

Resonant Reflection of Water Waves

by Periodic Sandbars

多段砂州による表面波の共鳴反射

by

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Abstract

A theory has been put forth by Mei(1985) that strong reflection of incident waves can be induced by bottom sandbars once the so-called Bragg resonance condition is met. We apply Mei's theory on a sloping bottom, and show that incident waves can be strongly reflected, and that the scattering process depends on whether the modulational frequency lies above or below a threshold frequency when incident waves are slightly detuned. The strong reflection and dispersion of incident wave packets are also shown. Mei's theory is extended up to the second order, and so-called wave set-down originally locked to wave envelopes is shown to be unlocked and to propagate as free long waves by the effect of sandbars. An experiment is carried out to observe the scattering process of steady incident waves and incident wave packets over artificial sandbars. The above theoretical results are strongly confirmed.

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List of Symbols

A	=	incident wave amplitude
A'	=	non-dimensional incident wave amplitude
A_m	=	maximum of incident wave amplitude
a	=	long wave amplitude
B	=	incident wave amplitude
B'	=	non-dimensional incident wave amplitude
b	=	long wave amplitude
C	=	wave celerity
C_g	=	group velocity
$\vec{C}_g^\pm$	=	group velocity vector
C_L	=	ratio of measured set-down to calculated set-down
C_n	=	coefficient of evanescent mode
C_o	=	non-dimensional long wave celerity
D	=	sandbar amplitude
D_n	=	coefficient of evanescent mode
F	=	Fourier transform of input signal
f_o, f_n	=	vertical profile of ϕ_{11}
g	=	gravity acceleration
h	=	water depth
K	=	modulational wave number
k, k_n	=	wave number
$\vec{k}^\pm$	=	wave number vector
k'	=	wave number of sandbars
L	=	length of sandbar region
L_o	=	distance between shoreline and sandbar region
l	=	L_o/L
m	=	number of sandbars
P, Q	=	modulational wave number
R, R_1, R_2	=	reflection coefficient

$S^{\pm}, S_n^{\pm}$ = phase
 S_{xx} = radiation stress
 s = bottom slope
 t = time
 $\bar{t}, t_1, t_2$ = slowly varying time
 t' = non-dimensional time
 X = Fourier transform of wave record
 x = horizontal coordinate
 $\bar{x}, x_1, x_2$ = slowly varying horizontal coordinate
 x' = non-dimensional horizontal coordinate
 x_0 = distance between wave generator and sandbar region
 y = horizontal coordinate
 $\bar{y}$ = slowly varying horizontal coordinate
 z = vertical coordinate
 α, α_n = wave number in x-direction
 β = wave number in y-direction
 γ = vertical profile of ϕ_{21}
 δ = deviation of bed surface from its mean level
 ε = small parameter
 ζ = water surface elevation
 ζ' = non-dimensional second order long wave
 ζ_n = n-th order portion of ζ
 ζ_{nm} = m-th mode of ζ_n in time
 $\bar{\zeta}_{20}$ = spatially averaged second order long wave
 ζ'_{20} = rapidly varying portion of ζ_{20} in space
 θ = local inclination of incident waves with respect to x-axis
 u = small parameter
 ξ = displacement of wave generator
 ξ_{nm} = n-th order m-th mode of ξ
 ρ = water density

ϕ = velocity potential
 ϕ_n = n-th order portion of ϕ
 ϕ_{nm} = m-th mode of ϕ_n in time
 ϕ_{nm}^k = k-th mode of ϕ_{nm} in space
 Ω, Ω_2 = modulational wave frequency
 Ω_0 = threshold frequency
 ω = wave angular frequency
 $O()$ = order of magnitude
 $*, ()^*$ = complex conjugate
 $(\bar{})$ = spatial mean over a wavelength

Chapter 1 Introduction

On natural beaches, periodic sandbars parallel to a coastline are often observed (Saylor & Hands 1970, Lau & Travis 1973, Short 1975, Dolan 1983, Ishida et al. 1984, Kato 1984). The mechanism of the formation of such kind of sandbars has been studied in various aspects, since it is one of the important topics in the recent study of the wave field, the nearshore current, and the deformation of topographies in the coastal zone. Though the actual process on natural coasts must be complicated, one of the promising interpretations is that the mass transport in a wave-induced boundary layer near the sea bottom contributes to the formation of longshore sandbars (Lau et al., Short, Kato).

The study of mass transport in a laminar boundary layer was first studied by Longuet-Higgins (1953). He showed that in a standing wave field, there appeared circulations of mass transport (Fig.1-1). Later it has been found that the eddy-viscosity model for the turbulent boundary layer does not change the results of mass transport qualitatively (Longuet-Higgins 1958, Johns 1970, Noda 1970). If the seabed is movable, heavy particles tend to drift towards the nodes and light particles in suspension towards the antinodes. Thus sandbars whose wavelength is one-half of the water waves can be formed. This feature has been observed experimentally by Herbich (1968), Noda (1970), Carter, Liu & Mei (1973), and Irie (1984). Carter et al. extended the study by Longuet-Higgins to partially standing waves, and found that when the reflection coefficient exceeds the critical value of 0.414, the Lagrangian drift converges somewhere between the nodes and their adjacent antinodes. If about one-fifth of the incident wave energy is reflected in some way, sandbars may be formed. However, the reflection at the shore is usually very small, because incoming waves lose most of their energy by breaking unless the wave slope is very small: $kA = O(s^2)$, k : wave number, A : wave amplitude, s : bottom slope.

Recently, Davies & Heathershaw (1984) put forth a theory that reflected waves can be resonated by sandbars themselves if the bar wavelength is one-half that of the incident waves. They put the deviation of the bed surface from its mean level ($z=-h$) as $\delta(x)$, and solved the following equations by the conventional perturbation method.

$$\phi_{xx} + \phi_{zz} = 0 \quad -h+\delta < z < \zeta \quad (1-1)$$

$$\zeta_t + \phi_z - \phi_x \zeta_x = 0 \quad z = \zeta \quad (1-2)$$

$$g\zeta - \phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) = 0 \quad z = \zeta \quad (1-3)$$

$$\phi_z - \phi_x \delta_x = 0 \quad z = -h+\delta \quad (1-4)$$

where ϕ is the velocity potential and ζ is the water surface elevation. They finally gave the reflection coefficient

$$R = \frac{2kD}{\sinh 2kh + 2kh} \frac{2k}{k'} \frac{|\sin (2k/k')m\pi|}{(2k/k')^2 - 1} \quad (1-5)$$

where D is the bar amplitude, m is the number of sandbars, and k' is their wave-number. When $2k/k' \rightarrow 1$, this reflection coefficient becomes unbounded and their theory breaks down. This is because when $2k/k' \rightarrow 1$, the amplitude of reflected waves (B) becomes comparable to that of incident waves (A), and their first assumption that $B/A \ll 1$ is not satisfied any more. This condition is known as the Bragg resonance. Mei (1985) treated this special resonance case. He assumed $2k \approx k'$ and $B/A = O(1)$, and introduced the multiple scale method. He showed that strong reflection can be induced by the sandbars, once the Bragg resonance condition is met. Furthermore, if the incident waves are slightly detuned or slowly modulated in time, the scattering process is found to depend critically on whether the modulational frequency lies above or below a threshold frequency. In addition, he found that for periodically modulated wave groups, nonlinear effects can radiate long waves over the bars far beyond the reach of the short waves themselves.

The purpose of the study in this thesis is essentially the extension of Mei's study. Since the governing equations for long waves are not correct in his paper, we first rederive the governing equations for first order short wave envelopes and second order long waves in a more rigorous way in Chapter 2. Then the solutions of wave envelopes for a detuned wave train and a wave packet on a slope are obtained. Furthermore long waves associated with a detuned wave train, a periodically modulated wave train, and a wave packet are treated. In Chapter 3, experimental results are shown to confirm the theory given in Chapter 2. First, detuned wave trains are used as the incident condition, and distinctive properties of wave height distributions for various modulational frequencies above or below the threshold frequency are demonstrated. Next, it is observed that a wave packet is strongly reflected and becomes dispersive in the sandbar region. At the same time, a wave set-down originally locked to a wave packet is seen to be unlocked as the packet enters the sandbar region and to propagate with the speed of long waves. In Chapter 4, the conclusion of this study is summarized.

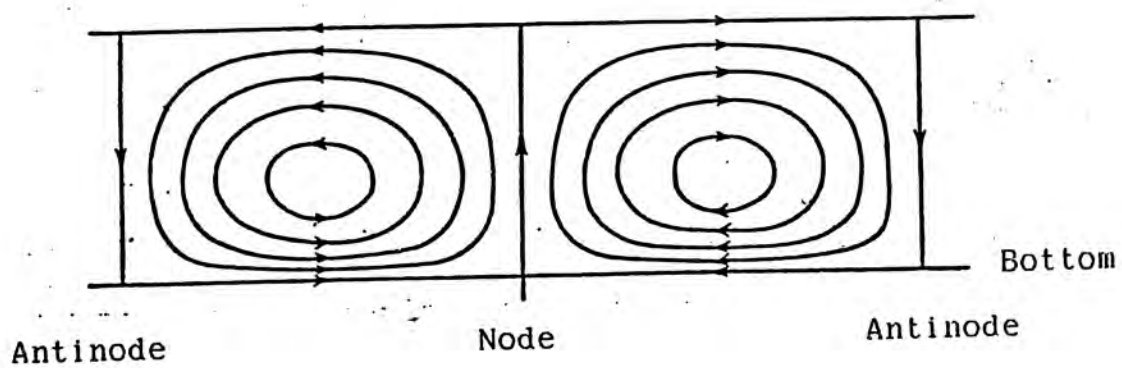


Fig. 1-1 Circulation of Mass Transport in a Boundary Layer
under a Standing Wave

Chapter 2 Theory

2-1 Approximate Equations

We first derive the approximate equations for wave envelopes and a mean sea level over spatially periodical sandbars. The slopes of the free surface, the mean bottom, and the sandbars are all assumed to be $O(\varepsilon)$ ($\varepsilon \ll 1$). Incident waves are allowed to be slowly modulated in time and space. The ratio of wavelength to group length is also assumed to be $O(\varepsilon)$. The velocity potential ϕ satisfies

$$\nabla^2 \phi + \phi_{zz} = 0 \quad -h + \delta < z < \zeta \quad (2-1)$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$$

where ζ is the water surface elevation, h is the mean depth, and δ is the bar height above the mean. The kinematic and dynamic boundary conditions on the free surface are

$$\zeta_t + \nabla \zeta \cdot \nabla \phi = \phi_z \quad z = \zeta \quad (2-2)$$

$$g\zeta + \phi_t + \frac{1}{2}\{(\nabla \phi)^2 + \phi_z^2\} = 0 \quad z = \zeta \quad (2-3)$$

On the sea bottom, the boundary condition is

$$\nabla \phi \cdot \nabla h - \nabla \phi \cdot \nabla \delta + \phi_z = 0 \quad z = -h + \delta \quad (2-4)$$

Eq.(2-3) can be applied at $z=0$,

$$g\zeta + \zeta_t \phi_z + \phi_t + \frac{1}{2}\{(\nabla \phi)^2 + \phi_z^2\} = O(\varepsilon^3) \quad z=0 \quad (2-5)$$

Eqs.(2-3) and (2-4) may be combined in terms of ϕ alone and applied at $z=0$,

$$\begin{aligned} \phi_{tt} + g\phi_z = & \left[-\frac{1}{2}\{(\nabla \phi)^2 + \phi_z^2\} + \frac{1}{g}\phi_t \phi_{zt} \right]_t - \nabla \cdot (\nabla \phi \cdot \phi_t) \\ & + \left[\frac{1}{2g}\{\phi_t (\nabla \phi)^2 + \phi_t \phi_z^2\}_z - \frac{1}{g^2}\phi_t \phi_{zt}^2 - \frac{1}{2g^2}\phi_{zzt} \phi_t^2 \right]_t \\ & + \nabla \cdot \left[-\frac{1}{2}\nabla \phi \{(\nabla \phi)^2 + \phi_z^2\} + \frac{1}{g}\nabla \phi \cdot \phi_t \phi_{zt} + \frac{1}{2g}\nabla \phi_z \phi_t^2 \right] + O(\varepsilon^4) \end{aligned} \quad (2-6)$$

Eq.(2-4) may also be rewritten as

$$\nabla \phi \cdot \nabla h + \nabla \phi_z \cdot \nabla h \cdot \delta - \nabla \phi \cdot \nabla \delta - \nabla \phi_z \cdot \nabla \delta \cdot \delta + \phi_z + \phi_{zz} \delta + \frac{1}{2}\phi_{zzz} \delta^2 = O(\varepsilon^4) \quad z = -h \quad (2-7)$$

We introduce slow coordinates

$$\bar{x} = \varepsilon x, \bar{y} = \varepsilon y, \bar{t} = \varepsilon t \quad (2-8)$$

and assume that

$$h = h(\bar{x}, \bar{y}), \delta = \delta(x, y, \bar{x}, \bar{y}), \zeta = \zeta(x, y, t, \bar{x}, \bar{y}, \bar{t}), \phi = \phi(x, y, z, t, \bar{x}, \bar{y}, \bar{t}) \quad (2-9)$$

We also use perturbation expansions for ϕ and ζ ,

$$\phi = \sum_{n=1}^{\infty} \varepsilon^n \phi_n, \quad \zeta = \sum_{n=1}^{\infty} \varepsilon^n \zeta_n \quad (2-10)$$

$$(\phi_n, \zeta_n) = \sum_{m=-n}^n \{ \phi_{nm}(x, y, z, \bar{x}, \bar{y}, \bar{t}), \zeta_{nm}(x, y, \bar{x}, \bar{y}, \bar{t}) \} e^{-im\omega t}$$

$$\phi_{nm} = \phi_{n,-m}^*, \quad \zeta_{nm} = \zeta_{n,-m}^* \quad (2-11)$$

where $()^*$ means the complex conjugate. Substituting Eqs.(2-10) and (2-11) into Eqs.(2-1), (2-6) and (2-7), and separating the orders, we obtain

$O(\varepsilon)$

$$\nabla^2 \phi_{10} + \phi_{10} \zeta \zeta = 0 \quad -h < z < 0 \quad (2-12)$$

$$\phi_{10z} = 0 \quad z=0 \quad (2-13)$$

$$\phi_{10z} = 0 \quad z=-h \quad (2-14)$$

$$\nabla^2 \phi_{11} + \phi_{11} \zeta \zeta = 0 \quad -h < z < 0 \quad (2-15)$$

$$-\omega^2 \phi_{11} + g \phi_{11} \zeta = 0 \quad z=0 \quad (2-16)$$

$$\phi_{11z} = 0 \quad z=-h \quad (2-17)$$

$O(\varepsilon^2)$

$$\nabla^2 \phi_{20} + \phi_{20} \zeta \zeta = -(\nabla \cdot \bar{\nabla} \phi_{10} + \bar{\nabla} \cdot \nabla \phi_{10}) \quad -h < z < 0 \quad (2-18)$$

$$\phi_{20z} = -\frac{i\omega}{g} \nabla \cdot (\nabla \phi_{11} \phi_{11}^* + \nabla \phi_{11}^* \phi_{11}) \quad z=0 \quad (2-19)$$

$$\phi_{20z} = -\nabla \phi_{10} \cdot \bar{\nabla} h + \nabla \phi_{10} \cdot \nabla \delta + \nabla^2 \phi_{10} \delta \quad z=-h \quad (2-20)$$

$$\nabla^2 \phi_{21} + \phi_{21} \nabla^2 Z = -(\nabla \cdot \bar{\nabla} \phi_{11} + \bar{\nabla} \cdot \nabla \phi_{11}) \quad -h < z < 0 \quad (2-21)$$

$$-\omega^2 \phi_{21} + g \phi_{21} z = 2i\omega \phi_{11} \bar{t} \quad z=0 \quad (2-22)$$

$$\phi_{21} z = -\nabla \phi_{11} \cdot \bar{\nabla} h + \nabla \phi_{11} \cdot \nabla \delta + \nabla^2 \phi_{11} \delta \quad z=-h \quad (2-23)$$

where

$$\bar{\nabla} = \left(\frac{\partial}{\partial \bar{x}}, \frac{\partial}{\partial \bar{y}} \right)$$

Let the sandbar region be between $x=-L$ and $x=0$. The boundary conditions then become

$$[\phi_t + \frac{1}{2}((\nabla \phi)^2 + \phi_z^2)]_{x=-L+0} = [\phi_t + \frac{1}{2}((\nabla \phi)^2 + \phi_z^2)]_{x=-L-0} \quad (2-24)$$

$$[\phi_t + \frac{1}{2}((\nabla \phi)^2 + \phi_z^2)]_{x=+0} = [\phi_t + \frac{1}{2}((\nabla \phi)^2 + \phi_z^2)]_{x=-0} \quad (2-25)$$

$$[\nabla \phi]_{x=-L+0} = [\nabla \phi]_{x=-L-0} \quad (2-26)$$

$$[\nabla \phi]_{x=+0} = [\nabla \phi]_{x=-0} \quad (2-27)$$

or at the first order,

$$[\phi_{11}]_{x=-L+0} = [\phi_{11}]_{x=-L-0} \quad (2-28)$$

$$[\phi_{11}]_{x=+0} = [\phi_{11}]_{x=-0} \quad (2-29)$$

$$[\nabla \phi_{11}]_{x=-L+0} = [\nabla \phi_{11}]_{x=-L-0} \quad (2-30)$$

$$[\nabla \phi_{11}]_{x=+0} = [\nabla \phi_{11}]_{x=-0} \quad (2-31)$$

$$[\nabla \phi_{10}]_{x=-L+0} = [\nabla \phi_{10}]_{x=-L-0} \quad (2-32)$$

$$[\nabla \phi_{10}]_{x=+0} = [\nabla \phi_{10}]_{x=-0} \quad (2-33)$$

From here on we assume that the mean bottom contours are parallel to y-axis, i.e. $h=h(x)$. The solution which satisfies Eqs.(2-15)-(2-17) is expressed as follows

$$\phi_{11} = A f_0(z) e^{iS^+} + B f_0(z) e^{iS^-} + \sum_{n=1}^{\infty} C_n f_n(z) e^{S_n^+} + \sum_{n=1}^{\infty} D_n f_n(z) e^{S_n^-}$$

(2-34)

$$S^{\pm} = \pm \int^x \alpha(\bar{x}) dx + \beta y \quad (2-35)$$

$$\alpha^2 + \beta^2 = k^2, \quad \vec{k}^{\pm} = \begin{pmatrix} \pm \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \pm k \cdot \cos \theta \\ k \cdot \sin \theta \end{pmatrix} \quad (2-36)$$

$$S_n^{\pm} = \pm \int^x \alpha_n(\bar{x}) dx + \beta y \quad (2-37)$$

$$\alpha_n^2 = \beta^2 + k_n^2 \quad (2-38)$$

$$f_0 = - \frac{ig}{2\omega} \frac{\cosh k(z+h)}{\cosh kh} \quad (2-39)$$

$$\omega^2 = gk \tanh kh \quad (2-40)$$

$$f_n = \cos k_n(z+h) \quad (2-41)$$

$$\omega^2 = -gk_n \tan k_n h \quad (2-42)$$

where θ is the local inclination of the incident wave with respect to the x-axis. Considering boundary conditions Eq.(2-28)-(2-31), we get

$$[A]_{x=-L+0} = [A]_{x=-L-0} \quad (2-43)$$

$$[A]_{x=+0} = [A]_{x=-0} \quad (2-44)$$

$$[B]_{x=-L+0} = [B]_{x=-L-0} \quad (2-45)$$

$$[B]_{x=+0} = [B]_{x=-0} \quad (2-46)$$

$$C_n = D_n = 0 \quad (2-47)$$

Thus

$$\phi_{11} = Af_0 e^{iS^+} + Bf_0 e^{iS^-} \quad (2-48)$$

From Eq.(2-5)

$$\zeta_{11} = \frac{1}{g} [\phi_{11}]_{z=0} = \frac{1}{2} (Ae^{iS^+} + Be^{iS^-}) \quad (2-49)$$

From Eqs.(2-12)-(2-14), (2-32) and (2-33) we obtain

$$\phi_{10} = \phi_{10}(\bar{x}, \bar{y}, \bar{t}) \quad (2-50)$$

These first order solutions are substituted into Eqs.(2-18)-(2-20) to give

$$\nabla^2 \phi_{20} + \phi_{20} \nabla^2 \phi_{10} = 0 \quad -h < z < 0 \quad (2-51)$$

$$\begin{aligned} \phi_{20z} &= -\frac{i\omega}{g} \nabla \cdot (\nabla \phi_{11} \phi_{11}^* + \nabla \phi_{11}^* \phi_{11}) \\ &= -\frac{i\omega}{g} \nabla \cdot \left(2i|A|^2 |f_0|^2 \left(\frac{\alpha}{g} \right) + 2i|B|^2 |f_0|^2 \left(\frac{-\alpha}{g} \right) \right. \\ &\quad \left. + 2iA^*B |f_0|^2 \left(\frac{0}{g} \right) e^{-2i \int^x \alpha dx} + 2iAB^* |f_0|^2 \left(\frac{0}{g} \right) e^{2i \int^x \alpha dx} \right) \\ &= 0 \quad z=0 \end{aligned} \quad (2-52)$$

$$\phi_{20z} = 0 \quad z=-h \quad (2-53)$$

From Eqs.(2-24)-(2-27) we get at the second order,

$$\begin{aligned} [\phi_{10} \bar{t} + |\nabla \phi_{11}|^2 + |\phi_{11z}|^2]_{x=-L+0} \\ = [\phi_{10} \bar{t} + |\nabla \phi_{11}|^2 + |\phi_{11z}|^2]_{x=-L-0} \end{aligned} \quad (2-54)$$

$$\begin{aligned} [\phi_{10} \bar{t} + |\nabla \phi_{11}|^2 + |\phi_{11z}|^2]_{x=+0} \\ = [\phi_{10} \bar{t} + |\nabla \phi_{11}|^2 + |\phi_{11z}|^2]_{x=-0} \end{aligned} \quad (2-55)$$

$$[\bar{\nabla} \phi_{10} + \nabla \phi_{20}]_{x=-L+0} = [\bar{\nabla} \phi_{10} + \nabla \phi_{20}]_{x=-L-0} \quad (2-56)$$

$$[\bar{\nabla} \phi_{10} + \nabla \phi_{20}]_{x=+0} = [\bar{\nabla} \phi_{10} + \nabla \phi_{20}]_{x=-0} \quad (2-57)$$

The solution of Eqs.(2-51)-(2-53) which satisfies the boundary conditions (2-56) and (2-57) should be

$$\phi_{20} = \phi_{20}(\bar{x}, \bar{y}, \bar{t}) \quad (2-58)$$

Thus

$$[\bar{\nabla} \phi_{10}]_{x=-L+0} = [\bar{\nabla} \phi_{10}]_{x=-L-0} \quad (2-59)$$

$$[\bar{\nabla} \phi_{10}]_{x=+0} = [\bar{\nabla} \phi_{10}]_{x=-0} \quad (2-60)$$

From Eqs.(2-54) and (2-55)

$$[\phi_{10}\bar{t}]_{x=-L+0} = [\phi_{10}\bar{t}]_{x=-L-0} \quad (2-61)$$

$$[\phi_{10}\bar{t}]_{x=+0} = [\phi_{10}\bar{t}]_{x=-0} \quad (2-62)$$

We assume the sandbars to be parallel to the y-axis and spaced at one-half of the local wavelength:

$$\delta = \frac{1}{2}D\{\exp[2i\int_{\alpha}^x dx] + \exp[-2i\int_{\alpha}^x dx]\} \quad (2-63)$$

where the real amplitude $D(\bar{x}, \bar{y})$ can vary slowly in both horizontal directions. Substituting Eqs.(2-48) and (2-63) into Eqs.(2-21)-(2-23), we obtain

$$\begin{aligned} \nabla^2 \phi_{21} + \phi_{21} \nabla^2 = & \{-i\vec{k}^+ \cdot \bar{\nabla}(Af_0) - i\bar{\nabla}(\vec{k}^+ Af_0)\} e^{iS^+} \\ & + \{-i\vec{k}^- \cdot \bar{\nabla}(Bf_0) - i\bar{\nabla}(\vec{k}^- Bf_0)\} e^{iS^-} \quad -h < z < 0 \end{aligned} \quad (2-64)$$

$$-\omega^2 \phi_{21} + g \phi_{21z} = 2i\omega(Af_0)\bar{t} e^{iS^+} + 2i\omega(Bf_0)\bar{t} e^{iS^-} \quad z=0 \quad (2-65)$$

$$\begin{aligned} \phi_{21z} = & \{-i\bar{\nabla}h(\vec{k}^+ Af_0) + \frac{1}{2}(\alpha^2 - \beta^2)D(Bf_0)\} e^{iS^+} \\ & + \{-i\bar{\nabla}h(\vec{k}^- Bf_0) + \frac{1}{2}(\alpha^2 - \beta^2)D(Af_0)\} e^{iS^-} \\ & + \dots \quad z=-h \end{aligned} \quad (2-66)$$

Let

$$\phi_{21} = i\gamma^+ e^{iS^+} + i\gamma^- e^{iS^-} + \dots \quad (2-67)$$

then Eqs.(2-64)-(2-66) become

$$-k^2 \gamma^+ + \gamma^+_{zz} = -\vec{k}^+ \cdot \bar{\nabla}(Af_0) - \bar{\nabla}(\vec{k}^+ Af_0) \quad -h < z < 0 \quad (2-68)$$

$$-\frac{\omega^2}{g} \gamma^+ + \gamma^+_z = \frac{2\omega}{g}(Af_0)\bar{t} \quad z=0 \quad (2-69)$$

$$\gamma^+_z = -\bar{\nabla}h(\vec{k}^+ Af_0) - \frac{iD}{2}(\alpha^2 - \beta^2)(Bf_0) \quad z=-h \quad (2-70)$$

Similar equations are obtained for γ^- . Let us consider the

solvability condition for γ^+ . We apply Green's formula to γ^+ and (Af_0) .

$$\int_{-h}^0 dz [(Af_0)(\gamma_{zz}^+ - k^2 \gamma^+) - \gamma^+ \{ (Af_0)_{zz} - k^2 (Af_0) \}]$$

$$= [(Af_0)\gamma_z^+ - \gamma^+(Af_0)_z]_{-h}^0 \quad (2-71)$$

Introducing Eqs.(2-68)-(2-70), we obtain

$$\int_{-h}^0 dz \bar{\nabla} \{ \vec{k}^+ (Af_0)^2 \} + [\bar{\nabla} h \cdot \vec{k}^+ (Af_0)^2]_{z=-h}$$

$$= -[\frac{2\omega}{g}(Af_0)(Af_0)_{\bar{t}}]_{z=0} - [\frac{iD}{2}(\alpha^2 - \beta^2)(Af_0)(Bf_0)]_{z=-h} \quad (2-72)$$

By Leibniz's rule,

$$\bar{\nabla} \left[\int_{-h}^0 dz \{ \vec{k}^+ (Af_0)^2 \} \right]$$

$$= -[\frac{2\omega}{g}(Af_0)(Af_0)_{\bar{t}}]_{z=0} - [\frac{iD}{2}(\alpha^2 - \beta^2)(Af_0)(Bf_0)]_{z=-h} \quad (2-73)$$

Thus

$$\bar{\nabla} [\vec{C}_g^+ \frac{A^2}{2}] + (\frac{A^2}{2})_{\bar{t}} = -i\Omega_0 AB \cos 2\theta \quad (2-74)$$

where

$$\Omega_0 = \frac{\omega k D}{2 \sinh 2kh} \quad (2-75)$$

$$\vec{C}_g^{\pm} = \frac{\vec{k}^{\pm}}{k} \frac{\omega}{2k} (1 + \frac{2kh}{\sinh 2kh}) \quad (2-76)$$

Finally we obtain

$$A_{\bar{t}} + \vec{C}_g^+ \cdot \bar{\nabla} A + \frac{A}{2} \bar{\nabla} \cdot \vec{C}_g^+ = -i\Omega_0 B \cos 2\theta \quad (2-77)$$

$$B_{\bar{t}} + \vec{C}_g^- \cdot \bar{\nabla} B + \frac{B}{2} \bar{\nabla} \cdot \vec{C}_g^- = -i\Omega_0 A \cos 2\theta \quad (2-78)$$

With a similar technique, we obtain

$$\bar{\nabla} [\vec{C}_g^+ \frac{|A|^2}{2}] + (\frac{|A|^2}{2})_{\bar{t}} = -\frac{i}{2} \Omega_0 A^* B \cos 2\theta + * \quad (2-79)$$

$$\bar{\nabla} [\vec{C}_g^- \frac{|B|^2}{2}] + (\frac{|B|^2}{2})_{\bar{t}} = +\frac{i}{2} \Omega_0 A^* B \cos 2\theta + * \quad (2-80)$$

These results are the same as those by Mei(1985). At the third order, Eqs.(2-1),(2-6) and (2-7) give:

$$O(\varepsilon^3)$$

$$\nabla^2 \phi_{30} + \phi_{30} \nabla \nabla = -\bar{\nabla}^2 \phi_{10} - (\nabla \bar{\nabla} \phi_{20} + \bar{\nabla} \nabla \phi_{20}) = -\bar{\nabla}^2 \phi_{10} \quad -h < z < 0 \quad (2-81)$$

$$\begin{aligned} \phi_{30z} = & -\frac{1}{g} \phi_{10} \bar{t} \bar{t} + \frac{1}{g} [-|\nabla \phi_{11}|^2 - |\nabla \phi_{11z}|^2 + \frac{\omega^2}{g} (\phi_{11} \phi_{11z}^* + \phi_{11}^* \phi_{11z})] \bar{t} \\ & - \frac{1}{g} \bar{\nabla} \cdot (i\omega \nabla \phi_{11} \cdot \phi_{11}^* - i\omega \nabla \phi_{11}^* \cdot \phi_{11}) - \frac{1}{g} \nabla \cdot (i\omega \bar{\nabla} \phi_{11} \cdot \phi_{11}^* - i\omega \bar{\nabla} \phi_{11}^* \cdot \phi_{11}) \\ & - \frac{1}{g} \nabla \cdot (i\omega \nabla \phi_{21} \cdot \phi_{11}^* - i\omega \nabla \phi_{21}^* \cdot \phi_{11}) - \frac{1}{g} \nabla \cdot (i\omega \nabla \phi_{11} \cdot \phi_{21}^* - i\omega \nabla \phi_{11}^* \cdot \phi_{21}) \\ & - \frac{1}{g} \nabla \cdot (\nabla \phi_{11} \cdot \phi_{11}^* \bar{t} + \nabla \phi_{11}^* \cdot \phi_{11} \bar{t}) \quad z=0 \quad (2-82) \end{aligned}$$

$$\begin{aligned} \phi_{30z} = & -\nabla \phi_{20} \bar{\nabla} h - \bar{\nabla} \phi_{10} \bar{\nabla} h + \nabla \phi_{20} \nabla \delta + \bar{\nabla} \phi_{10} \nabla \delta - \phi_{20} \nabla \nabla \delta \\ = & -\bar{\nabla} \phi_{10} \bar{\nabla} h + \bar{\nabla} \phi_{10} \nabla \delta \quad z=-h \quad (2-83) \end{aligned}$$

The solvability condition for ϕ_{30} gives

$$\int_{-h}^0 \phi_{30z} dz = - \int_{-h}^0 \nabla^2 \phi_{30} dz - \int_{-h}^0 \bar{\nabla}^2 \phi_{10} dz = [\phi_{30z}]_{-h}^0 \quad (2-84)$$

where Eqs. (2-82) and (2-83) should be substituted into the r.h.s. Taking spatial average in wavelength and neglecting evanescent mode, we obtain

$$\begin{aligned} \phi_{10} \bar{t} \bar{t} - g \bar{\nabla} \cdot (h \bar{\nabla} \phi_{10}) = & \overline{-\bar{\nabla} \cdot (i\omega \nabla \phi_{11} \cdot \phi_{11}^* - i\omega \nabla \phi_{11}^* \cdot \phi_{11})} \\ & - \overline{(|\nabla \phi_{11}|^2 + |\nabla \phi_{11z}|^2)} \bar{t} + \frac{\omega^2}{g} \overline{(\phi_{11} \phi_{11z}^* + \phi_{11}^* \phi_{11z})} \bar{t} \quad z=0 \quad (2-85) \end{aligned}$$

where overbars show the spatial average.
From Eq. (2-5)

$$\begin{aligned} \zeta_{20} = & -\frac{1}{g} (i\omega \zeta_{11} \phi_{11z}^* - i\omega \zeta_{11}^* \phi_{11z}) - \frac{1}{g} \phi_{10} \bar{t} - \frac{1}{g} \overline{(|\nabla \phi_{11}|^2 + |\nabla \phi_{11z}|^2)} \\ = & \frac{\omega^2}{g^2} \overline{(\phi_{11} \phi_{11z}^* + \phi_{11}^* \phi_{11z})} - \frac{1}{g} \phi_{10} \bar{t} - \frac{1}{g} \overline{(|\nabla \phi_{11}|^2 + |\nabla \phi_{11z}|^2)} \quad z=0 \quad (2-86) \end{aligned}$$

Taking spatial average,

$$\bar{\epsilon}_{20} = \frac{\omega^2}{g^2} \overline{(\phi_{11}\phi_{11Z}^* + \phi_{11}^*\phi_{11Z})} - \frac{1}{g}\phi_{10}\bar{t} - \frac{1}{g}(\overline{|\nabla\phi_{11}|^2} + \overline{|\phi_{11Z}|^2})$$

$z=0 \quad (2-87)$

Substituting Eq.(2-87) into Eq.(2-85), we get

$$\begin{aligned} \bar{\epsilon}_{20}\bar{t}\bar{t} &= g\bar{\nabla}[h\bar{\nabla}\{\bar{\epsilon}_{20} - \frac{\omega^2}{g^2} \overline{(\phi_{11}\phi_{11Z}^* + \phi_{11}^*\phi_{11Z})} + \frac{1}{g}(\overline{|\nabla\phi_{11}|^2} + \overline{|\phi_{11Z}|^2})\}] \\ &= \frac{1}{g} \bar{\nabla}(i\omega\nabla\phi_{11}\phi_{11}^* - i\omega\nabla\phi_{11}^*\phi_{11})\bar{t} \quad z=0 \end{aligned} \quad (2-88)$$

Now we introduce Eqs.(2-48) and (2-49) into Eqs.(2-86)-(2-88).

$$\begin{aligned} \bar{\epsilon}_{20} &= -\frac{1}{g}\phi_{10}\bar{t} - \frac{1}{4h} \frac{2kh}{\sinh 2kh} (|A|^2 + |B|^2) \\ &\quad + \frac{1}{4h} \frac{2kh}{\sinh 2kh} (\sinh^2 kh - \frac{\beta^2 - \alpha^2}{k^2} \cosh^2 kh) \{AB^* e^{2i} \int_{\alpha}^x dx_{+} (*)\} \end{aligned} \quad (2-89)$$

$$\bar{\epsilon}_{20} = -\frac{1}{g}\phi_{10}\bar{t} - \frac{1}{4h} \frac{2kh}{\sinh 2kh} (|A|^2 + |B|^2) \quad (2-90)$$

$$\begin{aligned} \bar{\epsilon}_{20}\bar{t}\bar{t} &= g\bar{\nabla}[h\bar{\nabla}\{\bar{\epsilon}_{20} + \frac{1}{4h} \frac{2kh}{\sinh 2kh} (|A|^2 + |B|^2)\}] \\ &= -\frac{g}{2\omega} \bar{\nabla}\left\{ \frac{\alpha(|A|^2 - |B|^2)}{\beta(|A|^2 + |B|^2)} \right\} \bar{t} \end{aligned} \quad (2-91)$$

From Eqs.(2-59)-(2-62) and (2-90), the boundary conditions for $\bar{\epsilon}_{20}$ become

$$[\bar{\epsilon}_{20}]_{x=-L+0} = [\bar{\epsilon}_{20}]_{x=-L-0} \quad (2-92)$$

$$[\bar{\epsilon}_{20}]_{x=+0} = [\bar{\epsilon}_{20}]_{x=-0} \quad (2-93)$$

$$\begin{aligned} [\bar{\nabla}\{\bar{\epsilon}_{20} + \frac{1}{4h} \frac{2kh}{\sinh 2kh} (|A|^2 + |B|^2)\}]_{x=-L+0} \\ = [\bar{\nabla}\{\bar{\epsilon}_{20} + \frac{1}{4h} \frac{2kh}{\sinh 2kh} (|A|^2 + |B|^2)\}]_{x=-L-0} \end{aligned} \quad (2-94)$$

$$\begin{aligned}
& [\bar{\nabla}(\bar{\epsilon}_{20} + \frac{1}{4h} \frac{2kh}{\sinh 2kh} (|A|^2 + |B|^2))]_{x=+0} \\
& = [\bar{\nabla}(\bar{\epsilon}_{20} + \frac{1}{4h} \frac{2kh}{\sinh 2kh} (|A|^2 + |B|^2))]_{x=+0} \quad (2-95)
\end{aligned}$$

When

$$h = \text{const.}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \bar{y}} = 0$$

$$\alpha = k, \beta = 0$$

Eq.(2-91) becomes

$$\begin{aligned}
\bar{\epsilon}_{20} \bar{t} \bar{t} - gh \bar{\epsilon}_{20} \bar{x} \bar{x} &= \left\{ \frac{g^2 C_g}{4C} - 1 \right\} (|A|^2 + |B|^2)_{\bar{x} \bar{x}} \\
&- \left\{ \frac{gk}{2\omega} (|A|^2 - |B|^2) \right\}_{\bar{x} \bar{t}} \quad (2-96)
\end{aligned}$$

Introducing Eq.(2-79), (2-80), we obtain

$$\bar{\epsilon}_{20} \bar{t} \bar{t} - gh \bar{\epsilon}_{20} \bar{x} \bar{x} = \frac{1}{\rho} S_{\bar{x} \bar{x}} + \left\{ \frac{g \Omega_0}{C} (iA^* B - iAB^*) \right\}_{\bar{x}} \quad (2-97)$$

where

$$S_{\bar{x} \bar{x}} = \frac{\rho g}{4} \left(\frac{4C_g}{C} - 1 \right) (|A|^2 + |B|^2) \quad (2-98)$$

is the radiation stress, and the second term in the r.h.s. of Eq.(2-97) is an additional term due to the existence of sandbars.

