On Nonlinear Ekman Surface-Layer Pumping

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ABSTRACT

Simple expressions are presented for the corrections to the classic Ekman pumping law $W_e = \hat{\mathbf{z}}\cdot\mathrm{curl}(\tau_0/f)$ due to nonlinear advection effects in the surface boundary layer. These involve products of the surface Reynolds stress τ_0 and the underlying ocean currents $\mathbf{v}_0(x, y, t)$ and their derivatives, and products of $\tau_0(x, y, t)$ and its own derivatives. The former interaction is independent of the turbulence closure, while the latter is obtained using solutions for a constant eddy viscosity. The corrections are usually small, as is assumed when the linear Ekman pumping relation is applied in ocean modeling. However, they can become significant in circumstances involving very high wind stresses (e.g., a hurricane), or in situations where a strong narrow oceanic current flows under a region of moderate but perhaps relatively uniform surface stress.

1. Introduction

A fundamental concept of dynamic oceanography is the notion of Ekman pumping, where horizontal divergence of the vertically integrated flow in the subsurface turbulent boundary layer generates by mass continuity a weak vertical suction velocity W_e at the bottom of the layer. This pumping velocity is usually derived by considering a neutrally stratified turbulent layer extending a small distance δ down into an ocean that is subject to Coriolis accelerations proportional to $f(y)$, where f is the Coriolis parameter and y is the meridional distance referred to a nonequatorial base latitude. The boundary conditions are that the mean velocities approach the interior values upon leaving the boundary layer, and that the turbulent momentum fluxes at the top of the boundary layer match those accompanying the wind stress above. This situation is illustrated schematically in Fig. 1.

A central feature of the derivation of the well-known pumping rule

$$W_e(z \approx -\delta) = \hat{\mathbf{z}}\cdot\mathrm{curl}(\tau_0/f), \qquad (1)$$

which may be found in most standard textbooks (e.g., Gill 1982; Pedlosky 1987; Tomczak and Godfrey 1994), is the assumption of weak wind stresses and low Rossby number flow in and below the boundary layer. This makes the Ekman layer problem linear and allows, as a consequence, a direct integration across the boundary layer that gives (1) without the necessity of introducing any turbulence closure assumptions. Although the Rossby number $\epsilon \equiv U/fL$, where U is a characteristic flow velocity, and L is the horizontal length scale, is usually quite small in the ocean interior, there are local regions where it may approach a few tenths. Under conditions of high surface wind stress the wind-driven mean currents in the boundary layer itself can have substantial Rossby number as well. Therefore, it is of some practical importance, as well as of academic interest, to determine the precise form of corrections to Eq. (1) when finite Rossby number effects are included in the boundary-layer dynamics. Following our derivation, which yields an analytical result through application of regular perturbation theory, some oceanographic situations with significant second-order corrections are cited. In addition to these specific illustrations, it is useful to have a more accurate surface layer pumping formula for use in conceptual and computational models.

In a previous paper (Hart 1995), the question of nonlinear corrections to bottom-layer Ekman pumping was addressed by a regular perturbation method. The modified Ekman suction law for W_e above a rigid stationary bottom wall is

$$W_e = \sqrt{\frac{\nu}{f_0}}\left\{\frac{\omega_0}{\sqrt{2}} - \frac{\sqrt{2}}{40f_0}\left(10\frac{\partial\omega_0}{\partial t} + 7\omega_0^2 + 13J(\Psi, \omega_0) + 7\nabla\Psi\cdot\nabla\omega_0 - 12J\left(\frac{\partial\Psi}{\partial x}, \frac{\partial\Psi}{\partial y}\right)\right)\right\}, \qquad (2)$$

where $\Psi(x, y, t)$ is the streamfunction for the interior flow such that $\mathbf{v} = -\nabla\times\hat{\mathbf{z}}\Psi$, $\omega_0 = \nabla^2\Psi$ is the vertical vorticity of the interior motion, and J is the Jacobian operator defined by $J(g, h) = g_x h_y - g_y h_x$. Thus, as

