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# A BAYESIAN APPROACH FOR ESTIMATION OF DIRECTIONAL WAVE SPECTRA WITH HF RADAR

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A new method for estimating directional wave spectra from the Doppler spectra by HF radar is proposed. This method is developed by introducing a Bayesian approach, previously proposed by one of the authors (Hashimoto *et al.*, 1987), as one of the most accurate and reliable methods for estimating directional wave spectra for *in situ* measurements. The principal advantage of the new method is that it can be applied without introducing empirical weighting coefficients. Applicability, validity and accuracy of the proposed method are demonstrated with numerically simulated data for various wave conditions.

*Keywords*: HF radar, VHF radar, directional wave spectrum, spectrum, wave observation, wave data analysis, currents measurement.

## 1. Introduction

HF radar, for measuring ocean surface currents, has been in practical use, and several examples have been reported so far. A method for estimating directional wave spectra from the data obtained by HF radar has not yet been established, although a few expedient methods have been reported. If an accurate and reliable method for estimating directional wave spectrum from the HF radar data is developed, it should enable the development of a more accurate and reliable wave model based on the data obtained.

The difficulty in estimating directional wave spectra from HF radar is that the fundamental equation to be solved is a nonlinear integral equation with respect to the directional wave spectrum. Even when a solution can be obtained for the nonlinear integral equation, there is still ambiguity because the solution obtained may not be unique.

Recently, a method for estimating directional wave spectra using HF radar was proposed by Hisaki (1996). He estimated directional wave spectra by solving the nonlinear integral equation iteratively with additional conditions. In Hisaki's method, he introduced *a priori* condition where the directional wave spectrum is assumed to be a smooth and continuous function. In addition, he also introduced other conditions that the directional wave spectrum has a value greater than zero and that the directional wave spectrum changes according to the known ratio in both frequency and directional angle. This can cause the *a priori* conditions to be in excess of the number of unknown parameters and fundamental equations. There still remains the issue of setting the empirical weighting coefficients imposed on each of the additional conditions mentioned above.

In this study, a new method for estimating directional wave spectra from the Doppler spectra measured by HF radar is proposed. This method was developed by introducing a Bayesian approach, previously proposed by one of the authors (Hashimoto *et al.*, 1987), as one of the most accurate and reliable methods for estimating directional wave spectra for *in situ* measurements. In the formulation of the equations from a Bayesian approach, a parameter which is called a hyperparameter is introduced to consider the balance of the two requirements imposed on the estimate of the directional wave spectrum: (1) maximizing the likelihood of the estimate and (2) maintaining the smoothness of the estimate. In order to select the most suitable value of the hyperparameter for the given Doppler spectra, ABIC (Akaike's Bayesian Information Criterion, Akaike, 1980) is introduced as a criterion to determine the most suitable estimate of the directional wave spectrum. Thus, the proposed method can be applied to the data measured by HF radar without introducing empirical weighting coefficients. Applicability, validity and accuracy of the proposed method are demonstrated with numerically simulated data for various wave conditions.

## 2. Formulation of Equations

The Doppler spectrum, $\sigma(\omega)$, obtained by HF radar represents the energy distribution of the radio wave signal backward-scattered by the ocean surface waves at the angular frequency $\omega$, and is expressed by the summation of the first-order component, $\sigma^{(1)}(\omega)$, and the second-order component, $\sigma^{(2)}(\omega)$, i.e. $\sigma(\omega) \approx \sigma^{(1)}(\omega)+\sigma^{(2)}(\omega)$. Each component can be expressed by the following equations for deep water conditions (Barrick, 1972):

$$\sigma^{(1)}(\omega) = 2^6\pi k_0^4 \sum_{m=\pm1} S(-2mk_0, 0)\delta(\omega - m\omega_B) \tag{1}$$

$$\sigma^{(2)}(\omega) = 2^6\pi k_0^4 \sum_{m_1,m_2=\pm1} \int\int_{-\infty}^{\infty} |\Gamma|^2 S(m_1\mathbf{k}_1)S(m_2\mathbf{k}_2) \times \delta(\omega - m_1\sqrt{gk_1} - m_2\sqrt{gk_2})dpdq \tag{2}$$