2-2 Method of Solution, Results, and Discussions

2-2-1 First Order Short Waves

We assume that the mean bottom has a constant slope, the sandbar region is $-L < x < 0$, and the shoreline is at $x = L_0$. The governing equations for first order short wave envelopes are given as follows:

$$\frac{\partial A}{\partial t} + C_g \cos \theta \frac{\partial A}{\partial x} + C_g \sin \theta \frac{\partial A}{\partial y} + \frac{A}{2} \frac{\partial C_g}{\partial x} \cos \theta = -i\Omega_0 B \cos 2\theta \quad (2-99)$$

$$\frac{\partial B}{\partial t} - C_g \cos \theta \frac{\partial B}{\partial x} + C_g \sin \theta \frac{\partial B}{\partial y} + \frac{B}{2} \frac{\partial C_g}{\partial x} \cos \theta = -i\Omega_0 A \cos 2\theta \quad (2-100)$$

$-L < x < 0$

$$\frac{\partial A}{\partial t} + C_g \cos \theta \frac{\partial A}{\partial x} + C_g \sin \theta \frac{\partial A}{\partial y} + \frac{A}{2} \frac{\partial C_g}{\partial x} \cos \theta = 0 \quad (2-101)$$

$$\frac{\partial B}{\partial t} - C_g \cos \theta \frac{\partial B}{\partial x} + C_g \sin \theta \frac{\partial B}{\partial y} + \frac{B}{2} \frac{\partial C_g}{\partial x} \cos \theta = 0 \quad (2-102)$$

$x < -L, 0 < x$

where all overbars for variables are omitted. Boundary conditions are

$$[A]_{x=-L+0} = [A]_{x=-L-0} \quad (2-103)$$

$$[A]_{x=+0} = [A]_{x=-0} \quad (2-104)$$

$$[B]_{x=-L+0} = [B]_{x=-L-0} \quad (2-105)$$

$$[B]_{x=+0} = [B]_{x=-0} \quad (2-106)$$

Let

$$\bar{A} = A C_g^{\frac{1}{2}}, \quad \bar{B} = B C_g^{\frac{1}{2}} \quad (2-107)$$

then Eqs. (2-99) and (2-100) become

$$\frac{\partial \bar{A}}{\partial t} + C_g (\cos \theta \frac{\partial \bar{A}}{\partial x} + \sin \theta \frac{\partial \bar{A}}{\partial y}) = -i\Omega_0 \bar{B} \cos 2\theta \quad (2-108)$$

$$\frac{\partial \bar{B}}{\partial t} + C_g (-\cos \theta \frac{\partial \bar{B}}{\partial x} + \sin \theta \frac{\partial \bar{B}}{\partial y}) = -i\Omega_0 \bar{A} \cos 2\theta \quad (2-109)$$

a) Normal Incidence of a Detuned Wave Train

Since

$$\theta = 0 \quad (2-110)$$

Eqs.(2-108) and (2-109) become

$$\bar{A}_t + C_g \bar{A}_x = -i\Omega_0 \bar{B} \quad (2-111)$$

$$\bar{B}_t - C_g \bar{B}_x = -i\Omega_0 \bar{A} \quad (2-112)$$

Let

$$\bar{A} = \bar{A} e^{-i\Omega t} \quad (2-113)$$

$$\bar{B} = \bar{B} e^{-i\Omega t} \quad (2-114)$$

then

$$-i\Omega \bar{A} + C_g \bar{A}_x = -i\Omega_0 \bar{B} \quad (2-115)$$

$$-i\Omega \bar{B} - C_g \bar{B}_x = -i\Omega_0 \bar{A} \quad (2-116)$$

We introduce dimensionless variable

$$x' = \frac{x}{L} \quad (2-117)$$

and combine Eqs.(2-115) and (2-116), obtaining

$$\left[\bar{B}_{x'}, \frac{C_g}{\Omega_0 L} \right]_{x'} + \left[\frac{\Omega_0 L}{C_g} \left\{ \left(\frac{\Omega}{\Omega_0} \right)^2 - 1 \right\} + i \left(\frac{\Omega}{\Omega_0} \right)_{x'} \right] \bar{B} = 0 \quad -1 < x' < 0 \quad (2-118)$$

Assume that there are no incoming waves from the x-positive direction. Boundary conditions then become

$$\bar{B} = 0 \quad x' = 0 \quad (2-119)$$

$$\frac{\Omega}{\Omega_0} \bar{B} - i \frac{C_g}{\Omega_0 L} \bar{B}_{x'} = \bar{A} \quad x' = -1 \quad (2-120)$$

Now we consider the Sturm-Liouville Equation

$$Lu = \left[-\frac{d}{dx} \left\{ p(x) \frac{d}{dx} \right\} + q(x) \right] u = f(x) \quad (2-121)$$

$$u(0) = g \quad (2-122)$$

$$\frac{du}{dx}(-1) + \alpha u(-1) = b \quad (2-123)$$

Its functional is defined as

$$I(u) = \int_0^{-1} \{p \cdot (\frac{du}{dx})^2 + q \cdot u^2 - 2fu\} dx + \alpha \cdot p(-1) \cdot \{u(-1)\}^2 - 2b \cdot p(-1) \cdot u(-1) \quad (2-124)$$

Taking the variation of Eq.(2-124)

$$\delta I(u) = 2\delta u \int_0^{-1} (Lu - f) dx + [2p \cdot (\alpha u - b + \frac{du}{dx}) \delta u]_{x=-1} \quad (2-125)$$

we obtain the following condition

$$\left. \begin{array}{l} \delta I(u) = 0 \\ \delta u(0) = 0 \\ u(0) = g \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} Lu = f \\ u(0) = g \\ \frac{du}{dx}(-1) + \alpha u(-1) = b \end{array} \right. \quad (2-126)$$

In this case we put

$$p(x') = - \frac{C_g}{\Omega_0 L} \quad (2-127)$$

$$q(x') = \frac{\Omega_0 L}{C_g} \{ (\frac{\Omega}{\Omega_0})^2 - 1 \} + i (\frac{\Omega}{\Omega_0})_{x'} \quad (2-128)$$

$$u(x') = \bar{B} \quad (2-129)$$

$$g = 0 \quad (2-130)$$

$$\alpha = i \left[\frac{\Omega_0 L}{C_g} \left(\frac{\Omega}{\Omega_0} \right) \right]_{x'} = -1 \quad (2-131)$$

$$b = i \left[\frac{\Omega_0 L}{C_g} \bar{A} \right]_{x'} = -1 \quad (2-132)$$

and we can solve Eq.(2-118) by the finite element method. The

solution only depends on $\frac{\Omega_0 L}{C_g}$ and $\frac{\Omega}{\Omega_0}$. Figs.(2-1) and (2-2) show

how $\frac{\Omega_0 L}{C_g}$ and $\frac{\Omega}{\Omega_0}$ change as water depth changes. When kh is large, $\frac{\Omega_0 L}{C_g}$ and $(\frac{\Omega}{\Omega_0})^{-1}$ decreases quickly as the water depth increases.

We first consider a perfectly tuned wave train ($\frac{\Omega}{\Omega_0} = 0$).

In Fig.(2-3) the reflected wave intensity at $x' = -1$ ($|R(-1)|$) is plotted as a function of $[\frac{\Omega_0 L}{C_g}]_{x' = 0}$ for several values of

$l = \frac{L_0}{L}$. Fig.(2-4) shows $|R(-1)|$ as a function of $\frac{1}{l} = \frac{L}{L_0}$ for

several values of $[\frac{\Omega_0 L}{C_g}]_{x' = 0}$. The reflection intensity

increases as $[\frac{\Omega_0 L}{C_g}]_{x' = 0}$ increases and approaches to unity. The

intensity $|R(-1)|$ also increases as $\frac{1}{l}$ increases, but not always approaches to unity. This is because, when the water depth is large enough, the effect of sandbars becomes negligible, and

$|R(-1)|$ does not increase any more even if $\frac{1}{l}$ increases.

We next consider a detuned wave train. Fig.(2-5) shows the incident wave amplitude A and the reflected wave amplitude B versus x' . As in the case of a constant depth (Mei 1985),

for $\frac{\Omega}{\Omega_0} < 1$, A and B decays monotonically, while for $\frac{\Omega}{\Omega_0} > 1$, A

and B changes oscillatory. (Note that $\frac{\Omega}{\Omega_0}$ becomes larger than

$[\frac{\Omega}{\Omega_0}]_{x' = 0}$ for a greater depth.) In Fig.(2-6), the reflected

wave intensity at $x' = -1$ ($|R(-1)|$) is plotted as a function of

$[\frac{\Omega_0 L}{C_g}]_{x' = 0}$ for several values of $[\frac{\Omega}{\Omega_0}]_{x' = 0}$. Fig.(2-7) shows

$|R(-1)|$ as a function of $[\frac{\Omega}{\Omega_0}]_{x' = 0}$ for several values of

$[\frac{\Omega_0 L}{C_g}]_{x' = 0}$. As $[\frac{\Omega}{\Omega_0}]_{x' = 0}$ increases, $|R(-1)|$ decays

monotonically for $\frac{\Omega}{\Omega_0} < 1$, but changes oscillatory for $\frac{\Omega}{\Omega_0} > 1$. We

can say that a sandbar region works as a kind of filter for incident waves with continuous spectrum, whose rate of reflection changes as in Fig.(2-7).

b) Normal Incidence of Transient Waves

We use the same governing equations as Eqs.(2-111) and (2-112),

$$\bar{A}_t + C_g \bar{A}_x = -i\Omega_0 \bar{B} \quad (2-133)$$

$$\bar{B}_t - C_g \bar{B}_x = -i\Omega_0 \bar{A} \quad (2-134)$$

and introduce dimensionless variables

$$x' = \frac{x}{L} \quad (2-135)$$

$$t' = [\Omega_0]_{x=-L} \cdot t \quad (2-136)$$

then Eqs.(2-133) and (2-134) become

$$\frac{\Omega_0(-L)}{\Omega_0(x)} \frac{\partial \bar{A}}{\partial t'} + \frac{C_g}{\Omega_0 L} \frac{\partial \bar{A}}{\partial x'} = -i\bar{B} \quad (2-137)$$

$$\frac{\Omega_0(-L)}{\Omega_0(x)} \frac{\partial \bar{B}}{\partial t'} - \frac{C_g}{\Omega_0 L} \frac{\partial \bar{B}}{\partial x'} = -i\bar{A} \quad (2-138)$$

These equations are equivalent to

$$\frac{\partial \bar{B}}{\partial x'} = -i \frac{\Omega_0 L}{C_g} \bar{A} \quad (2-139)$$

$$\text{on } \frac{dt'}{dx'} = - \frac{\Omega_0(-L) \cdot L}{C_g(x)} \quad (2-140)$$

$$\frac{\partial \bar{A}}{\partial x'} = -i \frac{\Omega_0 L}{C_g} \bar{B} \quad (2-141)$$

$$\text{on } \frac{dt'}{dx'} = \frac{\Omega_0(-L) \cdot L}{C_g(x)} \quad (2-142)$$

where Eqs.(2-140) and (2-142) show characteristic curves. We use third-order Adams-Bashforth difference scheme

$$\begin{aligned} \frac{A_j^{n+1} - A_j^n}{\Delta t} \left[\frac{\Omega_0(-L)}{\Omega_0} \right]_j + \left[\frac{C_g}{\Omega_0 L} \right]_j \left(\frac{23}{12} \frac{A_{j+1}^n - A_{j-1}^n}{2\Delta x} - \frac{16}{12} \frac{A_{j+1}^{n-1} - A_{j-1}^{n-1}}{2\Delta x} \right. \\ \left. + \frac{5}{12} \frac{A_{j+1}^{n-2} - A_{j-1}^{n-2}}{2\Delta x} \right) = -i \left(\frac{23}{12} B_j^n - \frac{16}{12} B_j^{n-1} + \frac{5}{12} B_j^{n-2} \right) \quad (2-143) \end{aligned}$$

$$\begin{aligned} \frac{B_j^{n+1} - B_j^n}{\Delta t} \left[\frac{\Omega_0(-L)}{\Omega_0} \right]_j - \left[\frac{C_g}{\Omega_0 L} \right]_j \left(\frac{23}{12} \frac{B_{j+1}^n - B_{j-1}^n}{2\Delta x} - \frac{16}{12} \frac{B_{j+1}^{n-1} - B_{j-1}^{n-1}}{2\Delta x} \right. \\ \left. + \frac{5}{12} \frac{B_{j+1}^{n-2} - B_{j-1}^{n-2}}{2\Delta x} \right) = -i \left(\frac{23}{12} A_j^n - \frac{16}{12} A_j^{n-1} + \frac{5}{12} A_j^{n-2} \right) \quad (2-144) \end{aligned}$$

where $A_j^n = A\{x=(j-1)\Delta x, t=n\Delta t\}$.

The error of this scheme is $O(\Delta x^2, \Delta t^3)$. This scheme is absolutely stable for

$$\frac{\Delta t}{\Delta x} \frac{C_g(x)}{\Omega_0(-L) \cdot L} < 0.75 \quad (2-145)$$

with very little dispersion and slight damping. Boundary conditions are

$$\bar{B} = 0 \quad \text{at } x' = 0 \quad (2-146)$$

$$\bar{A} : \text{given} \quad \text{at } x' = -1 \quad (2-147)$$

Spurious boundary conditions should be used for $\bar{B}$ at $x' = -1$, and $\bar{A}$ at $x' = 0$. Outside the sandbar region,

$$\bar{B} = \text{constant} \quad \text{on} \quad \frac{dt'}{dx'} = - \frac{\Omega_0(-L) \cdot L}{C_g(x)} \quad (2-148)$$

$$\bar{A} = \text{constant} \quad \text{on} \quad \frac{dt'}{dx'} = \frac{\Omega_0(-L) \cdot L}{C_g(x)} \quad (2-149)$$

We only need to calculate characteristic curves.

Figs.(2-8)-(2-10) show A and B versus t' at $x' = 0$ and $x' = -1$. The conditions at $x' = -1$ are the same in each figure. The dispersion of the incident wave envelope is clear at $x' = 0$. As the slope becomes steeper, the reflection becomes larger and the dispersion becomes clearer, since the water depth gets smaller at $x' = 0$.

2-2-2 Second Order Long Waves

We only consider a constant depth and normal incident waves. Governing equations are then

$$\bar{\epsilon}_{20} t t - g h \bar{\epsilon}_{20} x x = \frac{g}{4} \left(\frac{4C_g}{C} - 1 \right) (|A|^2 + |B|^2)_{xx} + \frac{g\Omega_0}{C} (iA^*B - iAB^*)_x \quad (2-150)$$

or

$$\bar{\epsilon}_{20} = - \frac{1}{4h} \frac{2kh}{\sinh 2kh} (|A|^2 + |B|^2) \quad (2-151)$$

when $\bar{\epsilon}_{20t} = \bar{\epsilon}_{20x} = 0$. At the edge of the sandbar region, boundary conditions are

$$\bar{\epsilon}_{20} : \text{continuous} \quad (2-152)$$

$$\{\bar{\epsilon}_{20} + \frac{1}{4h} \frac{2kh}{\sinh 2kh} (|A|^2 + |B|^2)\}_x : \text{continuous} \quad (2-153)$$

a) Detuned Wave Trains

We only consider a semi-infinite bar region ($x > 0$). Let the incident wave envelope A be given as

$$A = A_0 e^{i(Kx - \Omega t)} \quad \Omega = C_g K \quad x < 0 \quad (2-154)$$

$$(i) \left| \frac{\Omega}{\Omega_0} \right| < 1$$

The solutions of the first order short wave envelopes are

$$B = A_0 R_1 e^{i(-Kx - \Omega t)} \quad x < 0 \quad (2-155)$$

$$A = A_0 e^{-Qx - i\Omega t} \quad x > 0 \quad (2-156)$$

$$B = A_0 R_1 e^{-Qx - i\Omega t} \quad x > 0 \quad (2-157)$$

$$R_1 = \frac{\Omega}{\Omega_0} - i \left\{ 1 - \left(\frac{\Omega}{\Omega_0} \right)^2 \right\}^{\frac{1}{2}} \quad (2-158)$$

$$QC_g = (\Omega_0^2 - \Omega^2)^{\frac{1}{2}} \quad (2-159)$$

From Eqs. (2-150) and (2-151), we obtain

$$\bar{\epsilon}_{20} = - \frac{1}{4h} \frac{2kh}{\sinh 2kh} 2|A_0|^2 \quad x < 0 \quad (2-160)$$

$$\bar{\epsilon}_{20} = - \frac{1}{4h} \frac{2kh}{\sinh 2kh} 2|A_0|^2 e^{-2Qx} \quad x > 0 \quad (2-161)$$

$$(ii) \left| \frac{\Omega}{\Omega_0} \right| > 1$$

The solutions of the first order short wave envelopes are

$$B = A_0 R_2 e^{i(-Kx - \Omega t)} \quad x < 0 \quad (2-162)$$

$$A = A_0 e^{iPx - i\Omega t} \quad x > 0 \quad (2-163)$$

$$B = A_0 R_2 e^{iPx - i\Omega t} \quad x > 0 \quad (2-164)$$

$$R_2 = \left[\frac{\Omega}{\Omega_0} + \left\{ \left(\frac{\Omega}{\Omega_0} \right)^2 - 1 \right\}^{\frac{1}{2}} \right]^{-1} \quad (2-165)$$

$$PC_g = (\Omega^2 - \Omega_0^2)^{\frac{1}{2}} \quad (2-166)$$

From Eq. (2-151), we obtain

$$\bar{\epsilon}_{20} = - \frac{1}{4h} \frac{2kh}{\sinh 2kh} |A_0|^2 (1 + R_2^2) \quad x > 0 \quad (2-167)$$

b) Periodically Modulated Wave Envelope

Let us consider a periodically modulated incident wave envelope

$$A = A_0 e^{i(Kx - \Omega t)} + * \quad x < 0 \quad (2-168)$$

$$(i) \left| \frac{\Omega}{\Omega_0} \right| < 1$$

The solutions of the first order short wave envelopes are

$$B = A_0 R_1 e^{i(-Kx - \Omega t)} - * \quad x < 0 \quad (2-169)$$

$$A = A_0 e^{-Qx - i\Omega t} + * \quad x > 0 \quad (2-170)$$

$$B = A_0 R_1 e^{-Qx - i\Omega t} - * \quad x > 0 \quad (2-171)$$

$\bar{\epsilon}_{20}$ consists of two parts

$$\bar{\epsilon}_{20} = \bar{\epsilon}_S + (\bar{\epsilon}_U e^{-2i\Omega t} + *) \quad (2-172)$$

where $\bar{\xi}_s$ is the steady part of the solution, and may be given as

$$\bar{\xi}_s = -\frac{1}{4h} \frac{2kh}{\sinh 2kh} 4|A_0|^2 \quad x < 0 \quad (2-173)$$

$$\bar{\xi}_s = -\frac{1}{4h} \frac{2kh}{\sinh 2kh} 4|A_0|^2 e^{-2Qx} \quad x > 0 \quad (2-174)$$

Let

$$\bar{\xi}_u = \bar{\xi}_p + \bar{\xi}_h \quad (2-175)$$

where $\bar{\xi}_p$ is the particular solution of Eq.(2-150), and $\bar{\xi}_h$ is the homogeneous solution of Eq.(2-150). The solution $\bar{\xi}_p$ is easily calculated as

$$\bar{\xi}_p = \frac{-1}{gh - C_g^2} \frac{g}{4} \left(\frac{4C_g}{C} - 1 \right) A_0^2 (e^{2iKx} - R_1^2 e^{-2iKx}) \quad x < 0 \quad (2-176)$$

$$\bar{\xi}_p = \frac{-1}{gh + \frac{\Omega^2}{Q^2}} \left\{ \frac{g}{4} \left(\frac{4C_g}{C} - 1 \right) (1 - R_1^2) - i \frac{g\Omega_0}{CQ} R_1 \right\} A_0^2 e^{-2Qx} \quad x > 0 \quad (2-177)$$

We put

$$\bar{\xi}_h = a \cdot \exp(-2i \frac{\Omega}{\sqrt{gh}} x) \quad x < 0 \quad (2-178)$$

$$\bar{\xi}_h = b \cdot \exp(2i \frac{\Omega}{\sqrt{gh}} x) \quad x > 0 \quad (2-179)$$

then a and b are obtained using boundary conditions Eqs.(2-152) and (2-153),

$$\begin{aligned} \frac{1}{kA_0} \begin{pmatrix} a \\ b \end{pmatrix} &= \frac{1}{8} \frac{1}{\frac{gh}{C_g^2} - 1} \left(\frac{C}{C_g} \right)^2 \frac{1}{\tanh kh} \left(\frac{4C_g}{C} - 1 \right) \left\{ -\frac{\sqrt{gh}}{C_g} (1 + e^{-2i\gamma}) \pm (1 - e^{-2i\gamma}) \right\} \\ &+ \frac{1}{4} \frac{-1}{\pm 1 + i \frac{\sqrt{gh}}{C_g} \tan \gamma} \frac{i \cdot \sin \theta e^{-i\gamma}}{\cos^2 \gamma} \left(\frac{C}{C_g} \right)^2 \frac{1}{\tanh kh} \left\{ \sin^2 \gamma \left(\frac{4C_g}{C} - 1 \right) - \frac{2C_g}{C} \right\} \end{aligned}$$

$$+ \frac{1}{4} \left(\frac{2C_g}{C} - 1 \right) \frac{C}{C_g} \frac{1}{\sqrt{kh} \tanh kh} \frac{e^{-i\gamma}}{\cos \gamma} \quad (2-180)$$

where

$$R_1 = \frac{\Omega}{\Omega_0} - i \left\{ 1 - \left(\frac{\Omega}{\Omega_0} \right)^2 \right\}^{\frac{1}{2}} = \pm (\cos \gamma - i \sin \gamma) \quad \begin{cases} 0 < \gamma < \frac{\pi}{2} \\ -\frac{\pi}{2} < \gamma < 0 \end{cases} \quad (2-181)$$

$$(ii) \left| \frac{\Omega}{\Omega_0} \right| > 1$$

In the same way we obtain

$$\bar{\epsilon}_{20} = \bar{\epsilon}_S + (\bar{\epsilon}_U e^{-2i\Omega t} + *) \quad (2-182)$$

$$\bar{\epsilon}_S = - \frac{1}{4h} \frac{2kh}{\sinh 2kh} 2|A_0|^2 (1+R_2^2) \quad (2-183)$$

$$\bar{\epsilon}_U = \bar{\epsilon}_P + \bar{\epsilon}_h \quad (2-184)$$

$$\bar{\epsilon}_P = \frac{-1}{gh - C_g^2} \frac{g}{4} \left(\frac{4C_g}{C} - 1 \right) A_0^2 (e^{2iKx} - R_2^2 e^{-2iKx}) \quad x < 0 \quad (2-185)$$

$$\bar{\epsilon}_P = \frac{-1}{gh - \frac{\Omega^2}{P^2}} \left\{ \frac{g}{4} \left(\frac{4C_g}{C} - 1 \right) (1 - R_2^2) + \frac{g\Omega_0}{CP} R_2 \right\} A_0^2 e^{2iPx} \quad x > 0 \quad (2-186)$$

$$\bar{\epsilon}_h = a \cdot \exp(-2i \frac{\Omega}{\sqrt{gh}} x) \quad x < 0 \quad (2-187)$$

$$\bar{\epsilon}_h = b \cdot \exp(2i \frac{\Omega}{\sqrt{gh}} x) \quad x > 0 \quad (2-188)$$

$$\frac{1}{kA_0^2} \left(\frac{a}{b} \right) = \frac{1}{8} \frac{1}{\frac{gh}{C_g^2} - 1} \left(\frac{C}{C_g} \right)^2 \frac{1}{\tanh kh} \left(\frac{4C_g}{C} - 1 \right) \left(- \frac{\sqrt{gh}}{C_g} (1+e^{-2\gamma}) \pm (1-e^{-2\gamma}) \right)$$

$$\begin{aligned}
& + \frac{1}{4} \frac{-1}{\pm 1 + \frac{\sqrt{gh}}{C_g} \tanh \gamma} \frac{\sinh \theta e^{-\gamma}}{\cosh^2 \gamma} \left(\frac{C}{C_g}\right)^2 \frac{1}{\tanh kh} \left\{ -\sinh^2 \gamma \left(\frac{4C_g}{C} - 1\right) - \frac{2C_g}{C} \right\} \\
& + \frac{1}{4} \left(\frac{2C_g}{C} - 1\right) \frac{C}{C_g} \frac{1}{\sqrt{kh \tanh kh}} \frac{e^{-\gamma}}{\cosh \gamma}
\end{aligned} \tag{2-189}$$

where

$$R_2 = \frac{\Omega}{\Omega_0} - \left\{ \left(\frac{\Omega}{\Omega_0}\right)^2 - 1 \right\}^{\frac{1}{2}} = \pm (\cosh \gamma - \sinh \gamma) \quad \begin{cases} 0 < \gamma \\ 0 > \gamma \end{cases} \tag{2-190}$$

In Fig.(2-11) non-dimensional amplitudes of free long waves $\frac{a}{kA_0^2}$ and $\frac{b}{kA_0^2}$ are plotted as functions of $\frac{\Omega}{\Omega_0}$ for several values of kh .

