \equiv (i, z, \omega) \quad \text{and} \quad \overline{\mu} \equiv (i, z, -\omega)$$

Let us assume now that the external interactions can be expanded, in the same way as the internal interactions, in a power series. Then the equations of motion, including both internal and external interactions, have the general form

$$\begin{aligned} \dot{a}^{\nu}_{\mathbf{k}} + \mathrm{i}\omega^{\nu}_{\mathbf{k}} a^{\nu}_{\mathbf{k}} = &-3\mathrm{i}\omega^{\nu}_{\mathbf{k}} \Sigma D^{-\nu\nu_1\nu_2}_{-\mathbf{k}\mathbf{k}_1\mathbf{k}_2} a^{\nu_1}_{\mathbf{k}_1} a^{\nu_2}_{\mathbf{k}_2} - \ldots - \mathrm{i}(p+1)\,\omega^{\nu}_{\mathbf{k}} \Sigma D^{-\nu\nu_1\ldots\nu_p}_{-\mathbf{k}\mathbf{k}_1\ldots\mathbf{k}_p} a^{\nu_1}_{\mathbf{k}_1} \ldots a^{\nu_p}_{\mathbf{k}_p} \\ &\ldots - 2\mathrm{i}\omega^{\nu}_{\mathbf{k}} \Sigma E^{-\nu\nu_1}_{-\mathbf{k}\mathbf{k}} a^{\nu_1}_{\mathbf{k}} \ldots - 2\mathrm{i}\omega^{\nu}_{\mathbf{k}} \Sigma E^{-\nu\mu}_{-\mathbf{k}\mathbf{k}} b^{\mu}_{\mathbf{k}} \\ &\ldots - \mathrm{i}(p+1)\,\omega^{\nu}_{\mathbf{k}} \Sigma E^{-\nu\nu_1\ldots\nu_q\mu_{q+1}\ldots\mu_p}_{-\mathbf{k}\mathbf{k}_1\ldots\mathbf{k}_q\mathbf{k}_{q+1}\ldots\mathbf{k}_p} a^{\nu_1}_{\mathbf{k}_1} \ldots a^{\nu_q}_{\mathbf{k}_q} b^{\mu_{q+1}}_{\mathbf{k}_{q+1}} \ldots b^{\mu_p}_{\mathbf{k}_p} \ldots \end{aligned} \tag{2·14}$$

where $E^{\cdots}_{\cdots}$ are external interaction coefficients. The coefficients satisfy the reality and homogeneity conditions

$$E^{\nu-\nu_1\ldots-\nu_q\,\overline{\mu}_{q+1}\ldots\overline{\mu}_p}_{\mathbf{k}-\mathbf{k}_1\ldots-\mathbf{k}_q-\mathbf{k}_{q+1}\ldots-\mathbf{k}_p} = (E^{-\nu\nu_1\ldots\nu_q\mu_{q+1}\ldots\mu_p}_{-\mathbf{k}\mathbf{k}_1\ldots\mathbf{k}_q\mathbf{k}_{q+1}\ldots\mathbf{k}_p})^* \tag{2·15}$$

and

$$E^{\nu_1\nu_2\ldots\nu_q\mu_{q+1}\ldots\mu_p}_{\mathbf{k}_1\mathbf{k}_2\ldots\mathbf{k}_q\mathbf{k}_{q+1}\ldots\mathbf{k}_p} = 0 \quad \text{for} \quad k_1 + \ldots + k_\rho \neq 0, \tag{2·16}$$

in analogy with equations (2·7) and (2·8). But in contrast with the internal interaction coefficients $D^{\cdots}_{\cdots}$, they are not associated with a Hamiltonian and are therefore not symmetrical in all indices. This is the principal difference between internal and external interactions.

3. Interaction diagrams

We assume that the first-order solutions ${}_1a^\nu_{\mathbf{k}}$ and the external fields $b^\mu_{\mathbf{k}}$ are $O(\alpha)$, with $\alpha \ll 1$. The internal coupling coefficients $D^{\nu_1\dots\nu_n}_{\mathbf{k}_1\dots\mathbf{k}_n}$ are assumed to be $O(1)$, and the external coupling coefficients $E^{-\nu\nu_1\dots\nu_p\,\mu_{p+1}\dots\mu_q}_{-\mathbf{k}\mathbf{k}_1\dots\mathbf{k}_p\,\mathbf{k}_{p+1}\dots\mathbf{k}_q}$ to be $O(\beta)$ or smaller, with $\beta \ll 1$.

We can then construct a solution to equations (2·14) by expanding the wave fields in a perturbation series

$$a^\nu_{\mathbf{k}} = {}_1a^\nu_{\mathbf{k}} + {}_2a^\nu_{\mathbf{k}} + \dots, \tag{3·1}$$

where ${}_na^\nu_{\mathbf{k}} = O(\alpha^n)$. Each perturbation term ${}_na^\nu_{\mathbf{k}}$ can be expanded further with respect to β, but we shall not do this explicitly.

The ordering with respect to the parameters α and β applies to the ocean, but is not the most general case. A more consistent procedure would be to use a multi-parameter expansion in which each field and coupling coefficient is characterized by a different parameter. This is cumbersome, however, and not really necessary, since the implicit dependence on individual parameters can be readily recognized throughout the analysis.

Substituting the expansion (3·1) in the equations of motion (2·14), collecting the terms of nth order, and integrating, we obtain

$${}_na^\nu_{\mathbf{k}} = \int_0^t \{\dots - \mathrm{i}(p+1)\,\omega^\nu_{\mathbf{k}} \sum_{(n_1+\dots+n_q=n+p-q)} E^{-\nu\nu_1\dots\nu_q\,\mu_{q+1}\dots\mu_p}_{-\mathbf{k}\mathbf{k}_1\dots\mathbf{k}_q\,\mathbf{k}_{q+1}\dots\mathbf{k}_p} \times {}_{n_1}a^{\nu_1}_{\mathbf{k}_1}(t')\dots{}_{n_q}a^{\nu_q}_{\mathbf{k}_q}(t')\,b^{\mu_{q+1}}_{\mathbf{k}_{q+1}}(t')\dots b^{\mu_p}_{\mathbf{k}_p}(t')\dots\} \exp\{-\mathrm{i}\omega^\nu_{\mathbf{k}}(t-t')\}\,\mathrm{d}t'. \tag{3·2}$$

We have assumed that the linear solution (2·10) has been chosen to satisfy the initial conditions $a^\nu_{\mathbf{k}}(0) = A^\nu_{\mathbf{k}}$ rigorously, so that ${}_na^\nu_{\mathbf{k}}(0) = 0$ for $n \geqslant 2$. Equation (3·2) can be evaluated for successive n, since the right-hand side contains only perturbation terms of orders lower than n.

It is convenient to express the structure of the perturbation solution (3·2) in a more condensed form in terms of *interaction* diagrams.

We shall denote wave components $a^\nu_{\mathbf{k}}$ with $\nu > 0$, and external field components $b^\mu_{\mathbf{k}}$ with $\omega^\mu_{\mathbf{k}} > 0$, by directed arrows equal to $\mathbf{k}$. The complex conjugate ('anti-') components $a^{-\nu}_{-\mathbf{k}}\,(\nu > 0)$ and $b^{\overline{\mu}}_{-\mathbf{k}}(\omega^\mu_{\mathbf{k}} > 0)$ are denoted by cross-stroked arrows equal to $\mathbf{k}$. Terms with positive frequency are thus associated with 'components', terms with negative frequency with 'anti-components'. The sign convention is chosen such that in both cases the arrows point in the propagation direction of the waves. External fields are distinguished from wave fields by a cross at the base of the arrows (cf. figure 2).

The contribution

$$\delta_n a^\nu_{\mathbf{k}} = -\mathrm{i}(p+1)\,\omega^\nu_{\mathbf{k}} \int_0^t E^{-\nu\nu_1\dots\nu_q\,\mu_{q+1}\dots\mu_p}_{-\mathbf{k}\mathbf{k}_1\dots\mathbf{k}_q\,\mathbf{k}_{q+1}\dots\mathbf{k}_p}\, {}_{n_1}a^{\nu_1}_{\mathbf{k}_1}(t')\dots{}_{n_q}a^{\nu_q}_{\mathbf{k}_q}(t')\,b^{\mu_{q+1}}_{\mathbf{k}_{q+1}}(t')\dots b^{\mu_p}_{\mathbf{k}_p}(t') \times \exp\{-\mathrm{i}\omega^\nu_{\mathbf{k}}(t-t')\}\,\mathrm{d}t'$$

of the general term of equation (3·2) to the perturbation amplitude ${}_na^\nu_{\mathbf{k}}$ is represented by p arrows ${}_{n_1}a^{\nu_1}_{\mathbf{k}_1}\dots b^{\mu_p}_{\mathbf{k}_p}$ entering a vertex and a single arrow $\delta_n a^\nu_{\mathbf{k}}$ leaving the vertex (figure 2). On account of the homogeneity conditions (2·8) and (2·16), the vector sum of the components minus the anti-components entering the vertex is equal to the component (or minus the anti-component) leaving the vertex.

By successively representing the components entering a vertex in terms of lower order components, the component $\delta_n a_{\mathbf{k}}^{\nu}$ can be represented by a cascade diagram with n first-order inputs and one output. The component ${}_n a_{\mathbf{k}}^{\nu}$ is then given by the sum over all possible cascade diagrams with n first-order inputs.

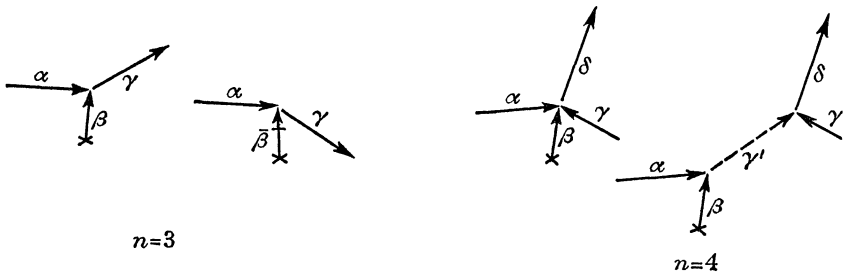

FIGURE 2. Examples of interaction diagrams for perturbations of 3rd and 4th order. External fields are denoted by crosses, anti-components by cross-bars and virtual components by dotted lines.