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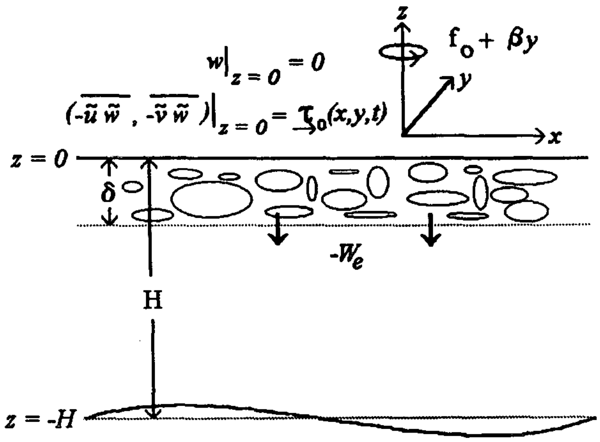

FIG. 1. Sketch of the upper boundary layer of thickness δ, in which turbulence is driven by a spatially variable surface stress $\boldsymbol{\tau}_0$. There is flow $\mathbf{v}_0(x, y, t)$ between the first isopycnal (or bottom) and $z = -\delta$.

expected, in regions of strong vorticity or vorticity gradient, the correction terms to the usual linear-in-vorticity suction law can be significant. For this bottom boundary-layer model, a turbulence closure is made. In particular, the result (2) reflects the analogy with laboratory flows where the momentum diffusion of the mean currents goes like $\rho\nu\nabla^2\mathbf{v}$, so that ν is to be interpreted as a constant eddy viscosity.

A similar second-order-accurate equation for the surface layer pumping is easier to derive, and many of the terms in it are be obtained without any turbulence closure assumptions. The purpose of this note is to describe this calculation, and to discuss briefly some consequences of the result.

2. Analysis

The equations of motion for the mean velocities in the neutrally stratified boundary layer are taken to be

$$\frac{Du}{Dt} - fv + \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial \tau_x}{\partial z} = \nu\frac{\partial^2 u}{\partial z^2} \tag{3a}$$

$$\frac{Dv}{Dt} + fu + \frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{\partial \tau_y}{\partial z} = \nu\frac{\partial^2 v}{\partial z^2} \tag{3b}$$

$$\frac{\partial p}{\partial z} = 0 \tag{3c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{3d}$$

where $\boldsymbol{\tau}(x, y, z, t) \equiv (\tau_x, \tau_y) \equiv -(\overline{\tilde{u}\tilde{w}}, \overline{\tilde{v}\tilde{w}})$ gives the turbulent stresses associated with deviations in velocity from the means $\mathbf{v}$, $f(y) = f_0 + \beta y$ is the Coriolis parameter on the usual β plane, p is the dynamic pressure, $D/Dt = \partial/\partial t + \mathbf{v}\cdot\nabla$, and ρ is the constant water density. The second equality in (3a)–(b) is the usual Newtonian friction assumption and is used in part of the derivation presented below.

The suction velocity W_e at the base of the Ekman layer may be determined by integrating the vertical vorticity equation formed from (3) across the boundary layer. This vorticity equation can be written as

$$-\frac{f_0\partial w}{\partial z} - \frac{\partial}{\partial z}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right) = -\frac{D}{Dt}(\omega + f) - (\omega + \beta y)\nabla_h\cdot\mathbf{v} + \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\frac{\partial v}{\partial z}. \tag{4}$$

When the Rossby number ϵ and the surface stress $\boldsymbol{\tau}_0 \equiv \boldsymbol{\tau}(x, y, z = 0, t)$ are small, and $\beta y/f_0 \approx \epsilon$, we can view the right-hand side of (4) as a small, but finite, correction to the usual linear balance obtained by setting the left-hand side equal to zero. We write the Ekman suction as this linear part w_0 plus a correction w_1, and represent the Ekman layer flow as a sum of its interior and boundary layer (primed) parts: $u = u_0(x, y, t) + u'(x, y, z, t)$, $v = v_0(x, y, t) + v'(x, y, z, t)$, and $w = w'(x, y, z, t)$. An integration from $z = 0$ to $z = -\delta$, followed by application of boundary conditions $\boldsymbol{\tau} = \boldsymbol{\tau}_0$, $w = 0$ at $z = 0$, and $u' = v' = 0$ at $z = -\delta$ gives