When $|\frac{\Omega}{\Omega_0}| < 1$, $\frac{a}{kA_0^2}$ and $\frac{b}{kA_0^2}$ are equal. We should notice that

$\left(\frac{b}{kA_0^2}\right)$ becomes infinite when $\frac{\Omega}{P} = \pm \sqrt{gh}$, i.e. the long waves have

the same wavelength as of wave groups in the sandbar region. For this particular case, a different kind of approach is necessary.

c) Transient Waves

We consider the same condition as in 2-2-1(b). Let us introduce dimensionless variables

$$x' = \frac{x}{L}, \quad t' = \Omega_0 t, \quad \zeta' = \frac{\bar{\zeta}_{20}}{kA_m^2}, \quad \begin{pmatrix} A' \\ B' \end{pmatrix} = \frac{1}{A_m} \begin{pmatrix} A \\ B \end{pmatrix} \tag{2-191}$$

where A_m is the maximum amplitude of incident waves. Eq.(2-150)

becomes

$$\begin{aligned} \zeta'_{t't'} - \left(\frac{C_g}{\Omega_0 L}\right)^2 \frac{gh}{C_g^2} \zeta'_{x'x'} &= \left(\frac{C_g}{\Omega_0 L}\right)^2 \left(\frac{C_g}{C} - \frac{1}{4}\right) \frac{g}{C_g^2 k} (|A'|^2 + |B'|^2)_{x'x'} \\ &+ \left(\frac{C_g}{\Omega_0 L}\right) \frac{g}{C \cdot C_g k} (iA'^* B' - iA' B'^*)_{x'} \end{aligned} \quad (2-192)$$

This equation can be solved by the integral method,

$$\begin{aligned} \zeta'(x', t') &= \frac{1}{2C_0} \int_0^{t'} dt'' \left\{ \left(\frac{C_g}{\Omega_0 L}\right)^2 \left(\frac{C_g}{C} - \frac{1}{4}\right) \frac{g}{C_g^2 k} (|A'|^2 + |B'|^2)_{x'x'} \right. \\ &\quad \left. + \left(\frac{C_g}{\Omega_0 L}\right) \frac{g}{C \cdot C_g k} (iA'^* B' - iA' B'^*)_{x'} \right\}_{x' \pm C_0(t' - t'')} \\ &+ \frac{1}{2} \{ \zeta'(x' - C_0 t', 0) + \zeta'(x' + C_0 t', 0) \} \\ &+ \frac{1}{2C_0} \int_{x' - C_0 t'}^{x' + C_0 t'} \zeta'_{t'}(x'', 0) dx'' \end{aligned} \quad (2-193)$$

where

$$C_0 = \left(\frac{C_g}{\Omega_0 L}\right) \sqrt{\frac{gh}{C_g}} \quad (2-194)$$

The initial condition is given as

$$\zeta' = \left\{ \left(\frac{C_g}{C}\right)^2 \tanh kh - kh \right\}^{-1} \left[\frac{C_g}{C} - \frac{1}{4} \right] |A'|^2 \quad (2-195)$$

Fig.(2-12) shows A' , B' , and ζ' versus t' and Fig.(2-13) shows A' , B' , and ζ' versus x' . It is clear that a wave set-down originally locked to an incident wave packet is unlocked as the packet enters the sandbar region, and propagates with the speed of long waves ($\sqrt{gh}$) in both directions. At the same time steady wave set-downs for the deformed incident and reflected wave packets appear.

$$\frac{k^2}{\omega^2} + kh = \frac{k^2}{gk} = \frac{k}{g}$$

$$\frac{\omega^2}{k^2} = \frac{gh}{k} \quad \frac{kh}{k^2} = \frac{gh}{k^2}$$

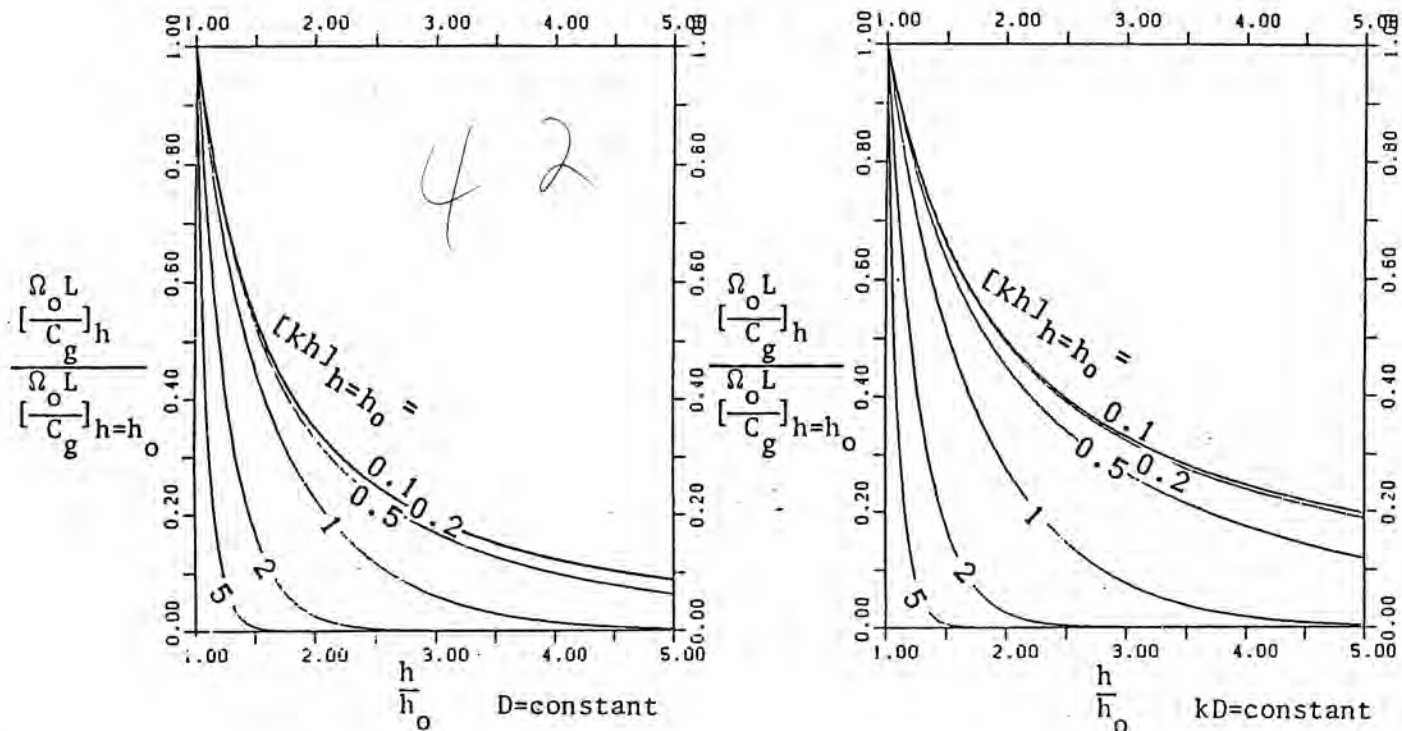


Fig. 2-1 Parameter $\frac{\Omega_0 L}{C_g}$ versus Water Depth

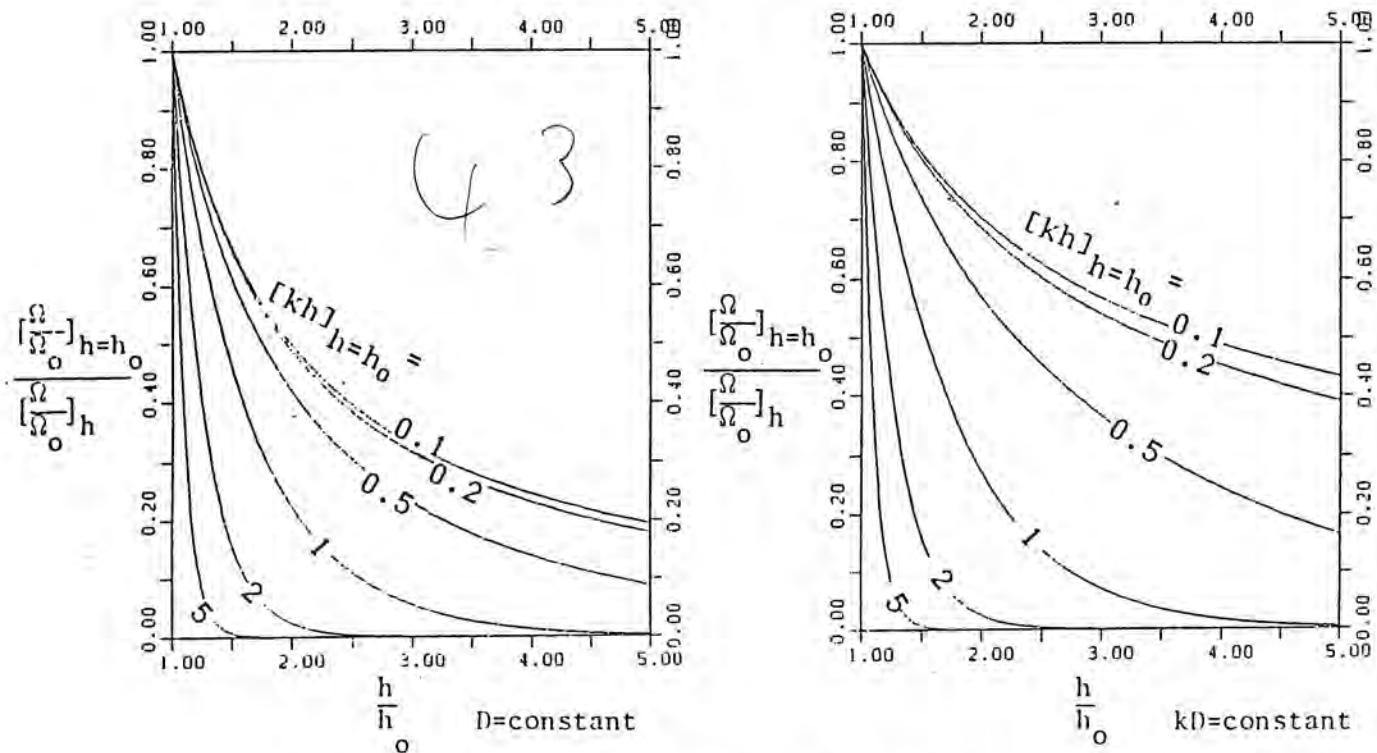


Fig. 2-2 Parameter $\frac{\Omega}{\Omega_0}$ versus Water Depth

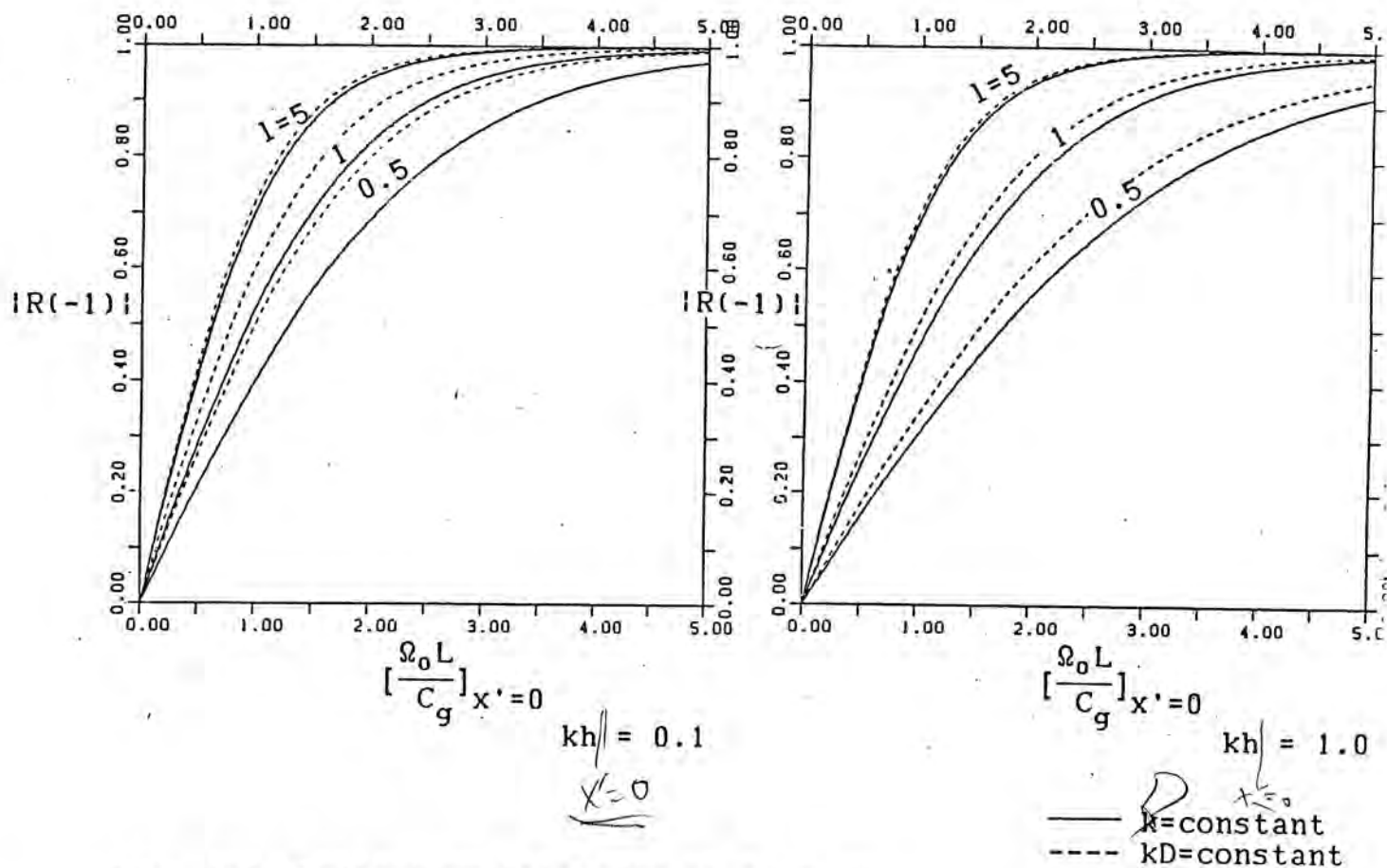


Fig. 2-3 Reflection Intensity at $x' = -1$

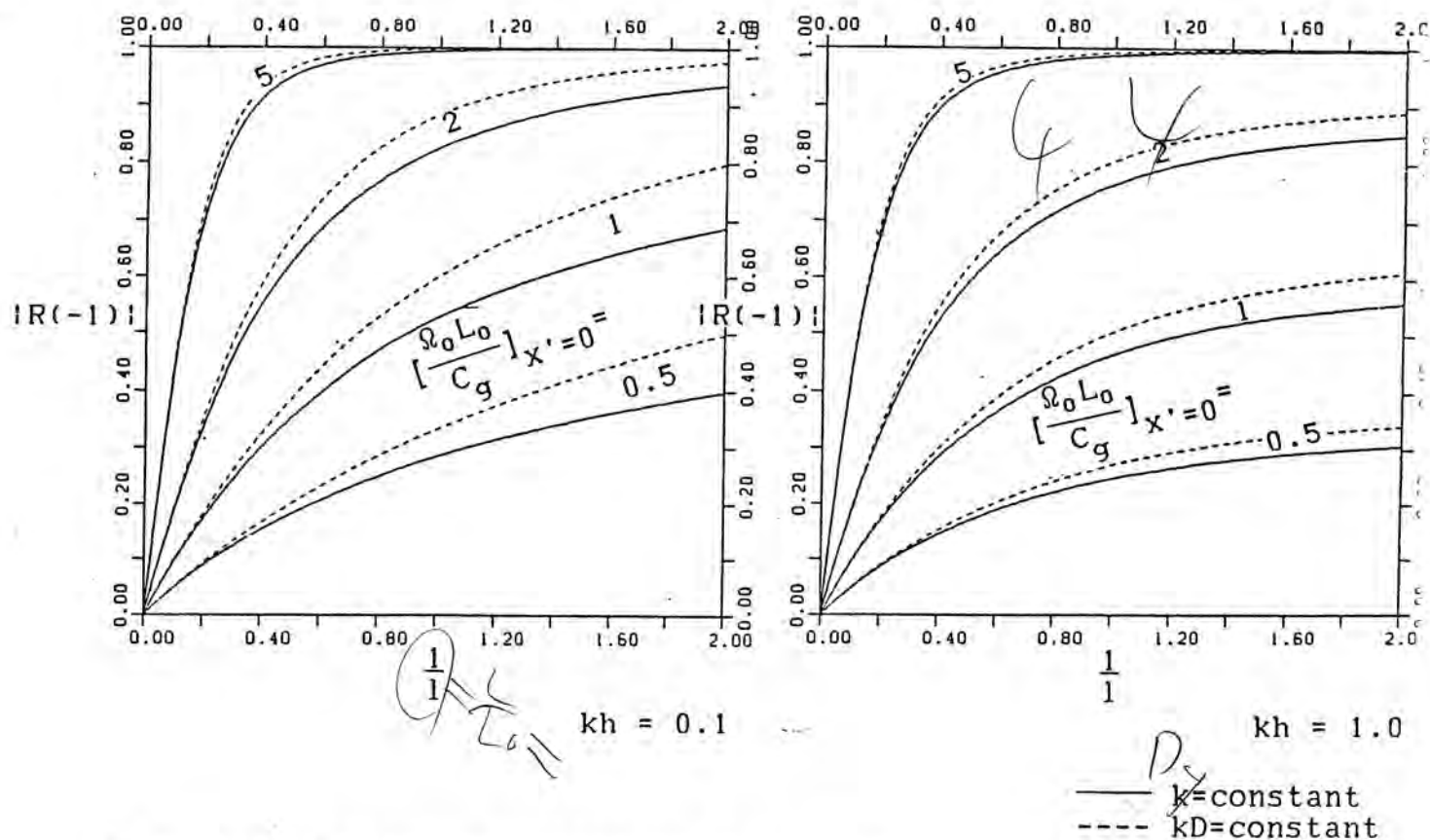
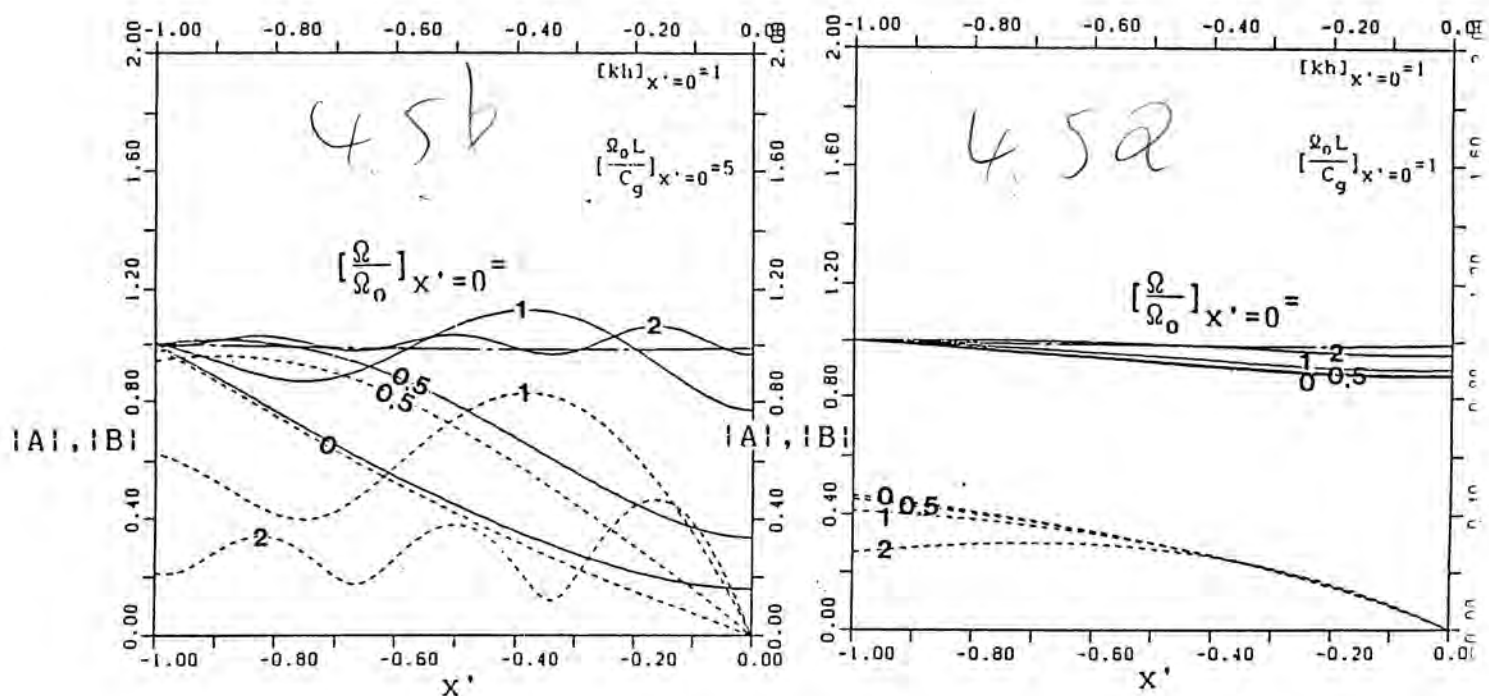
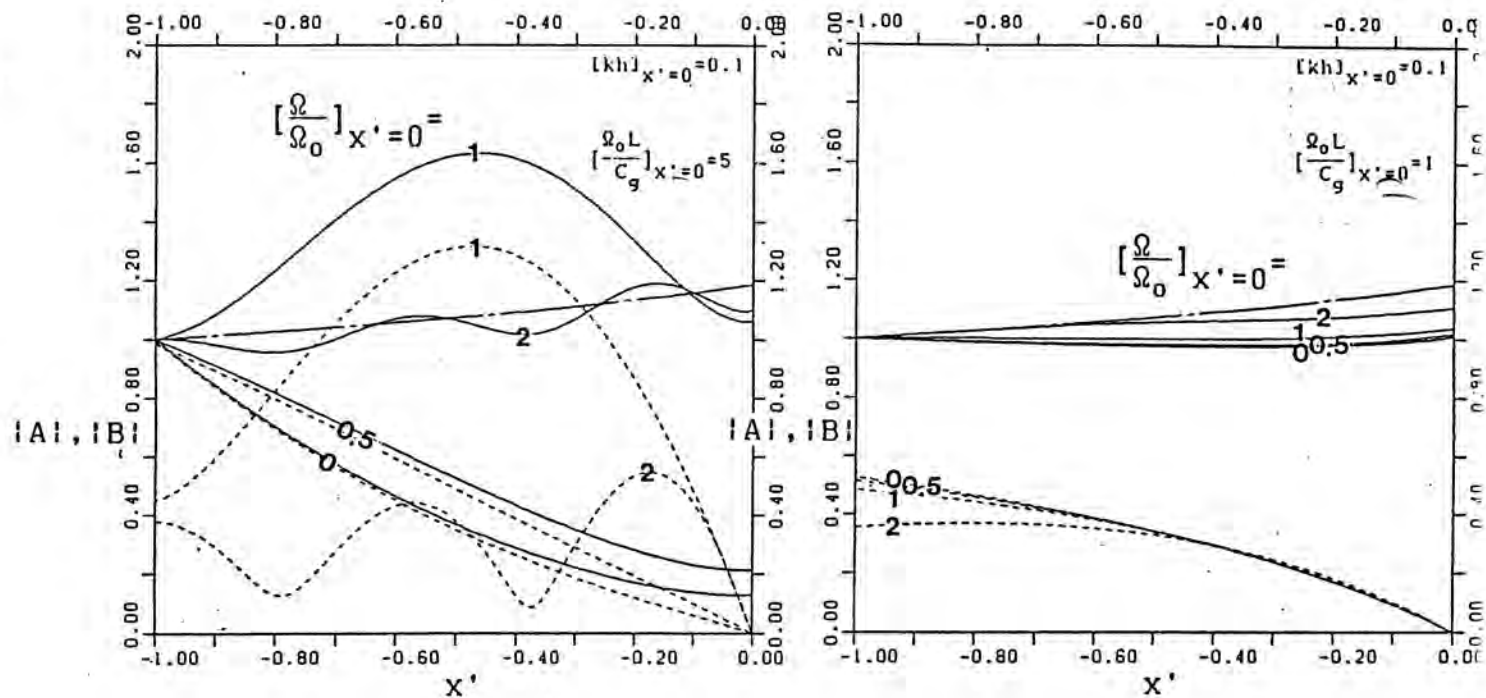
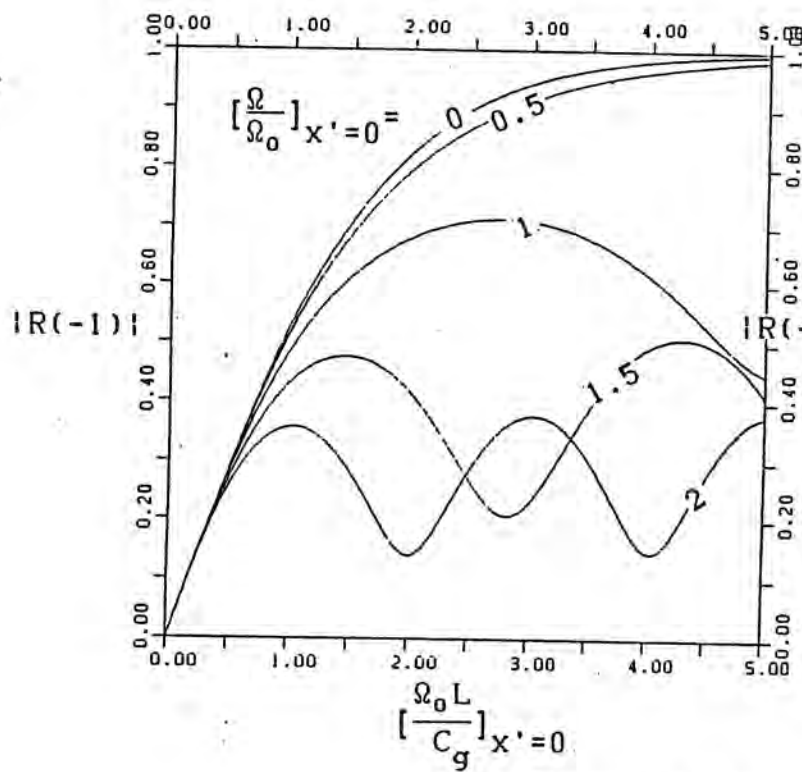


Fig. 2-4 Reflection Intensity at $x' = -1$

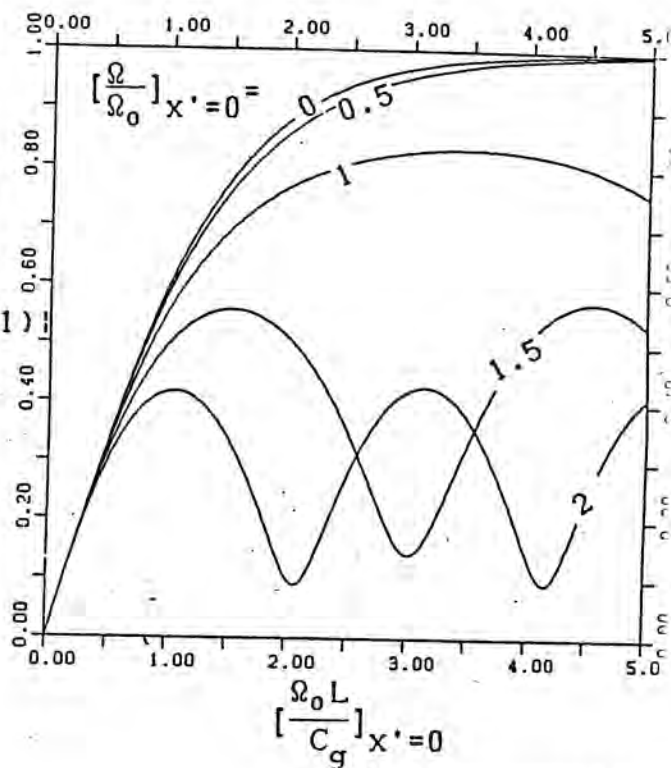


— Incident Wave Amplitude $|A|$
 - - - Reflected Wave Amplitude $|B|$
 — Incident Wave Amplitude without Sanchar

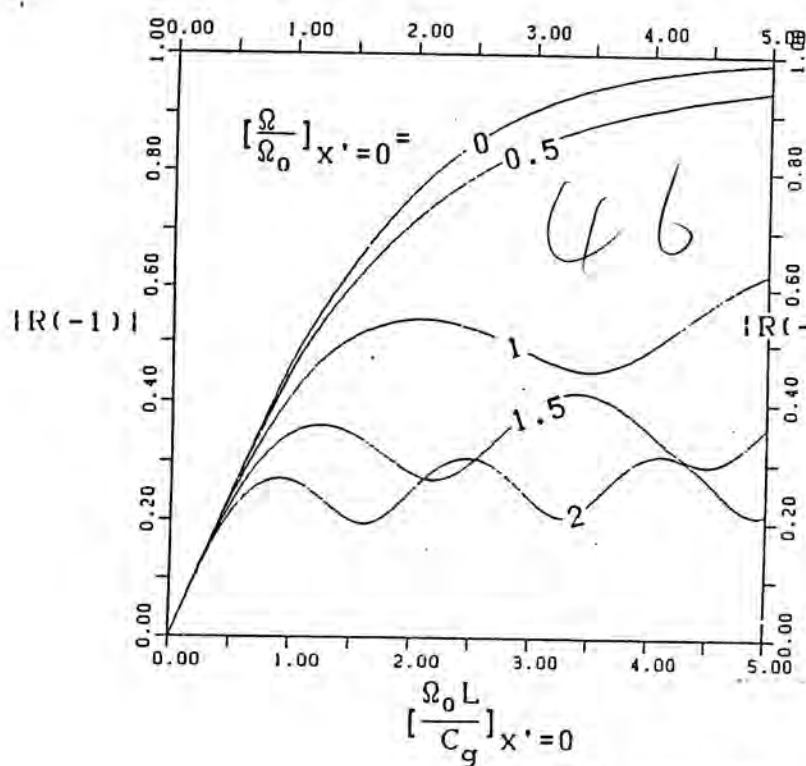
Fig. 2-5 Incident and Reflected Wave Amplitudes ($D=\text{constant}$, $l=1$)



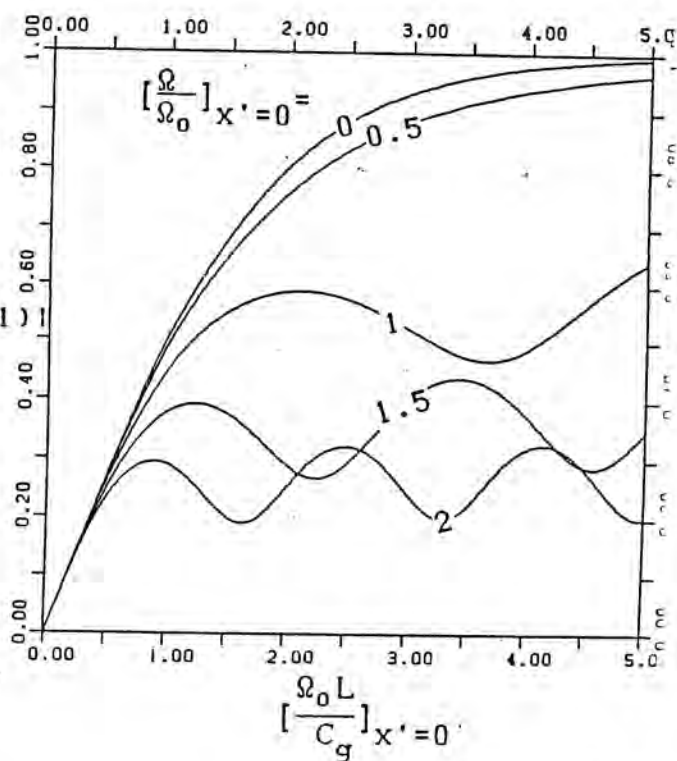
$Dk = \text{constant}$
 $[kh]_{x'=0} = 0.1$



$kD = \text{constant}$
 $[kh]_{x'=0} = 0$



$Dk = \text{constant}$
 $[kh]_{x'=0} = 1$



$kD = \text{constant}$
 $[kh]_{x'=0} = 1$

Fig. 2-6 Reflection Intensity at $x' = -1$ ($l = 1$)

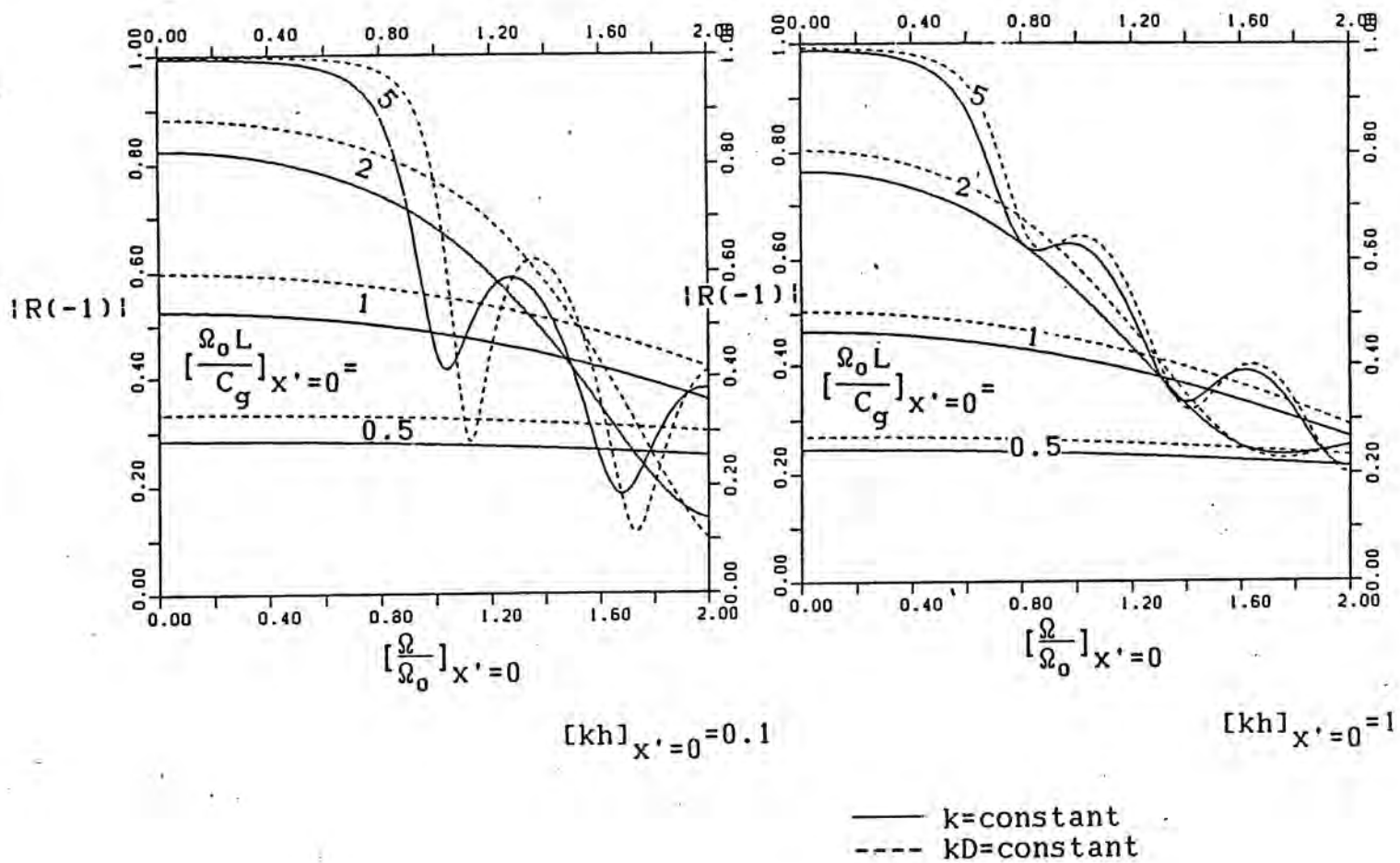
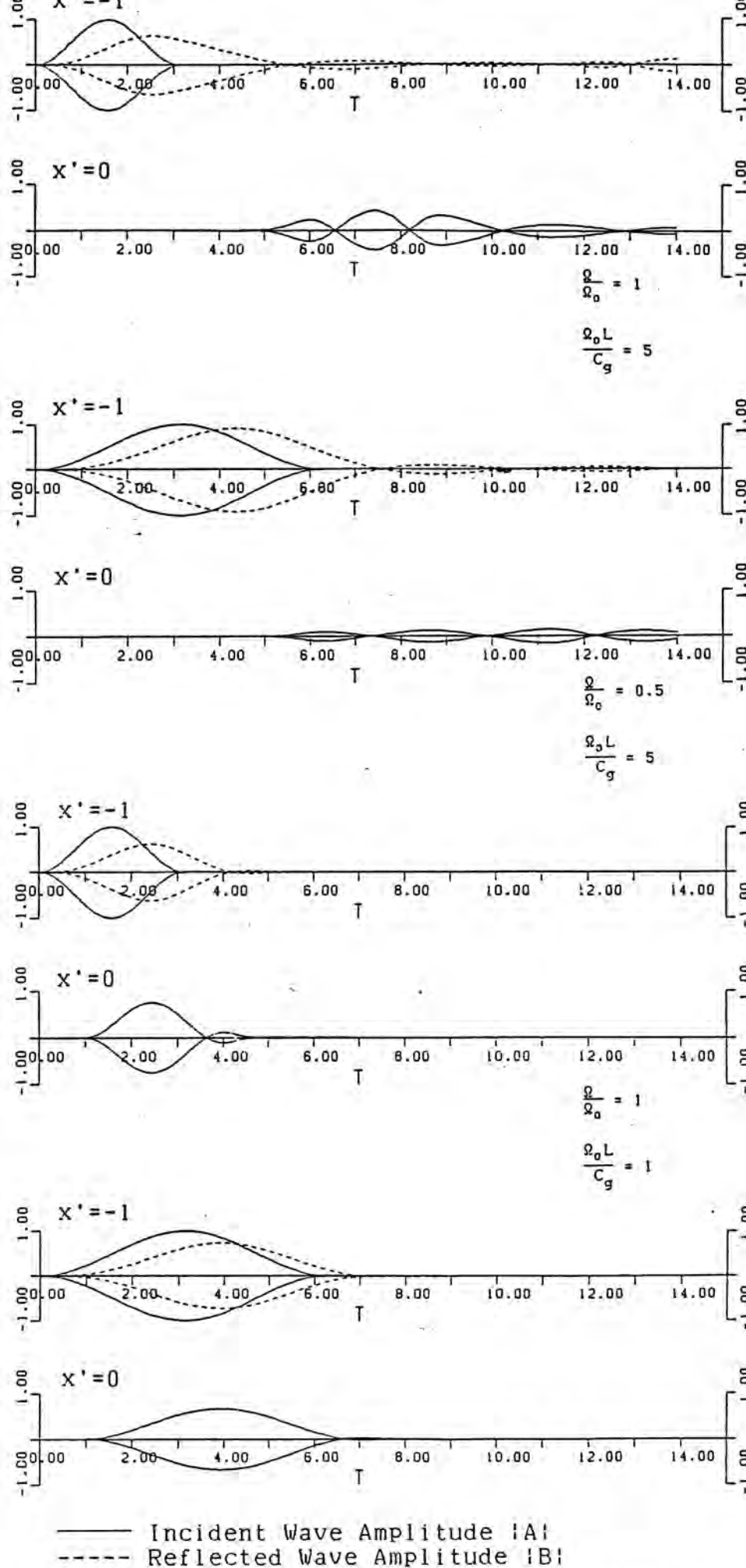
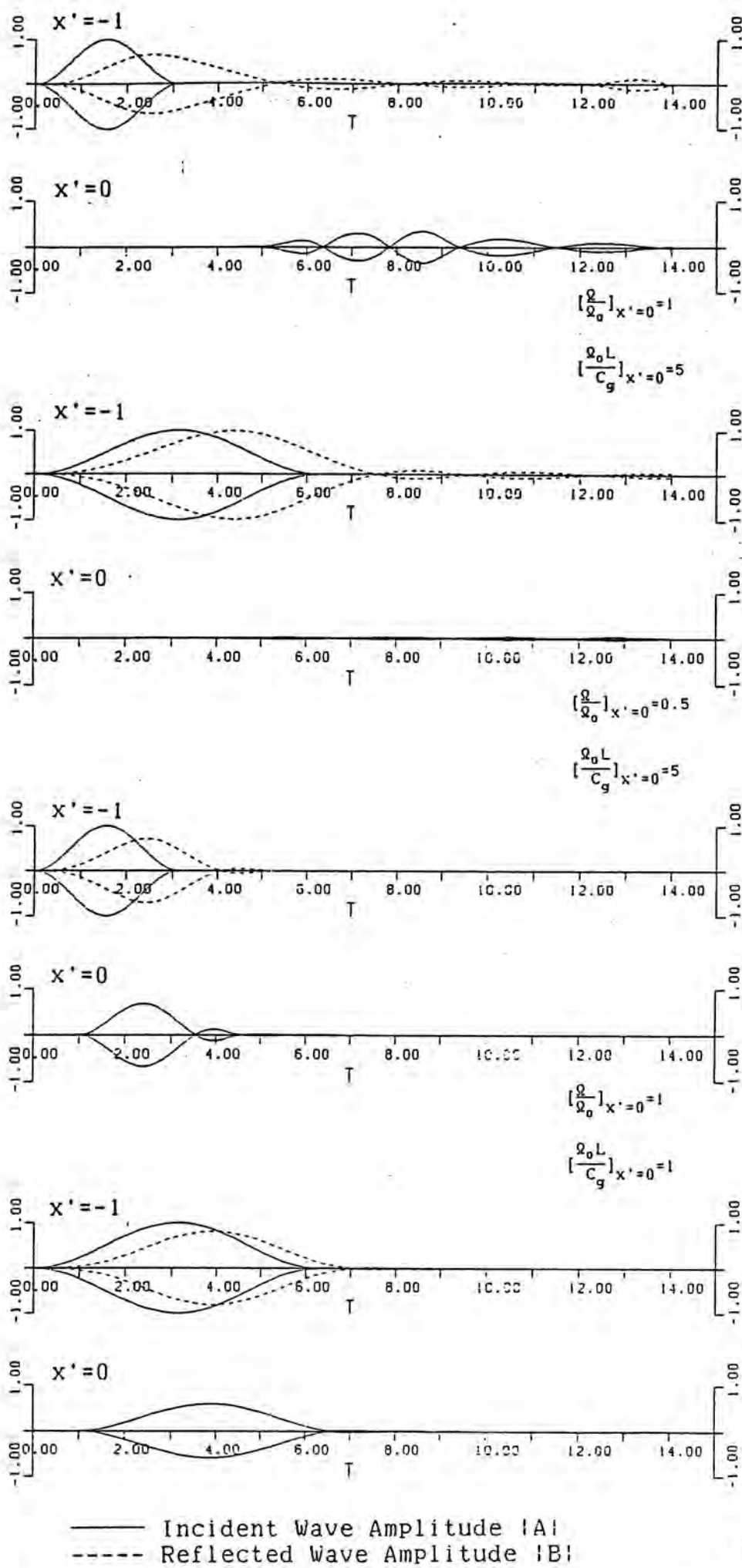