where $k_0$ is the absolute value of the wave number vector $\mathbf{k}_0$ of radio waves, $S(\mathbf{k}) = S(k_x, k_y)$ is the wave number spectrum of ocean surface waves, $\omega_B(= \sqrt{2gk_0})$ is the Bragg angular frequency. The independent variables, $p$ and $q$, of the integration represent coordinates, each of which is parallel to the axis of the radar beam and orthogonal to the radar beam, respectively. The wave number vectors for ocean waves, $\mathbf{k}_1$ and $\mathbf{k}_2$, are related to these variables by the following equations:

$$\mathbf{k}_1 = (p - k_0, q), \qquad \mathbf{k}_2 = (-p - k_0, -q) \tag{3}$$

These relations indicate the Bragg's resonance condition expressed by

$$\mathbf{k}_1 + \mathbf{k}_2 = -2\mathbf{k}_0 \tag{4}$$

The coupling coefficient, $\Gamma$, shows the degree of the contribution from the wave components having the wave number $\mathbf{k}_1$ and $\mathbf{k}_2$ to the second-order energy distribution of the backward-scattered radar signal, and is commonly expressed by the summation of the electromagnetic scattering effect, $\Gamma_E$, and the hydrodynamic scattering effect, $\Gamma_H$, i.e. $\Gamma = \Gamma_E + \Gamma_H$. Each is expressed by the following equations for deep water conditions (Barrick, 1972):

$$\Gamma_E = \frac{1}{2}\left[\frac{(\mathbf{k}_1 \cdot \mathbf{k}_0)(\mathbf{k}_2 \cdot \mathbf{k}_0)/k_0^2 - 2\mathbf{k}_1 \cdot \mathbf{k}_2}{\sqrt{\mathbf{k}_1 \cdot \mathbf{k}_2} - k_0\phi}\right] \tag{5}$$

$$\Gamma_H = \frac{-i}{2}\left[k_1 + k_2 - \frac{(k_1 k_2 - \mathbf{k}_1 \cdot \mathbf{k}_2)(\omega^2 + \omega_B^2)}{m_1 m_2 \sqrt{k_1 k_2}(\omega^2 - \omega_B^2)}\right] \tag{6}$$

where, $\phi$ is the complex impedance of the sea surface, the absolute value of which is small enough to be negligible.

Since the first-order scattering component $\sigma^{(1)}(\omega)$ and the second-order scattering component $\sigma^{(2)}(\omega)$ appear in different frequencies in the Doppler spectrum $\sigma(\omega)$, they can be easily separated even though they are small in magnitude. Consequently, valuable oceanographic information such as surface currents and waves can be obtained from the respective spectrum components.

As shown in Eq. (2), the two component waves having the wave number vector $\mathbf{k}_1$ and $\mathbf{k}_2$ are related to the second-order scattering component $\sigma^{(2)}(\omega)$. There are infinite combinations of $\mathbf{k}_1$ and $\mathbf{k}_2$ relevant to the Doppler frequency $\omega$ under the restriction condition of $\delta$ function included in Eq. (2) and the resonance condition of Eq. (4). This indicates that Eq. (2) includes the contributions of infinite numbers of component waves having different frequencies $\omega$ and propagation directions $\theta$; thus, we will focus on the second-order component and introduce a method for estimating directional wave spectra from $\sigma^{(2)}(\omega)$. In this study, deep water waves will be examined. The method developed for deep water waves can be easily extended to shallow water waves.

For convenience, the parameters are nondimensionalized by the Bragg angular frequency, $\omega_B$, and the doubled wave number of the radio wave, $2k_0$, as follows.

$$\left.\begin{aligned} \tilde{\omega} &= \omega/\omega_B & \tilde{\mathbf{k}} &= \mathbf{k}/(2k_0) \\ \tilde{\Gamma} &= \Gamma/(2k_0), & \tilde{S}(\tilde{\mathbf{k}}) &= (2k_0)^4 S(\mathbf{k}) \end{aligned}\right\} \tag{7}$$

The integration of Eq. (2) with respect to the two variables $p$ and $q$ can be transformed into a single variable since the integrand includes the delta function $\delta$. If the wave propagation direction $\theta_1$ of the wave number vector $\mathbf{k}_1$ is adopted as a single independent variable for the integration, Eq. (2) can be transformed as follows (Lipa and Barrick, 1982):

$$\tilde{\sigma}^{(2)}(\tilde{\omega}) = \int_0^{\theta_L} G(\theta_1, \tilde{\omega}) d\theta_1 \tag{8}$$