$$w_0(-\delta) = \frac{1}{f_0}\left(\frac{\partial \tau_{0y}}{\partial x} - \frac{\partial \tau_{0x}}{\partial y}\right) = \hat{\mathbf{z}}\cdot\nabla\times\left(\frac{\boldsymbol{\tau}_0}{f_0}\right) \tag{5}$$

and

$$w_1(-\delta) = \frac{1}{f_0}\int_0^{-\infty}\left\{\frac{\partial \omega'}{\partial t} + u_0\frac{\partial \omega'}{\partial x} + u'\frac{\partial}{\partial x}(\omega_0 + \omega') + w'\frac{\partial \omega'}{\partial z} + v_0\frac{\partial \omega'}{\partial y} + v'\frac{\partial}{\partial y}(\beta y + \omega_0 + \omega') + \frac{\partial w'}{\partial x}\frac{\partial v'}{\partial z} - \frac{\partial w'}{\partial y}\frac{\partial u'}{\partial z} + (\beta y + \omega_0 + \omega')\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right)\right\}dz - \delta\left[\frac{D_0(\omega_0 + \beta y)}{Dt} + (\omega_0 + \beta y)\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right)\right], \tag{6}$$

where the vertical integration has been extended to $-\infty$ when the integrands involve primed variables because these become zero below $z = -\delta$; D_0/Dt is the interior advection operator $\partial/\partial t + (\mathbf{v}_0\cdot\nabla_h)$.

The last term in (6) is canceled when the interior vertical vorticity equation is integrated from the first stratification level (or the bottom) $z = -H$, up to $z = -\delta$, in order to obtain the effect of the Ekman pumping on the interior vorticity. That is, this integration is usually done from $-H$ to 0, neglecting the thin contribution over the height δ. Considering first-order terms in δ/H, it is consistent to replace $f/(H - \delta)\partial w/\partial z$ in the interior vorticity equation with the usual $f/H\partial w/\partial z$ because the δ contribution to this stretching effect is balanced by the final term in (6). Thus, if we remember to use H as the stretching distance for the interior vertical vorticity, then the $\delta[\ldots]$ term in (6) may be dropped.

Many terms remain to be calculated. The lowest-order boundary layer dynamics gives

$$\int_0^{-\infty} v'dz = \frac{\tau_x}{f_0}, \quad \int_0^{-\infty} u'dz = -\frac{\tau_y}{f_0}, \qquad (7)$$

which here are written in a form that can be used in (6), that is, the left-hand sides are the negatives of the usual Ekman (1905) transports. Equation (7) can now be used to evaluate all the terms in (6) that are linear in the primed variables because the interior flow variables are independent of height and can be taken outside the vertical integration. Combining the resulting w_1 flow with (1) leads to the partial conclusion

$$W_e = \frac{\hat{\mathbf{z}}\cdot\nabla\times\left(\dfrac{\boldsymbol{\tau}_0}{f(y)}\right)}{1+\dfrac{\omega_0}{f_0}} + \frac{1}{f_0^2}\left\{\left(\frac{\partial}{\partial t}+\mathbf{v}_0\cdot\nabla\right)(\nabla\cdot\boldsymbol{\tau}_0) + \hat{\mathbf{z}}\cdot(\boldsymbol{\tau}_0\times\nabla\omega_0)\right\}. \qquad (8)$$

There is an interaction between the integrated Ekman transport in the boundary layer, which is proportional to $\boldsymbol{\tau}_0$, and the interior flow. If $\beta = 0$, the first term reduces to $\hat{\mathbf{z}}\cdot\nabla\times\boldsymbol{\tau}_0/(f_0 + \omega_0)$. For this term the net stretching of interior relative plus planetary vorticity, which is $(f_0 + \omega_0)W_e/H$, is exactly the same as before the correction. The time derivative is the so-called isallobaric effect (e.g., Gill 1982, p. 328). However, $\nabla\cdot\boldsymbol{\tau}_0$ is typically quite small and the Lagrangian derivative of the divergence of the surface stress may often be neglected. When this is the case, the most important correction to (1) represents an interaction between the wind stress and perpendicular gradients of the interior vorticity. Equation (8) reflects a net force balance across the boundary layer is obtained without the necessity to introduce any turbulence model or closure assumption!