The perturbation solution (3·2) represents the response of a linear oscillator to a superposition of sinusoidal forcing terms. It is therefore also a superposition of sinusoidal oscillations. Normally, these are small. However, the response is large if the forcing frequency is close to the resonant frequency, and grows indefinitely with time if the resonance is satisfied exactly. In diagram notation, resonance occurs if the frequency of a component in a diagram is equal to the frequency sum of any set of lower order components which generate that component (the frequencies of anti components being counted negatively). We shall denote resonant, *free* components in a diagram by full lines. Non-resonant, *virtual* components will be denoted by broken lines. Free components satisfy the two conditions

$$s\mathbf{k} = \sum_j s_j \mathbf{k}_j, \tag{3·3}$$

$$s\omega = \sum_j s_j \omega_j \quad (s, s_j = \pm 1), \tag{3·4}$$

where the sums on the right-hand sides are taken over the set of lower-order input components. Virtual components satisfy only the first condition.

4. THE ENERGY TRANSFER DUE TO INTERNAL INTERACTIONS

The resonant perturbations represent an energy transfer between wave components, in analogy with the beat phenomenon of linearly coupled, tuned oscillators. If the fields consist of a finite number of discrete components, the evolution of the fields can be determined by rewriting the secular terms in the perturbation expansion as the slow rate of change of time-dependent wave amplitudes (cf. Benney 1962; Ball 1964; Bretherton 1964). In the case of random fields, we are concerned with the evolution of the *spectra*. We shall adopt essentially the same approach: from the perturbation equations we determine first the secular terms in the

perturbation expansion of the spectra; these are then rewritten as the rate of change of slowly varying spectra. (Intermediate between these two cases lies the problem of the scattering of a discrete wave by random fields (cf. Chernov 1960; Tatarski 1961). Our theory yields the intensity relations for this problem, but not the phase fluctuations.)

The linear wave system is completely determined statistically by the set of energy spectra

$$F^{\nu}_{\mathbf{k}} = \tfrac{1}{2}\langle a^{\nu}_{\mathbf{k}} a^{-\nu}_{-\mathbf{k}}\rangle \quad (= \text{const.}), \tag{4·1}$$

where the cornered brackets denote ensemble mean values. This is valid in the coarse-grained sense (see appendix B). The wave fields are Gaussian, and are therefore determined by the covariance matrix $\langle a^{\nu}_{\mathbf{k}} a^{\mu}_{\mathbf{k}}\rangle$. They are furthermore stationary, so that only the diagonal terms $\langle a^{\nu}_{\mathbf{k}} a^{-\nu}_{\mathbf{k}}\rangle$ are nonzero. The energy of the linear field ν is

$$E^{\nu}_2 = H^{\nu}_2 = \sum_{\mathbf{k}} (F^{\nu}_{\mathbf{k}} + F^{-\nu}_{\mathbf{k}}) = 2\sum_{\mathbf{k}} F^{\nu}_{\mathbf{k}}. \tag{4·2}$$

To determine the evolution of the spectra in the nonlinear case we expand $F^{\nu}_{\mathbf{k}}$ in a perturbation series

$$F^{\nu}_{\mathbf{k}} = {}_2F^{\nu}_{\mathbf{k}} + {}_3F^{\nu}_{\mathbf{k}} + {}_4F^{\nu}_{\mathbf{k}} + \ldots,$$

where

$${}_2F^{\nu}_{\mathbf{k}} = \tfrac{1}{2}\langle {}_1a^{\nu}_{\mathbf{k}_1} a^{-\nu}_{-\mathbf{k}}\rangle = \text{const.}, \tag{4·3}$$

$${}_3F^{\nu}_{\mathbf{k}} = \mathscr{R}\langle {}_2a^{\nu}_{\mathbf{k}_1} a^{-\nu}_{-\mathbf{k}}\rangle, \tag{4·4}$$

$${}_4F^{\nu}_{\mathbf{k}} = \tfrac{1}{2}\langle {}_2a^{\nu}_{\mathbf{k}_2} a^{-\nu}_{-\mathbf{k}}\rangle + \mathscr{R}\langle {}_3a^{\nu}_{\mathbf{k}_1} a^{-\nu}_{-\mathbf{k}}\rangle. \tag{4·5}$$

The spectral perturbations can be expressed in terms of the known linear amplitudes ${}_1a^{\nu}_{\mathbf{k}}$ by substituting the solutions (3·2) for the perturbation amplitudes in expressions (4·3) to (4·5).

Retaining only the terms depending on the internal coupling coefficients $D^{\nu_1\ldots\nu_p}_{\mathbf{k}_1\ldots\mathbf{k}_p}$, we obtain

$${}_3F^{\nu}_{\mathbf{k}} = \mathscr{R}\{-3\mathrm{i}\omega\Sigma\, D^{-\nu\nu_1\nu_2}_{-\mathbf{k}\mathbf{k}_1\mathbf{k}_2}\langle A^{-\nu}_{-\mathbf{k}} A^{\nu_1}_{\mathbf{k}_1} A^{\nu_2}_{\mathbf{k}_2}\rangle \Delta_1(\omega-\omega_1-\omega_2)\}, \tag{4·6}$$

$$\begin{aligned}{}_4F^{\nu}_{\mathbf{k}} = {}& \tfrac{9}{2}\omega^2\Sigma\, D^{-\nu\nu_1\nu_2}_{-\mathbf{k}\mathbf{k}_1\mathbf{k}_2} D^{\nu-\nu_3-\nu_4}_{\mathbf{k}-\mathbf{k}_3-\mathbf{k}_4}\langle A^{\nu_1}_{\mathbf{k}_1} A^{\nu_2}_{\mathbf{k}_2} A^{-\nu_3}_{-\mathbf{k}_3} A^{-\nu_4}_{-\mathbf{k}_4}\rangle \Delta_1(\omega-\omega_1-\omega_2)\Delta_1^*(\omega-\omega_3-\omega_4)\\ & + \mathscr{R}\{\Sigma - 18\omega\omega_4 D^{-\nu\nu_3\nu_4}_{-\mathbf{k}\mathbf{k}_3\mathbf{k}_4} D^{-\nu_4\nu_1\nu_2}_{-\mathbf{k}_4\mathbf{k}_1\mathbf{k}_2}\langle A^{-\nu}_{-\mathbf{k}} A^{\nu_1}_{\mathbf{k}_1} A^{\nu_2}_{\mathbf{k}_2} A^{\nu_3}_{\mathbf{k}_3}\rangle \Delta_2(\omega-\omega_3-\omega_4, \omega_4-\omega_1-\omega_2)\}\\ & + \mathscr{R}\{\Sigma - 4\mathrm{i}\omega D^{-\nu\nu_1\nu_2\nu_3}_{-\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\langle A^{-\nu}_{-\mathbf{k}} A^{\nu_1}_{\mathbf{k}_1} A^{\nu_2}_{\mathbf{k}_2} A^{\nu_3}_{\mathbf{k}_3}\rangle \Delta_1(\omega-\omega_1-\omega_2-\omega_3)\},\end{aligned} \tag{4·7}$$

where

$$\Delta_1(\omega) = \frac{\mathrm{e}^{\mathrm{i}\omega t}-1}{\mathrm{i}\omega}, \tag{4·8}$$

$$\Delta_2(\omega,\omega') = \frac{\Delta_1(\omega+\omega')-\Delta_1(\omega)}{\mathrm{i}\omega'}, \tag{4·9}$$

and we have introduced the abbreviated notation $\omega = \omega^{\nu}_{\mathbf{k}}$, $\omega_j = \omega^{\nu_j}_{\mathbf{k}_j}$.

Each term in equations (4·6) and (4·7) represents a quadratic product of two perturbation components. The structure of the various terms is shown in diagram notation in figure 3.

The spectral perturbations involve mean products of three or more initial amplitudes $A^{\nu}_{\mathbf{k}}$. We shall assume that the fields are initially Gaussian and stationary;

the mean products can then be expressed in terms of the initial spectra ${}_2F^{\nu}_{\mathbf{k}}$. The assumption appears reasonable, since the linear fields tend to a Gaussian, stationary state, and the nonlinear interactions are weak. We shall discuss the basis of the hypothesis in more detail later.

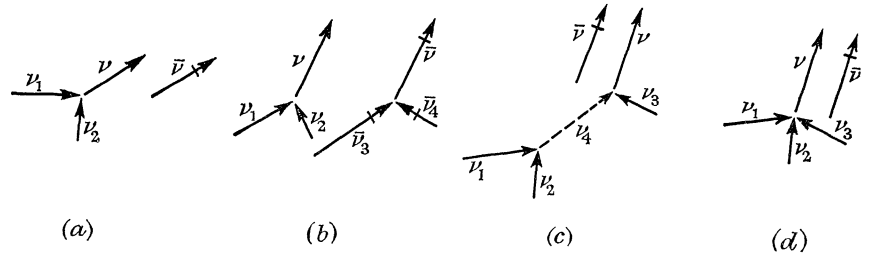

FIGURE 3. Interaction diagrams of the spectral perturbation ${}_3F^{\nu}_{\mathbf{k}}$ (equation (4·6)) and ${}_4F^{\nu}_{\mathbf{k}}$ (equation (4·7)). Diagram (*a*) represents the right-hand side of equation (4·6); diagrams (*b*), (*c*), and (*d*) represent the 1st 2nd and 3rd terms, respectively, in the right-hand side of equation (4·7).

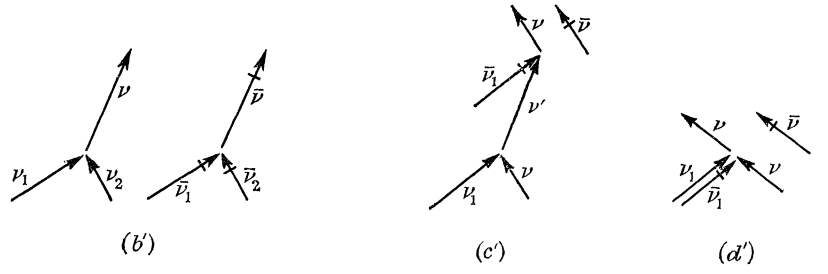

FIGURE 4. Interaction diagrams of the spectral perturbation ${}_4F^{\nu}_{\mathbf{k}}$ for an initially Gaussian field, equation (4·10). Diagrams (*b'*), (*c'*) and (*d'*) correspond to the 1st, 2nd and 3rd terms, respectively; ν' in diagram (*c'*) has been taken as a resonant component.