Fig. 2-7 Reflection Intensity at $x' = -1$ ($l = 1$)



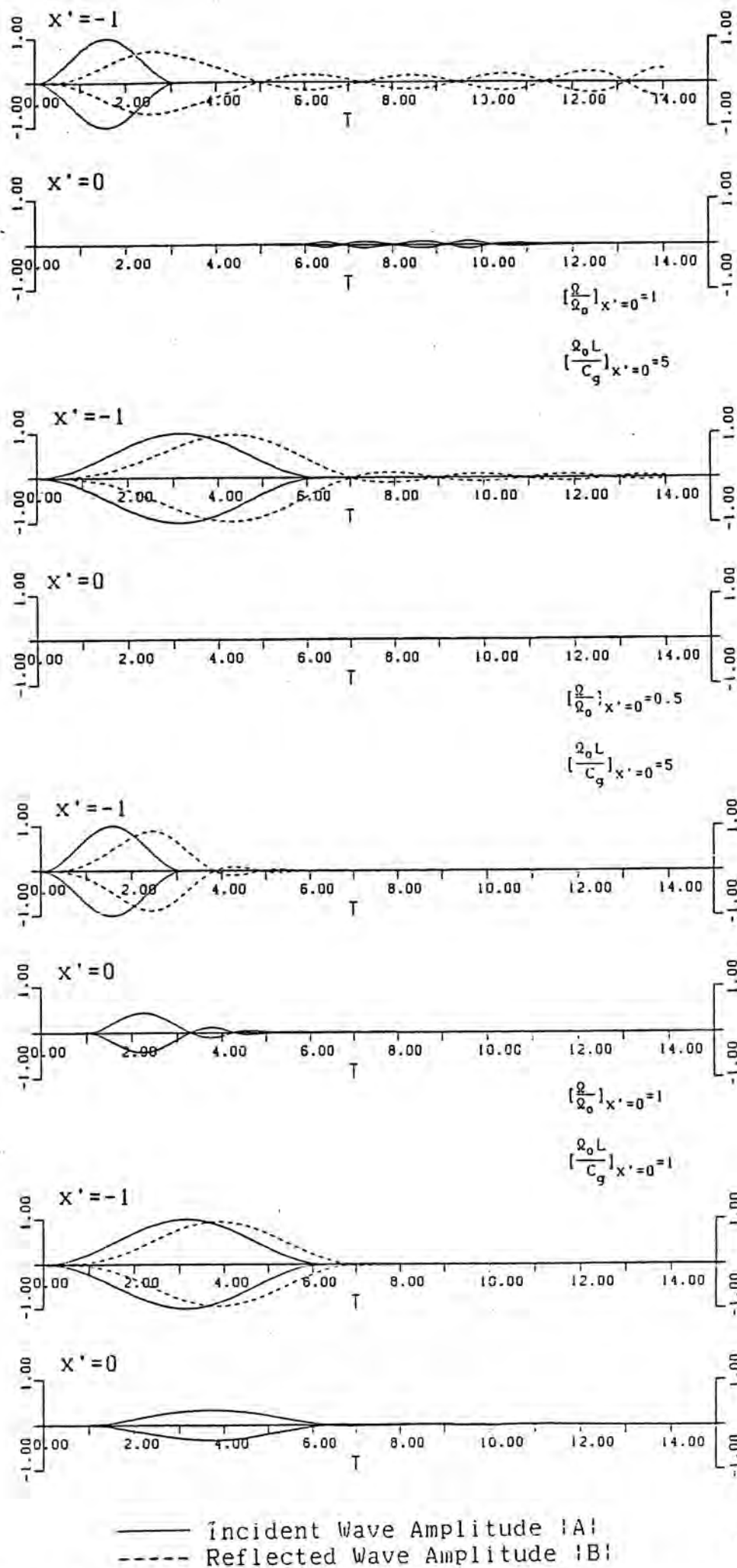
Incident Wave Amplitude at $x' = -1$
 $A = \sin^2 \Omega t = \sin^2 \left(\frac{\Omega}{\Omega_0} x' = 0 \right)$

Fig. 2-8 Incident and Reflected Wave Amplitudes
 $(kh = 1, \text{ constant depth})$



Incident Wave Amplitude at $x' = -1$
 $A = \sin^2 \Omega t = \sin^2 \left[\left(\frac{\Omega}{\Omega_0} \right) x' = 0 \right]$

Fig. 2-9 Incident and Reflected Wave Amplitudes



Incident Wave Amplitude at $x' = -1$
 $A = \sin^2 \Omega t = \sin^2 \left(\frac{\Omega}{\Omega_0} x' t' \right)$

Fig. 2-10 Incident and Reflected Wave Amplitudes

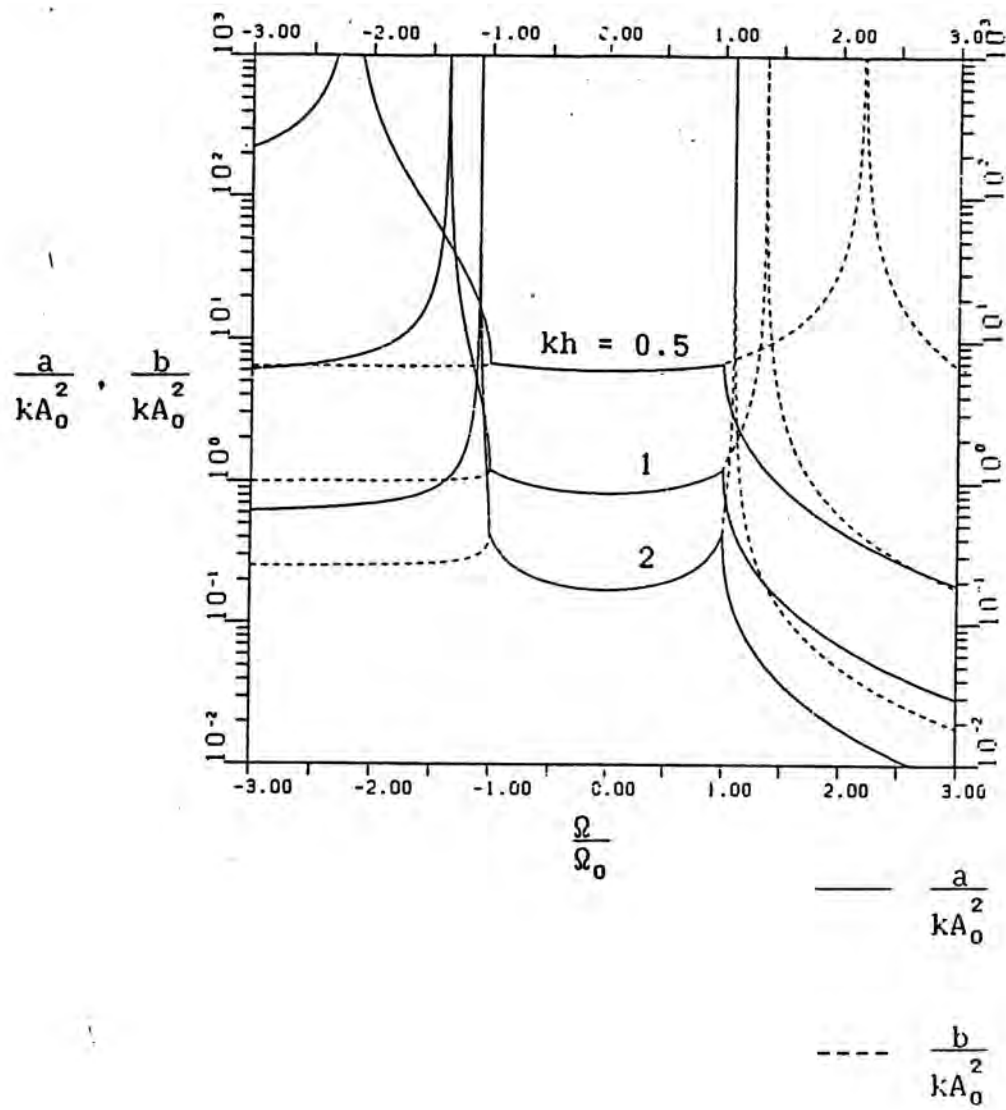
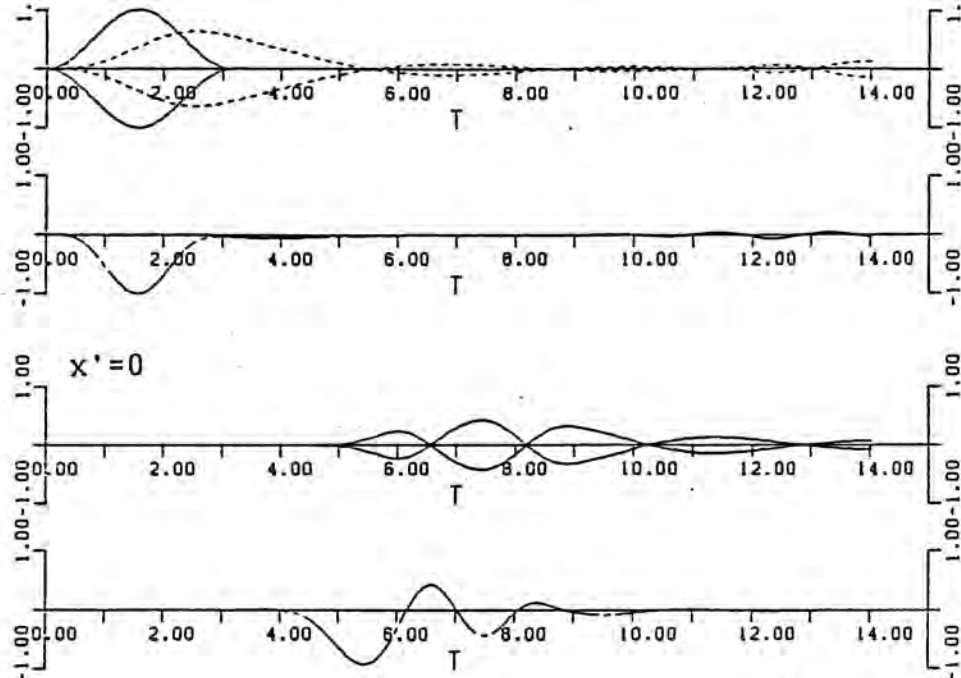


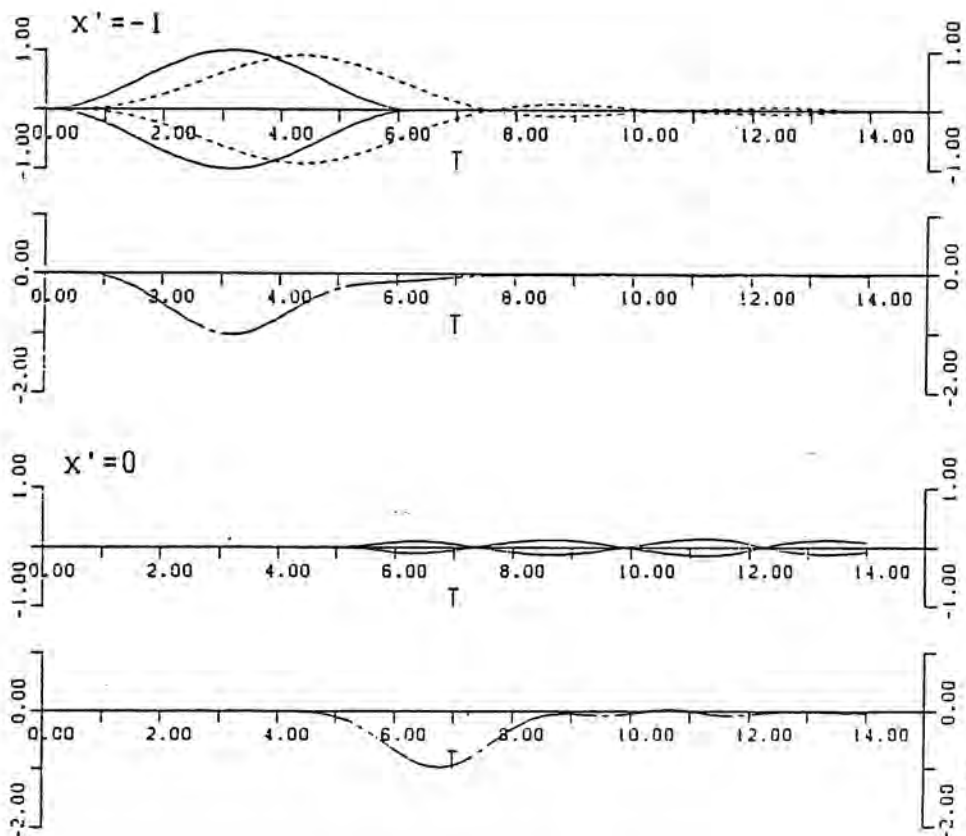
Fig. 2-11 Non-dimensional Amplitude of free long waves



$$\frac{\Omega}{\Omega_0} = 1$$

— Incident Wave Amplitude $|A|$
 --- Reflected Wave Amplitude $|B|$
 — Second Order Long Waves

$$\frac{\Omega_0 L}{C_g} = 5$$



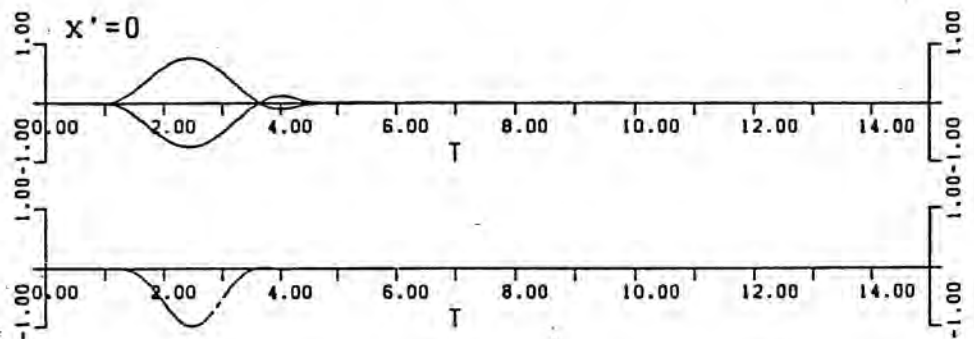
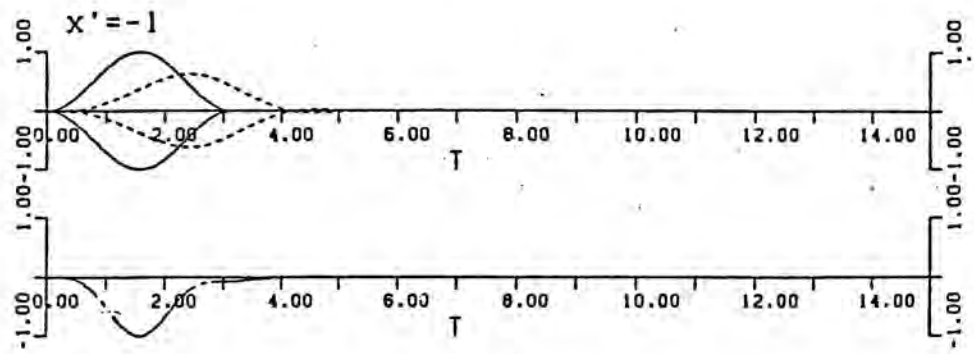
$$\frac{\Omega}{\Omega_0} = 0.5$$

$$\frac{\Omega_0 L}{C_g} = 5$$

Fig. 2-12 Incident & Reflected Wave Amplitudes,
 and Second Order Long Waves
 ($kh = 1$, constant depth)

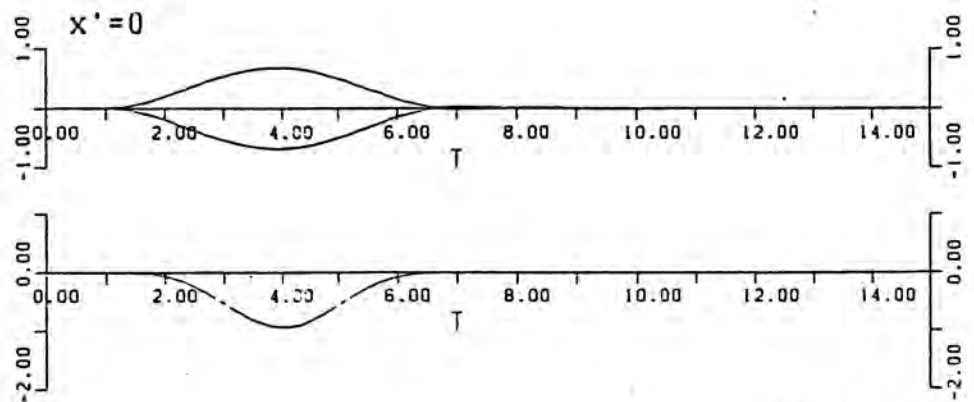
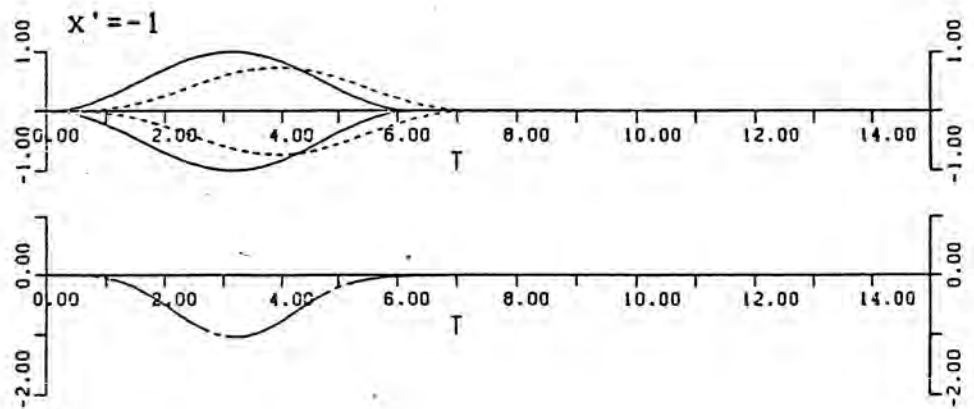
Incident Wave Amplitude at $x' = -1$

$$A = \sin^2 \Omega t = \sin^2 \left[\left(\frac{\Omega}{\Omega_0} \right)_{x'=-1} t' \right]$$



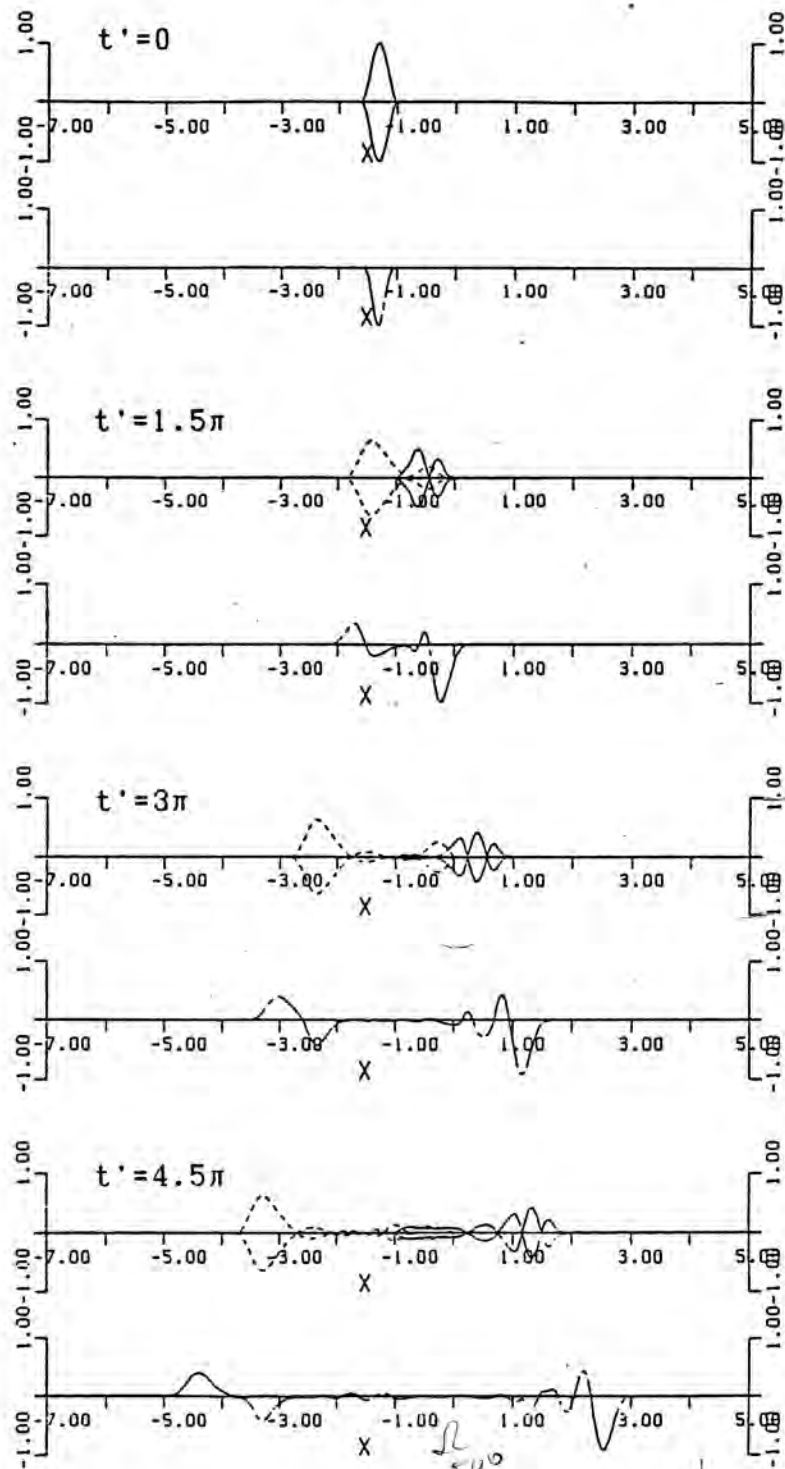
$$\frac{\Omega}{\Omega_0} = 1$$

$$\frac{\Omega_0 L}{C_g} = 1$$



$$\frac{\Omega}{\Omega_0} = 0.5$$

$$\frac{\Omega_0 L}{C_g} = 1$$

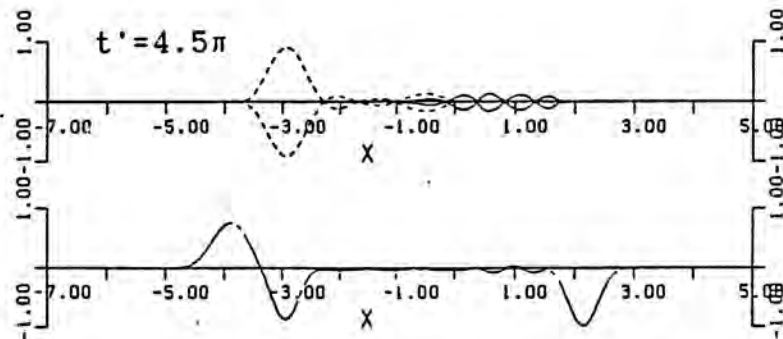
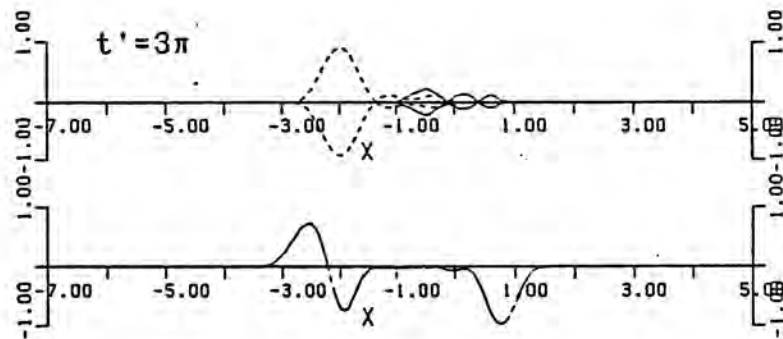
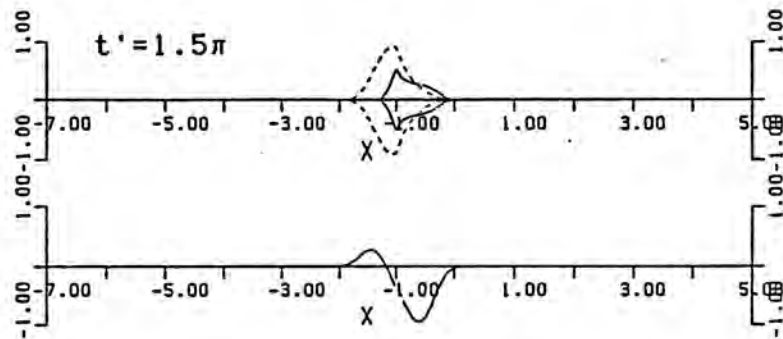
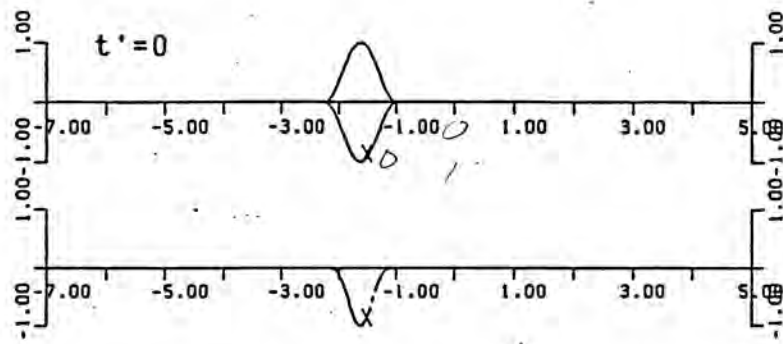


— Incident Wave Amplitude $|A|$
 - - - Reflected Wave Amplitude $|B|$
 - · - Second Order Long Waves

Incident Wave Amplitude at $x' = -1$
 $A = \sin^2 \Omega t = \sin^2 \left[\left(\frac{\Omega}{\Omega_0} \right)_{x'=0} t' \right]$

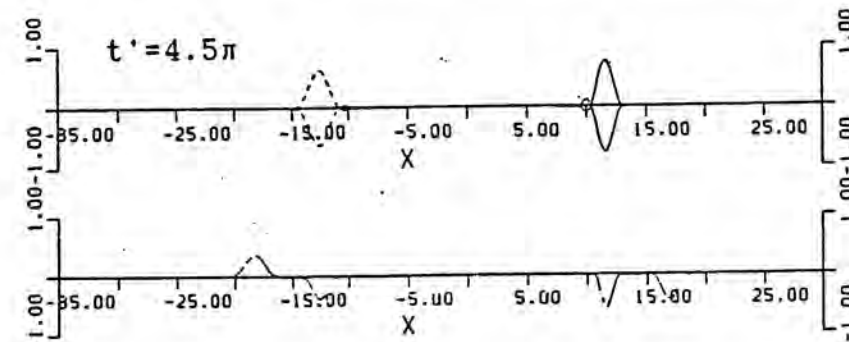
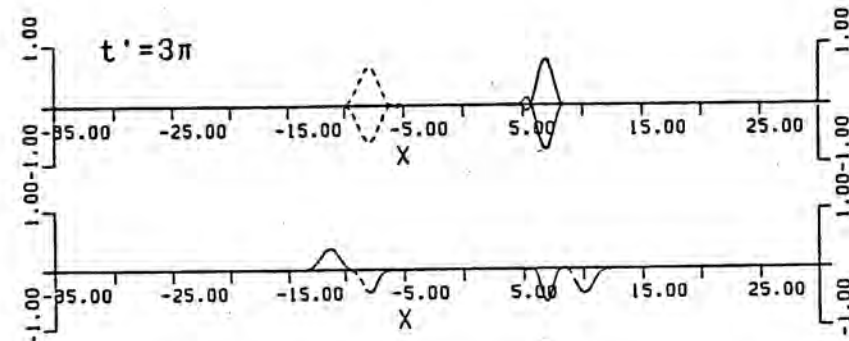
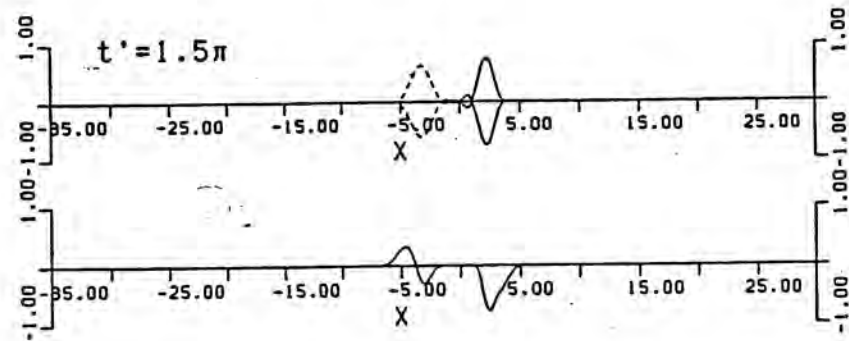
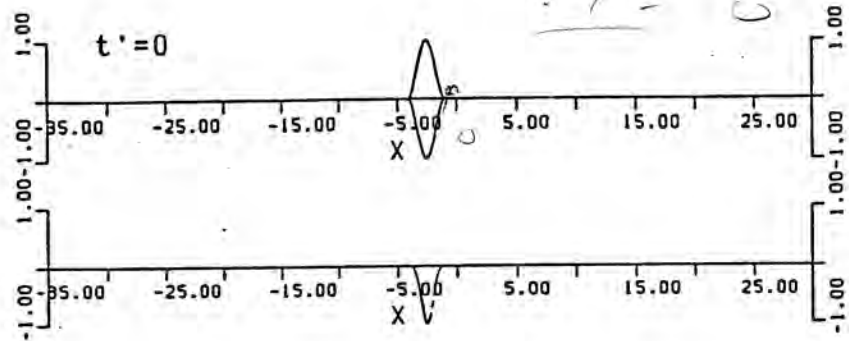
$\frac{\Omega}{\Omega_0} = 1$
 $\frac{\Omega_0 L}{C_g} = 5$

Fig. 2-13 Incident & Reflected Wave Amplitudes,
 and Second Order Long Waves
 ($kh = 1$, constant depth)



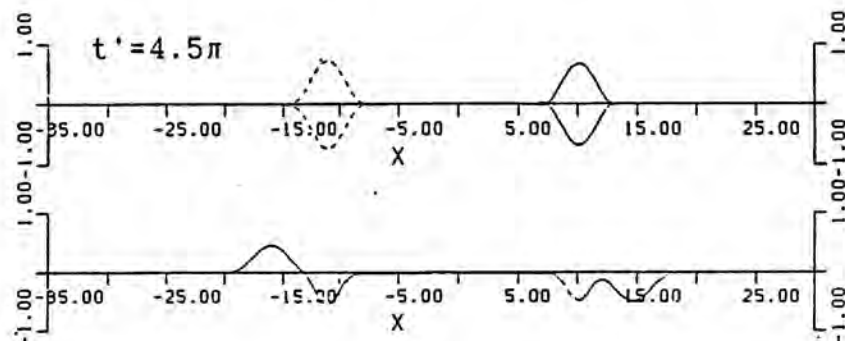
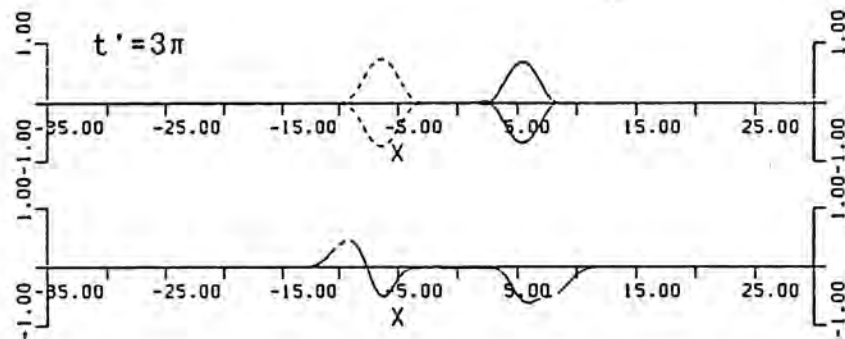
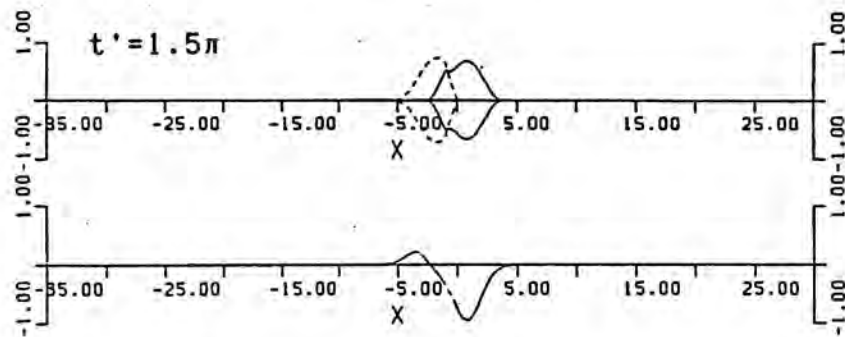
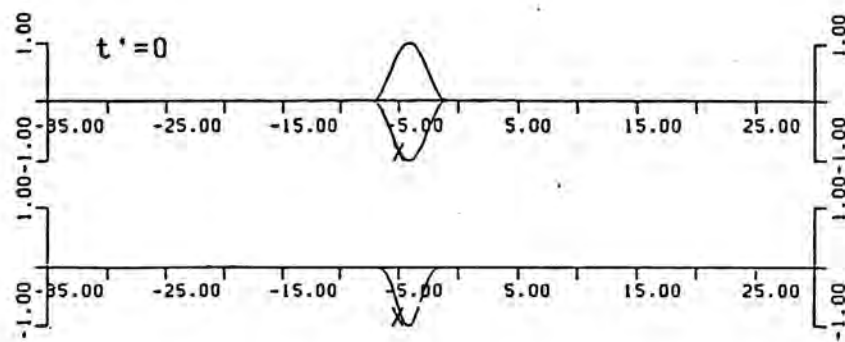
$$\frac{\Omega}{\Omega_0} = 0.5$$

$$\frac{\Omega_0 L}{C_g} = 5$$



$$\frac{\Omega}{\Omega_0} = 1$$

$$\frac{\Omega_0 L}{C_g} = 1$$



$$\frac{\Omega}{\Omega_0} = 0.5$$

$$\left(\frac{\Omega_0 L}{C_g} \right) = 1$$

Chapter 3 Experiment

3-1 Method of Experiment

3-1-1 Experimental Set-Up

To test the theoretical results in Chapter 2, experiments were carried out by the present author in Massachusetts Institute of Technology.