We are left with corrections arising from products of primed variables in (6), and these cannot be so simply integrated. However, the primed variables, resulting from the solution of the boundary layer equations at order ϵ^0, depend only on $\boldsymbol{\tau}_0$, not on $\mathbf{v}_0$. Nonetheless, an explicit model must be used to find their z distributions. One way to proceed, in order to get an indication of how stress interactions enter the pumping rule, is to use the simple closure expressed by the second equality in (3a,b). To lowest order the boundary layer dynamics are described by

$$-fv' = \nu\frac{\partial^2 u'}{\partial z^2} \qquad (9a)$$

$$fu' = \nu\frac{\partial^2 v'}{\partial z^2}, \qquad (9b)$$

with the standard solution

$$u' = ae^{\eta}\cos(\eta) + be^{\eta}\sin(\eta),$$
$$v' = ae^{\eta}\sin(\eta) - be^{\eta}\cos(\eta), \qquad (9c)$$

where

$$\eta = \sqrt{\frac{f_0}{2\nu}}z, \quad a = \frac{1}{\sqrt{2f_0\nu}}(\tau_x + \tau_y),$$
$$b = \frac{1}{\sqrt{2f_0\nu}}(\tau_x - \tau_y). \qquad (9d)$$

The continuity equation yields $w'(x, y, z, t)$, and then all the remaining terms in (6) that involve products of primed variables can be integrated. The integrations and subsequent algebraic reduction is expedited by using the symbolic manipulator MAPLE. Combining with a slightly more compact version of the previous partial pumping formula (8) yields the total ϵ-accurate (error of order ϵ^2) suction. Let $\xi \equiv \hat{\mathbf{z}}\cdot\nabla\times\boldsymbol{\tau}_0$ be the curl of the surface stress, and let $D \equiv \nabla\cdot\boldsymbol{\tau}_0$ be the divergence of said stress. The final result is

$$W_e = \hat{\mathbf{z}}\cdot\nabla\times\left(\frac{\boldsymbol{\tau}_0(x,y,t)}{f(y)+\omega_0(x,y,t)}\right) + \frac{1}{f_0^2}\left\{\left(\frac{\partial}{\partial t}+\mathbf{v}_0\cdot\nabla\right)D\right\}$$
$$+ \frac{1}{\sqrt{8\nu f_0^5}}\left\{\boldsymbol{\tau}_0\cdot\nabla(\xi + \nabla\times\hat{\mathbf{z}}\xi) + \nabla D\cdot(\boldsymbol{\tau}_0 - \mathbf{z}\times\boldsymbol{\tau}_0) - \xi^2 + D^2 + \xi D\right\}, \qquad (10)$$

which is written in a form to highlight the different contributions from the divergence and curl of the applied surface wind stress.

An alternate way to derive (10) is to vertically integrate the horizontal divergence of the Ekman transports, after finding these latter objects by solving the quasi-linear momentum equations in which $\mathbf{v}\cdot\nabla\mathbf{v} + (\mathbf{f} \times \mathbf{v})$ is replaced with $(\boldsymbol{\omega} + \mathbf{f}) \times \mathbf{v} + \nabla(\mathbf{v}\cdot\mathbf{v})/2$. The integrated ω' is proportional to $\nabla\cdot\boldsymbol{\tau}_0$, and this may usually be assumed small. The gradient term vanishes at lowest order when cross differentiating to form the horizontal divergence of the primed flow, so that the dominant correction term as far as the pumping goes is just $(\omega_0 + f)(\hat{\mathbf{z}} \times \mathbf{v}')$. The first term on the right side of (10) then falls out right away after solving the momentum equations for $\mathbf{v}$, then integrating the resulting horizontal divergence to get W_e. This shows that the boundary layer flow feels a Coriolis force associated with the planetary plus relative interior vorticity.