where

$$G(\theta, \tilde{\omega}) = 16\pi[|\tilde{\Gamma}|^2\{\tilde{S}(m_1\tilde{\mathbf{k}}_1)\tilde{S}(m_2\tilde{\mathbf{k}}_2) + \tilde{S}(m_1\tilde{\mathbf{k}}_1*)\tilde{S}(m_2\tilde{\mathbf{k}}_2*)\}y^3|dy/dh|]_{y=\hat{y}} \tag{9}$$

$$\left|\frac{dy}{dh}\right| = \left|1 + m_1 m_2 \frac{y(y^2 + \cos\theta_1)}{(y^4 + 2y^2\cos\theta_1 + 1)^{3/4}}\right|^{-1} \tag{10}$$

and $y = \sqrt{\tilde{k}_1}$. $\hat{y}$ can be obtained by solving Eq. (11).

$$\tilde{\omega} - m_1\hat{y} - m_2(\hat{y}^4 + 2\hat{y}^2\cos\theta_1 + 1)^{1/4} = 0 \tag{11}$$

$\tilde{\mathbf{k}}_i*$ is the nondimensional symmetry wave number vector of $\tilde{\mathbf{k}}_i$ with respect to the radar beam axis (the $p$-axis). An upper limit of integration $\theta_L$ can be given by $\theta_L = \pi$ when $\tilde{\omega} \leq 2$, and $\theta_L = \pi - \cos^{-1}(2/\tilde{\omega}^2)$ when $\tilde{\omega} > 2$, respectively.

The wave number spectrum $S(\mathbf{k})$ in Eq. (9) can be transformed into the frequency-direction spectrum (directional wave spectrum) $S(f, \theta)$ as follows:

$$S(\mathbf{k}) = \frac{g^2}{2^5\pi^4 f^3} S(f, \theta) \tag{12}$$

Thus, by imposing the directional wave spectrum $S(f, \theta)$, the second-order scattering component $\sigma^{(2)}(\omega)$ measured by the HF radar can be theoretically calculated by numerical integration of Eq. (8). For the integration of Eq. (8), however, special treatment is necessary around the singular point where the denominator of the electromagnetic coupling coefficient $\Gamma_E$ approaches zero. Smaller size segments in the numerical integration are therefore adopted around the singular point in an attempt to prevent the deterioration of accuracy of the integration. Confirmation that this approach is valid requires an analytical solution be obtained; however no such analytical solution is currently available.

Figure 1 shows examples of the second-order scattering components $\tilde{\sigma}^{(2)}(\tilde{\omega})$ of the Doppler spectrum calculated for the wave condition having a Bretschneider-Mitsuyasu-type spectrum (Mitsuyasu, 1971) with significant wave height $H_{1/3}$ =

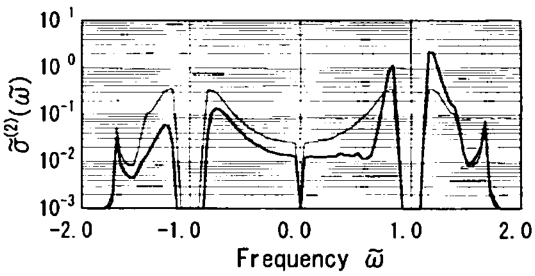


Fig. 1. Examples of the theoretically computed Doppler spectra. (Thick line: 0° crossing angle, thin line: 90° crossing angle).

3.0 m, significant wave period $T_{1/3} = 10.0$ seconds and Mitsuyasu-type directional spreading function (Mitsuyasu *et al.*, 1975) of the directional spreading parameter $S_{\max} = 10$. In the computations the radar signals are assumed to be transmitted with crossing angles of 0° and 90° to the mean wave propagation direction, respectively. The thick and thin lines show examples of the crossing angle of 0° and 90°, respectively. The frequency of the radar signal is assumed to be 24.515 MHz.

Contrary to the computation of $\tilde{\sigma}^{(2)}(\tilde{\omega})$ shown in Fig. 1, the problem for estimating directional wave spectrum with HF radar is to estimate a non-negative solution $S(f,\theta)$ based on simultaneous integral equations of Eq. (8) set up for $\tilde{\sigma}^{(2)}(\tilde{\omega})$.