For a Gaussian initial field the cubic term ${}_3F^{\nu}_{\mathbf{k}}$ vanishes. The only contribution to the fourth-order perturbations ${}_4F^{\nu}_{\mathbf{k}}$ are terms in which the four amplitudes $A^{\nu_j}_{\mathbf{k}_j}$ can be divided into two complex conjugate pairs. This yields

$$
\begin{aligned}
{}_4F^{\nu}_{\mathbf{k}} = {} & 36\omega^2 \sum_{\substack{\mathbf{k}_1,\mathbf{k}_2 \\ \nu_1,\nu_2 \gtrless 0}} {}_2F^{\nu_1}_{\mathbf{k}_1}\,{}_2F^{\nu_2}_{\mathbf{k}_2} |D^{-\nu\nu_1\nu_2}_{-\mathbf{k}\mathbf{k}_1\mathbf{k}_2}|^2 |\Delta_1(\omega-\omega_1-\omega_2)|^2 \\
& - 72\omega\,{}_2F^{\nu}_{\mathbf{k}} \sum_{\substack{\mathbf{k}_1 \\ \nu_1,\nu' \gtrless 0}} {}_2F^{\nu_1}_{\mathbf{k}_1} \omega' \mathscr{R}\{D^{-\nu-\nu_1\nu'}_{-\mathbf{k}-\mathbf{k}_1\mathbf{k}+\mathbf{k}_1} D^{-\nu'\nu\nu_1}_{-\mathbf{k}-\mathbf{k}_1\mathbf{k}\mathbf{k}_1} \Delta_2(\omega+\omega_1-\omega', -\omega-\omega_1+\omega')\} \\
& - 32\omega\,{}_2F^{\nu}_{\mathbf{k}} \sum_{\mathbf{k}_1,\nu_1 \gtrless 0} {}_2F^{\nu_1}_{\mathbf{k}_1} \mathscr{R}\{\mathrm{i}\, D^{-\nu\nu-\nu_1\nu_1}_{-\mathbf{k}\mathbf{k}-\mathbf{k}_1\mathbf{k}_1} \Delta_1(0)\},
\end{aligned}
\tag{4·10}
$$

where

$$\omega' \equiv \omega^{\nu'}_{\mathbf{k}+\mathbf{k}_1}.$$

The diagrams corresponding to the three terms on the right-hand side of equation (4·10) are shown in figure 4. We have assumed that the interaction coefficients vanish if one of the wavenumbers are zero, which excludes one of the three possible pair combinations for the second and third terms.

For large t, the dominant contributions are due to the secular terms arising from the resonant interactions. These are determined by the asymptotic behaviour of the response functions Δ_1 and Δ_2, namely

$$|\Delta_1(\omega)|^2 \to 2\pi t\delta(\omega) \tag{4·11}$$

and

$$\Delta_2(\omega, -\omega) \to t(\pi\delta(\omega)+\mathrm{i}P(1/\omega)). \tag{4·12}$$

Equation (4·12) is an abbreviated notation for the relation

$$\int_{-\infty}^{+\infty} \Delta_2(\omega, -\omega)f(\omega)\,\mathrm{d}\omega \to t\left(\pi f(0)+\mathrm{i}P\int\frac{f(\omega)}{\omega}\,\mathrm{d}\omega\right),$$

which is valid for any continuous $f(\omega)$.

In evaluating the asymptotic form of ${}_4F_{\mathbf{k}}^{\nu}$ with the aid of equations (4·11) and (4·12), we note that:

(*a*) on account of the symmetry of the coupling coefficients $D_{\cdots}^{\cdots}$ and the reality condition (2·7), the real part of the expression $\{\ldots\}$ in the second term in equation (4·10) involves only the δ function contribution of Δ_2, and

(*b*) the reality of the interaction Hamiltonian H_4 implies the reality of the coefficient $D_{-\mathbf{k}\mathbf{k}-\mathbf{k}_1\mathbf{k}_1}^{-\nu\nu-\nu_1\nu_1}$. Thus the third term in equation (4·10) vanishes. Both of these properties apply only for internal interactions.

The asymptotic relations (4·11) and (4·12) lead to an expression of the form

$${}_4F_{\mathbf{k}}^{\nu} = tI({}_2F_{\mathbf{k}'}^{\nu'}),$$

where I is a quadratic integral operator acting on the set of initial spectra ${}_2F_{\mathbf{k}'}^{\nu'}$. If the spectra are regarded as slowly varying functions, the equation may be rewritten

$$\partial F_{\mathbf{k}}^{\nu}/\partial t = I(F_{\mathbf{k}'}^{\nu'}).$$

Explicitly,

$$\frac{\partial n_\nu(\mathbf{k})}{\partial t} = \sum_{\nu_1,\nu_2>0}\iint \{T_+(n_1n_2-nn_1-nn_2)\,\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\,\delta(\omega_1+\omega_2-\omega)$$

$$+2T_-(n_1n_2+nn_1-nn_2)\,\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k})\,\delta(\omega_1-\omega_2-\omega)\}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2, \tag{4·13}$$

where

$$T_+ = 72\pi\omega\omega_1\omega_2|D_{-\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{-\nu\nu_1\nu_2}|^2, \tag{4·14}$$

$$T_- = 72\pi\omega\omega_1\omega_2|D_{-\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{-\nu\nu_1\nu_2}|^2, \tag{4·15}$$

and $n_\nu(\mathbf{k}) = F_\nu(\mathbf{k})/\omega_{\mathbf{k}}^{\nu}$. We have introduced continuous spectra according to the notation

$$\Sigma F_{\mathbf{k}}^{\nu} \to \int F_\nu(\mathbf{k})\,\mathrm{d}\mathbf{k}.$$

Equation (4·13) represents the lowest order energy transfer due to quadratic coupling. It was first derived by Peierls (1929) for interacting lattice vibrations. In certain cases, such as surface gravity waves, the lowest order resonant conditions have only trivial solutions, and the analysis has to be carried through to higher order

(Hasselmann 1962). The resulting transfer expressions are similar in structure to equation (4·13), but involve cubic rather than quadratic spectral products (cf. Hasselmann 1966*a*).

The transfer expressions are valid for all time, provided the basic hypothesis that the fields can be treated as approximately Gaussian remains valid. It has been shown by Prigogine (1962) that this is indeed the case, provided the fields were rigorously Gaussian initially, and the Gaussian hypothesis is applied only for the development of the fields *forwards* in time.

Benney & Saffman (1966) have shown that the Gaussian hypothesis is not necessary if the cumulants are assumed to be smooth in wavenumber space. It is shown in appendix A, however, that this assumption contradicts the irreversibility of the transfer expressions. The cumulants of an aged field are rapidly fluctuating fine structure functions (appendix B).

5. The energy transfer due to external interactions

We shall assume in this section that the external fields are statistically stationary, homogeneous and orthogonal, i.e. that

$$\langle b^{\mu}_{\mathbf{k}}(t+\tau)\, b^{\mu'}_{\mathbf{k}'}(t)\rangle = 0 \quad \text{for} \quad \mathbf{k}+\mathbf{k}' \neq 0 \quad \text{or} \quad \mu' \neq \bar{\mu}$$

and

$$\langle b^{\mu}_{\mathbf{k}}(t+\tau)\, b^{\bar{\mu}}_{-\mathbf{k}}(t)\rangle = 2G^{\mu}_{\mathbf{k}} \exp\{-\mathrm{i}\omega^{\mu}_{\mathbf{k}}\tau\},$$

where

$$G^{\mu}_{\mathbf{k}} = \tfrac{1}{2}\langle B^{\mu}_{\mathbf{k}} B^{\bar{\mu}}_{-\mathbf{k}}\rangle = \text{const.} \tag{5·1}$$

We have assumed orthogonality for simplicity of notation. An arbitrary representation will generally not lead to orthogonal external-field components. For example, the turbulence components $U_i(\mathbf{k},\omega,z)$ of the representation (2·12) do not satisfy the condition

$$\langle U_i(\mathbf{k},\omega,z)\, U_j(-\mathbf{k},-\omega,z')\rangle = 0 \quad \text{for} \quad i \neq j \quad \text{or} \quad z \neq z'.$$

Orthogonality can be attained by transforming to a new representation. Rather than do this explicitly, we simply note that if a nonorthogonal representation is used, expressions of the form $\sum_{\mu} G^{\mu}_{\mathbf{k}}$ have to be replaced in the following by expressions $\sum_{\mu\mu'} K^{\mu\mu'}_{\mathbf{k}} G^{\mu\mu'}_{\mathbf{k}}$, where $G^{\mu\mu'}_{\mathbf{k}} = \tfrac{1}{2}\langle b^{\mu}_{\mathbf{k}} b^{\mu'}_{-\mathbf{k}}\rangle$ and $K^{\mu\mu'}_{\mathbf{k}}$ is a coefficient matrix.

We shall assume further that the linear wave fields and external fields are statistically independent. This can be deduced in the coarse-grained sense from an extension of the analysis in appendix B.

(*a*) *Linear interactions*

The linear interaction equation

$$\dot{a}^{\nu}_{\mathbf{k}} + \mathrm{i}\omega^{\nu}_{\mathbf{k}} a^{\nu}_{\mathbf{k}} = -2\mathrm{i}\,\omega^{\nu}_{\mathbf{k}} \sum_{\mu} E^{-\nu\mu}_{-\mathbf{k}\mathbf{k}} b^{\mu}_{\mathbf{k}} \tag{5·2}$$

can be solved exactly, without expanding in perturbation series. For the general theory the right-hand side of equation (5·2) must none the less be assumed to be

a small perturbation (i.e. $E^{-\nu\mu}_{-\mathbf{k}\mathbf{k}} = O(\beta)$, with $\beta \ll 1$), since the analysis of the higher order interactions is based on the free-wave solution of the homogeneous equation.