Measurements were carried out in a glass-walled wave tank, 21.82m X 0.40m (Fig. 3-1). Twenty sandbars, a false bottom, and a slope were built in the tank. At the end of the tank, a vertical wall was set up, by which waves were assumed to be reflected completely.

To generate desired waves, a hydropressure-type wave generator was used, which was controlled by signals made in a connected micro-computer. Measurements of wave elevations were made using capacitance-type wave gauges. The output records were sampled by 100Hz in digital oscilloscope, and were analyzed by computers.

Since the shape of bottom sandbars is fixed, wave-number of surface waves is also fixed:

$$K = \frac{2\pi}{1.20} = 5.236 \text{ (m}^{-1}\text{)} \quad (3-1)$$

For given h , we can calculate ω and Ω_0 by

$$\omega^2 = gk \tanh kh \quad (3-2)$$

$$\Omega_0 = \frac{\omega k D}{2 \sinh 2kh} \quad (3-3)$$

3-1-2 Steady Incident Waves

The characters of incident waves are as follows.

Table 3-1 Characters of Incident Waves

	Case	L(m)	h(cm)	ω (S ⁻¹)	$\frac{\Omega_0}{\omega}$	Ω_0 (S ⁻¹)	$\omega + \Omega_0$ (S ⁻¹)	$ A_0 $ (cm)
Case 1	1-1	12.0	14.8	5.77	2.04	0.196	6.17	0.165
	1-2	12.0	14.8	5.77	1.02	0.196	5.97	0.164
	1-3	12.0	14.8	5.77	0.51	0.196	5.87	0.190
	1-4	12.0	14.8	5.77	0.0	0.196	5.77	0.203
	1-5	12.0	14.8	5.77	-0.51	0.196	5.67	0.234
	1-6	12.0	14.8	5.77	-1.02	0.196	5.57	0.735
	1-7	12.0	14.8	5.77	-2.04	0.196	5.37	0.232

Case 2	2-1	6.0	14.8	5.77	2.04	0.196	6.17	0.262
	2-2	6.0	14.8	5.77	1.02	0.196	5.97	0.186
	2-3	6.0	14.8	5.77	0.0	0.196	5.77	0.235
	2-4	6.0	14.8	5.77	-1.02	0.196	5.57	0.355
	2-5	6.0	14.8	5.77	-2.04	0.196	5.37	0.316
Case 3		12.0	20.0	6.33	2.0	0.121	6.57	0.245

where $A = A_0 e^{-i\Omega t}$ $\chi = 0$ (3-4)

In theory, complete standing wave field is to be observed in the whole tank. At first, nodes and antinodes were searched for by moving a wave gauge 1cm by 1cm along the tank. Then average heights of continuous 30 waves were measured at antinodes, and were compared with the numerical results.

3-1-3 Incident Wave Packet

Before sandbars were built, measurements were carried out to make sure that the incident wave packet had a desired shape, and to observe the deformation of the packet due to dissipation and dispersion. The following wave packets were generated.

Table 3-2 Characters of Incident Wave Packets

	Case	h(cm)	$\frac{\Omega}{\Omega_0}$	ω (s ⁻¹)	Ω_0 (s ⁻¹)	$ A_0 $ cm
Case 4	4-1	15.0	1.0	5.80	0.192	1.45
	4-2	15.0	2.0	5.80	0.192	1.32
	4-3	20.0	0.5	6.33	0.121	2.04
	4-4	20.0	0.75	6.33	0.121	2.06
	4-5	20.0	1.0	6.33	0.121	2.01

where $A = A_0 \sin^2(\Omega t)$ $0 \leq \Omega t \leq \pi$ $\chi = 0$ (3-5)

Incident and reflected waves were measured at the midst of wave tank for 66 seconds (first run: 0s-36s, second run: 30s-66s). The output records for the first 5 seconds (before the first wave reaches) were averaged to obtain the mean water level. Fast Fourier Transform Method was then used to get second order long wave component (0-0.25Hz) and first order short wave component (0.25-1.4Hz) out of the records.

In the experiment with sandbars, the following wave packets were used.

Table 3-3 Characters of Incident Wave Packets

	Case	h(cm)	$\frac{\Omega}{\Omega_0}$	$\frac{\Omega_2}{\Omega_0}$	$\omega (s^{-1})$	$\Omega_0 (s^{-1})$	$ A _{cm}$
Case 5	5-1	14.8	1.0	0.0	5.77	0.196	1.475
	5-2	14.8	0.667	0.0	5.77	0.196	1.454
	5-3	14.8	0.5	0.0	5.77	0.196	1.454
	5-4	14.8	1.0	1.0	5.77	0.196	1.474
	5-5	14.8	1.0	-1.0	5.77	0.196	1.455
	5-6	20.0	1.0	0.0	6.33	0.121	1.992

where $A = A_0 \sin^2(\Omega t) e^{-i\Omega_2 t}$ $0 \leq \Omega t \leq \pi$ $\chi = \chi_1$ (3-6)

The displacement of wave generator was assumed to be linear to the input signal, and the following signal was used to make wave packets.

$$\text{signal} = g_{aim} \times \left[\sin^2(\Omega t) \sin\{(\omega + \Omega_2)t\} + \frac{1}{4} \left(\frac{c}{c_g}\right)^2 \frac{\omega}{\Omega} \frac{|A|}{h} \left(\frac{4c_g}{c} - 1\right) \times \right. \\ \left. \times \frac{1}{1 - 2h/c_g^2} \left(\frac{3}{8} \Omega t - \frac{\sin 2\Omega t}{4} + \frac{\sin 4\Omega t}{32} \right) \right] \quad 0 \leq \Omega t \leq \pi \quad (3-7)$$

This equation is derived theoretically in Chapter 3-2.

The signal was made to satisfy this equation at larger depth (before the slope). Wave packets are deformed slightly by the effect of the slope. In particular, actual wave set-down becomes much smaller than the steady solution after propagating above the slope.

In each case, the gain of the output signal was controlled so that

$$A_0 \approx \frac{1}{10} h \quad (3-8)$$

was satisfied.

Water surface elevations were measured at

x=-600	all cases
x=1140, 1185	all cases
x=-55, -50, -45, ..., +10	all cases
x=545, 550, 555, ..., 610	Case 5-1
x=1125, 1130, 1135, ..., 1190 (cm)	Case 5-1

for 40 seconds. Mean water level and long and short wave components of the record were obtained in the same way as the experiment without sandbars.

The Fourier transforms of incident signal (short wave component) are shown in Fig. 3-2. In Case 5-1~5-3, the width of the peak of Fourier transform changes, while in Case 5-1, 5-4 and

5-5, the whole transform shifts to the left or right keeping the same shape.

3-2 Numerical Simulation

3-2-1 Governing equations

Let the sandbar region be between $x=0$ and $x=L$, the wave generator be at $x=-x_0$, and the vertical wall be at $x=L$. In principle, we can use the same equations as in section 2. Since the shape of sandbars is expressed as:

$$\delta = \frac{D}{2} (e^{2iS^+} + e^{2iS^-}) = \sin 2kx \quad (3-9)$$

we put

$$S^\pm = \pm (kx - \frac{\pi}{4}) \quad (3-10)$$

a) Boundary condition at the end wall ($x=L$)

Since water velocity normal to the wall is zero, we have

$$\phi_x = 0 \quad (3-11)$$

or at each order,

$$O(\varepsilon) : \phi_{11x} = 0 \quad (3-12)$$

$$O(\varepsilon^2) : \phi_{10\bar{x}} + \phi_{20x} = 0 \quad (3-13)$$

From Eq.(3-12) we get

$$ikf_0 \left\{ A e^{i(kL - \frac{\pi}{4})} - B e^{-i(kL - \frac{\pi}{4})} \right\} = 0 \quad (3-14)$$

thus

$$A = iB \quad (3-15)$$

Eqs.(2-79), (2-80) and (3-15) are combined as

$$\frac{C_g}{2} (|A|^2 + |B|^2)_{\bar{x}} + \frac{1}{2} (|A|^2 - |B|^2)_{\bar{x}} = -i\Omega_0 A^* B + * = -2\Omega_0 |A|^2 \quad (3-16)$$

From Eqs.(2-90), (3-13) and (3-16) we obtain

$$\begin{aligned} \bar{\phi}_{20\bar{x}} &= -\frac{1}{g} \phi_{10\bar{x}} - \frac{1}{4h} \frac{2kh}{\sinh 2kh} (|A|^2 + |B|^2)_{\bar{x}} \\ &= \frac{\Omega_0}{c_g} \frac{2k}{\sinh 2kh} |A|^2 \end{aligned} \quad (3-17)$$

b) Boundary condition at the wave generator ($x=-x_0$)

Let the displacement of the wave generator as

$$\xi = \sum_{n=1}^{\infty} \sum_{m=-n}^m \xi_{nm} e^{-im\omega t} \quad (3-18)$$

Kinematic boundary condition is given as follows:

$$\frac{\partial \phi}{\partial x} = \frac{\partial \xi}{\partial t}$$

$$x = -x_0 + \xi \quad (3-19)$$

At $O(\varepsilon)$

$$\begin{aligned} -i\omega \xi_{11} &= \phi_{11x} \\ &= ikf_0 \left\{ A e^{i(-kx_0 - \frac{\pi}{4})} - B e^{-i(-kx_0 - \frac{\pi}{4})} \right\} \\ &\quad + \sum_{n=1}^{\infty} (-k_n) D_n f_n e^{-i(-k_n x_0 - \frac{\pi}{4})} \end{aligned} \quad (3-20)$$

where

$$f_0 = -\frac{i g}{2\omega} \frac{\cosh k(z+h)}{\cosh kh}, \quad \omega^2 = gk \tanh kh \quad (3-21)$$

$$f_n = \cos k_n(z+h), \quad \omega^2 = -gk_n \tanh k_n h \quad (3-22)$$

Multiply Eq.(3-20) by f_0 and integrate it from $z=-h$ to $z=0$,

$$-i\omega \xi_{11} \int_{-h}^0 f_0 dz = ik \left\{ A e^{i(-kx_0 - \frac{\pi}{4})} - B e^{-i(-kx_0 - \frac{\pi}{4})} \right\} \int_{-h}^0 f_0^2 dz \quad (3-23)$$

Thus

$$A e^{i(-kx_0 - \frac{\pi}{4})} - B e^{-i(-kx_0 - \frac{\pi}{4})} = -4i \xi_{11} \tanh kh \left(1 + \frac{2kh}{\sinh 2kh} \right)^{-1} \quad (3-24)$$

At $O(\varepsilon^2)$

$$\xi_{10\bar{z}} = \phi_{10\bar{z}} + \phi_{20x} + (\xi_{11} \phi_{11x\bar{z}} + *) \quad (3-25)$$

Integrate Eq.(3-25) from $z=-h$ to $z=0$,

$$h \xi_{10\bar{z}} = h \phi_{10\bar{z}} + \int_{-h}^0 \phi_{20x} dz + \int_{-h}^0 (\xi_{11} \phi_{11x\bar{z}} + *) dz \quad (3-26)$$

where

$$\begin{aligned} \int_{-h}^0 (\xi_{11} \phi_{11x\bar{z}} + *) dz &= - \int_{-h}^0 (\xi_{11} \phi_{11z\bar{z}} + *) dz \\ &= - \left[\xi_{11} \phi_{11\bar{z}} + * \right]_{-h}^0 \\ &= - \left[-\frac{1}{i\omega} \phi_{11x} \frac{\omega^2}{g} \phi_{11\bar{z}} + * \right]_{z=0} \\ &= \left[-i \frac{\omega}{g} \phi_{11x} \phi_{11\bar{z}} + * \right]_{z=0} \end{aligned} \quad (3-27)$$

From Eqs.(2-18)-(2-20) we get

$$\phi_{20x\bar{z}} + \phi_{20z\bar{z}} = 0 \quad -h < z < 0 \quad (3-28)$$

$$\phi_{20\bar{z}} = \left(-\frac{i\omega}{g} \phi_{11x} \phi_{11\bar{z}} + * \right)_x \quad z = 0 \quad (3-29)$$

$$\phi_{20z} = 0$$

$$z = -h$$

$$(3-30)$$

Consider a control volume to the right of the wave maker and bounded by the horizontal lines $z=0$ and $z=-h$, and by the vertical lines $x=-x_0$ and $x \rightarrow -x_0 + \infty$, where $x \rightarrow -x_0 + \infty$ is in the region $0 \{ \bar{x} - (-\bar{x}_0) \} \ll 1$, but outside the influence of the evanescent mode.

Applying Green's formula to ϕ_{20} and any constant, we obtain

$$\int_{-h}^0 \phi_{20x} dz \Big|_{x=-x_0}^{x \rightarrow -x_0 + \infty} = \int_{-h}^0 \phi_{20x} dz \Big|_{x \rightarrow -x_0 + \infty} + \left[-\frac{i\omega}{g} \phi_{11x} \phi_{11}^* + * \right]_{x=-x_0}^{x \rightarrow -x_0 + \infty} \quad (3-31)$$

From Eqs. (3-26), (3-27) and (3-31) we finally obtain

$$\begin{aligned} \xi_{10} \bar{\epsilon} &= \phi_{10} \bar{x} + \frac{1}{h} \left[-\frac{i\omega}{g} \phi_{11x} \phi_{11} + * \right]_{x \rightarrow -x_0 + \infty}^{z=0} (x = -x_0) \\ &= \phi_{10} \bar{x} + \frac{2k}{2\omega h} (|A|^2 - |B|^2) \end{aligned} \quad (3-32)$$

Combining this equation and Eq. (2-96)

$$g \bar{\epsilon}_{20} \bar{x} + \xi_{10} \bar{\epsilon} \bar{\epsilon} = \frac{2k}{2\omega h} (|A|^2 - |B|^2) \bar{\epsilon} - \frac{g}{4h} \frac{2kh}{\sinh 2kh} (|A|^2 + |B|^2) \bar{x} \quad (3-33)$$

Summary of the governing equations is as follows:

$$x = -x_0$$

$$A e^{i(kx_0 - \frac{\pi}{4})} - B e^{-i(-kx_0 - \frac{\pi}{4})} = -4i \xi_{11} \tanh kh \left(1 + \frac{2kh}{\sinh 2kh} \right)^{-1} \quad (3-34)$$

$$g \bar{\epsilon}_{20} \bar{x} + \xi_{10} \bar{\epsilon} \bar{\epsilon} = \frac{2k}{2\omega h} (|A|^2 - |B|^2) \bar{\epsilon} - \frac{g}{4h} \frac{2kh}{\sinh 2kh} (|A|^2 + |B|^2) \bar{x} \quad (3-35)$$

$$-x_0 < x < 0$$

$$A \bar{\epsilon} + C_2 A \bar{x} = 0 \quad (3-36)$$

$$B \bar{\epsilon} - C_2 B \bar{x} = 0 \quad (3-37)$$

$$\bar{\epsilon}_{10} \bar{\epsilon} \bar{\epsilon} - 2h \bar{\epsilon}_{20} \bar{x} \bar{x} = \frac{g}{4} \left(\frac{4C_2}{C} - 1 \right) (|A|^2 + |B|^2) \bar{x} \bar{x} \quad (3-38)$$

$$x = 0$$

$$A_+ = A_- \quad (3-39)$$

$$B_+ = B_- \quad (3-40)$$

$$\bar{\epsilon}_{20+} = \bar{\epsilon}_{20-} \quad (3-41)$$

$$\left[\bar{\epsilon}_{20} \bar{x} + \frac{1}{4h} \frac{2kh}{\sinh 2kh} (|A|^2 + |B|^2) \bar{x} \right]_+ = \left[\quad \cdot \quad \right]_- \quad (3-42)$$

$$0 < x < L$$

$$A \bar{\epsilon} + C_2 A \bar{x} = -i \Omega_0 B \quad (3-43)$$

$$B_{\bar{x}} - C_2 B_x = -i\Omega_0 A \quad (3-44)$$

$$\bar{Y}_{20\bar{x}\bar{x}} - 2k \bar{Y}_{20\bar{x}} = \frac{2}{4} \left(\frac{4C_2}{C} - 1 \right) (|A|^2 + |B|^2)_{\bar{x}\bar{x}} + \frac{2\Omega_0}{C} (iA^*B + *) \quad (3-45)$$

$x = L$

$$A = iB \quad (3-46)$$

$$\bar{Y}_{20\bar{x}} = \frac{\Omega_0}{C_2} \frac{2k}{\sinh 2kh} |A|^2 \quad (3-47)$$

3-2-2 Steady Incident Wave

Let

$$A = \bar{A}(\bar{x}) e^{-i\Omega\bar{t}} \quad 0 < x < L \quad (3-48)$$

$$B = \bar{B}(\bar{x}) e^{-i\Omega\bar{t}} \quad 0 < x < L \quad (3-49)$$

From Eqs. (3-43) and (3-44)

$$-i\Omega \bar{A} + C_2 \bar{A}_{\bar{x}} = -i\Omega_0 \bar{B} \quad (3-50)$$

$$-i\Omega \bar{B} - C_2 \bar{B}_{\bar{x}} = -i\Omega_0 \bar{A} \quad (3-51)$$

Combining Eqs. (3-50) and (3-51)

$$\bar{A}_{\bar{x}\bar{x}} + \left(\frac{\Omega_0}{C_2} \right) \left\{ \left(\frac{\Omega_0}{\Omega} \right)^2 - 1 \right\} \bar{A} = 0 \quad (3-52)$$

$$a) \left| \frac{\Omega}{\Omega_0} \right| < 1$$

Let

$$Q^2 = \left(\frac{\Omega_0}{C_2} \right)^2 \left\{ 1 - \left(\frac{\Omega}{\Omega_0} \right)^2 \right\} \quad (3-53)$$

then

$$\bar{A}_{\bar{x}\bar{x}} - Q^2 \bar{A} = 0 \quad (3-54)$$

We put

$$\bar{A} = \bar{A}_1 e^{Q\bar{x}} + \bar{A}_2 e^{-Q\bar{x}} \quad (3-55)$$

From Eq. (3-50)

$$\bar{B} = R_1^* \bar{A}_1 e^{Q\bar{x}} + R_2 \bar{A}_2 e^{-Q\bar{x}} \quad (3-56)$$

where

$$R_1 = \frac{\Omega}{\Omega_0} + i \left\{ 1 - \left(\frac{\Omega}{\Omega_0} \right)^2 \right\}^{\frac{1}{2}} \quad (3-57)$$

From Eq. (3-46)

$$\bar{A}_1 e^{QL} + \bar{A}_2 e^{-QL} = i(R_1^* \bar{A}_1 e^{QL} + R_1 \bar{A}_2 e^{-QL}) \quad (3-58)$$

thus

$$\bar{A}_2 = \bar{A}_1 e^{2QL} \frac{iR_1^* - 1}{1 - iR_1} \quad (3-59)$$

Finally we obtain

$$\bar{A} = \bar{A}_1 e^{QL} \left\{ e^{Q(\bar{x}-L)} + \frac{iR_1^* - 1}{1 - iR_1} e^{-Q(\bar{x}-L)} \right\} \quad (3-60)$$

$$\bar{B} = \bar{A}_1 e^{QL} \left\{ R_1^* e^{Q(\bar{x}-L)} + R_1 \frac{iR_1^* - 1}{1 - iR_1} e^{-Q(\bar{x}-L)} \right\} \quad (3-61)$$

b) $|\frac{\Omega}{\Omega_0}| > 1$

Let

$$P^2 = \left(\frac{\Omega_0}{\Omega}\right)^2 \left\{ \left(\frac{\Omega}{\Omega_0}\right)^2 - 1 \right\} \quad (3-62)$$

We put

$$\bar{A} = \bar{A}_1 e^{iP\bar{x}} + \bar{A}_2 e^{-iP\bar{x}} \quad (3-63)$$

From Eq. (3-50)

$$\bar{B} = \bar{A}_1 R_2 e^{iP\bar{x}} + \bar{A}_2 \frac{1}{R_2} e^{-iP\bar{x}} \quad (3-64)$$

where

$$R_2 = \frac{\Omega_0}{\Omega} - \left\{ \left(\frac{\Omega}{\Omega_0}\right)^2 - 1 \right\}^{\frac{1}{2}} \quad (3-65)$$

From Eq. (3-46)

$$\bar{A}_1 e^{iPL} + \bar{A}_2 e^{-iPL} = i(\bar{A}_1 R_2 e^{iPL} + \bar{A}_2 \frac{1}{R_2} e^{-iPL}) \quad (3-66)$$

thus

$$\bar{A}_2 = \bar{A}_1 e^{2iPL} \frac{iR_2 - 1}{1 - i\frac{1}{R_2}} \quad (3-67)$$

Finally we obtain

$$\bar{A} = \bar{A}_1 e^{iPL} \left\{ e^{iP(\bar{x}-L)} + R_2 \frac{iR_2 - 1}{R_2 - i} e^{-iP(\bar{x}-L)} \right\} \quad (3-68)$$

$$\bar{B} = \bar{A}_1 e^{iPL} \left\{ R_2 e^{iP(\bar{x}-L)} + \frac{iR_2 - 1}{R_2 - i} e^{-iP(\bar{x}-L)} \right\} \quad (3-69)$$

It is to be noticed that

$$|A| = |B| \quad (3-70)$$

is always satisfied.

The place of antinodes is calculated by

$$\overline{A} e^{i(kx - \frac{\pi}{4})} = \overline{B} e^{-i(kx - \frac{\pi}{4})} \quad (3-71)$$

The wave amplitude at antinodes is $2|A|$.

3-2-3 Incident Wave Packet

Let the initial condition to be given at $x = -x_1$ as follows:

$$A = A_0 \sin^2(\Omega \bar{t}) e^{-i\Omega_2 \bar{t}} \quad 0 < \Omega \bar{t} < \pi \quad (3-72)$$

The solution outside the sandbar region is given as

$$A = A_0 \sin^2 \left[\Omega \left\{ \bar{t} - \frac{1}{C_g} (\bar{x} + \bar{x}_1) \right\} \right] e^{-i\Omega_2 \left\{ \bar{t} - \frac{1}{C_g} (\bar{x} + \bar{x}_1) \right\}} \quad \bar{x} < 0, \quad 0 < \Omega \left\{ \bar{t} - \frac{1}{C_g} (\bar{x} + \bar{x}_1) \right\} < \pi \quad (3-73)$$

The steady solution of $\bar{\gamma}_{20}$ associated with this short waves is given as

$$\bar{\gamma}_{20s} = \frac{1}{C_g^2 - gk} \frac{g}{4} \left(\frac{4C_g}{C} - 1 \right) |A|^2 \quad (3-74)$$

At $x = -x_1$,

$$\bar{\gamma}_{20s} = \frac{1}{C_g^2 - gk} \frac{g}{4} \left(\frac{4C_g}{C} - 1 \right) |A_0|^2 \sin^4(\Omega \bar{t}) \quad 0 < \Omega \bar{t} < \pi \quad (3-75)$$

Since the measured $\bar{\gamma}_{20}$ was always smaller than this solution, we assume

$$\bar{\gamma}_{20} = C_L \times \bar{\gamma}_{20s} \quad x = -x_1 \quad (3-76)$$

Let

$$\bar{\gamma}_{20} = \bar{\gamma}_{20s} + \bar{\gamma}_{20f} \quad (3-77)$$

where $\bar{\gamma}_{20s}$ is steady solution shown above. Then governing equations of $\bar{\gamma}_{20f}$ are given as follows:

$x = -x_1$,

$$\bar{\gamma}_{20f} = (C_L - 1) \bar{\gamma}_{20s} = (C_L - 1) \frac{1}{C_g^2 - gk} \frac{g}{4} \left(\frac{4C_g}{C} - 1 \right) |A_0|^2 \sin^4(\Omega \bar{t}) \quad 0 < \Omega \bar{t} < \pi \quad (3-78)$$

$-x_1 < x < 0$

$$\bar{\gamma}_{20f} \bar{t} \bar{t} - gk \bar{\gamma}_{20f} \bar{x} \bar{x} = 0 \quad (3-79)$$

$$x = 0$$

$$\bar{\xi}_{20f+} = \bar{\xi}_{20f-} \quad (3-80)$$

$$\bar{\xi}_{20f\bar{x}+} = \bar{\xi}_{20f\bar{x}-} \quad (3-81)$$

$$0 < x < L$$

$$\bar{\xi}_{20f} \bar{\xi} \bar{\xi} - gh \bar{\xi}_{20f} \bar{x} \bar{x} = 0 \quad (3-82)$$

$$x = L$$

$$\bar{\xi}_{20f} \bar{x} = 0 \quad (3-83)$$

$\bar{\xi}_{20f}$ is a free wave which propagates with the speed of long wave, on which the sandbars do not have any effect. The solution of $\bar{\xi}_{20f}$ is

$$\begin{aligned} \bar{\xi}_{20f} = (C_L - 1) \frac{1}{C_g^2 - gh} \frac{g}{4} \left(\frac{4C_2}{C} - 1 \right) |A_0|^2 x \\ \times \left[\sin^4 \left[\Omega \left\{ \bar{x} - \frac{1}{\sqrt{gh}} (\bar{x} + \bar{x}_1) \right\} \right] + \sin^4 \left[\Omega \left\{ \bar{x} + \frac{1}{\sqrt{gh}} (\bar{x} - \bar{x}_1 - 2L) \right\} \right] \right] \end{aligned} \quad (3-84)$$

We can solve A, B, and $\bar{\xi}_{20f}$ in the sandbar region in the same way as in Chapter 2.

From Eqs. (3-34) and (3-35), we can decide the input signal for the wave maker. Since $B=0$,

$$\xi_{11} = \frac{A}{-4i \tanh kh} \left(1 + \frac{2kh}{\sinh 2kh} \right) e^{i(-kx_0 - \frac{\pi}{4})} \quad (3-85)$$

$$\xi_{10} \bar{\xi} \bar{\xi} = -g \bar{\xi}_{20} \bar{x} + \frac{gk}{2\omega h} |A|^2_{\bar{x}} - \frac{g}{4h} \frac{2kh}{\sinh 2kh} |A|^2_{\bar{x}} \quad (3-86)$$

Introducing Eqs. (3-72) and (3-75),

$$\xi_{11} = \frac{1}{-4i \tanh kh} \left(1 + \frac{2kh}{\sinh 2kh} \right) A_0 \sin^2(\Omega \bar{x}) e^{-i\Omega_1 \bar{x}} e^{i(-kx_0 - \frac{\pi}{4})} \quad (3-87)$$

$$\begin{aligned} \xi_{10} \bar{\xi} \bar{\xi} &= -g \frac{1}{C_g^2 - gh} \frac{g}{4} \left(\frac{4C_2}{C} - 1 \right) \frac{1}{-C_g} |A|^2_{\bar{x}} + \frac{gk}{2\omega h} |A|^2_{\bar{x}} - \frac{1}{4h} \frac{2kh}{\sinh 2kh} \frac{1}{-C_g} |A|^2_{\bar{x}} \\ &= \frac{1}{4h} \frac{g}{C_g} \left(\frac{4C_2}{C} - 1 \right) \frac{C_g^2}{C_g^2 - gh} |A|^2_{\bar{x}} \end{aligned} \quad (3-88)$$

Thus

$$\bar{\xi}_{10} = \frac{1}{4h} \frac{g}{C_g} \left(\frac{4C_2}{C} - 1 \right) \frac{C_g^2}{C_g^2 - gh} \int_0^{\bar{x}} |A|^2 d\bar{x} + C_1 \bar{x} + C_2 \quad (3-89)$$

Since $\bar{\xi}_{10} = \bar{\xi}_{10} \bar{x} = 0$ at $\bar{x} = 0$

$$C_1 = C_2 = 0 \quad (3-90)$$

$$\bar{\xi}_{10} = \frac{1}{4h} \frac{g}{C_2} \left(\frac{4C_2}{C} - 1 \right) \frac{C_2^2}{C^2 - gh} |A_0|^2 \left(\frac{3}{8} \bar{t} - \frac{\sin 2\Omega \bar{t}}{4\Omega} + \frac{\sin 4\Omega \bar{t}}{32\Omega} \right) \quad (3-91)$$

Finally we obtain

$$\begin{aligned} \xi &= \bar{\xi}_{10} + (\bar{\xi}_{11} e^{-i\omega t} + *) \\ &= \frac{1}{2 \tanh kh} \left(1 + \frac{2kh}{\sinh 2kh} \right) |A_0|^2 \left[\sin^2(\Omega \bar{t}) \sin \{ (\omega + \Omega_2) t - \alpha \} \right. \\ &\quad \left. + \frac{1}{4h} \left(\frac{C}{C_2} \right)^2 \frac{\omega}{\Omega} \frac{|A_0|^2}{h} \left(\frac{4C_2}{C} - 1 \right) \frac{1}{1 - gh/C^2} \left(\frac{3}{8} \Omega \bar{t} - \frac{\sin 2\Omega \bar{t}}{4} + \frac{\sin 4\Omega \bar{t}}{32} \right) \right] \quad (3-92) \end{aligned}$$

Since the displacement of the wave generator is assumed to be linear to the input signal, the above equation can be used for the input signal as well.

3-3 Results and Discussions

3-3-1 Steady Incident Waves

In Fig. 3-3 experimental and numerical results of wave height distribution at antinodes are shown. All the wave heights are normalized by those at the antinode nearest to $x=0$. When $|\frac{\Omega}{\Omega_0}| < 1$ i.e. near the resonant condition, both results decay exponentially and agree with each other very well. However when $|\frac{\Omega}{\Omega_0}| > 1$, they agree only qualitatively but not quantitatively. Experimental results clearly show oscillatory wave height distribution, which is an important character when $|\frac{\Omega}{\Omega_0}| > 1$. But phase and amplitude of the oscillation is rather different between the theory and the experiment.