Although the directional wave spectrum $S(f,\theta)$ is $S(f,\theta) \geq 0$ in general, here it is treated as $S(f,\theta) > 0$, being assumed to be exponential piecewise-constant function over the directional range from 0 to $2\pi$ and the frequency range from $f_{\min}$ to $f_{\max}$ (Hashimoto, 1987). This assumption is commonly employed in numerically generating random waves.

$$S(f,\theta) = \alpha \sum_{i=1}^{I} \sum_{j=1}^{J} \exp(x_{i,j}) \delta_{i,j}(f,\theta) \tag{13}$$

where $x_{i,j} = \ln\{S(f_i,\theta_j)/\alpha\}$, $I$ is the number of segments $\Delta f$ of frequency $f$, $J$ is the number of segments $\Delta\theta$ of direction $\theta$, and

$$\delta_{i,j}(f,\theta) = \begin{cases} 1 : f_{i-1} \leq f < f_i \text{ and } \theta_{j-1} \leq \theta < \theta_j \\ 0 : \text{otherwise} \end{cases} \tag{14}$$

$\alpha$ is a parameter introduced for normalizing the magnitude of $x_{i,j}$, and is given by

$$\alpha = \frac{\int_{f_{\min}}^{f_{\max}} \int_0^{2\pi} S(f,\theta) df d\theta}{\int_{f_{\min}}^{f_{\max}} \int_0^{2\pi} df d\theta} \tag{15}$$

The numerator on the right hand side of Eq. (15) is approximately given by the following equation (Barrick, 1977):

$$\int_{f_{\min}}^{f_{\max}} \int_0^{2\pi} S(f,\theta) df d\theta \approx \frac{2\int_{-\infty}^{\infty} \{\sigma^{(2)}(\omega)/W(\omega/\omega_B)\} d\omega}{k_0^2 \int_{-\infty}^{\infty} \sigma^{(1)}(\omega) d\omega} \tag{16}$$

where $W(\omega/\omega_B) = 8|\bar{\Gamma}^2|/k_0^2$ is a weighting function and $\bar{\Gamma}$ is an approximate coupling coefficient of $\Gamma$ (Barrick, 1977).

The frequency $f$ and the direction $\theta$ are discretized by the following equations, respectively.

$$\mu_i = \ln f_i = \ln f_{i-1} + \Delta f, \qquad \theta_j = \theta_{j-1} + \Delta\theta \tag{17}$$

Substituting Eq. (13) into Eq. (8) yields an integral equation including unknown variables, $\mathbf{X} = (x_{1,1}, \ldots, x_{I,J})^t$. After digitizing the Eq. (8) by replacing the integration with the summation $\sum$, the integral equation can be approximated by the non-linear algebraic equation.

The integral Eq. (8) is, however, a curvilinear integral where the integration must be performed along a special path in $(f, \theta)$-plane due to the restrictions of Eqs. (4) and (11). As mentioned earlier, Eq. (8) includes a singular point, and has to be integrated with smaller segments around the singular point. In discretizing Eq. (8), the value of the directional wave spectrum along the path in $(f, \theta)$-plane is linearly interpolated by the neighboring grid point values of the directional wave spectrum in the same way as Hisaki (1996), and is expressed by

$$\begin{aligned} S(\mu, \theta) = {} & (1-\xi)(1-\zeta)S(\mu_i, \theta_j) + \xi(1-\zeta)S(\mu_{i+1}, \theta_j) \\ & + (1-\xi)\zeta S(\mu_i, \theta_{j+1}) + \xi\zeta S(\mu_{i+1}, \theta_{j+1}) \end{aligned} \tag{18}$$

where $\mu = \ln f$, $0 \leq \xi$ and $\zeta \leq 1$. Equation (8) can therefore be digitized with respect to the grid point values of $S(\mu_i, \theta_j)$ with the desired degree of accuracy.

Finally, by taking into account the errors $\varepsilon_k$ of the Doppler spectrum, the integral Eq. (8) can be approximated by the non-linear algebraic equation including the unknown $\mathbf{X} = (x_{1,1}, \ldots, x_{I,J})^t$, and is expressed by

$$\tilde{\sigma}_k^{(2)} = F_k(\mathbf{X}) + \varepsilon_k \tag{19}$$

where the suffix $k$ indicates a value of the Doppler frequency $\tilde{\omega}_k (k = 1, \ldots, K)$.