The solution of equation (5·2) is

$$a^{\nu}_{\mathbf{k}} = -2\mathrm{i}\omega^{\nu}_{\mathbf{k}} \sum_{\mu} E^{-\nu\mu}_{-\mathbf{k}\mathbf{k}} B^{\mu}_{\mathbf{k}} \Delta_1(\omega^{\nu}_{\mathbf{k}} - \omega^{\mu}_{\mathbf{k}}) \exp\{-\mathrm{i}\omega^{\mu}_{\mathbf{k}} t\}.$$

This yields a spectrum which grows linearly in time (equation 4·11). In accordance with out previous notation, we may write this in the differential form

$$\frac{\partial F_{\nu}(\mathbf{k})}{\partial t} = 8\pi(\omega^{\nu}_{\mathbf{k}})^2 \sum_{\mu} |E^{-\nu\mu}_{-\mathbf{k}\mathbf{k}}|^2 G_{\mu}(\mathbf{k})\, \delta(\omega^{\nu}_{\mathbf{k}} - \omega^{\mu}_{\mathbf{k}}), \tag{5·3}$$

where we have used the continuous notation

$$\int G_{\mu}(\mathbf{k})\, \mathrm{d}\mathbf{k} \quad \text{for} \quad \sum_{\mathbf{k}} G^{\mu}_{\mathbf{k}}.$$

The nonconservative linear wave interactions

$$\dot{a}^{\nu}_{\mathbf{k}} + \mathrm{i}\omega^{\nu}_{\mathbf{k}} a^{\nu}_{\mathbf{k}} = -2\mathrm{i}\omega^{\nu}_{\mathbf{k}} \sum_{\nu'} E^{-\nu\nu'}_{-\mathbf{k}\mathbf{k}} a^{\nu'}_{\mathbf{k}} \tag{5·4}$$

yield a slightly different expression. Multiplying equation (5·4) by $a^{-\nu}_{-\mathbf{k}}$, adding the complex conjugate equation, and taking mean values, we obtain

$$\frac{\partial F_{\nu}(\mathbf{k})}{\partial t} = 4\omega^{\nu}_{\mathbf{k}} \mathscr{I}(E^{-\nu\nu}_{-\mathbf{k}\mathbf{k}})\, F_{\nu}(\mathbf{k}), \tag{5·5}$$

Examples of transfer processes of the form (5·3) and (5·5) are Phillips's (1957) and Miles's (1957) theories of wave generation by wind (§ 7).

(*b*) *Nonlinear interactions between waves and external fields*

The analysis of nonlinear external interactions is similar to the analysis of wave–wave interactions, except for some additional modifications due to the fact that the properties (*a*) and (*b*), which led to the simple transfer expression (4·13) from equation (4·10), no longer apply.

A complication also arises if the expressions corresponding to equations (4·6) and (4·7) contain more than two external field components. Although wave fields can be treated as Gaussian, and wave and external fields as statistically independent, the external fields are generally not Gaussian. Hence mean products containing more than two external field components cannot be reduced to the external-field spectra. This is not a basic difficulty, as the statistical structure of the external fields may be assumed to be known, but it leads to more complicated transfer expressions. In many cases, however, the Gaussian hypothesis may still be used to estimate the order of magnitude of the energy transfer (for example, in the generation of sound by turbulence, cf. Lighthill 1963). As we shall not be needing the exact expressions in the following applications, we shall assume here for simplicity that the external fields are also Gaussian. This means that the following expressions are not rigorously correct whenever products of two external-field spectra appear (equation (5·8)). Other expressions are not affected by the assumption.

It is convenient to divide the total transfer into three expressions according to

the spectral products occurring in the integrals. The first expression involves bilinear products of a wave spectrum and an external field spectrum,

$$\frac{\partial F_\nu(\mathbf{k})}{\partial t} = \iint \sum_{\substack{\nu_1 > 0 \\ \mu(\omega^\mu_{\mathbf{k}} > 0)}} \{(T^a_1 F_1 G_2 - T^b_1 F G_2)\,\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\,\delta(\omega_1+\omega_2-\omega)$$
$$+ (T^a_2 F_1 G_2 - T^b_2 F G_2)\,\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k})\,\delta(\omega_1-\omega_2-\omega)$$
$$+ (T^a_3 F_1 G_2 + T^b_3 F G_2)\,\delta(\mathbf{k}_2-\mathbf{k}_1-\mathbf{k})\,\delta(\omega_2-\omega_1-\omega) - T_4 F G_2\}\,\mathrm{d}\mathbf{k}, \mathrm{d}\mathbf{k}_2, \quad (5{\cdot}6)$$

where $F_1 = F_{\nu_1}(\mathbf{k}_1)$, $G_2 = G_\mu(\mathbf{k}_2)$, $F = F_\nu(\mathbf{k})$, $\omega_1 = \omega^{\nu_1}_{\mathbf{k}_1}$, $\omega_2 = \omega^{\mu}_{\mathbf{k}_2}$, $\omega = \omega^{\nu}_{\mathbf{k}}$

and
$$T^a_1 = 36\pi\omega^2 |E^{-\nu\nu_1\mu}_{-\mathbf{k}\mathbf{k}_1\mathbf{k}_2}|^2,$$
$$T^a_2 = 36\pi\omega^2 |E^{-\nu\nu_1\overline{\mu}}_{-\mathbf{k}\mathbf{k}_1-\mathbf{k}_2}|^2,$$
$$T^a_3 = 36\pi\omega^2 |E^{-\nu-\nu_1\mu}_{-\mathbf{k}-\mathbf{k}_1\mathbf{k}_2}|^2,$$
$$T^b_1 = 36\pi\omega\omega_1 \mathscr{R}\{E^{-\nu\nu_1\mu}_{-\mathbf{k}\mathbf{k}_1\mathbf{k}_2} E^{-\nu_1\nu\overline{\mu}}_{-\mathbf{k}_1\mathbf{k}-\mathbf{k}_2}\},$$
$$T^b_2 = 36\pi\omega\,\omega_1 \mathscr{R}\{E^{-\nu\nu_1\overline{\mu}}_{-\mathbf{k}\mathbf{k}_1-\mathbf{k}_2} E^{-\nu_1\nu\mu}_{-\mathbf{k}_1\mathbf{k}\mathbf{k}_2}\},$$
$$T^b_3 = 36\pi\omega\omega_1 \mathscr{R}\{E^{-\nu-\nu_1\mu}_{-\mathbf{k}-\mathbf{k}_1\mathbf{k}_2} E^{\nu_1\nu\overline{\mu}}_{\mathbf{k}_1\mathbf{k}-\mathbf{k}_2}\},$$
$$T_4 = +36\omega \sum_{\nu_1>0}\left[\omega_1 P\mathscr{I}\left\{\frac{E^{-\nu\nu_1\mu}_{-\mathbf{k}\mathbf{k}_1\mathbf{k}_2} E^{-\nu_1\nu\overline{\mu}}_{-\mathbf{k}_1\mathbf{k}-\mathbf{k}_2}}{\omega_1+\omega_2-\omega} + \frac{E^{-\nu\nu_1\overline{\mu}}_{-\mathbf{k}\mathbf{k}_1-\mathbf{k}_2} E^{-\nu_1\nu\mu}_{-\mathbf{k}_1\mathbf{k}\mathbf{k}_2}}{\omega_1-\omega_2-\omega} + \frac{E^{-\nu-\nu_1\overline{\mu}_2}_{-\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2} E^{\nu_1\nu\nu_2}_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2}}{\omega_1+\omega_2+\omega}\right.\right.$$
$$\left.\left. - \frac{E^{-\nu-\nu_1\mu}_{-\mathbf{k}-\mathbf{k}_1\mathbf{k}_2} E^{\nu_1\nu\overline{\mu}}_{\mathbf{k}_1\mathbf{k}-\mathbf{k}_2}}{\omega_2-\omega_1-\omega}\right\}\right] - 32\omega\mathscr{I}\{E^{-\nu\nu\overline{\mu}\mu}_{-\mathbf{k}\mathbf{k}-\mathbf{k}_2\mathbf{k}_2}\}.$$

The second expression represents the energy transfer due to nonconservative wave–wave interactions. It is similar to equation (5·6) except that the index μ is replaced by ν_2 and the external spectrum G_2 by $F_2 = F_{\nu_2}(\mathbf{k}_2)$. The number of pairing combinations is larger for two wave spectra than for one wave spectrum and one external-field spectrum; this leads to additional factors 2 and 4, respectively, in the transfer functions T^a_j and T^b_j ($j = 1, 2, 3$). Similarly, the first and second terms of T_4 are multiplied by factors 4 and 3, respectively. By reordering, the transfer expression can be written in a form analogous to the transfer equation (4·13) for conservative wave–wave interactions,

$$\frac{\partial F_\nu(\mathbf{k})}{\partial t} = \sum_{\nu_1\nu_2>0} \iint (T^\alpha_1 F_1 F_2 - T^\beta_1 F F_1 - T^\gamma_1 F F_2)\,\delta(\omega_1+\omega_2-\omega)\,\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})$$
$$+ 2(T^\alpha_2 F_1 F_2 + T^\beta_2 F F_1 - T^\gamma_2 F F_2)\,\delta(\omega_1-\omega_2-\omega)\,\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k})$$
$$- T_3 F F_1\}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2, \quad (5{\cdot}7)$$