Fig. 3-4 shows how wave height distribution changes as the parameter $\frac{\Omega}{\Omega_0}$ changes. Using this figure, we try to fit the numerical curve to experimental data by changing the parameter as follows.

Table 3-4 Parameter $\frac{\Omega}{\Omega_0}$

Case	$\frac{\Omega}{\Omega_0}(\text{old})$	$\frac{\Omega}{\Omega_0}(\text{new})$	$\frac{\Omega}{\Omega_0}(\text{new}) - \frac{\Omega}{\Omega_0}(\text{old})$
1-1	2.04	2.3	0.26
1-2	1.02	1.23	0.21
1-3	0.51	0.7	0.19
1-4	0.0	0.2	0.20
1-5	-0.51	-0.3	0.21
1-6	-1.02	-0.8	0.22
1-7	-2.04	-1.85	0.19
2-1	2.04	2.3	0.26
2-2	1.02	1.3	0.28
2-3	0.0	0.3	0.3
2-4	-1.02	-0.9	0.12
2-5	-2.04	-1.8	0.24
3	2.0	2.32	0.32

The results are shown in Fig. 3-5. Clearly the agreement between numerical and experimental results is much better.

Observed position of the node near $x=0$ is compared with calculated one for each case in Table 3-5. The discrepancy between them is larger as $|\frac{\Omega}{\Omega_0}|$ increases. We change the parameter $\frac{\Omega}{\Omega_0}$ by the same magnitude as above. The results are shown in the rightmost column of Table 3-5. It is evident that they agree with each other much better.

Table 3-5 Position of Node

Case	measured x= (cm)	calculated x= (cm)	calculated(corrected) x= (cm)
1-1	+2	-11	+1
1-2	-12	-18	-12
1-3	-21	-25	-23
1-4	-28	-30	-28
1-5	+26	+25	+27
1-6	+20	+18	+21
1-7	+18	+12	+19

Considering all these results and the fact that the frequency of input signal for the wave generator was very accurate, we can suppose that the actual $\frac{\omega}{\Omega_0}$ was always larger than calculated $\frac{\omega}{\Omega_0}$, that is, the resonant frequency ω was larger than the estimated one. Since we estimated resonant frequency ω by

$$\omega^2 = gk \tanh kh \quad (3-93)$$

the error should be due to one of the following reasons.

1. error in k
2. error in h
3. error in g
4. error in Eq.(3-93)

The errors in k and g are at most 0.1%. The error in the equation itself may be assumed to be $O(ka)^2$, which is less than 0.1% in this case. It is likely that measured h included error of $O(1\%)$, since the bottom of wave tank was not completely flat, and the artificial flat or wavy bottom could not be so accurate. Measured D may also include similar magnitude of error. Thus, the error in Ω_0 , which depends on both h and D, can be $O(1\%)$. We try to change h and D (ω and Ω_0) so that we can obtain the corrected $\frac{\omega}{\Omega_0}$ shown above. The results is as follows.

Table 3-6 Corrected h and D

D=2.8(cm)

Case	h(cm)	Case	h(cm)
1-1	14.4	2-1	14.4
1-2	14.5	2-2	14.4
1-3	14.5	2-3	14.3
1-4	14.5	2-4	14.6
1-5	14.4	2-5	14.4
1-6	14.4		
1-7	14.5	3	19.5

It seems that the actual water depth was smaller than the estimated one by about 0.4cm. The scattering of h is $\pm 1\text{mm}$, and may be due to the measurement error in each case. In Fig. 3-6 wave height distributions calculated with the condition ($h=14.4\text{cm}$ for Cases 1 and 2, $h=19.6\text{cm}$ for Case 3, $D=2.8\text{cm}$) are compared with the experimental data. The agreement is much better than that in Fig. 3-3.

Though errors in h and D explain the disagreement between the experiment and the simulation very well, 4mm error in h is too large. It seems that measurement error was at most $\pm 1\text{mm}$ (total 2mm) in this experiment. Another possible explanation is that the governing equations were not accurate enough. We try to consider higher order terms neglected so far. Since kD is much greater than kA in this case, we only consider higher order terms of kD . The governing equations become as follows: (see Appendix)

$$A_z + C_g A_{\bar{z}} = -i\Omega_0 B + \frac{i}{2} \frac{\partial \omega}{\partial k^2} A_{\bar{z}\bar{z}} + \frac{\Omega_0}{K} (-2 \sinh^2 kh) B_{\bar{z}} - \frac{gk}{4\omega} \frac{1 - \sinh^2 kh}{\cosh^2 kh} D_{\bar{z}} B + \frac{i}{2\omega} \left(1 + 4 \sinh^2 kh + 6 \sinh 2kh \frac{\cosh 4kh + 2 \cosh 2kh}{\sinh 4kh + 2 \sinh 2kh} \right) \quad (3-94)$$

$$B_z - C_g B_{\bar{z}} = -i\Omega_0 A + \frac{i}{2} \frac{\partial \omega}{\partial k^2} B_{\bar{z}\bar{z}} + \frac{\Omega_0}{K} (-2 \sinh^2 kh) A_{\bar{z}} - \frac{gk}{4\omega} \frac{1 - \sinh^2 kh}{\cosh^2 kh} D_{\bar{z}} A + \frac{i}{2\omega} \left(1 + 4 \sinh^2 kh + 6 \sinh 2kh \frac{\cosh 4kh + 2 \cosh 2kh}{\sinh 4kh + 2 \sinh 2kh} \right) \quad (3-95)$$

We put the solutions as follows:

$$A = e^{-i\Omega_0 t} (\bar{A}_1 e^{i(Q+\Delta Q_1)\bar{x}} + \bar{A}_2 e^{-(Q+\Delta Q_2)\bar{x}}) \quad (3-96)$$

$$A = e^{-i\Omega_0 t} (\bar{A}_1 e^{i(P+\Delta P_1)\bar{x}} + \bar{A}_2 e^{-i(P+\Delta P_2)\bar{x}}) \quad (3-97)$$

$$B = e^{-i\Omega_0 t} \left\{ \bar{A}_1 (R_1 + \Delta R_{11}) e^{i(Q+\Delta Q_1)\bar{x}} + \bar{A}_2 (R_1 + \Delta R_{12}) e^{-(Q+\Delta Q_2)\bar{x}} \right\} \quad (3-98)$$

$$B = e^{-i\Omega_0 t} \left\{ \bar{A}_1 (R_2 + \Delta R_{21}) e^{i(P+\Delta P_1)\bar{x}} - \bar{A}_2 \frac{1}{R_2 + \Delta R_{22}} e^{-i(P+\Delta P_2)\bar{x}} \right\} \quad (3-99)$$

Introducing Eqs. (3-96)-(3-99) into Eqs. (3-94) and (3-95), we obtain

$$\left. \begin{aligned} \Delta P_1 \\ \Delta P_2 \end{aligned} \right\} = -\frac{1}{2} \frac{\partial \omega}{\partial k^2} \frac{\Omega_0}{\Omega_0} \left(\frac{Q_0}{C_g} \right)^2 \frac{1}{C_g \sqrt{1 - \left(\frac{\Omega_0}{\Omega_0} \right)^2}} + \left(1 + 4 \sinh^2 kh + 6 \sinh 2kh \frac{\cosh 4kh + 2 \cosh 2kh}{\sinh 4kh + 2 \sinh 2kh} \right) \times \frac{\Omega_0^2}{2 C_g \omega} \frac{\Omega_0}{\Omega_0} \frac{1}{\sqrt{1 - \left(\frac{\Omega_0}{\Omega_0} \right)^2}} + 2 \Omega_0 \sinh^2 kh \frac{\Omega_0}{K C_g^2} \quad (3-100)$$

$$\left. \begin{aligned} \Delta Q_1 \\ \Delta Q_2 \end{aligned} \right\} = -\frac{1}{2} \frac{\partial \omega}{\partial k^2} \frac{\Omega_0}{\Omega_0} \left(\frac{Q_0}{C_g} \right)^2 \frac{1}{C_g \sqrt{1 - \left(\frac{\Omega_0}{\Omega_0} \right)^2}} - \left(1 + 4 \sinh^2 kh + 6 \sinh 2kh \frac{\cosh 4kh + 2 \cosh 2kh}{\sinh 4kh + 2 \sinh 2kh} \right) \times \frac{\Omega_0^2}{2 C_g \omega} \frac{\Omega_0}{\Omega_0} \frac{1}{\sqrt{1 - \left(\frac{\Omega_0}{\Omega_0} \right)^2}} + 2 i \Omega_0 \sinh^2 kh \frac{\Omega_0}{K C_g^2} \quad (3-101)$$

where the last term in each equation only changes the phases of A and B, and may be neglected in the following discussion. In Fig. 3-7, P, Q and P+ΔP, Q+ΔQ are plotted versus $\frac{\Omega}{\Omega_0}$. This results show that for a given wave height distribution (for given P or Q), apparent $\frac{\Omega}{\Omega_0}$ is always larger than actual $\frac{\Omega}{\Omega_0}$ by 0.2~0.3. This is consistent to our experimental results.

We found two possible reasons for the discrepancy between experimental and simulated results. In any case, we should notice that the phenomena is very sensitive to the change of $\frac{\Omega}{\Omega_0}$.

The following conclusions can be made for this steady case experiment.

1. In general, the theory derived in Chapter 2 is valid at least when the deviation of the incident wave frequency from the resonant frequency is less than $2\Omega_0$.
2. Higher order terms or measurement error of only a few percent may have considerable effects on the results.

3-3-2 Incident Wave Packet

a) Without Sandbars

Fig. 3-8 shows the incident and reflected wave records in the experiment without sandbars. From the record of incident wave packet, we can say that the input signal was good enough to generate desired wave packet. As a packet propagates, the whole shape is flattened and the maximum wave height decreases. This deformation is clearer as h decreases, or as $\frac{\Omega}{\Omega_0}$ increases. Mase et al.(1985) reported similar experimental results. They explained the flattening of packet by solving

$$-i\left(\frac{\partial A}{\partial t} + C_g \frac{\partial A}{\partial x}\right) + \alpha \frac{\partial^2 A}{\partial x^2} + \beta |A|^2 A = 0 \quad (3-102)$$

where

$$\alpha = -\frac{i}{2} \frac{\partial^2 \omega}{\partial k^2}$$

$$\beta = \frac{\omega k^2}{16 \sinh^4 kh} (\cosh 4kh + 8 - 2 \tanh^2 kh) - \frac{\omega}{2 \sinh^2 2kh} \frac{(2\omega \cosh^2 kh + k C_g)^2}{gh - C_g^2}$$

numerically. They could not clearly explain the reason for the decay of maximum wave height. But it is probably due to both dispersion (estimated by the above equation) and dissipation by friction at the bottom and side walls.

b) With Sandbars

Fig. 3-9 shows typical Fourier coefficients of measured output at $x=-600\text{cm}$. The peaks of 0th-mode (long wave), 1st-mode (1st order short wave), 2nd, 3rd....mode are clearly observed. From these results, cut off frequencies for short and long waves (0.25 and 1.4 Hz) were determined.

In Fig. 3-10 the Fourier coefficients of first-order incident waves at $x=0$ are shown for each case. Incident and reflected waves were divided by the method of Goda et al. (1977). Good agreement between Fig. 3-10 and Fig. 3-2 shows that the initial condition used in the simulation was accurate enough.

Short wave profiles during one wave period are plotted in the region $-55 < x < 10$, $545 < x < 610$, and $1125 < x < 1190$ (cm), so as to know the position of nodes and antinodes (Fig. 3-11). It is to be noticed that the antinode shifts from $x=595\text{cm}$ ($t=26.0\text{s}$) to $x=565\text{cm}$ ($t=29.0\text{s}$) and to $x=580\text{cm}$ ($t=34.0\text{s}$) in Case 5-1. This seems to have happened because the wave reflected by the end wall reached. Near the end wall antinodes were always observed at $x=1200\text{cm}$ and near $x=1140\text{cm}$.

The second order long wave ζ_{20} consists of two component:

$$\zeta_{20} = \bar{\zeta}_{20} + \zeta'_{20} \quad (3-103)$$

where $\bar{\zeta}_{20}$ is the average of ζ_{20} over one wavelength in space. Theoretically ζ'_{20} only depends on local A and B:

$$\zeta'_{20} = \frac{1}{4h} \frac{2kh}{\sinh 2kh} (1 + 2 \sinh^2 kh) (AB^* e^{2i(kx - \frac{\pi}{4})} + *) \quad (3-104)$$

and is maximum at antinodes, minimum at nodes, and zero at the midst between nodes and antinodes. Fig. 3-12 shows measured ζ_{20} versus x when $|\zeta_{20}|$ is maximum. It is evident that ζ'_{20} is maximum at antinodes and minimum at nodes. Magnitude of ζ'_{20} agrees with the computed one by Eq. (3-104) very well (Table 3-7).

Table 3-7 Measured and Calculated ($\zeta'_{20\text{max}} - \zeta'_{20\text{min}}$)

		Measured (mm)	Calculated (mm)	
Case 5-1	$x=0\text{cm}$	0.74	0.78	(Fig. 3-12(a))
	$x=600\text{cm}$	0.26	0.28	(Fig. 3-12(b))
	$x=1200\text{cm}$	0.24	0.24	(Fig. 3-12(c))
Case 5-6	$x=0\text{cm}$	1.43	1.42	(Fig. 3-12(i))

Experimental results of short and long waves are compared with simulated results in Figs. (3-13)-(3-18). The wave generator started to move when $t=0\text{s}$. It is to be noticed that

the short wave data can be compared with simulated results for about 20 seconds after the first wave reaches, because waves reflected by the wave generator are not considered in simulation. Similarly long wave data can be compared for about 15 seconds. Measured ξ_{z_0} are compared with computational results at the midst between nodes and antinodes so as to exclude ξ'_{z_0} .

In case 5-1, the experimental results agree with the theory very well. As to short waves, reflection by sandbars is clear at $x=0\text{cm}$, and dispersion of incident envelope is clear at $x=600\text{cm}$ and at $x=1200\text{cm}$. At $x=600\text{cm}$ the incident wave envelope agrees with the theory very well. However reflected wave envelope does not agree very well after $t=33\text{s}$. At $x=1200\text{cm}$ the magnitudes of incident and reflected waves are the same. The coincidence between the experiment and theory is rather good. But the first wave group is a little longer than the theory, and the second one is shorter. These discrepancies may be due to higher order terms in Eqs.(3-94)and(3-95).

As to long waves, agreement between the measured and simulated results is very good. Especially at $x=1200\text{cm}$, large set-down is clearly observed. This set-down was originally locked to short wave envelope, but was unlocked as the packet entered sandbar region, and propagated with the speed of long wave ($\sqrt{gh}$) without decaying, though first order incident short waves decayed quickly due to reflection.

In Cases 5-2 and 5-3, the spectrum of incident wave packet is narrower, but the results is no better than in Case 5-1. When the spectrum shifts in Cases 5-4 and 5-5, the results become much worse. Especially in Case 5-5, the shape of wave packet at $x=1200\text{cm}$ is very different. From these results, we can say that our theory is good when the major part of spectrum is as centered on ω as that in Fig. 3-10(a).

In Case 5-6, the depth is greater than in Case 5-1. To compare the results in these cases, we should use similarity law. When we normalize variables as

$$t' = \Omega_0 t \quad (3-105)$$

$$x' = \frac{x}{L} \quad (3-106)$$

governing equations become

$$A_t + \frac{C_g}{\Omega_0 L} A_{x'} = -iB \quad (3-107)$$

$$B_t - \frac{C_g}{\Omega_0 L} B_{x'} = -iA \quad (3-108)$$

To make the conditions similar in both cases, we make

$$\frac{C_R}{\Omega_0 L} = \text{constant}. \quad (3-109)$$

or

$$\frac{L(\text{Case 5-6})}{L(\text{Case 5-1})} = \frac{\Omega_0/C_g(\text{Case 5-1})}{\Omega_0/C_g(\text{Case 5-6})} \doteq 1.6 \quad (3-110)$$

The results at $x=1200\text{cm}$ in Case 5-6 should be compared with the results at $x=750\text{cm}$ in Case 5-1.

If we compare Fig. 3-13(c) and Fig. 3-18(c), the results in Fig. 3-18(c) seems a little better than that in Fig. 3-13(c), though corresponding x is larger. We may say that the coincidence is better as kh is larger.

Since we assumed that measured h and D were not accurate in steady case, we also attempt to use condition " $h=14.4(\text{cm})$ and $D=2.8(\text{cm})$ " in the packet case. Fig. 3-19 shows the results at $x=600\text{cm}$ and $x=1200\text{cm}$. In this case, the change of h and D makes the results no better (even worse in some cases).

The conclusions deduced from the packet case experiment are as follows:

1. The theory for both first order short wave and second order long wave is valid, when the main part of the spectrum of incident wave is as close to ω as in Fig. 3-10(a).
2. The effect of higher order terms in governing equations becomes significant when the packet propagates for a long distance.
3. Agreement between the experiment and simulation seems better as kh becomes larger.

Appendix

We assume that the slope of the sandbars is $O(u)$ ($\varepsilon \ll u \ll 1$), the water depth is constant, and $\frac{\partial}{\partial y} = 0$. The governing equation and the boundary conditions become

$$\phi_{xx} + \phi_{zz} = 0 \quad -h < z < 0 \quad (A-1)$$

$$\phi_{tt} + g\phi_z = O(\varepsilon^2) \quad z=0 \quad (A-2)$$

$$-\phi_x \delta_x - \phi_{xz} \delta_x \delta + \phi_z + \phi_{zz} \delta + \frac{1}{2} \phi_{zzz} \delta^2 = O(\varepsilon^2, \varepsilon u^3) \quad z=-h \quad (A-3)$$

We introduce slow coordinates

$$x_1 = ux, \quad x_2 = u^2 x, \quad t_1 = ut, \quad t_2 = u^2 t \quad (A-4)$$

and use perturbation expansions for ϕ ,

$$\phi = \sum_{n=1} \varepsilon u^{n-1} \sum_{m=-n}^n \phi_{nm} e^{-im\omega t}, \quad \phi_{nm} = \phi_{n,-m}^* \quad (A-5)$$

Substituting Eqs.(A-4) and (A-5) into Eqs.(A-1)-(A-3), and separating the orders, we obtain

$O(\varepsilon)$

$$\phi_{11xx} + \phi_{11zz} = 0 \quad -h < z < 0 \quad (A-6)$$

$$-\omega^2 \phi_{11} + g\phi_{11z} = 0 \quad z=0 \quad (A-7)$$

$$\phi_{11z} = 0 \quad z=-h \quad (A-8)$$

$O(\varepsilon u)$

$$\phi_{21xx} + \phi_{21zz} = -2\phi_{11x_1x} \quad -h < z < 0 \quad (A-9)$$

$$-\omega^2 \phi_{21} + g\phi_{21z} = 2i\omega\phi_{11t_1} \quad z=0 \quad (A-10)$$

$$\phi_{21z} = \phi_{11x} \delta_x + \phi_{11xx} \delta \quad z=-h \quad (A-11)$$

$O(\varepsilon u^2)$

$$\phi_{31xx} + \phi_{31zz} = -2\phi_{21x_1x} - 2\phi_{11x_2x} - \phi_{11x_1x_1} \quad -h < z < 0 \quad (A-12)$$

$$-\omega^2 \phi_{31} + g\phi_{31z} = 2i\omega\phi_{11t_2} + 2i\omega\phi_{21t_1} - \phi_{11t_1t_1} \quad z=0 \quad (A-13)$$

$$\begin{aligned}\phi_{31Z} = & (\phi_{21XX} + 2\phi_{11X_1X})\delta + (\phi_{11X_1} + \phi_{21X})\delta_X + \phi_{11X}\delta_{X_1} \\ & + \frac{1}{2}(\phi_{11XZ}\delta^2)_X \quad z=-h \quad (A-14)\end{aligned}$$

The solution which satisfies Eqs.(A-6)-(A-8) is expressed as follows

$$\phi_{11} = \phi_{11}^1 e^{iS^+} + \phi_{11}^{-1} e^{iS^-} \quad (A-15)$$

$$\phi_{11}^1 = Af_0, \quad \phi_{11}^{-1} = Bf_0 \quad (A-16)$$

$$S^\pm = \pm(kx - \frac{z}{4}) \quad (A-17)$$

$$f_0 = -\frac{ig}{2\omega} \frac{\cosh k(z+h)}{\cosh kh} \quad (A-18)$$

Let

$$\phi_{21} = \phi_{21}^1 e^{iS^+} + \phi_{21}^{-1} e^{iS^-} + \phi_{21}^3 e^{3iS^+} + \phi_{21}^{-3} e^{3iS^-} \quad (A-19)$$

Introducing Eq.(A-19) into Eqs.(A-9)-(A-11), we obtain

$$-k^2 \phi_{21}^1 + \phi_{21}^1_{ZZ} = -2ik\phi_{11}^1_{X_1} \quad -h < z < 0 \quad (A-20)$$

$$-\omega^2 \phi_{21}^1 + g\phi_{21}^1_Z = 2i\omega\phi_{11}^1_{t_1} \quad z=0 \quad (A-21)$$

$$\phi_{21Z}^1 = \frac{k^2 D}{2} \phi_{11}^{-1} \quad z=-h \quad (A-22)$$

$$-9k^2 \phi_{21}^3 + \phi_{21}^3_{ZZ} = 0 \quad -h < z < 0 \quad (A-23)$$

$$-\omega^2 \phi_{21}^3 + g\phi_{21}^3_Z = 0 \quad z=0 \quad (A-24)$$

$$\phi_{21Z}^3 = -\frac{3}{2} k^2 D \phi_{11}^1 \quad z=-h \quad (A-25)$$

The solvability condition for ϕ_{21}^1 gives

$$A_{t_1} + C_g A_{x_1} = -i\Omega_0 B \quad (A-26)$$

$$\Omega_0 = \frac{\omega k D}{2 \sinh 2kh} \quad (A-27)$$

$$C_g = \frac{\omega}{2k} \left(1 + \frac{2kh}{\sinh 2kh} \right) \quad (A-28)$$

The solutions of ϕ_{21}^1 and ϕ_{21}^3 are obtained as follows:

$$\phi_{21}^1 = -\frac{g}{2\omega k} \frac{Q \sinh Q}{\cosh kh} A_{x_1} - \frac{igkD \sinh Q}{4\omega \cosh kh} B \quad (A-29)$$

$$\phi_{21}^3 = \frac{i\omega AD}{4 \sinh kh} \left(\sinh 3Q - \frac{\cosh 4kh + 2 \cosh 2kh}{\sinh 4kh + 2 \sinh 2kh} \cosh 3Q \right) \quad (A-30)$$

where

$$Q = k(z+h) \quad (A-31)$$

Similarly we get

$$B_{t_1} - C_g B_{x_1} = -i\Omega_0 A \quad (A-32)$$

$$\phi_{21}^{-1} = \frac{g}{2\omega k} \frac{Q \sinh Q}{\cosh kh} B_{x_1} - \frac{igkD \sinh Q}{4\omega \cosh kh} A \quad (A-33)$$

$$\phi_{21}^{-3} = \frac{i\omega BD}{4 \sinh kh} \left(\sinh 3Q - \frac{\cosh 4kh + 2 \cosh 2kh}{\sinh 4kh + 2 \sinh 2kh} \cosh 3Q \right) \quad (A-34)$$

At the next order, we put

$$\phi_{31} = \phi_{31}^1 e^{iS^+} + \phi_{31}^{-1} e^{iS^-} + \dots \quad (A-35)$$

and obtain the following equations for ϕ_{31}^1 :

$O(\varepsilon u^2)$

$$-k^2 \phi_{31}^1 + \phi_{31}^1 z z = -2ik \phi_{21}^1 x_1 - 2ik \phi_{11}^1 x_2 - \phi_{11}^1 x_1 x_1 \quad -h < z < 0 \quad (A-36)$$

$$-\omega^2 \phi_{31}^1 + g \phi_{31}^1 z = 2i\omega \phi_{11}^1 t_2 + 2i\omega \phi_{21}^1 t_1 - \phi_{11}^1 t_1 t_1 \quad z=0 \quad (A-37)$$

$$\phi_{31}^1 z = \frac{k^2 D}{2} \phi_{21}^{-1} - \frac{3k^2 D}{2} \phi_{21}^3 - \frac{ik}{2} \phi_{11}^{-1} D_{x_1} \quad z=-h \quad (A-38)$$

The solvability condition for Φ_{31}^1 gives

$$\begin{aligned}
 A_{t_2} + C_g A_{x_2} &= \frac{i}{2} \frac{\partial^2 \omega}{\partial k^2} A_{x_1 x_1} - \frac{2\Omega_0}{k} \sinh^2 kh B_{x_1} \\
 &+ \frac{i}{2\omega} (1 + 4\sinh^2 kh + 6\sinh 2kh \frac{\cosh 4kh + 2\cosh 2kh}{\sinh 4kh + 2\sinh 2kh}) \Omega_0^2 A \\
 &- \frac{gk}{4\omega} \frac{1 - \sinh^2 kh}{\cosh^2 kh} D_{x_1} B
 \end{aligned} \tag{A-39}$$

Finally we add Eqs. (A-26) and (A-39), and consider A and B as functions of x_1 and t_1 only, yielding

$$\begin{aligned}
 A_{t_1} + C_g A_{x_1} &= -i\Omega_0 B + \frac{i}{2} \frac{\partial^2 \omega}{\partial k^2} A_{x_1 x_1} - \frac{2\Omega_0}{k} \sinh^2 kh B_{x_1} \\
 &+ \frac{i}{2\omega} (1 + 4\sinh^2 kh + 6\sinh 2kh \frac{\cosh 4kh + 2\cosh 2kh}{\sinh 4kh + 2\sinh 2kh}) \Omega_0^2 A \\
 &- \frac{gk}{4\omega} \frac{1 - \sinh^2 kh}{\cosh^2 kh} D_{x_1} B
 \end{aligned} \tag{A-40}$$

Similarly we obtain

$$\begin{aligned}
 B_{t_1} - C_g B_{x_1} &= -i\Omega_0 A + \frac{i}{2} \frac{\partial^2 \omega}{\partial k^2} B_{x_1 x_1} - \frac{2\Omega_0}{k} \sinh^2 kh A_{x_1} \\
 &+ \frac{i}{2\omega} (1 + 4\sinh^2 kh + 6\sinh 2kh \frac{\cosh 4kh + 2\cosh 2kh}{\sinh 4kh + 2\sinh 2kh}) \Omega_0^2 B \\
 &- \frac{gk}{4\omega} \frac{-1 - \sinh^2 kh}{\cosh^2 kh} D_{x_1} A
 \end{aligned} \tag{A-41}$$

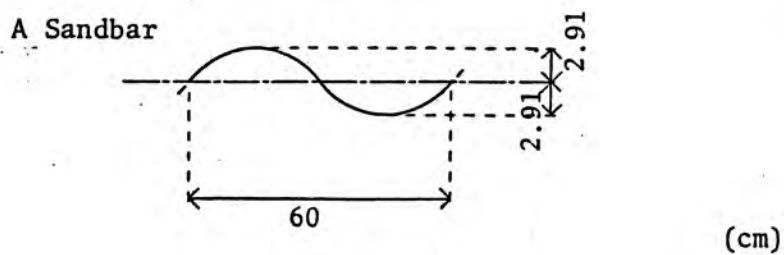
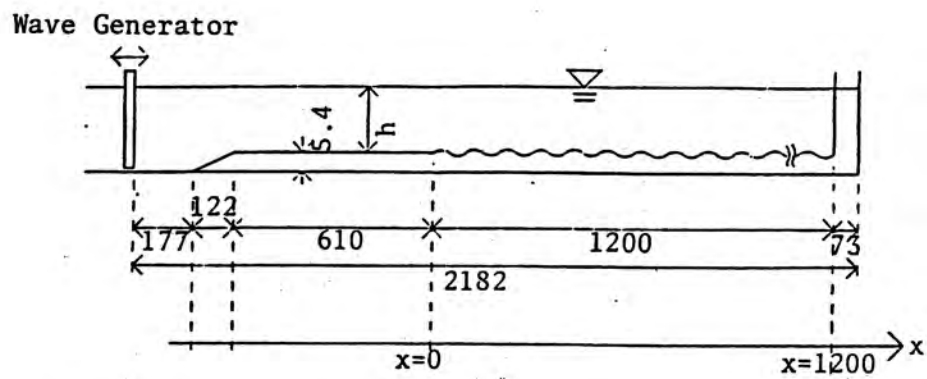