The errors $\varepsilon_k (k = 1, \ldots, K)$ of every Doppler frequencies $\tilde{\omega}_k$ are assumed to be independent of each other and their occurrence probabilities can be expressed by a normal distribution having a zero mean and variance $\lambda^2$. Then, for a given $\tilde{\sigma}_k^{(2)}$ $(k = 1, \ldots, K)$, the likelihood function of $\mathbf{X}$ and $\lambda^2$ is given by

$$L(\mathbf{X}; \lambda^2) = \frac{1}{(\sqrt{2\pi}\lambda)^K} \exp\left[-\frac{1}{2\lambda^2} \sum_{k=1}^{K} \{\tilde{\sigma}_k^{(2)} - F_k(\mathbf{X})\}^2\right] \tag{20}$$

Note that the directional wave spectrum $S(f, \theta)$ has thus far been expressed by a piecewise-constant function, with the correlation between the wave energy of each segment of $\Delta f \times \Delta\theta$ not yet having been taken into account. As directional wave analysis is commonly based on the linear wave theory, it can be assumed that each energy on each segment is independent of each other. In actuality, however, it is not realistic to assume that the energy distribution over wave frequency $f$ and wave propagation direction $\theta$ can be discontinuous. Thus, $S(f, \theta)$ is generally considered to be a continuous and smooth function. This allows an introduction of an additional condition that the local variation of $x_{i,j}(i = 1, \ldots, I; j = 1, \ldots, J)$ can be well-approximated by a smooth surface so that the value given in Eq. (21) is expected to be small (Hashimoto *et al.*, 1987, 1990).

$$x_{i,j+1} + x_{i+1,j} + x_{i,j-1} + x_{i-1,j} - 4x_{i,j} \tag{21}$$

In the upper boundary ($i = I$) and the lower boundary ($i = 1$) of the frequency $f$, the value given in Eq. (22) is expected to be small as *a priori* condition.

$$x_{i,j+1} - 2x_{i,j} + x_{i,j-1} \tag{22}$$

These additional conditions lead to

$$\sum_{i=2}^{I-1}\sum_{j}(x_{i,j+1} + x_{i+1,j} + x_{i,j-1} + x_{i-1,j} - 4x_{i,j})^2 + \sum_{j}(x_{1,j+1} - 2x_{1,j} + x_{1,j-1})^2$$
$$+ \sum_{j}(x_{I,j+1} - 2x_{I,j} + x_{I,j-1})^2 \to \text{small} \quad (\text{where } x_{i,0} = x_{i,J},\ x_{i,-1} = x_{i,J-1}) \tag{23}$$

In the matrix form, Eq. (23) can be written as

$$\|\mathbf{DX}\|^2 \to \text{small} \tag{24}$$

where $\mathbf{D}$ is the coefficient matrix of Eq. (23), and $\| \ \|$ is the Euclid norm.

It is, therefore, surmised that the optimal estimate of $S(f, \theta)$ is the one maximizing the likelihood function of Eq. (20) under the condition of Eq. (24). More precisely, the most suitable estimate is given as a set of $\mathbf{X} = (x_{1,1}, \ldots, x_{I,J})^t$ which maximizes the following equation for a given hyperparameter $u$.

$$\ln L(\mathbf{X}; \lambda^2) - \frac{u^2}{2\lambda^2}\|\mathbf{DX}\|^2 \tag{25}$$

The hyperparameter $u$ is a type of weighting coefficient which represents the smoothness of $\mathbf{X}$, where large or small values of $u$, respectively, give an estimate of the directional wave spectrum having either smooth or rough shapes.

It should be noted that Eq. (25) corresponds to the Bayesian relationship expressed by the following equation when we consider the exponential function having the power of Eq. (25).

$$p_{\text{POST}}(\mathbf{X}|u^2, \lambda^2) = L(\mathbf{X}; \lambda^2)p(\mathbf{X}|u^2, \lambda^2) \tag{26}$$

where $p_{\text{POST}}(\mathbf{X}|u^2, \lambda^2)$ is the posterior distribution, and $p(\mathbf{X}|u^2, \lambda^2)$ is the prior distribution of $\mathbf{X} = (x_{1,1}, \ldots, x_{I,J})^t$ expressed by

$$p(\mathbf{X}|u^2, \lambda^2) = \left(\frac{u}{\sqrt{2\pi}\lambda}\right)^M \exp\left\{-\frac{u^2}{2\lambda^2}\|\mathbf{DX}\|^2\right\} \tag{27}$$

The estimate $\mathbf{X}$ obtained by maximizing Eq. (25) can be considered as the mode of the posterior distribution $p_{\mathrm{POST}}(\mathbf{X}|u^2, \lambda^2)$.