where
$$T^\alpha_1 = 72\pi\omega^2 |E^{-\nu\nu_1\nu_2}_{-\mathbf{k}\mathbf{k}_1\mathbf{k}_2}|^2,$$
$$T^\alpha_2 = 72\pi\omega^2 |E^{-\nu\nu_1-\nu_2}_{-\mathbf{k}\mathbf{k}_1-\mathbf{k}_2}|,^2$$
$$T^\beta_1 = 72\pi\omega\omega_2 \mathscr{R}\{E^{-\nu\nu_1\nu_2}_{-\mathbf{k}\mathbf{k}_1\mathbf{k}_2} E^{-\nu_2\nu-\nu_1}_{-\mathbf{k}_2\mathbf{k}-\mathbf{k}_1}\},$$
$$T^\beta_2 = 72\pi\omega\omega_2 \mathscr{R}\{E^{-\nu\nu_1-\nu_2}_{-\mathbf{k}\mathbf{k}_1-\mathbf{k}_2} E^{\nu_2\nu-\nu_1}_{\mathbf{k}_2\mathbf{k}-\mathbf{k}_1}\},$$
$$T^\gamma_1 = 72\pi\omega\omega_1 \mathscr{R}\{E^{-\nu\nu_1\nu_2}_{-\mathbf{k}\mathbf{k}_1\mathbf{k}_2} E^{-\nu_1\nu-\nu_2}_{-\mathbf{k}_1\mathbf{k}-\mathbf{k}_2}\},$$
$$T^\gamma_2 = 72\pi\omega\omega_1 \mathscr{R}\{E^{-\nu\nu_1-\nu_2}_{-\mathbf{k}\mathbf{k}_1-\mathbf{k}_2} E^{-\nu_1\nu\nu_2}_{-\mathbf{k}_1\mathbf{k}\mathbf{k}_2}\},$$
$$T_3 = 144\omega \sum_{\nu_2>0}\left[\omega_2 P\mathscr{I}\left\{\frac{E^{-\nu\nu_1\nu_2}_{-\mathbf{k}\mathbf{k}_1\mathbf{k}_2} E^{-\nu_2\nu-\nu_1}_{-\mathbf{k}_2\mathbf{k}-\mathbf{k}_1}}{\omega_1+\omega_2-\omega} + \frac{E^{-\nu-\nu_1\nu_2}_{-\mathbf{k}-\mathbf{k}_1\mathbf{k}_2} E^{-\nu_2\nu\nu_1}_{-\mathbf{k}_2\mathbf{k}\mathbf{k}_1}}{\omega_2-\omega_1-\omega} + \frac{E^{-\nu-\nu_1-\nu_2}_{-\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2} E^{\nu_2\nu\nu_1}_{\mathbf{k}_2\mathbf{k}\mathbf{k}_1}}{\omega_1+\omega_2+\omega}\right.\right.$$
$$\left.\left. - \frac{E^{-\nu\nu_1-\nu_2}_{-\mathbf{k}\mathbf{k}_1-\mathbf{k}_2} E^{\nu_2\nu-\nu_1}_{\mathbf{k}_2\mathbf{k}-\mathbf{k}_1}}{\omega_1-\omega_2-\omega}\right\}\right] - 96\omega\mathscr{I}\{E^{-\nu\nu-\nu_1\nu_1}_{-\mathbf{k}\mathbf{k}-\mathbf{k}_1\mathbf{k}_1}\}.$$

For symmetrical coefficients, equation (5·7) becomes identical with equation (4·13).

The third expression represents the transfer due to external field interactions only,

$$\frac{\partial F_\nu(\mathbf{k})}{\partial t} = \iint \sum_{\substack{\mu_1 \mu_2 \\ (\omega_1, \omega_2 > 0)}} \{T_1 G_1 G_2 \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})\, \delta(\omega_1 + \omega_2 - \omega) + 2T_2 G_1 G_2 \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k})\, \delta(\omega_1 - \omega_2 - \omega)\}\, \mathrm{d}\mathbf{k}_1\, \mathrm{d}\mathbf{k}_2, \quad (5{\cdot}8)$$

where $T_1 = 72\pi\omega^2 |E^{-\nu\mu_1\mu_2}_{-\mathbf{k}\mathbf{k}_1\mathbf{k}_2}|^2$,

$T_2 = 72\pi\omega^2 |E^{-\nu\mu_1\bar{\mu}_2}_{-\mathbf{k}\mathbf{k}_1-\mathbf{k}_2}|^2$.

Equations (5·3), (5·5), (5·6), (5·7) and (5·8) are the complete set of lowest order transfer expressions involving products of not more than two spectra.

6. Transfer diagrams

It is convenient to summarize the transfer expressions derived in §§4 and 5 in terms of energy *transfer* diagrams. In the special case of conservative wave–wave interactions, these can be interpreted as collision diagrams in a particle picture (Hasselmann 1966*a*). In the general case of nonconservative interactions, the particle interpretation is no longer applicable, but several useful features of the diagrams remain valid.

Each term in the transfer integrals can be associated with an arrow component in a transfer diagram. Terms containing δ function factors, and the linear transfer expressions (5·3) and (5·5), are represented by a diagram in which n components enter a vertex and a single component leaves the vertex. (We have considered in detail only the linear case $n = 1$ and the quadratic case $n = 2$.) Different sign combinations in the δ functions are distinguished by components and anti-components, in the same way as in the interaction diagrams (§3).

Thus the factor $\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})\, \delta(\omega_1 + \omega_2 - \omega)$ is represented by two arrows equal to $\mathbf{k}_1$ and $\mathbf{k}_2$ entering a vertex and an arrow equal to $\mathbf{k}$ leaving the vertex. The factor $\delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k})\ \delta(\omega_1 - \omega_2 - \omega)$ is represented by the same diagram, except that the component $\mathbf{k}_2$ is replaced by an anti-component. The summation rule for wavenumbers and frequencies is the same as for interaction diagrams. Both δ function factors can be represented by two further diagrams, in which either $\mathbf{k}_1$ or $\mathbf{k}_2$ is chosen as the resultant component. We shall refer to the three diagrams of a given δ function factor as a diagram *set*. (The three diagrams which have outgoing anti-components may be ignored; they are identical with the other diagrams except that the signs of all components are changed.) Each diagram of a set represents an energy transfer from the ingoing components to the outgoing component. If the outgoing component represents a fixed external field, however, the transfer is zero; these diagrams can be excluded. As in the interaction diagrams, the arrows point always in the propagation direction of the waves. (To this end, the transfer expressions were written in a form which contains only positive frequencies.)

The last terms in equations (5·6) and (5·7), which contain no δ function factors, are represented simply by two arrows entering a vertex, without a resultant

component. In this case no distinction is made between components and anti-components.

The structure of the transfer expressions can be determined from the diagrams by a single rule: *the rate of change of the spectrum of any wave component or anti-component in a transfer diagram is proportional to the product of the spectral densities of the ingoing components.*

Thus the linear transfer expressions (5·3) and (5·5) are represented by the linear diagrams 5(*a*) and (*b*), respectively. The first two terms associated with the factor $\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\,\delta(\omega_1+\omega_2-\omega)$ in equation (5·6) are represented by the two diagrams (*c*). The third diagram of the set represents the generation of the external component μ_2 and can be ignored. If μ_2 is replaced by ν_2, one obtains the first three terms in equation (5·7). In this case all three diagrams of the set contribute to the energy transfer. If ν_1 is replaced by μ_1, one obtains similarly the first term in equation (5·8).

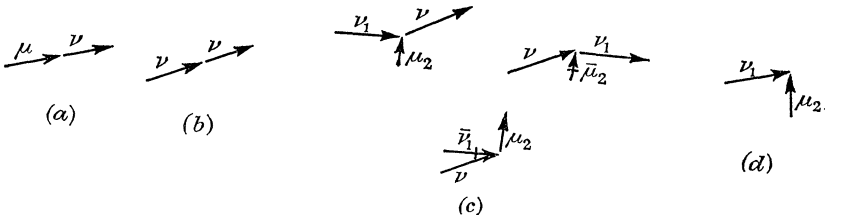

FIGURE 5. Transfer diagram notation of energy transfer expressions. Diagrams (*a*) and (*b*) represent the linear transfer expressions (5·3) and (5·5), respectively. Diagrams (*c*) represent the first two terms, containing the factor $\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\,\delta(\omega_1+\omega_2-\omega)$, in equation (5·6). The third diagram of the set generates an external component, which yields zero transfer. Other δ function terms in equations (5·6), (5·7) and (5·8) are obtained by interchanging either components and anti-components or wave and external-field components. Diagram (*d*) represents the last term in equation (5·6).

Here only one of the diagrams contributes. The remaining terms of the transfer expressions (5·6), (5·7) and (5·8) are obtained by replacing one of the ingoing components by an anti-component. Diagram (*d*) represents the last term in equation (5·6). The diagrams (*a*), (*c*), (*d*), etc., will be referred to as the interactions $\mu \to \nu$, $\nu_1\mu_2 \to \nu$, $\nu_1\mu_2$, etc.

In the case of conservative wave–wave interactions, the transfer rates for all components of a diagram can be characterized by a single transfer coefficient, and the same coefficient applies to all diagrams of a given set. Furthermore, there is a close correspondence between transfer and interaction diagrams; the various contributions to the transfer expressions can be classified directly in terms of interaction (Feynman) diagrams, and there is no need to distinguish explicitly between the two types of diagram (Hasselmann 1966*a*).

This simplicity is lost in the general case. The transfer rate is different for each component of a diagram. The relation between the transfer diagrams and the interaction diagrams, which describe the perturbation structure of the transfer terms, is more involved. (For example, the transfer term corresponding to diagram 5(*d*) contains a fourth order coupling coefficient.) But the transfer diagrams none

the less remain a useful tool for discussing the structure of complicated interacting systems.

The application of transfer (Feynman) diagrams for conservative wave–wave scattering processes is illustrated in a number of examples in Hasselmann (1966*a*). In the following we shall consider a case involving nonconservative external interactions.

7. The generation of waves by turbulent wind

Let us apply the foregoing results to interactions between gravity waves g and the atmospheric turbulent boundary layer. We assume that the boundary layer consists of a mean flow m, which depends only on the vertical coordinate, and a statistically stationary, horizontally homogeneous, fluctuating field f.

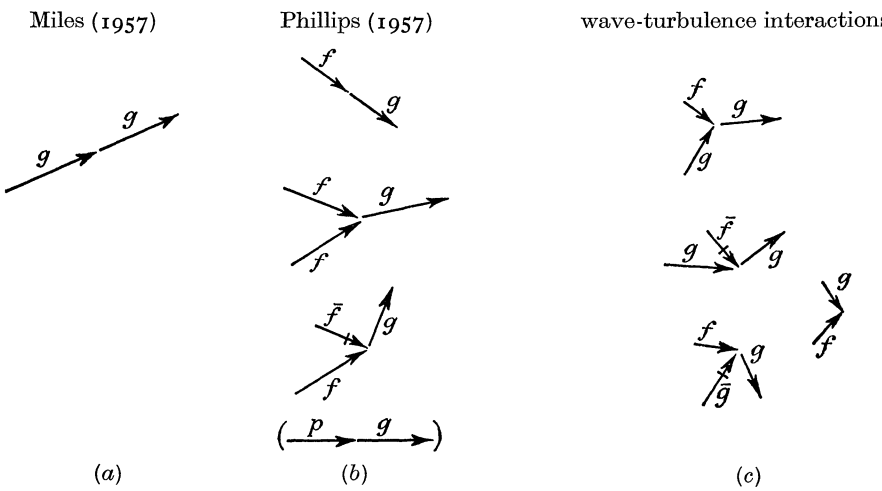

FIGURE 6. Transfer diagrams for interactions between gravity waves and the atmospheric boundary layer.