3. Discussion

For most oceanic conditions, the last bracketed term in (10) is quite small. If we calibrate ν by setting the boundary layer depth $\delta \approx (\nu/f_0)^{1/2}$, then the ratio of these stress-squared interaction terms to the leading pumping velocity ξ/f_0 is of order $\xi/\delta f_0^2$ (assuming $\xi > D$). Large-scale curls of $\boldsymbol{\tau}_0$ are about 2×10^{-8} $cm^2\ s^{-1}$ on a 1000-km wind scale. Taking a surface-layer thickness $\delta \approx 20$ m shows that this ratio is around 10^{-3}. So unless the wind stress curl is locally very much bigger than the value used here, the effect of nonlinear self-advection of the boundary layer component of the flow on the Ekman pumping is quite small. For the classic nondivergent westerly wind stress model $\boldsymbol{\tau} = -\tau_0\hat{\mathbf{x}}\cos(\pi y/L)$, on $y = [0, L]$, the f-plane pumping is

$$W_e = -\frac{\tau_0\pi}{Lf_0}\sin\left(\frac{\pi y}{L}\right) + \frac{\tau_0^2\pi^2}{\sqrt{8}L^2 f_0^3\delta}\cos\left(\frac{2\pi y}{L}\right). \quad (11)$$

Thus, with negative wind stress curls ($\tau_0 > 0$) the correction induces more downward pumping at the meridional midpoint of the gyre and shrinks the meridional extent of the gyre (a defined zero line of W_e) by a kilometer or so.

More significant stress–stress interactions are possible at larger values of $\boldsymbol{\tau}_0$, such as occur locally under strong cyclonic storms. The problem of the oceanic response to hurricane-force winds has been studied using numerical calculation (O'Brien and Reid 1967) and linear theory (Geisler 1970), among others. Equation (10) allows a direct estimate of the importance of nonlinear Ekman transports in such situations (assuming a modest Rossby number for the ocean response and a slow wind evolution timescale, relative to f^{-1}). We compute the predicted Ekman flux for a boundary layer 30-m deep at 25° latitude, in a resting ($\mathbf{v}_0 = 0$) ocean under a cyclonic surface wind consisting of an azimuthal flow that has a radial structure given by $r/L \exp(-r/L)$. The peak wind speed is set to 35 m s^{-1}. We use $L = 50$ km and let the actual wind vector include a component with identical r dependence that is oriented at an angle of 20 deg into the low. The wind stress is calculated using the drag rule $\boldsymbol{\tau}_0 = -10^{-6}(0.75 + 0.067v)v\mathbf{v}_0$ (Garratt 1992) with $v \equiv \mathrm{abs}(\mathbf{v})$. Figure 2 shows that although the suction is still dominated by the standard relation [i.e., Eq. (1)], the nonlinear corrections are significant. The curve labeled w_1 reflects the { }-terms in (10) arising from wind stress curls ξ (i.e., what is left after setting $D = 0$), while that labeled w_1' shows the suction velocity arising from nonzero divergences D of the surface stress. Both corrections reduce the upwelling below the core of the storm. The negative divergence term comes from the ξD contribution to (10). An anticyclone with the same structure has enhanced downwelling, since both second-order corrections are unchanged upon reversal of the sign of $\boldsymbol{\tau}_0$, while the lowest-order term is opposite to that for a cyclonic (positive ξ) excitation.