Now, if the value of $u$ is given, then regardless of the value of $\lambda^2$, the values of $\mathbf{X}$ that maximize Eq. (25) can be determined by minimizing

$$\sum_{k=1}^{K} \{\tilde{\sigma}_k^{(2)} - F_k(\mathbf{X})\}^2 + u^2 \|\mathbf{DX}\|^2 \tag{28}$$

The determination of $u$ and the estimation of $\lambda^2$ can be automatically performed by minimizing the following ABIC (Akaike's Bayesian Information Criterion, Akaike, 1980) from the view point of the suitability and smoothness of the estimate of $\mathbf{X}$.

$$\mathrm{ABIC} = -2 \ln \int L(\mathbf{X}|\lambda^2) p(\mathbf{X}|u^2, \lambda^2) d\mathbf{X} \tag{29}$$

## 3. Numerical Computation

Numerical computation to estimate the directional wave spectrum using a Bayesian approach requires minimization of Eqs. (28) and (29). However, it is impossible to minimize them analytically. Therefore, in the same way as Hashimoto (1987), the linearization and iteration are applied for Eq. (19) to obtain the optimal estimate of $\mathbf{X}$.

Since the first term on the right-hand side of Eq. (19) is non-linear with respect to $\mathbf{X}$, it is linearized using the Taylor expansion of $F_k(\mathbf{X})$ around $\mathbf{X}_0$, with $\mathbf{X}_0$ being a value close to the estimated solution of $\mathbf{X} = (x_{1,1}, \ldots, x_{I,J})^t$, called estimate $\hat{\mathbf{X}}$. It is expressed as

$$F_k(\mathbf{X}) = F_k(\mathbf{X}_0) + \mathbf{G}_k(\mathbf{X}_0)(\mathbf{X} - \mathbf{X}_0) \tag{30}$$

where

$$\mathbf{G}_k(\mathbf{X}_0) = [\partial F(\mathbf{X})/\partial x_{1,1}, \ldots, \partial F(\mathbf{X})/\partial x_{I,J}]_{\mathbf{x}=\mathbf{x}_0} \tag{31}$$

Substitution of Eq. (30) into Eq. (19) and rearrangement in the matrix form give the following linearized equation with respect to $\mathbf{X}$.

$$\mathbf{B} = \mathbf{AX} + \mathbf{E} \tag{32}$$

where

$$\left.\begin{aligned} \mathbf{A} &= [\mathbf{G}_1(\mathbf{X}_0), \ldots, \mathbf{G}_K(\mathbf{X}_0)] \\ \mathbf{B} &= [\tilde{\sigma}_1^{(2)} - F_1(\mathbf{X}_0) + \mathbf{G}_1(\mathbf{X}_0)\mathbf{X}_0, \ldots, \tilde{\sigma}_K^{(2)} - F_K(\mathbf{X}_0) + \mathbf{G}_K(\mathbf{X}_0)\mathbf{X}_0]^t \\ \mathbf{E} &= [\varepsilon_1, \ldots, \varepsilon_K]^t \end{aligned}\right\} \tag{33}$$

Consequently, the optimal solution $\hat{\mathbf{X}}$ can be estimated by the following procedure.

(1) For a hyperparameter $u$ and the initial value $\mathbf{X}_0$ of $\mathbf{X}$, compute $\tilde{\mathbf{X}}$ using the least-squares method to iteratively minimize $W(\mathbf{X})$ defined by Eq. (34). That is, for $\mathbf{X}_0$, a new value of $\mathbf{X}^{(1)}$ is obtained by applying the least-squares method to Eq. (34). Then, by replacing $\mathbf{X}_0$ in Eq. (33) with $\mathbf{X}^{(1)}$ and repeating the same process, a new value of $\mathbf{X}^{(2)}$ is obtained. The iteration of Eqs. (33) and (34) is terminated when the convergence condition is satisfied. Thus, the iteration of these processes continues until $\mathbf{X}$ converges $\tilde{\mathbf{X}}$ for the given $u$.

$$W(\mathbf{X}) = \|\mathbf{AX} - \mathbf{B}\|^2 + u^2\|\mathbf{DX}\|^2 \tag{34}$$

(2) Using the given $u$ and $\tilde{\mathbf{X}}$ obtained in (1), compute the ABIC by the following equation:

$$\text{ABIC} = K\{1 + \ln(2\pi\hat{\lambda}^2)\} + \ln\{\det(\mathbf{A}^t\mathbf{A} + u^2\mathbf{D}^t\mathbf{D})\} - K\ln(u^2) \tag{35}$$

where

$$\hat{\lambda}^2 = \frac{1}{K}\{\|\mathbf{A}\tilde{\mathbf{X}} - \mathbf{B}\|^2 + u^2\|\mathbf{D}\tilde{\mathbf{X}}\|^2\} \tag{36}$$

(3) Repeat (1) and (2) after changing $u$.