The details of the analysis will be presented elsewhere. One finds that the interactions between g and f can be expanded in a perturbation series of the form (2·14). The mean flow m determines the form of the $g-f$ coupling coefficients, but does not enter as a perturbation. Hence we are concerned formally only with interactions between the components g and f.

The complete set of lowest order transfer diagrams, containing not more than two ingoing components, is shown in figure 6. (The diagrams $g_1 g_2 \rightarrow g_3$ and $g_1 g_2$, which should be included in the complete set, are not shown. The first diagram cannot satisfy the resonance conditions (Phillips 1960). The second diagram can be shown to be a higher order effect as compared with diagram (*a*) and can therefore be neglected.)

The linear diagram (*a*), $g \rightarrow g$, corresponds to Miles's (1957) theory of wave generation through linear coupling between the waves and the mean boundary layer flow. Each wave component produces a perturbation of the mean flow. The associated pressure perturbation feeds back positively into the wave component, making it grow. The physical mechanism of this feed back has been discussed by Lighthill

(1962). According to the transfer rule, the rate of growth of the spectral density $F(\mathbf{k})$ of the component g is proportional to $F(\mathbf{k})$, so that

$$\partial F(\mathbf{k})/\partial t = \beta F(\mathbf{k}), \tag{7·1}$$

where β depends on the mean boundary layer profile. The wave growth is exponential.

The second set of diagrams (*b*) represents Phillips's (1957) theory of forced wave generation by turbulent pressure fluctuations p. The process can be represented more simply by the linear diagram $p \to g$. However, the pressure is a derived field; it can be expressed in terms of interacting velocity components, which yields the three diagrams (*b*). The transfer rate is proportional to the three-dimensional spectrum G_p of the turbulent pressure at the sea surface (Hasselmann 1960)

$$\frac{\partial F(\mathbf{k})}{\partial t} = \frac{\pi\omega^2}{2\rho g} G_p(\mathbf{k}, -\omega), \tag{7·2}$$

where ω is the frequency of the wave component $\mathbf{k}$ and ρ is the density of water. The spectra are normalized such that

$$\iint F(\mathbf{k})\,\mathrm{d}\mathbf{k} = \tfrac{1}{2}\rho g\langle\zeta^2\rangle = \tfrac{1}{2}E,$$

where ζ is the surface displacement and E the wave energy, and

$$\iiint G_p(\mathbf{k}, \omega)\,\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2\mathrm{d}\omega = \langle p^2\rangle.$$

Equation (7·2) yields a linear wave growth.

The remaining interactions, diagrams (*c*), have not been considered previously. In the simplest case they represent a disturbance of the mean boundary layer flow by the component g', an interaction between this disturbance and a turbulence component f, and a feedback of the pressure fluctuation due to this interaction into the component g. According to the transfer rule, the resultant energy transfer is of the form

$$\partial F(\mathbf{k})/\partial t = -\gamma F(\mathbf{k}) + \int \delta(\mathbf{k}, \mathbf{k}')\, F(\mathbf{k}')\,\mathrm{d}\mathbf{k}'. \tag{7·3}$$

The first term represents the energy loss (or gain) of the component g'. It is proportional to the spectral density of the component g' itself. The second term represents the energy gained by the component g, which is proportional to the energy spectrum of the components g' at other wavenumbers. The functions γ and δ are linear functionals of the turbulence spectrum at wavenumbers corresponding to the turbulence components in the diagrams. The second term is always positive, whereas the first term can have either sign.

It is difficult to decide from present data which of the three mechanisms is the most important, as very few quantitative measurements of wave growth exist. Snyder & Cox (1966) have measured the growth of waves of period 3·3 s under

various wind conditions. They find that the wave growth is initially linear, and that the initial growth rate is not inconsistent with Phillips's mechanism, if one assumes that Priestley's (1965) measurements of pressure fluctuations over land are representative also for the ocean. The initial linear stage is followed by a period of more rapid exponential growth, which is responsible for the major part of the wave energy. The growth rate in this period is almost an order of magnitude larger than predicted by Miles or by the earlier sheltering theory of Jeffreys (1925).

It is natural to suspect then that the wave-turbulence interactions, which complete the lowest order interaction set, are the principal source of wave energy. Although it is difficult to estimate the transfer rate (7·3) on the basis of present turbulence measurements, the expression has certain qualitative features which appear to agree with the observations. The turbulence spectra are peaked at lower frequencies than the wave spectra, so that the frequency difference between the component g and g' in the interactions $g'f \rightarrow g$, etc. will generally be small. Hence the energy transfer is confined to neighbouring frequency bands of the wave spectrum. If the waves are generated first at high frequencies, then the second term in equation (7·3) will tend to generate waves of slightly lower frequency. When these are sufficiently large, the adjacent lower frequency band will be generated, and so forth. Observations seem to support this sequential development of the wave spectrum, but other explanations have also been put forward (cf. Phillips & Katz 1961; Hasselmann 1962).

Once the waves have been generated by the second term in equation (7·3), the first term will either enhance or counteract their further development, depending on the sign of δ. The latter case is of some interest, as it is then possible to explain the observed equilibrium of the spectrum without invoking a strong momentum transfer to ocean currents. It is usually assumed that in the equilibrium state the transfer of momentum from the atmospheric boundary layer to the waves is of the same order of magnitude as during wave growth, but is balanced by an equally strong transfer of momentum from the waves to the mean ocean currents. On this assumption, Snyder & Cox find that their observed wave growth rates yield an effective shear stress several times the tangential wall stress found for boundary layers at fixed walls. On the other hand, various measurements of the boundary layer over the ocean have not indicated appreciable deviations (under conditions of neutral buoyancy) from the stress and velocity profiles of the logarithmic 'law of the wall'. This difficulty would not arise if the equilibrium could be largely explained in terms of the wave-atmosphere and nonlinear wave–wave interactions (Phillips 1960; Hasselmann 1962)†, without invoking a strong dissipative process.

Further investigations of the atmospheric boundary layer, correlated more closely with measurements of both wave spectra and spectral growth rates, would help to resolve these questions.

† The significance of wave–wave interactions for the energy balance of gravity waves is indicated by the recent swell-propagation observations of Snodgrass *et al.* (1966).

Appendix A

Benney & Saffman (1966) have shown that the transfer expressions can be derived without invoking the Gaussian hypothesis if the cumulants are assumed to be smooth in wavenumber space (or contain only certain δ function singularities). However, Benney & Saffman's derivation (if assumed to apply for all time) contradicts the irreversibility of the transfer expression. It is impossible to derive the transfer expression from purely local considerations without introducing a statistical hypothesis which, in effect, determines the direction of time. This can be deduced from the well known discussion of the paradox of irreversibility for the Boltzmann equation.

Suppose that the transfer expressions are valid for all time $t > 0$ for a certain ensemble E of wave fields. Then determine the spectrum $F(\mathbf{k}, t_1 - \tau)$ at a time $t_1 - \tau > 0$ earlier than a given time $t_1 > 0$ by solving the equations of motion backwards in time, using the ensemble of states at time t_1 as initial value. Since the equations of motion are reversible, one obtains the same spectrum if one reverses the sign of the velocities $\dot{q}''_{\mathbf{k}}$ at time t_1 and then determines $F(\mathbf{k}, t_1 + \tau)$ by solving the initial value problem for this new ensemble $\bar{E}$ forwards in time. If the transfer expressions are assumed to be valid for *both* ensembles E and $\bar{E}$, one obtains the same spectrum at $t_1 + \tau$ for both E and $\bar{E}$. For the transfer expressions depend only on the spectra, and these are independent of the sign of the velocity field. It follows that $F(\mathbf{k}, t_1 - \tau) = F(\mathbf{k}, t_1 + \tau)$ for the ensemble E. Thus the time evolution is symmetric about t_1. In particular, there is a discontinuity of slope at t_1, with

$$\frac{\partial F}{\partial t}(\mathbf{k}, t_1 - 0) = -\frac{\partial F}{\partial t}(\mathbf{k}, t_1 + 0). \tag{A 1}$$

This clearly contradicts the assumption that the transfer expressions are valid for all $t > 0$. Hence if the transfer expressions are valid for an ensemble E, they cannot be valid for the ensemble $\bar{E}$. The assumption that a given ensemble of velocity fields corresponds to E rather than $\bar{E}$ is then necessarily a hypothesis which cannot be decided without analysing the history of the field.

Clearly, any attempt to derive the transfer expressions locally at time t_1 must be based on the distinction between the ensembles E and $\bar{E}$. If the class of initial conditions allowed at time t_1 includes both E and $\bar{E}$, the analysis must lead to a cusp. Both Gaussian fields and fields with smooth cumulants represent symmetrical classes in this sense.† Since Prigogine (1962) has proved the validity of the transfer expressions for all $t > 0$, it follows that in general the cumulants are neither zero nor remain smooth.

The behaviour of the cumulants is best understood from the analysis of the linear field (appendix B). The continuous mixing of wave groups propagating in

† The relation (A 1) can be verified directly in Benney & Saffman's case. The right-hand side of Benney & Saffman's equation (2·51) and similarly the first term on the right-hand side of equation (2·56) take a negative sign if one goes to the limit $t \to -\infty$ instead of $t \to \infty$. This affects the right hand side of equation (2·44) for the energy transfer rate. Thus the spectral derivative takes opposite signs depending on whether it is determined from past or future states. The same conclusion follows for a rigorously Gaussian field.

different directions leads to the degeneration of an initially smooth cumulant field into a rapidly fluctuating fine structure in wavenumber space. In the nonlinear case, the combination of mixing and weak coupling leads to a complicated fine structure containing singularities.† The Gaussian hypothesis is equivalent to the assumption that the fine structure can be ignored if one goes forwards in time. But it cannot be ignored if one attempts to reconstruct the past. The connexion between irreversibility of the coarse grained distributions and the time asymmetry of fine grained distributions is well known from other problems of statistical mechanics.