The ratio of the surface-stress vorticity-gradient interaction, which from (10) is proportional to $\nabla\tau_0 \times \nabla\omega_0$, to the leading pumping velocity (1) is of order $\chi \equiv UL_s/f_0L_c{}^2$. Here L_s is the scale of variation for the stress and L_c is the scale of variation for the interior oceanic current, which has characteristic velocity U. In a 1 m s^{-1} boundary current that is 40 km wide, χ is considerably greater than unity, indicating that the suction velocity may be dominated by the stress–current interaction rather than the lowest-order curl-stress/planetary vorticity term. This will certainly be true in regions where a strong wind stress is relatively curl free. Consider a simple case with τ_x = const, $\tau_y = 0$, and $u_0 = U(y)$. Then the surface-stress vorticity-gradient term leads to a suction velocity

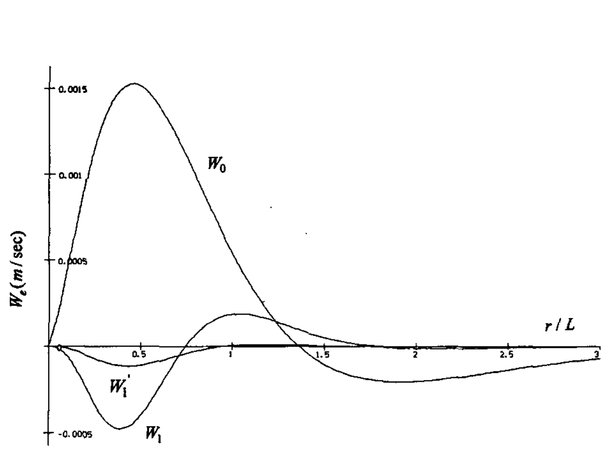

FIG. 2. Vertical suction velocity into the surface Ekman layer under a cyclonically swirling and weakly converging wind: w_0 is the lowest-order suction [Eq. (1)], w_1 is the correction due to the rotational part of the wind stress ξ, while w_1' is the correction due to the divergent part of the stress D. See text for parameter values.

$$W_e = \frac{\tau_x}{f_0^2}\frac{\partial^2 U}{\partial y^2}. \quad (12)$$

If both the stress and the jet (e.g., $U = 1$ m s^{-1} sech$^2(y/L_c)$ are eastward, then there is upwelling above the $y = 0$ axis of the jet into the boundary layer and downwelling out in the wings beyond the locations of zero curvature of $U(y)$. For $L_c = 20$ km (a 40-km jet width), an anemometer height wind of 15 m s^{-1} at 45°N, and a drag law as described above, the peak vertical velocity above the core is about 0.2 mm s^{-1} (16 m day^{-1}), while the downwelling in the wings peaks at about half this value. From (11) the gyre-scale Ekman suction peaks at about 10^{-2} mm s^{-1} for $L = 10^6$ m and the same wind. Thus, the stress vorticity-gradient interaction can produce significant upwelling velocities in situations like this, which has a maximum local Rossby number of about one-third.

The way in which this part of the nonlinear Ekman pumping affects the ocean currents depends on the other terms in the interior vorticity equation. However, if the current has depth H, is over a strongly stratified thermocline that isolates it from drag on the bottom, and is parallel and oriented east–west, the full vorticity equation for ocean flow under a uniform east–west stress $\tau_x(y)$ becomes

$$\frac{\partial \omega}{\partial t} = \frac{\tau_x}{Hf_0}\frac{\partial \omega}{\partial y} = C_d\frac{|U_w|U_w}{Hf_0}\frac{\partial \omega}{\partial y} \equiv c\frac{\partial \omega}{\partial y}. \quad (13)$$

The general exact solution for constant U_w is simply $\omega = \omega_i(y + ct)$, where ω_i is the initial vorticity distribution of the jet. The nonlinear suction therefore causes the current system to drift south (for $U_w > 0$) at rate $c = C_d|U_w|U_w/Hf_0$. For the parameters used above and $H = 800$ m, the drift is about one-half kilometer per day.

These examples suggest that although the nonlinear corrections to Ekman surface layer pumping are generally small, there are special situations involving high wind stresses and/or narrow currents where the second-order effects described analytically in this paper are important. Equation (10) may be useful in theory or simulations of ocean circulations that require a more accurate surface layer suction law than that represented by the classic linear Ekman pumping rule expressed in (1).

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