Fig. 3-1 Experimental Set-Up

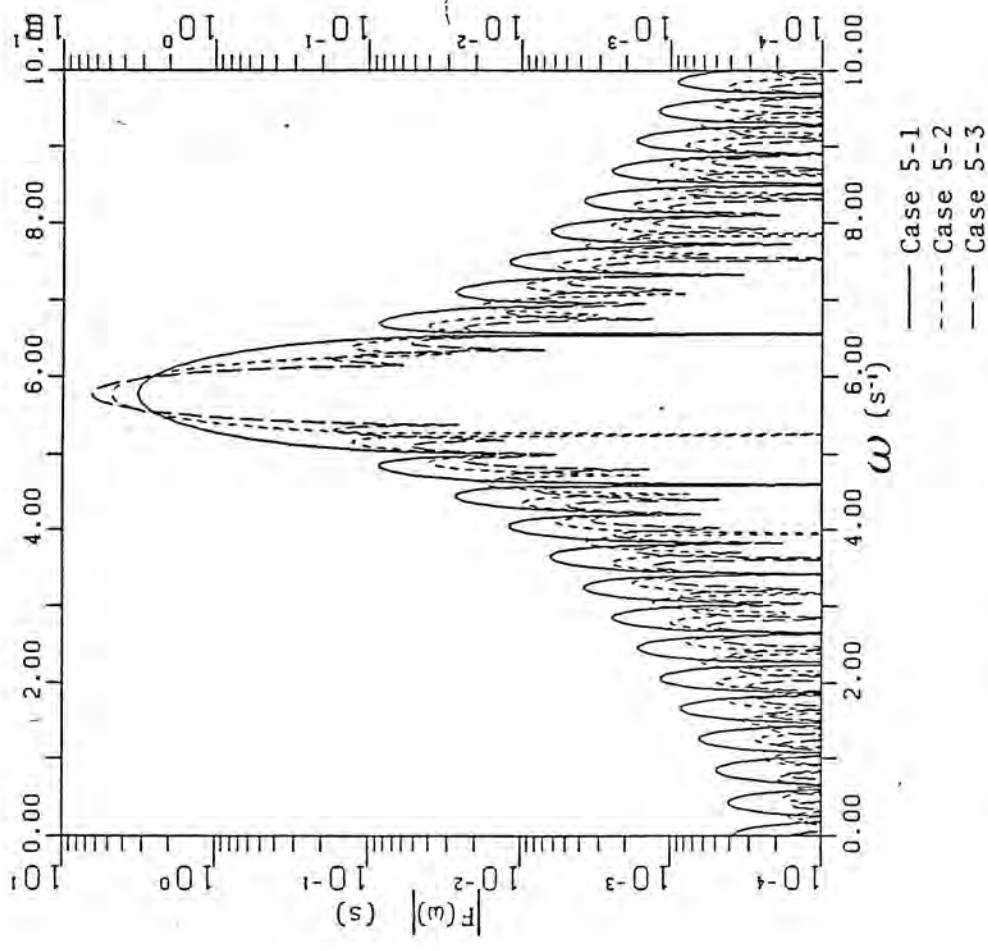


Fig. 3-2 (a) Fourier Transform of Input Signal

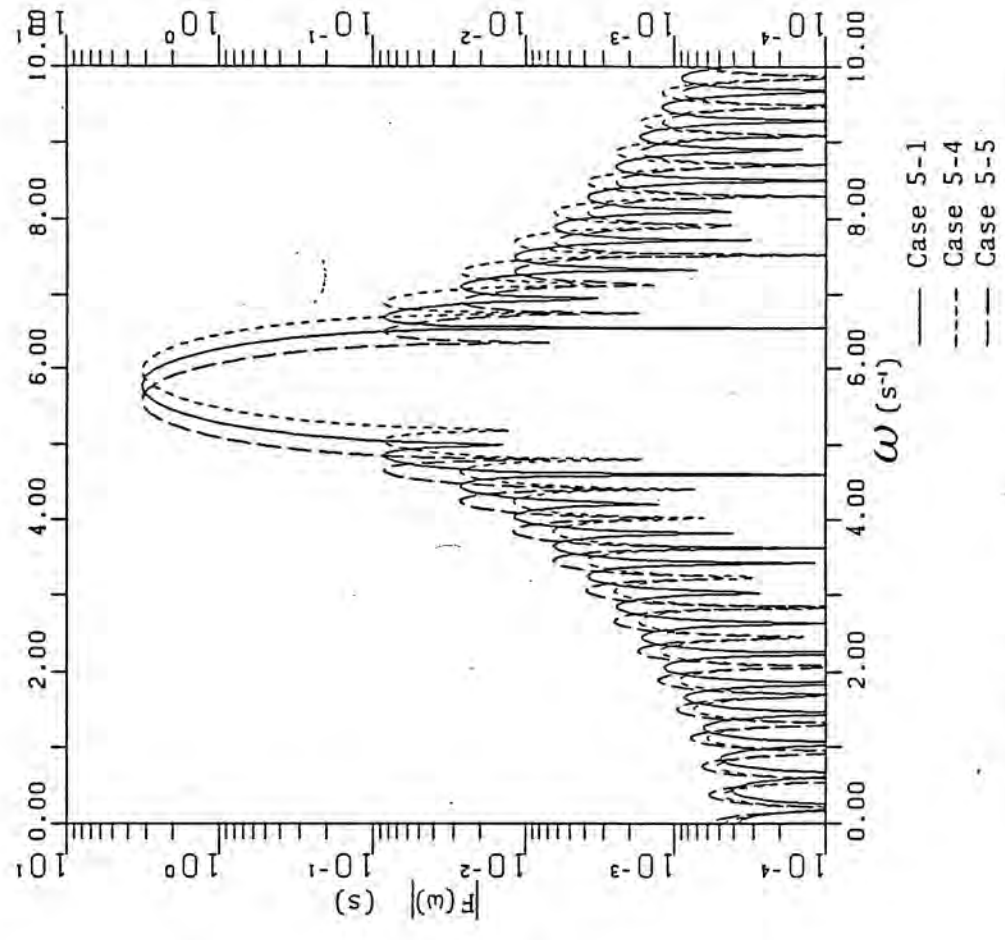


Fig. 3-2(b) Fourier Transform of Input Signal

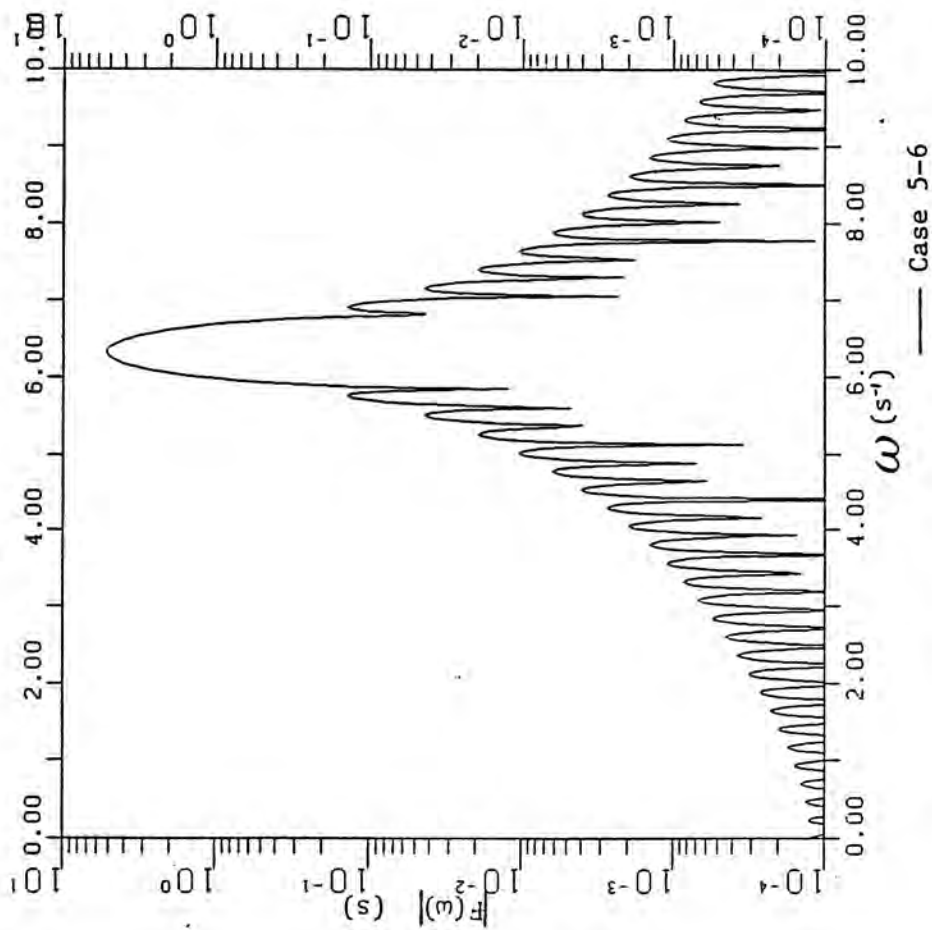


Fig. 3-2(c) Fourier Transform of Input Signal

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} f(t) dt$$

$$f(t) = \sin^2(\Omega_0 t) \sin(\omega + \Omega_0) t$$

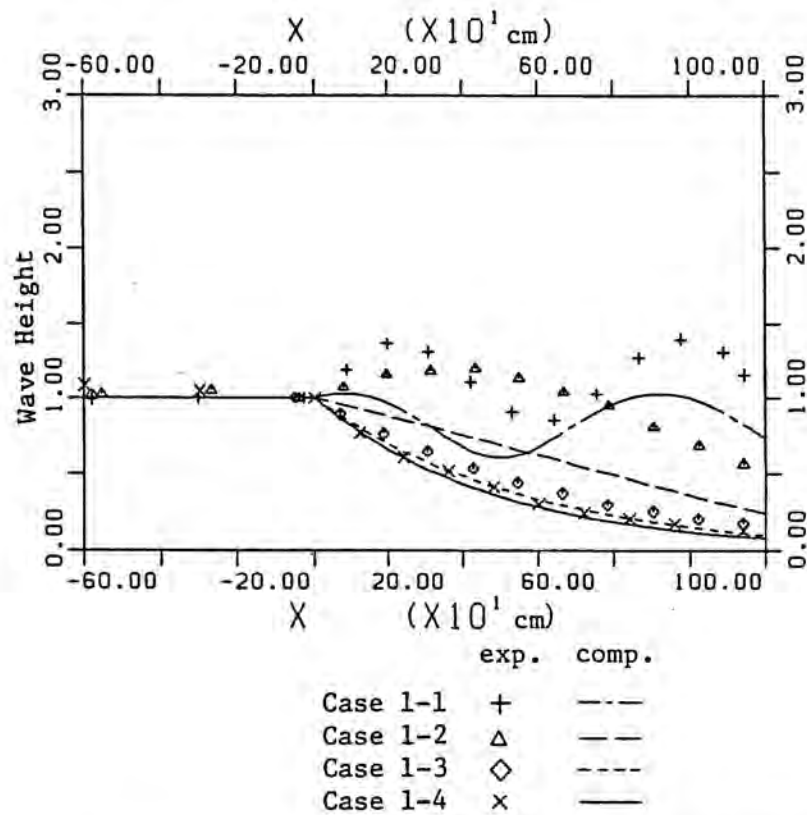


Fig. 3-3(a) Wave Height Distribution at Antinodes

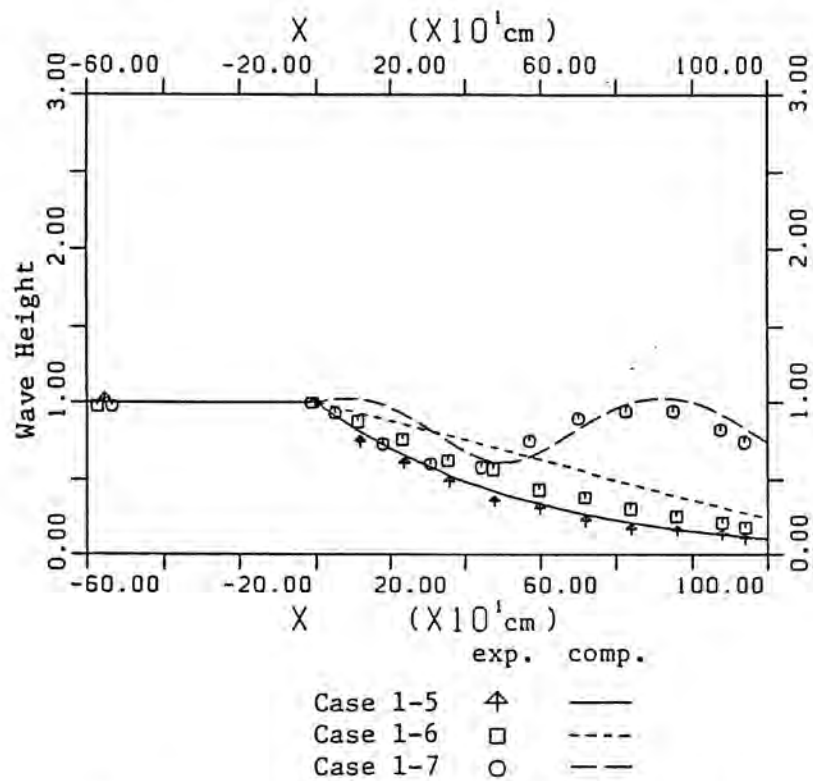


Fig. 3-3(b) Wave Height Distribution at Antinodes

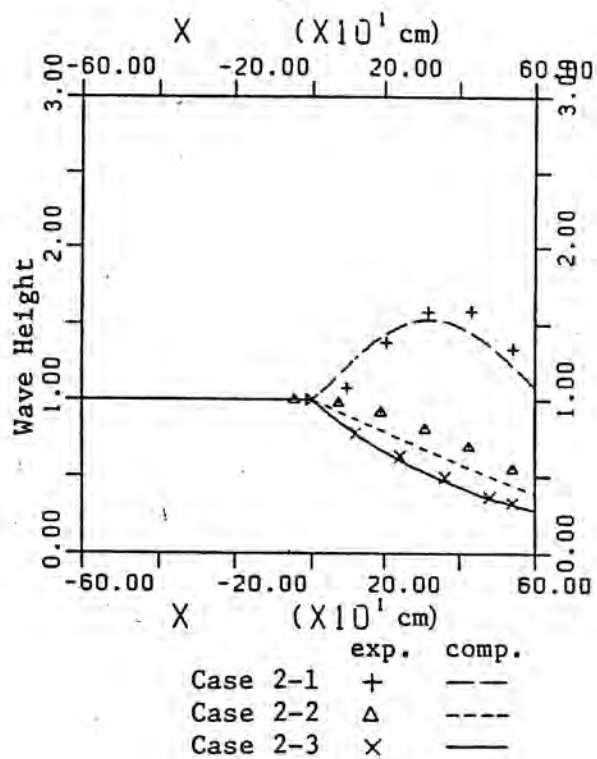


Fig. 3-3(c) Wave Height Distribution at Antinodes

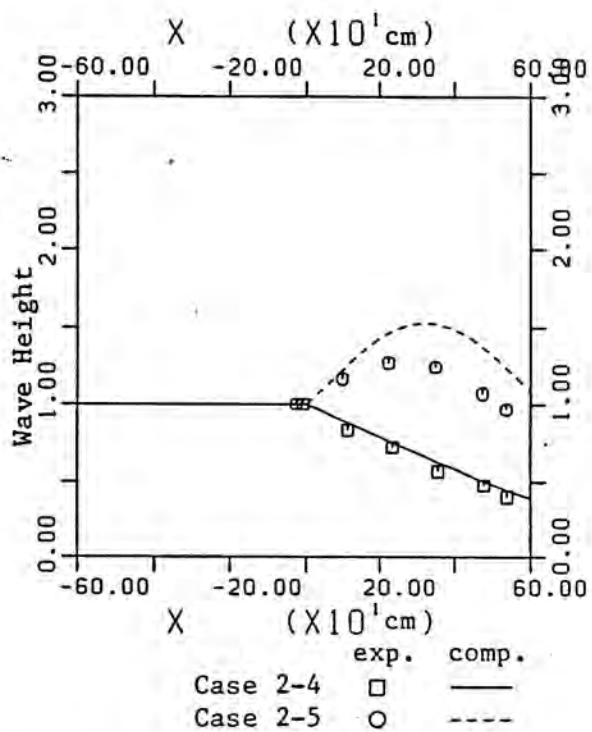


Fig. 3-3(d) Wave Height Distribution at Antinodes

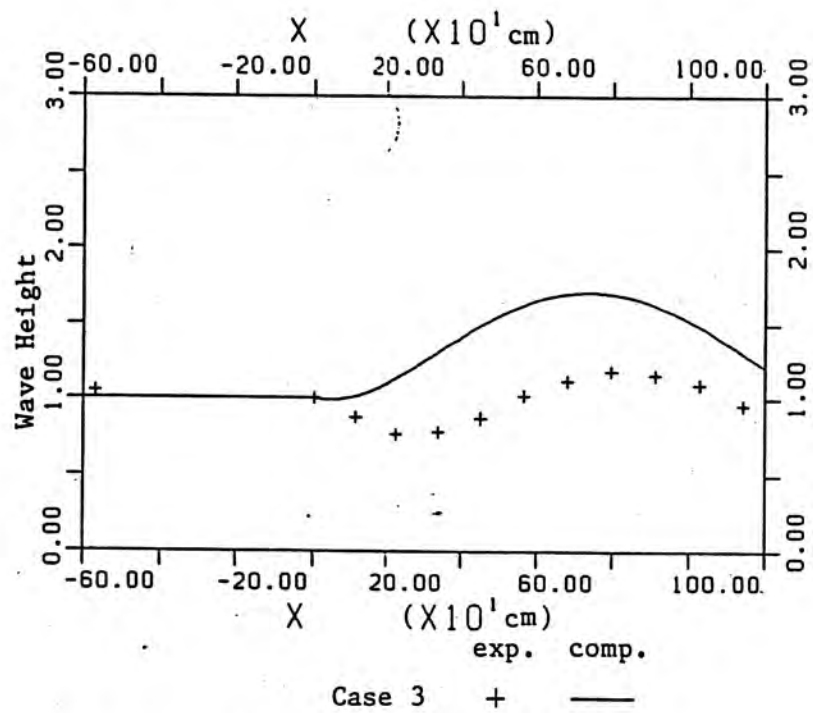


Fig. 3-3(e) Wave Height Distribution
at Antinodes

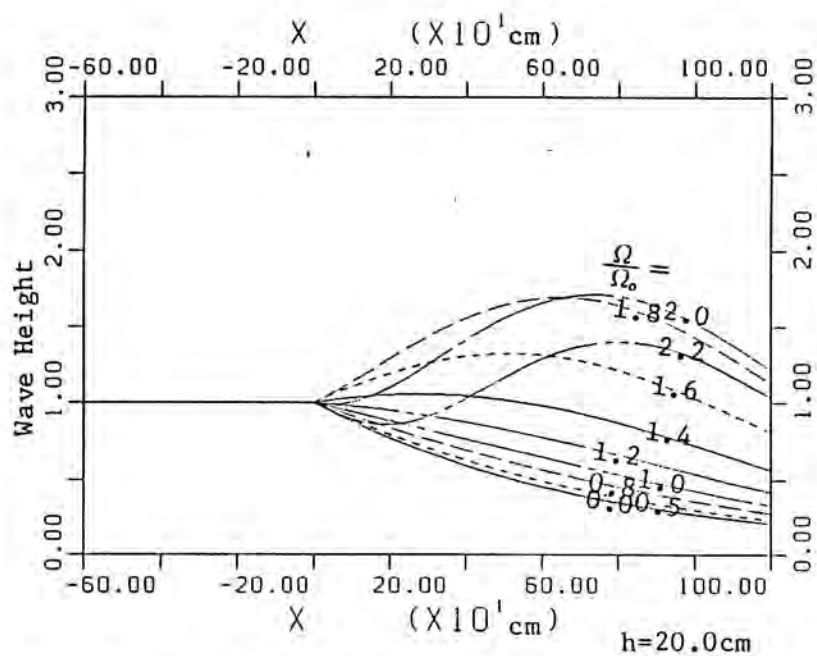
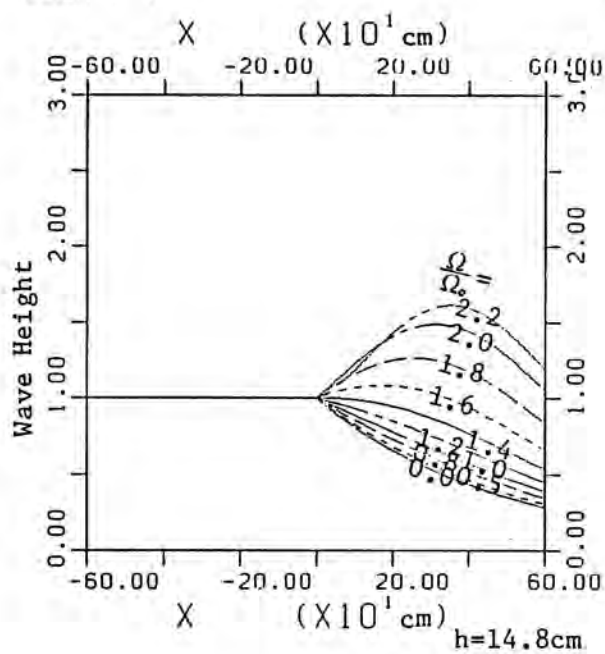
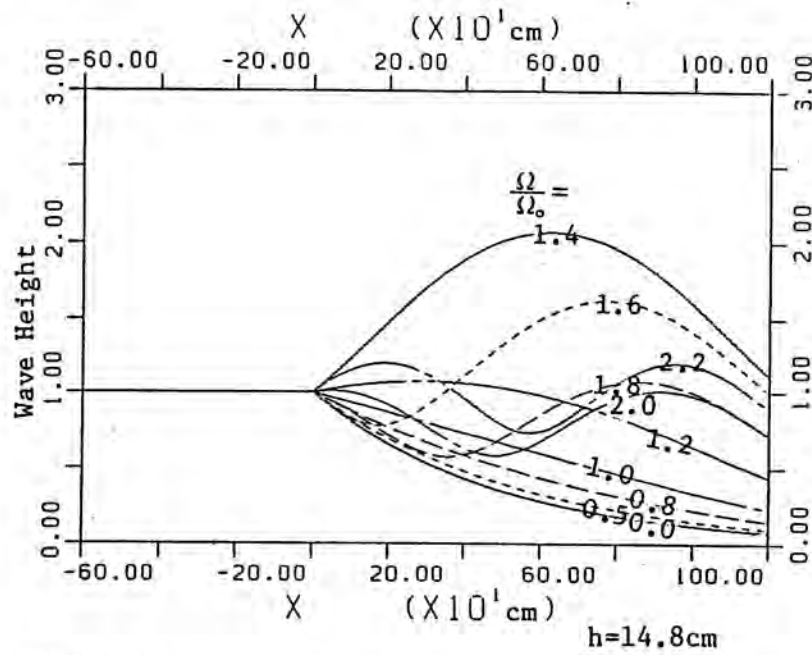


Fig. 3-4 Calculated Wave Height Distribution at Antinodes

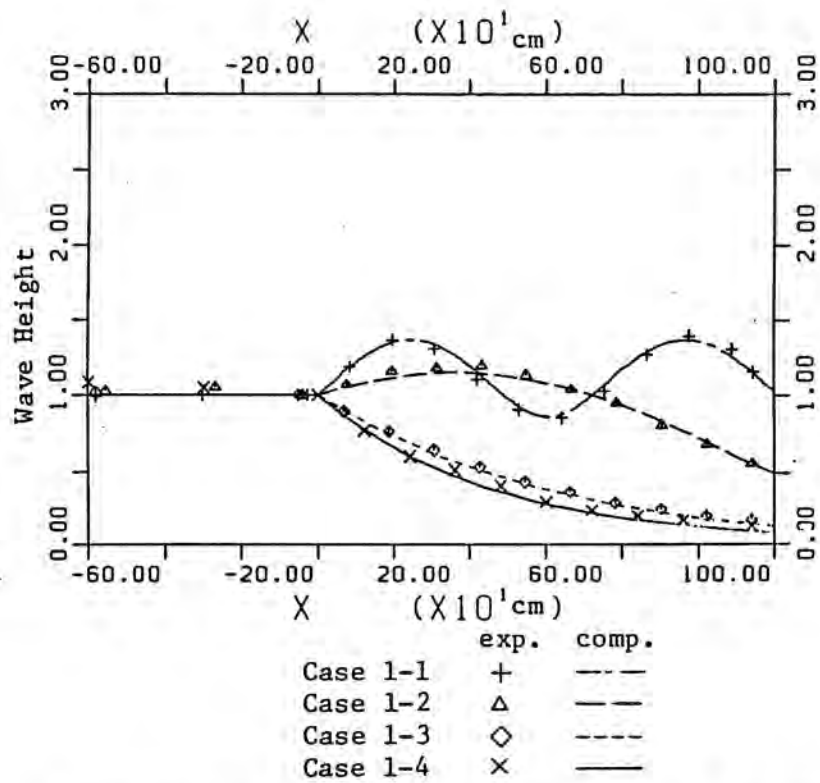


Fig. 3-5(a) Wave Height Distribution at Antinodes
(corrected $\frac{\Omega}{\Omega_0}$)

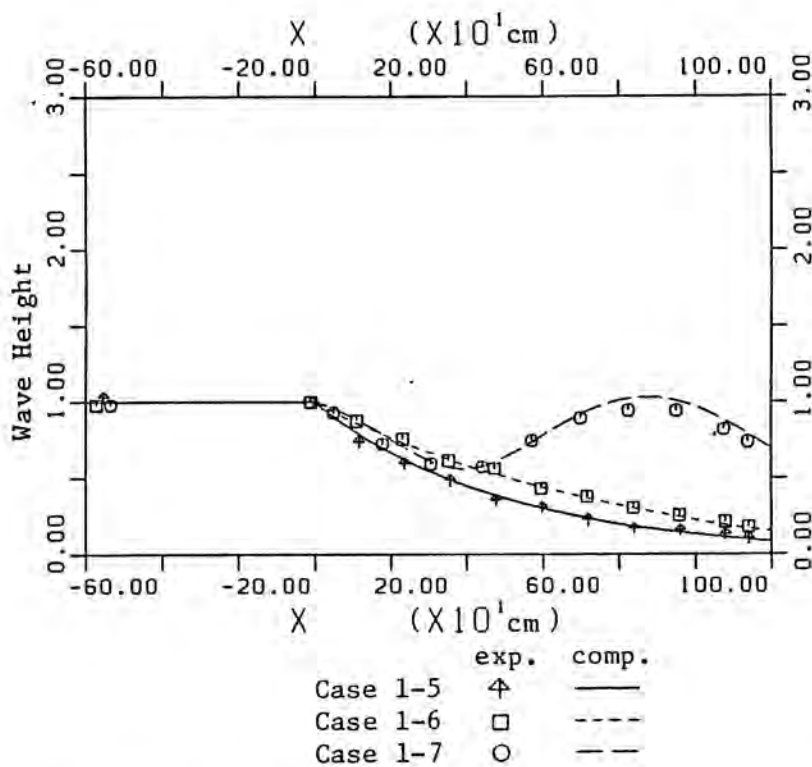


Fig. 3-5(b) Wave Height Distribution at Antinodes
(corrected $\frac{\Omega}{\Omega_0}$)

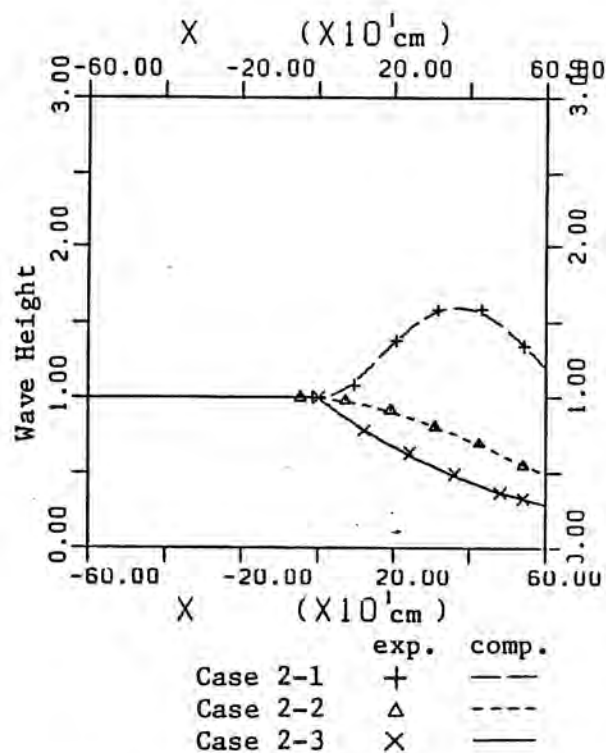


Fig. 3-5(c) Wave Height Distribution
at Antinodes (corrected $\frac{\Omega}{\Omega_0}$)

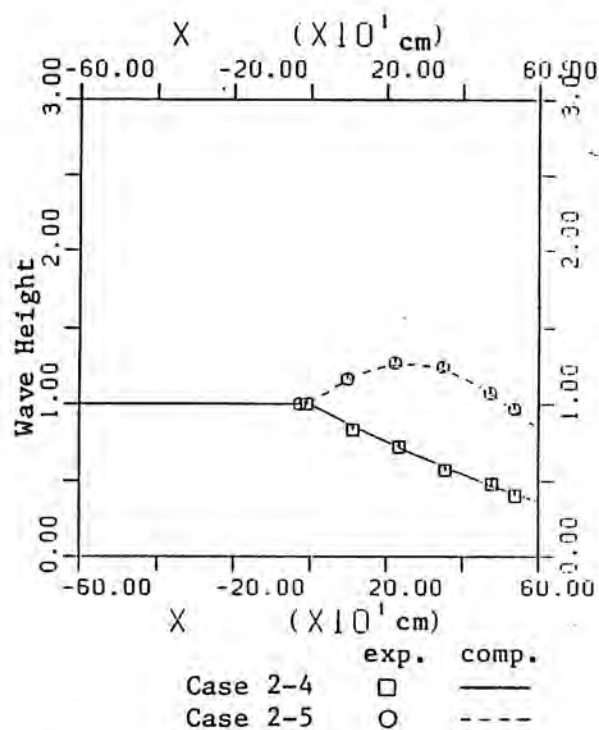


Fig. 3-5(d) Wave Height Distribution
at Antinodes (corrected $\frac{\Omega}{\Omega_0}$)

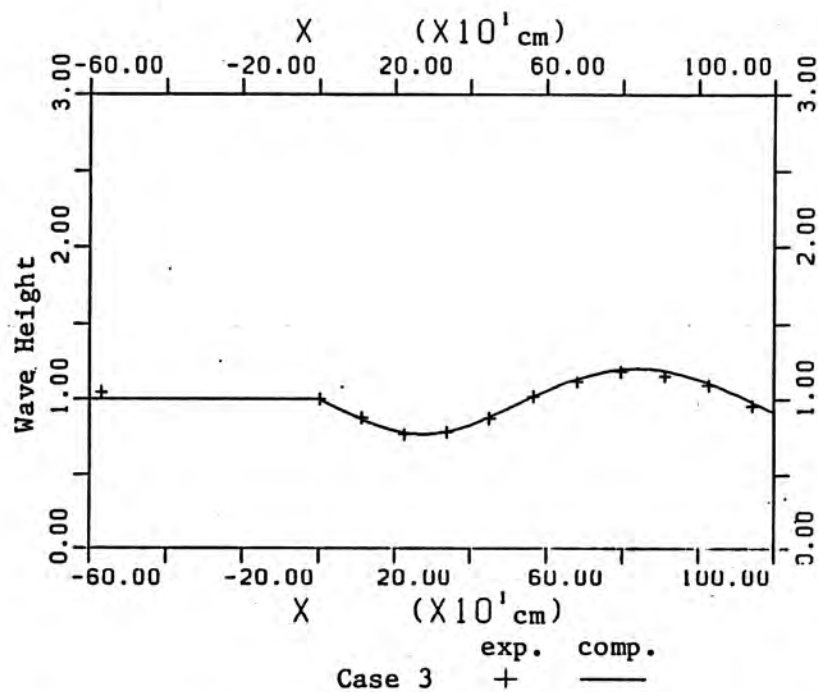


Fig. 3-5(e) Wave Height Distribution at Antinodes
(corrected $\frac{\Omega}{\Omega_0}$)

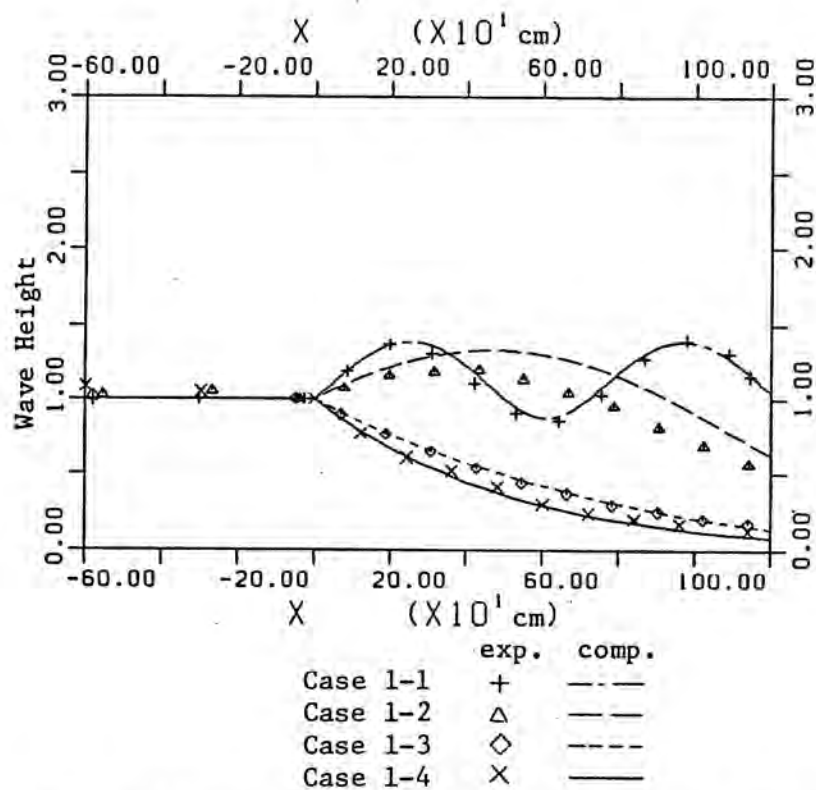


Fig. 3-6(a) Wave Height Distribution at Antinodes
(corrected h and D)

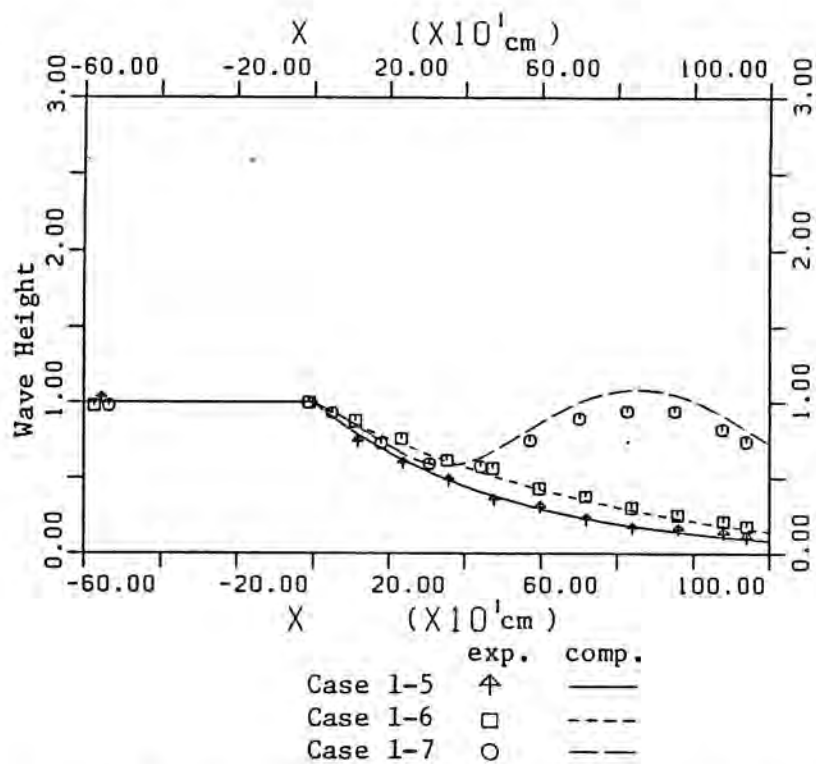


Fig. 3-6(b) Wave Height Distribution at Antinodes
(corrected h and D)

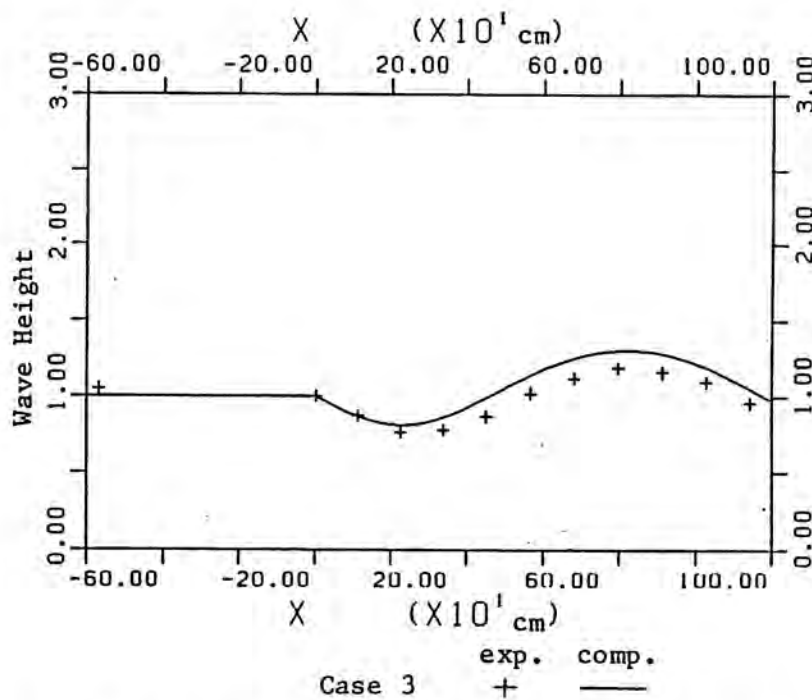


Fig. 3-6(c) Wave Height Distribution at Antinodes
(corrected h and D)