Physically, the significance of the fine structure can best be understood by regarding the interactions as taking place between large, but finite wave groups, rather than infinite wave trains. The situation is then closely analogous to the case of interacting particles (Appendix B).

Appendix B. The Gaussian property of random linear wave fields

It is generally assumed that random, linear wave fields can be regarded as Gaussian. The hypothesis has widespread support from observations and can be justified intuitively by a loose application of the central limit theorem. A proof of the Gaussian property, however, and a general understanding of the conditions on which it is based, appear to be lacking.

The question has gained particular significance through recent work on nonlinear interactions in random wave fields. Spectral analysis techniques have been extended to higher orders to determine nonlinear transfer functions (Hasselmann, Munk & MacDonald 1963; Hasselmann 1966*b*). In contrast with the linear case, the higher order methods depend heavily on the assumption that the first order, linear fields are Gaussian. In the present context, however, we shall be concerned primarily with recent discussions of the Gaussian hypothesis in connexion with the derivation of the transfer expressions of §§4 and 5. Several misunderstandings in this discussion can be traced directly to a misconception of the nature of the Gaussian property of linear wave fields.

It has occasionally been stated (cf. Benney & Saffman 1966) that the Gaussian hypothesis appears no more tenable for interacting wave fields than for a turbulence field; in both cases the interactions destroy the Gaussian property of a given initial state in a time scale comparable with the time scale of the energy transfer. The analogy ignores an important property of linear wave fields: it is shown in the following that a set of homogeneous, linear wave fields which are non-Gaussian initially, and which have smooth cumulants in wavenumber space, asymptotically approach a Gaussian state (and furthermore become stationary and mutually independent). Thus the Gaussian property is not merely consistent with, but is a consequence of the linearity of the wave fields. In the case of weak nonlinear interactions, the linear tendency to a Gaussian state may then be expected

† The author derived the same result as Benney & Saffman in the course of a discussion about the Gaussian property with F. P. Bretherton three years ago. It was pointed out by Bretherton that the singularities generated by the resonant interactions violated the assumption of smooth cumulants. The inconsistency of the assumption follows still more simply from the development of a fine structure.

to maintain the fields approximately Gaussian, despite the counteracting influence of the nonlinearities.

The approach of the linear fields to a Gaussian state holds in the 'coarse-grained' sense: it applies to spectral moments that have been smoothed by convolution with an arbitrarily narrow, but finite filter. The fine structure remains nonGaussian, but the frequency resolution required to detect it increases indefinitely with time and ultimately exceeds all attainable resolutions. Although unobservable, the fine structure is none the less important, for it explains the irreversible behaviour of the coarse grained distributions in both the linear and nonlinear case.

Representation of the fields

Consider a set of random, real fields $\phi_\nu(\mathbf{x}, t)$, where $\mathbf{x}$ is the coordinate vector in a space of dimension δ. The fields can be characterized by the set of mean products

$$R_{\nu_1 \dots \nu_n} = \langle \varphi_{\nu_1}(\mathbf{x}_1, t_1) \dots \varphi_{\nu_n}(\mathbf{x}_n, t_n) \rangle, \tag{B 1}$$

where the cornered brackets denote ensemble mean values. We assume that the fields are statistically homogeneous, so that

$$R_{\nu_1 \dots \nu_n} = R_{\nu_1 \dots \nu_n}(\boldsymbol{\xi}_1 \dots \boldsymbol{\xi}_{n-1}, t_1 \dots t_n)$$

where $\boldsymbol{\xi}_m = \mathbf{x}_m - \mathbf{x}_n$, $m = 1, \dots, n-1$ is the difference coordinate. For simplicity, we set $\langle \varphi \rangle = 0$. Alternatively, we can introduce the Fourier–Stieltjes representations

$$\varphi_\nu(\mathbf{x}, t) = \int \mathrm{d}\phi_\nu(\mathbf{k}, t)\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \tag{B 2}$$

and describe the field by the set of mean products

$$\langle \mathrm{d}\phi_{\nu_1}(\mathbf{k}_1, t_1) \dots \mathrm{d}\phi_{\nu_n}(\mathbf{k}_n, t_n) \rangle = G_{\nu_1 \dots \nu_n}(\mathbf{k}_1 \dots \mathbf{k}_{n-1}, t_1 \dots t_n)\, \delta(\mathbf{k}_1 + \dots + \mathbf{k}_n)\, \mathrm{d}\mathbf{k}_1 \dots \mathrm{d}\mathbf{k}_n, \tag{B 3}$$

where

$$G_{\nu_1 \dots \nu_n}(\mathbf{k}_1 \dots \mathbf{k}_{n-1}, t_1 \dots t_n) = (2\pi)^{-(n-1)\delta} \int \dots \int R_{\nu_1 \dots \nu_n}(\boldsymbol{\xi}_1 \dots \boldsymbol{\xi}_{n-1}, t_1 \dots t_n) \exp\{-\mathrm{i}(\mathbf{k}_1 \cdot \boldsymbol{\xi}_1 + \dots + \mathbf{k}_{n-1} \cdot \boldsymbol{\xi}_{n-1})\}\, \mathrm{d}\boldsymbol{\xi}_1 \dots \mathrm{d}\boldsymbol{\xi}_{n-1} \tag{B 4}$$

is the Fourier transform of $R_{\nu_1 \dots \nu_n}$.

The δ function in equation (B 3) is characteristic for homogeneous fields. It arises formally because the mean products $\langle \varphi_{\nu_1}(\mathbf{x}_1, t_1) \dots \varphi_{\nu_n}(\mathbf{x}_n, t_n) \rangle$ remain finite as $\mathbf{x}_n \to \infty$ for fixed $\boldsymbol{\xi}_m$. For similar reasons, the mean products $\langle \mathrm{d}\phi_{\nu_1}(\mathbf{k}_1, t_1) \dots \mathrm{d}\phi_{\nu_n}(\mathbf{k}_n, t_n) \rangle$ generally contain further δ functions which may be associated with the asymptotic behaviour of $\langle \varphi_{\nu_1}(\mathbf{x}_1, t_1) \dots \varphi_{\nu_n}(\mathbf{x}_n, t_n) \rangle$ for large $\mathbf{x}_j$. The singularities may be factorized out by expanding the moments in terms of cluster functions, or cumulants.

We shall assume that the mean products satisfy the asymptotic condition

$$\langle \varphi(1) \dots \varphi(n) \rangle \to \langle \varphi(1) \dots \varphi(p) \rangle \langle \varphi(p+1) \dots \varphi(q) \rangle \dots \langle \varphi(s+1) \dots \varphi(n) \rangle, \tag{B 5}$$

with $\varphi(\alpha) \equiv \varphi_{\nu_\alpha}(\mathbf{x}_\alpha, t_\alpha)$ if the separations between individual clusters of an arbitrary *cluster set* $(\mathbf{x}_1 \ldots \mathbf{x}_p)$, $(\mathbf{x}_{p+1} \ldots \mathbf{x}_q) \ldots (\mathbf{x}_{s+1} \ldots \mathbf{x}_n)$ tend to infinity. The *cluster functions* S are then defined by the recursive relations

$$\left.\begin{aligned} S_\nu &= \langle \varphi_\nu \rangle (= 0), \\ S_{\nu_1 \nu_2} &= R_{\nu_1 \nu_2} - S_{\nu_1} S_{\nu_2}, \\ S_{\nu_1 \nu_2 \nu_3} &= R_{\nu_1 \nu_2 \nu_3} - S_{\nu_1} S_{\nu_2 \nu_3} - S_{\nu_2} S_{\nu_1 \nu_3} - S_{\nu_3} S_{\nu_1 \nu_2} - S_{\nu_1} S_{\nu_2} S_{\nu_3} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ S_{\nu_1 \ldots \nu_n} &= R_{\nu_1 \ldots \nu_n} - \sum_{\substack{\text{cluster} \\ \text{sets}}} S_{\alpha_1 \ldots \alpha_p} S_{\beta_1 \ldots \beta_q} \ldots S_{\gamma_1 \ldots \gamma_s}, \end{aligned}\right\} \tag{B 6}$$

which are constructed such that under the condition (B 5) all $S_{\nu_1 \ldots \nu_n}$ tend to zero if any difference coordinate $\boldsymbol{\xi}_m$ approaches infinity. We shall assume further that $S_{\nu_1 \ldots \nu_n}$ approaches zero sufficiently smoothly to ensure a continuous Fourier transform

$$H_{\nu_1 \ldots \nu_n}(\mathbf{k}_1 \ldots \mathbf{k}_{n-1}, t_1 \ldots t_n) = (2\pi)^{-(n-1)\delta} \int \ldots \int S_{\nu_1 \ldots \nu_n}(\boldsymbol{\xi}_1 \ldots \boldsymbol{\xi}_{n-1}, t_1 \ldots t_n) \times \exp\{-\mathrm{i}(\mathbf{k}_1 . \boldsymbol{\xi}_1 + \ldots + \mathbf{k}_{n-1} . \boldsymbol{\xi}_{n-1})\} \mathrm{d}\boldsymbol{\xi}_1 \ldots \mathrm{d}\boldsymbol{\xi}_{n-1}. \tag{B 7}$$

Comparing equations (B 6), (B 7) with equations (B 3) and (B 4) we can then write

$$\begin{aligned} \langle \mathrm{d}\phi_{\nu_1}(\mathbf{k}_1, t_1) \ldots \mathrm{d}\phi_{\nu_n}(\mathbf{k}_n, t_n) \rangle &= \mathrm{d}\mathbf{k}_1 \ldots \mathrm{d}\mathbf{k}_n \\ &\times \Big\{ \sum_{\substack{\text{cluster} \\ \text{sets}}} \delta(\mathbf{k}_{\mu_1} + \ldots + \mathbf{k}_{\mu_p}) \ldots \delta(\mathbf{k}_{\lambda_1} + \ldots + \mathbf{k}_{\lambda_s}) \\ &\times H_{\alpha_1 \ldots \alpha_p}(\mathbf{k}_{\mu_1} \ldots \mathbf{k}_{\mu_{p-1}}, t_{\mu_1} \ldots t_{\mu_p}) \ldots H_{\gamma_1 \ldots \gamma_s}(\mathbf{k}_{\lambda_1} \ldots \mathbf{k}_{\lambda_{s-1}}, t_{\lambda_1} \ldots t_{\lambda_p}) \Big\} \end{aligned}$$

where $\alpha_j \equiv \nu_{\mu j} \ldots \gamma_j \equiv \nu_{\lambda_j}$. (B 8)