(4) From estimates for each $u$ obtained in (1) $\sim$ (3), select the value $\hat{u}$ and $\hat{\lambda}^2$, as well as $\hat{\mathbf{X}}$, which yields the minimum ABIC.

(5) Substitute $\hat{\mathbf{X}}$ obtained in (4) into Eq. (13) to determine $\hat{S}(f, \theta)$.

In this study, for the purpose of the practical computation, the initial value $\mathbf{X}_0$ is simply set to be zero in all cases to confirm the stability of the computation with respect to the initial value $\mathbf{X}_0$. Note that introducing a parameter $\alpha$ defined by Eq. (15) allows the initial value $\mathbf{X}_0$ to be set to zero in a rational way. As the convergence condition of the iterative computation of Eq. (34), when the ratio of the value of $\|\mathbf{X}^{(k+1)} - \mathbf{X}^{(k)}\|$ of $(k+1)$th step to the value of $\|\mathbf{X}^{(k)}\|$ of $(k)$th step is less than $10^{-2}$, then the computation is considered to be converged and the estimate $\tilde{\mathbf{X}}$ is determined.

The optimal hyperparameter $u$ that minimizes the ABIC is determined via trial and error by changing $m$ in the following equation.

$$u = ab^m \qquad (m = 1, 2, \ldots) \tag{37}$$

where $a$ and $b$ are the search coefficients, here chosen for convenience as $a = 0.1$ and $b = 0.5$.

## 4. Examination by the Numerical Simulation

Numerical simulation was carried out to examine the validity and accuracy of the proposed method for estimating the directional wave spectrum from a Baysian approach described in Secs. 2 and 3. In the numerical simulation, first, the second-order scattering component $\sigma^{(2)}(\omega)$ was calculated by numerically integrating Eq. (2)

for the benchmark directional wave spectrum. Then, an inverse estimation of the directional wave spectrum was carried out based on the second-order scattering component $\sigma^{(2)}(\omega)$ by using the method described in Sec. 3.

The frequency of HF radar was assumed to be 24.515 MHz. Various layouts of the two sets of radar array were examined in the numerical simulations. Besides, various sea conditions of wind waves and swell or their combination were also examined by assuming the various shapes of directional wave spectra.

In addition, a single Doppler spectrum obtained by a single radar array cannot distinguish between waves coming from the symmetrical direction with respect to the beam axis. In such a case, an apparent energy peak may appear in the directional wave spectrum as shown in Hisaki (1997). However, according to the proposed method, an apparent energy peak barely appears because of the restriction of the additional condition of Eq. (23) although the true peak cannot be distinguished from the apparent peak and vice versa.

Figure 2 shows an example of the bi-directional wave field where the dominant energy peaks of the directional wave spectrum are assumed to be in different frequencies. The significant wave period $T_{1/3}$ of each wave field is 5 seconds and 12 seconds, respectively. Each directional wave spectrum in Fig. 2 was estimated from the two Doppler spectra obtained under the conditions where the radio signals were transmitted toward the different directions on the sea surface using the two sets of the radar arrays. The upper left panel in Fig. 2 shows the true directional wave spectrum (True), and Case 1 to Case 5 are the directional wave spectra estimated for the conditions where the crossing angle of the two beam axis of the radar signals are $\delta\theta = 15°$, $45°$, $90°$, $105°$ and $135°$, respectively. Each estimated directional wave spectrum shows good agreement with the benchmark directional wave spectrum (True).

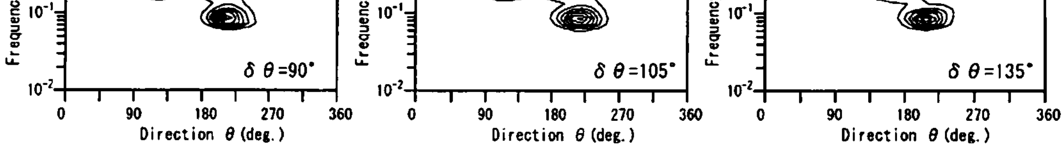


Fig. 2. Examples of the estimated directional wave spectra where the dominant energy peaks of the directional wave spectrum are assumed to be in different frequencies.