Asymptotic properties of wave fields

Let us assume now that the fields φ_ν are wave fields whose Fourier components satisfy a harmonic oscillator equation

$$\mathrm{d}\ddot{\phi}_\nu(\mathbf{k}) + \omega_\nu^2(\mathbf{k})\, \mathrm{d}\phi_\nu(\mathbf{k}) = 0, \tag{B 9}$$

with solutions

$$\mathrm{d}\phi_\nu(\mathbf{k}, t) = \mathrm{d}\phi_\nu^+(\mathbf{k})\, \mathrm{e}^{-\mathrm{i}\omega_\nu t} + \mathrm{d}\phi_\nu^-(\mathbf{k})\, \mathrm{e}^{\mathrm{i}\omega_\nu t}. \tag{B 10}$$

For real φ_ν,

$$\mathrm{d}\phi_\nu^+ = (\mathrm{d}\phi_\nu^-(-\mathbf{k}))^*. \tag{B 11}$$

The mean product (B 3) then takes the form

$$\langle \mathrm{d}\phi_{\nu_1}(\mathbf{k}_1, t_1) \ldots \mathrm{d}\phi_{\nu_n}(\mathbf{k}_n, t_n) \rangle = \sum_{\text{signs } sj} \langle \mathrm{d}\phi_{\nu_1}^{s_1}(\mathbf{k}_1) \ldots \mathrm{d}\phi_{\nu_n}^{s_n}(\mathbf{k}_n) \rangle \times \exp\{-\mathrm{i}(s_1 \omega_{\nu_1} t_1 + \ldots + s_n \omega_{\nu_n} t_n)\} \tag{B 12}$$

where

$$\langle \mathrm{d}\phi_{\nu_1}^{s_1}(\mathbf{k}_1) \ldots \mathrm{d}\phi_{\nu_n}^{s_n}(\mathbf{k}_n) \rangle = G_{\nu_1 \ldots \nu_n}^{s_1 \ldots s_n}(\mathbf{k}_1 \ldots \mathbf{k}_{n-1})\, \delta(\mathbf{k}_1 + \ldots + \mathbf{k}_n)\, \mathrm{d}\mathbf{k}_1 \ldots \mathrm{d}\mathbf{k}_n \tag{B 13}$$

is given by the initial statistical distribution.

From equations (B 3) and (B 8) we obtain similar relations for the cumulants

$$H_{\nu_1 \ldots \nu_n}(\mathbf{k}_1 \ldots \mathbf{k}_{n-1}, t_1 \ldots t_n) = \sum_{s_1 \ldots s_n} H_{\nu_1 \ldots \nu_n}^{s_1 \ldots s_n}(\mathbf{k}_1 \ldots \mathbf{k}_{n-1}) \exp\{-\mathrm{i}(s_1 \omega_{\nu_1} t_1 + \ldots + s_n \omega_{\nu_n} t_n)\}, \tag{B 14}$$

where $H^{s_1\ldots s_n}_{\nu_1\ldots\nu_n}$ is the set of initial cumulants associated with the set of initial moments $G^{s_1\ldots s_n}_{\nu_1\ldots\nu_n}$.

We shall assume that the initial distributions are arbitrary, except for the restriction that the cumulants $H^{s_1\ldots s_n}_{\nu_1\ldots\nu_n}$ are continuous.

For Gaussian fields, all cumulants higher than the second vanish. It follows immediately from equation (B 14) that for arbitrary initial conditions a set of wave fields cannot approach an asymptotic Gaussian state in the strict sense, since the right-hand side of equation (B 14) contains a finite number of constant-modulus terms, which in general do not cancel. Consider, however, the *observable* spectral moments, which are obtained by convolution of the theoretical moments with finite-width filters,

$$\bar{H}_{\nu_1\ldots\nu_n}(\mathbf{k}_1\ldots k_{n-1},\tau_1\ldots\tau_{n-1},t_n) = \int\ldots\int H_{\nu_1\ldots\nu_n}(\mathbf{k}'_1\ldots\mathbf{k}'_{n-1},\tau_1+t_n,\ldots,\tau_{n-1}+t_n,t_n)$$
$$\gamma(\mathbf{k}'_1-\mathbf{k}_1,\ldots,\mathbf{k}'_{n-1}-k_{n-1})\,\mathrm{d}\mathbf{k}'_1\ldots\mathrm{d}\mathbf{k}'_{n-1}, \quad \text{(B 15)}$$

where $\gamma(\mathbf{k}_1\ldots\mathbf{k}_{n-1})$ is a finite, continuous filter function normalized such that

$$\int\ldots\int\gamma(\mathbf{k}_1\ldots\mathbf{k}_{n-1})\,\mathrm{d}\mathbf{k}_1\ldots\mathrm{d}\mathbf{k}_{n-1}=1.$$

In practice, γ is effectively non-zero only in a small region around the origin $\mathbf{k}_1=0,\ldots,\mathbf{k}_{n-1}=0$.

Let us investigate the limit of $\bar{H}_{\nu_1\ldots\nu_n}$ as $t_n\to\infty$ for fixed time lags

$$\tau_j = t_j - t_n (j=1,\ldots,n-1).$$

According to equation (B 14),

$$\bar{H}_{\nu_1\ldots\nu_n} = \sum_{s_1\ldots s_n}\int\ldots\int H^{s_1\ldots s_n}_{\nu_1\ldots\nu_n}(\mathbf{k}'_1\ldots\mathbf{k}'_{n-1})\,\gamma(\mathbf{k}'_1-\mathbf{k}_1\ldots\mathbf{k}'_{n-1}-\mathbf{k}_n)$$
$$\times\exp\{-\mathrm{i}(s_1\omega'_{\nu_1}+\ldots+s_n\omega'_{\nu_n})t_n\}\exp\{-\mathrm{i}(s_1\omega'_{\nu_1}\tau_1+\ldots+s_{n-1}\omega'_{\nu_{n-1}}\tau_{n-1})\}\,\mathrm{d}\mathbf{k}'_1\ldots\mathrm{d}\mathbf{k}'_{n-1}.$$

Since $H^{s_1\ldots s_n}_{\nu_1\ldots\nu_n}$ and γ are continuous functions, the integral over the rapidly fluctuating factor $\exp\{-\mathrm{i}(s_1\omega'_{\nu_1}+\ldots+s_n\omega'_{\nu_n})t_n\}$ tends to zero as $t_n\to\infty$ unless the subspace $s_1\omega'_{\nu_1}+\ldots+s_n\omega'_{\nu_n}=0$ yields a finite contribution to the integral. Assuming nondegenerate modes,

$$\omega_\nu(\mathbf{k})\neq\omega_\mu(\mathbf{k}) \quad \text{for} \quad \nu\neq\mu, \quad \text{and} \quad \partial\omega_\nu/\partial k_i\neq 0,$$

this is the case only if (*a*) $\delta=1$, $\omega_\nu/k=c=$ const. for some ν, $\nu_1=\nu_2\ldots=\nu_n=\nu_1$ and $s_1=\ldots=s_n=s_\nu$ or (*b*) $n=2$, $s_1=-s_2$, $\nu_1=\nu_2$. Case (*a*) represents a one-dimensional, nondispersive wave field. The wave field is then a superposition of two wavetrains which propagate in opposite directions without changing shape. An asymptotic approach to a Gaussian state is clearly not possible.

Excluding this case, only the second cumulants given by the index combinations (*b*) remain finite as $t_n\to\infty$. Thus the observable fields tend to an asymptotic Gaussian state. The conditions $s_1=-s_2$ and $\nu_1=\nu_2$ imply further that the fields are stationary and that different modes are statistically independent.

If we had investigated the cumulants in physical space, instead of wavenumber

space, the smoothing problem would not have arisen. The integral transform from $\mathbf{k}$ to $\mathbf{x}$ space automatically yields smoothed cumulants S in $\mathbf{x}$ space. The difficulty is transformed into an anomalous behaviour of S for large separations $\boldsymbol{\xi}_m$, such that in the limit $t_n \to \infty$ the Fourier transforms of S no longer exist.

The asymptotic behaviour of the cumulants S and H is best understood by regarding the random wave fields as a superposition of a large ensemble of finite wave *groups*, rather than infinite wavetrains.

Gaussian fields are equivalent to statistically independent wave groups. The assumption that the cumulants H of an initially non-Gaussian state are continuous implies that the initial dependence between wave groups approaches zero smoothly as the separations between groups tend to infinity. Later, the wave groups occupy different positions in space, but retain their statistical dependence. Hence the fields cannot approach a Gaussian state rigorously. But the separations between dependent wave groups increase indefinitely with time, so that the statistical information disperses to infinity and can be recovered only by continually extending the spacial domain of the analysis. This is equivalent to increasing the spectral resolution.

Essentially the same situation applies if the wave groups interact with one another. The coupling between interacting wave groups which satisfy the resonance conditions leads to a small energy transfer and a weak statistical dependence between the wave groups. After interacting, the wave groups propagate away from one another and the statistical correlations disperse into a fine structure. The Gaussian hypothesis implies that the fine structure can be ignored for the development of the field forwards in time. Or in other words, in subsequent interactions (involving new sets of wave groups which interact for the first time) the interacting components can be regarded as statistically independent. The hypothesis is clearly closely analogous to the Boltzmann hypothesis of statistical independence of interacting particles.

If the field is allowed to interact for a certain time and then all velocities are suddenly reversed, the field will develop back into its original state. After the reversal, the wave components are no longer statistically independent before interacting, and the transfer expressions are not valid. Similarly, in the linear case the fine structure cannot be ignored after the reversal of the velocities, and it is possible for an apparently Gaussian field to develop back into a non-Gaussian field.

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