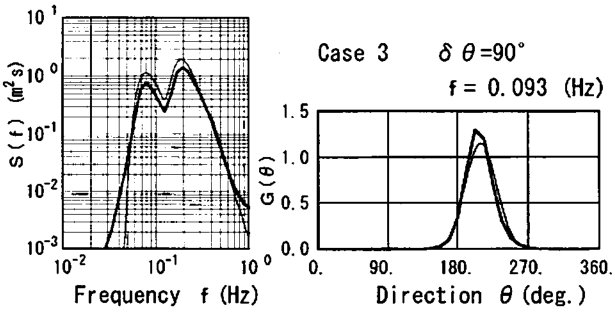


Fig. 3. Examples of the estimated frequency spectrum and directional distribution function (Thin line: benchmark spectrum, thick line: estimated spectrum).

The left panel in Fig. 3 shows the frequency spectrum $S(f)$ of Case 3 in Fig. 2, and the right panel shows the directional distribution function $G(\theta)$ at the frequency $f = 0.093$ (Hz) of the same case. The thin lines in Fig. 3 represent the true frequency spectrum and the directional distribution function, and the thick lines represent the estimated ones. Although the estimated frequency spectrum is underestimated around the energy peak and the estimated directional distribution function is overestimated around the energy peak, the locations of the peaks are properly estimated.

Figure 4 also shows an example of the bi-directional wave fields where the dominant energy peaks are assumed to be in the same frequency. The other conditions are set to be equal to those of Fig. 2. In this example, the accuracy of the estimated directional wave spectra depends on the observation conditions because of the complexity of the wave field. For example, a uni-directional wave field is estimated instead of a bi-directional field in Case 1 where the narrow crossing angle of radio signals of $\delta\theta = 15°$ is assumed. Figure 5 shows the frequency spectrum $S(f)$

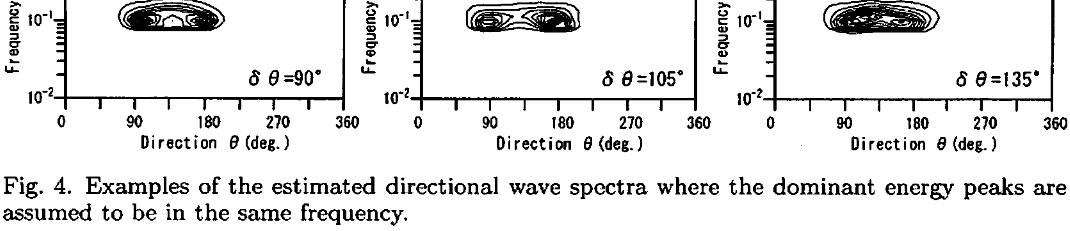


Fig. 4. Examples of the estimated directional wave spectra where the dominant energy peaks are assumed to be in the same frequency.

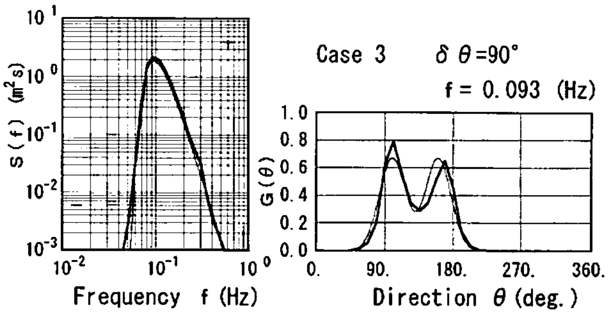


Fig. 5. Examples of the estimated frequency spectrum and directional distribution function. (Thin line: benchmark spectrum, thick line: estimated spectrum).

and the directional distribution function $G(\theta)$ in the frequency $f = 0.093$ (Hz) of Case 3 in Fig. 4. A proper frequency spectrum and a directional distribution function can be seen when a proper crossing angle of radio signals ($\delta\theta = 90°$) are used.

## 5. Concluding Remarks

We proposed a method for estimating directional wave spectra using HF radar, developed from a Bayesian approach. Accuracy, validity and applicability were examined for numerically simulated data. The results demonstrate that the directional wave spectra can be estimated with high accuracy on the basis of the Doppler spectra obtained by HF radar, theoretically. However, for applying the proposed method to the real Doppler spectrum acquired by field observations, more examination must be carried out since there may be uncertain factors which were not considered in this study such as observational errors, estimation errors of the Doppler spectra and others. We are now preparing for applying the proposed method to field data and carrying out further studies for enhancing the practical use of this method.

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