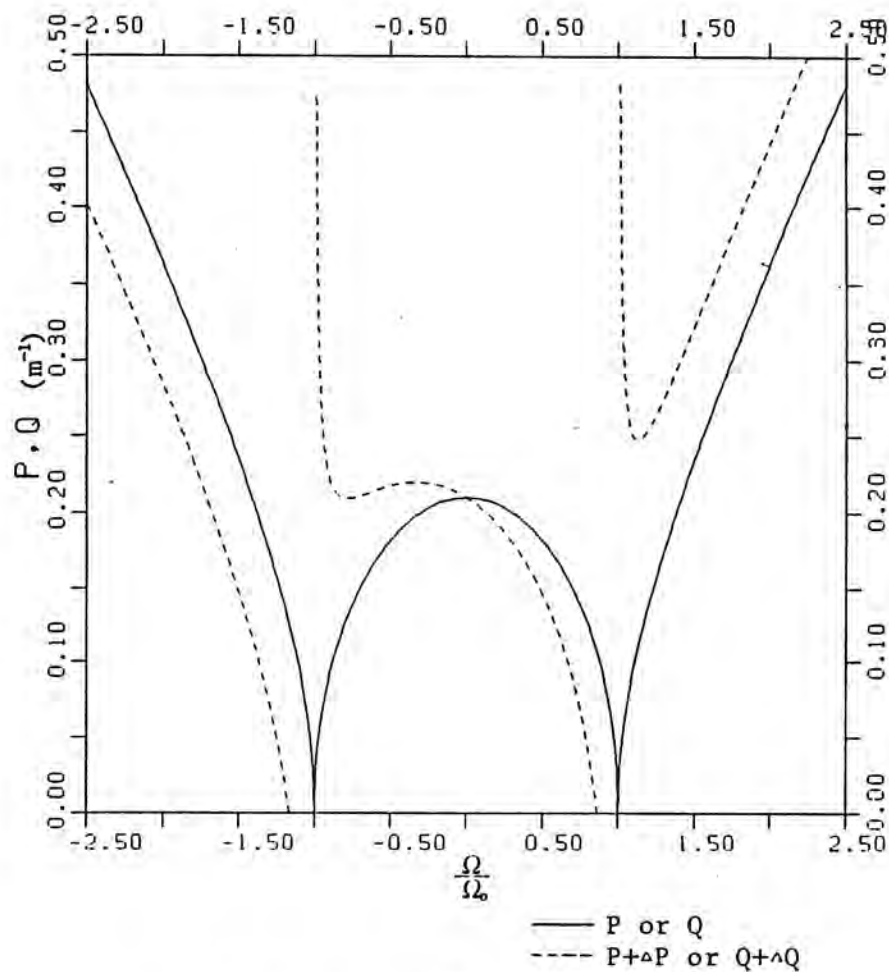


Fig. 3-7 P, Q versus $\frac{\Omega}{\Omega_0}$

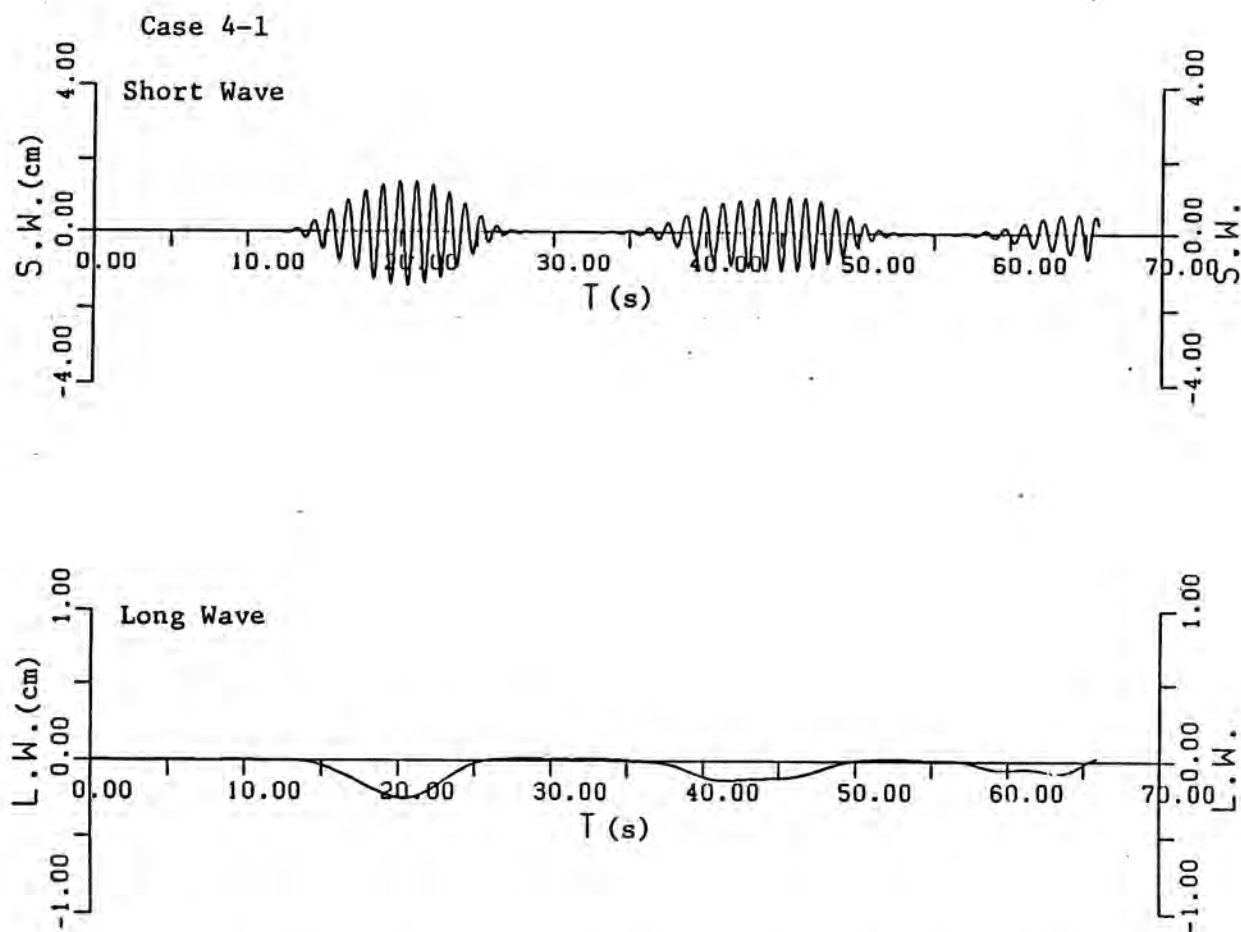


Fig. 3-8(a) Incident and Reflected Waves (without sandbars)

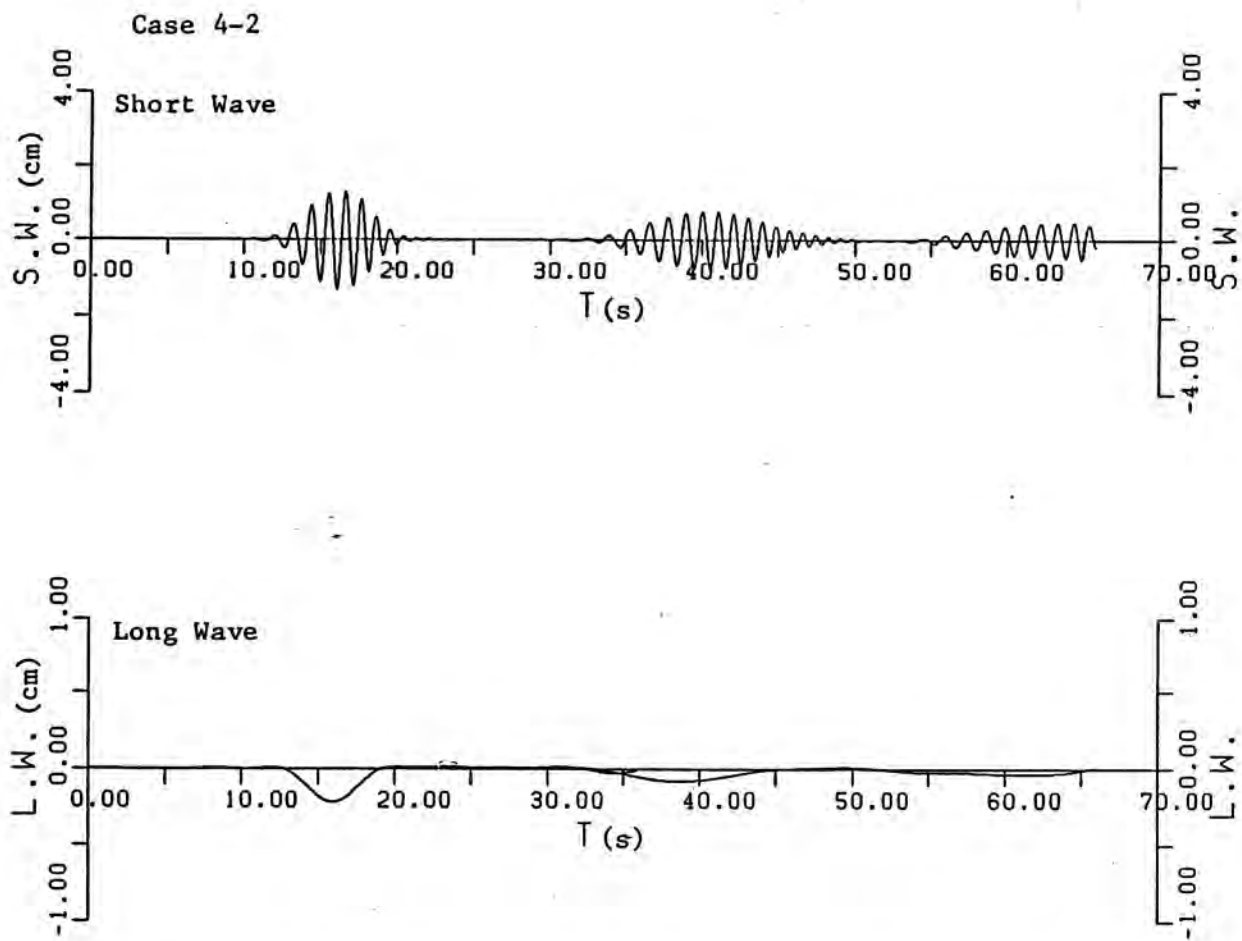


Fig. 3-8(b) Incident and Reflected Waves (without sandbars)

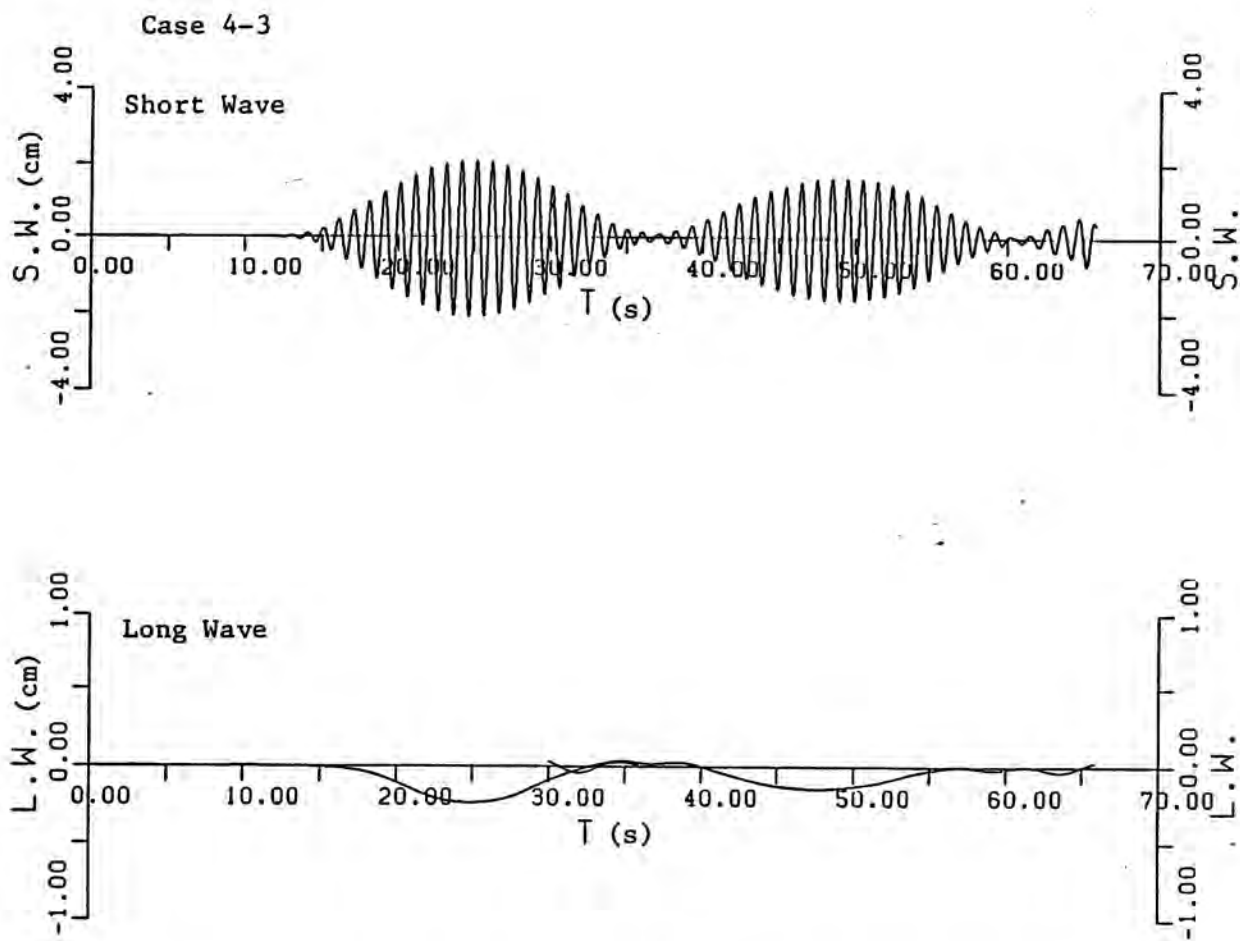


Fig. 3-8(c) Incident and Reflected Waves (without sandbars)

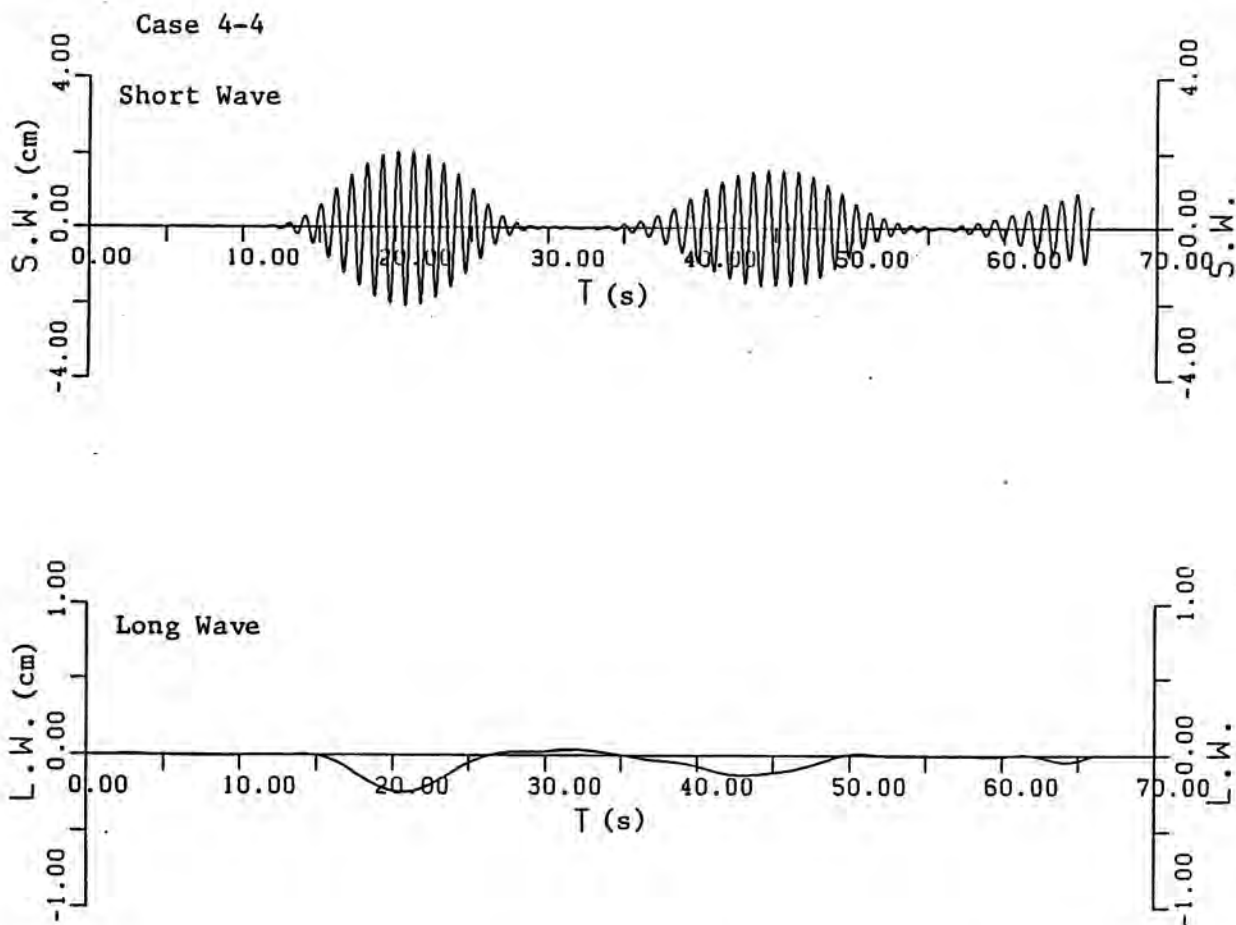


Fig. 3-8(d) Incident and Reflected Waves (without sandbars)

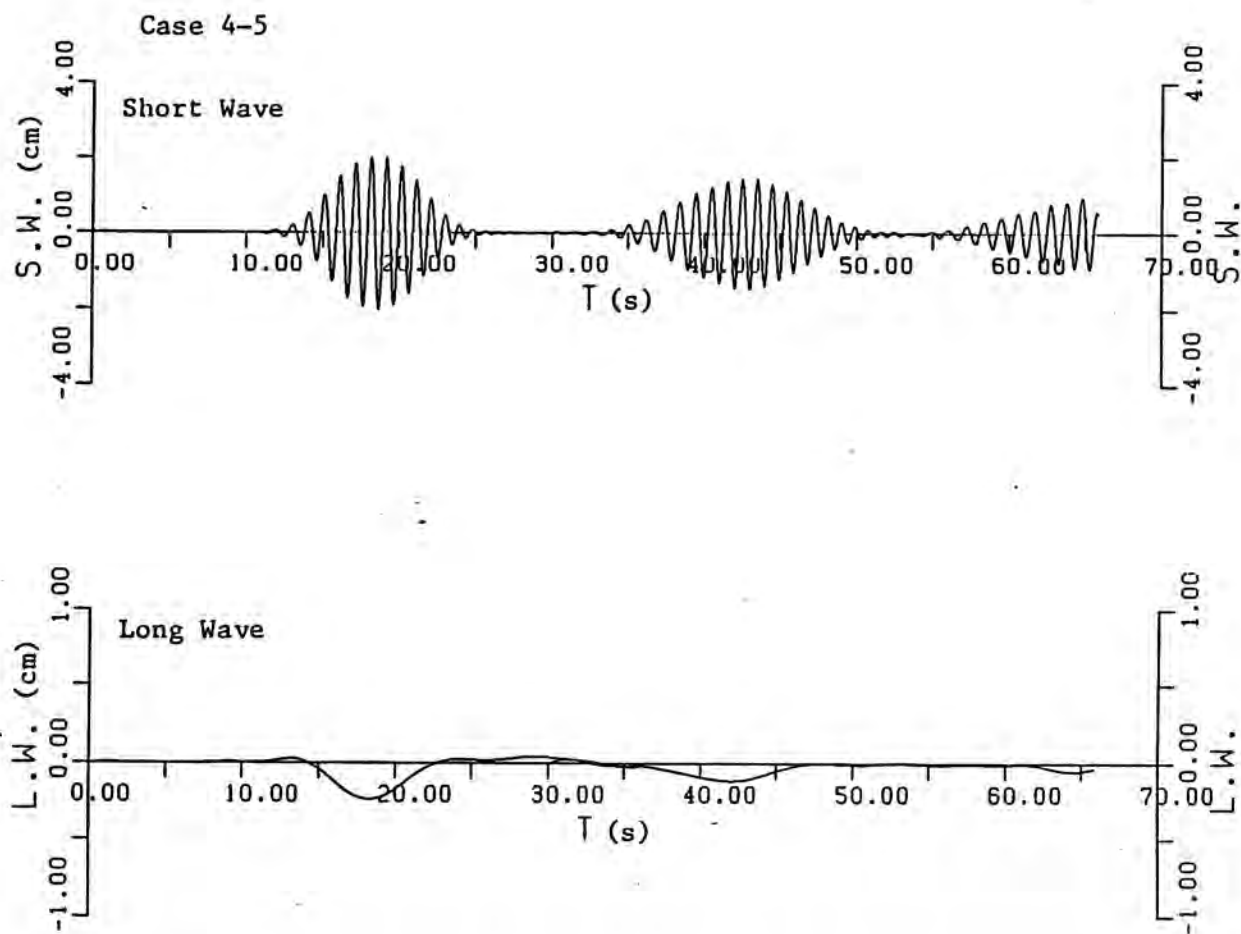


Fig. 3-8(e) Incident and Reflected Waves (without sandbars)

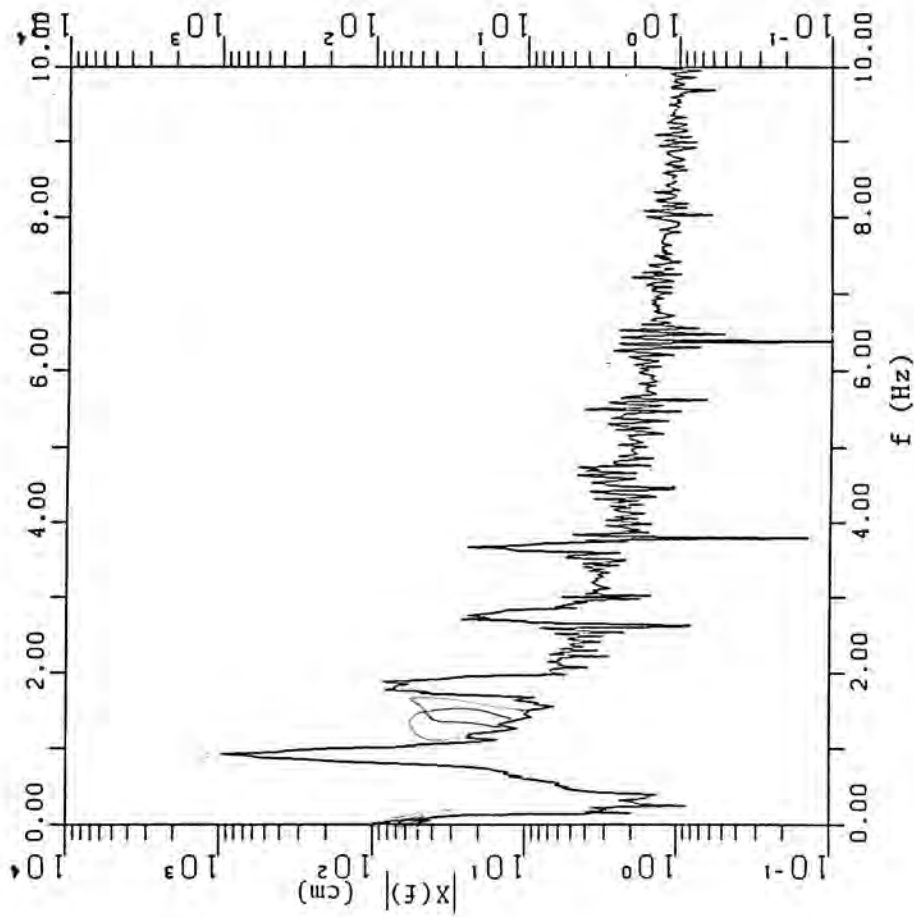


Fig. 3-9(a) Fourier Transform of Wave Record at $x=-600$
(Case 5-1)

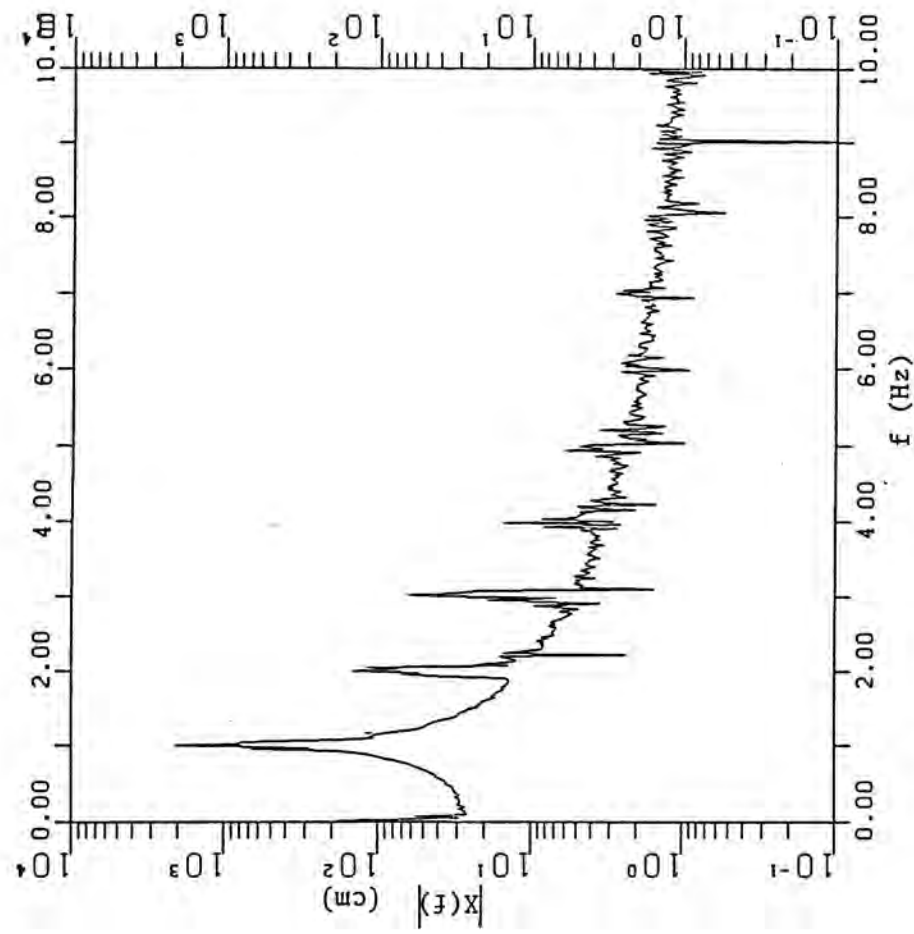


Fig. 3-9(b) Fourier Transform of Wave Record at $x=-600$
(Case 5-6)

$$X(f) = \sum_{m=0}^{4095} \zeta(\tau) e^{-i(2\pi f)(m \Delta t)}$$

$$\Delta t = 0.01(s)$$

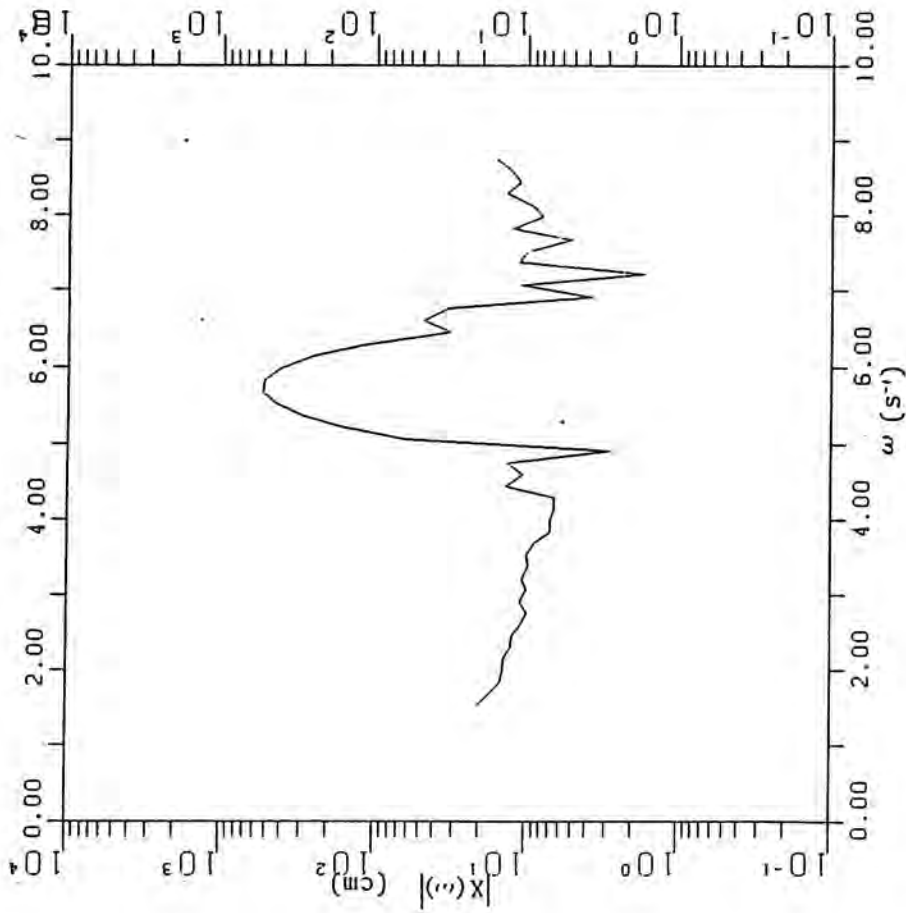


Fig. 3-10(a) Fourier Transform of First Order Incident Wave Record at $x=0$ (Case 5-1)

$$X(\omega) = \sum_{m=0}^{4095} f(t) e^{-i\omega(mt)}$$

$$\Delta t = 0.01 \text{ (s)}$$

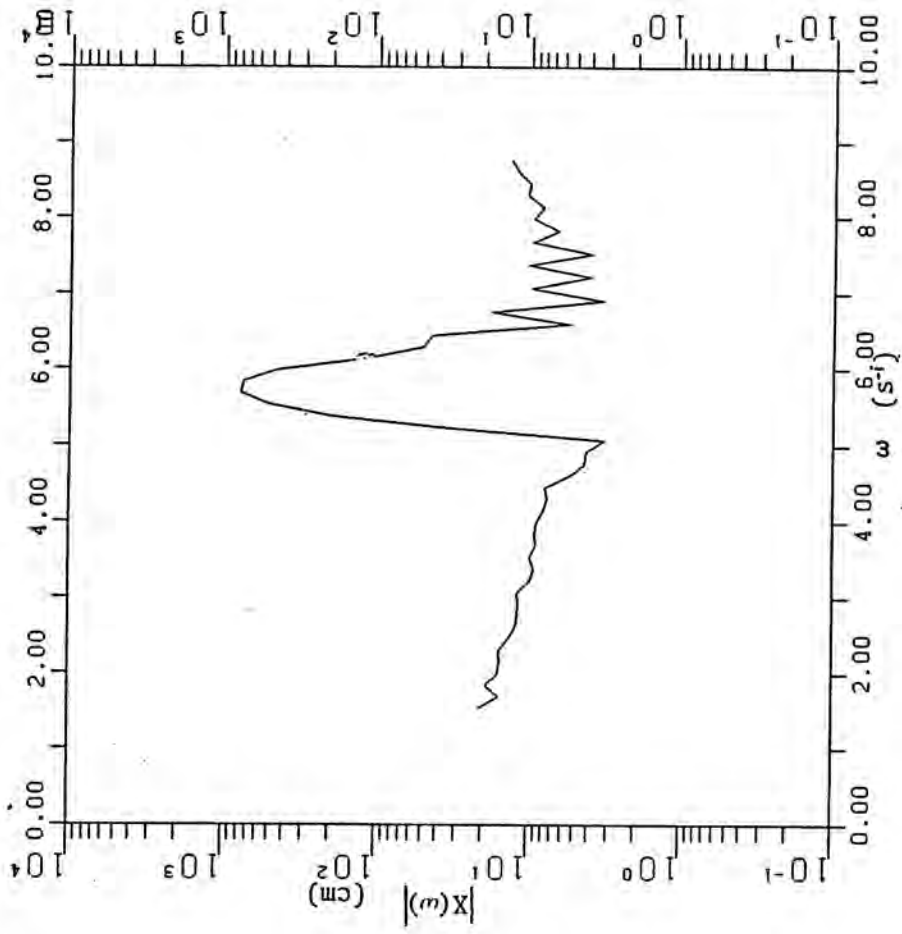


Fig. 3-10(b) Fourier Transform of First Order Incident Wave Record at $x=0$ (Case 5-2)

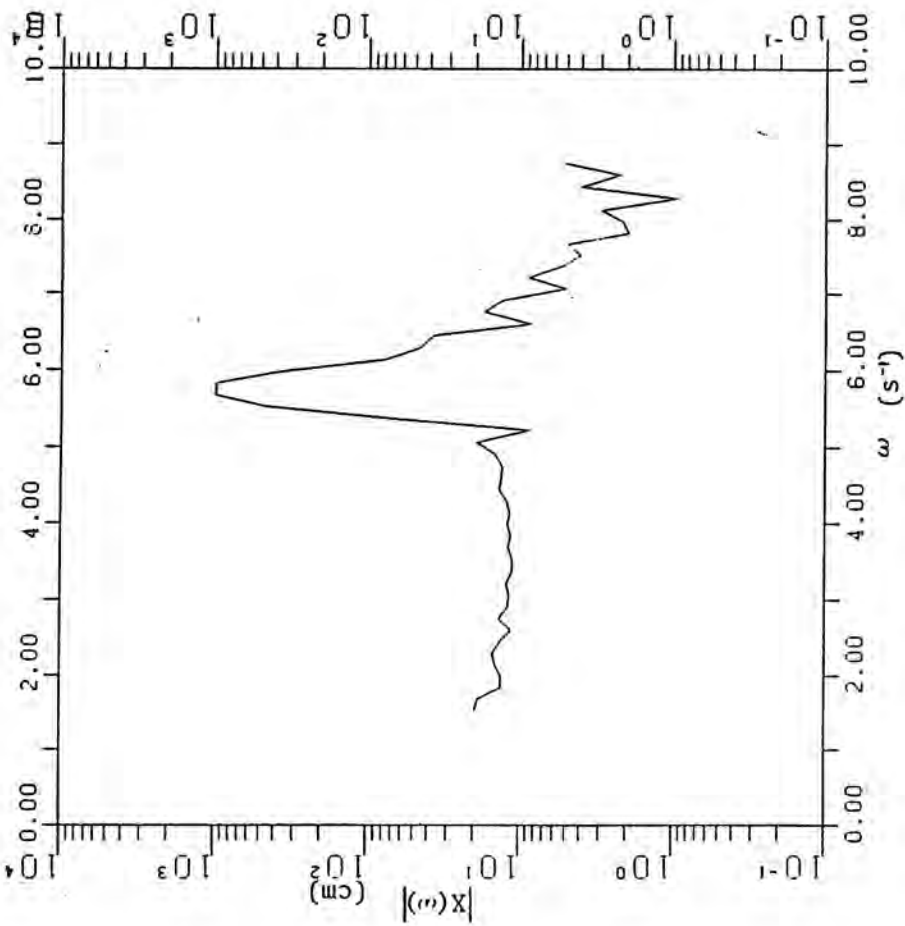


Fig. 3-10(c) Fourier Transform of First Order Incident Wave Record at $x=0$ (Case 5-3)

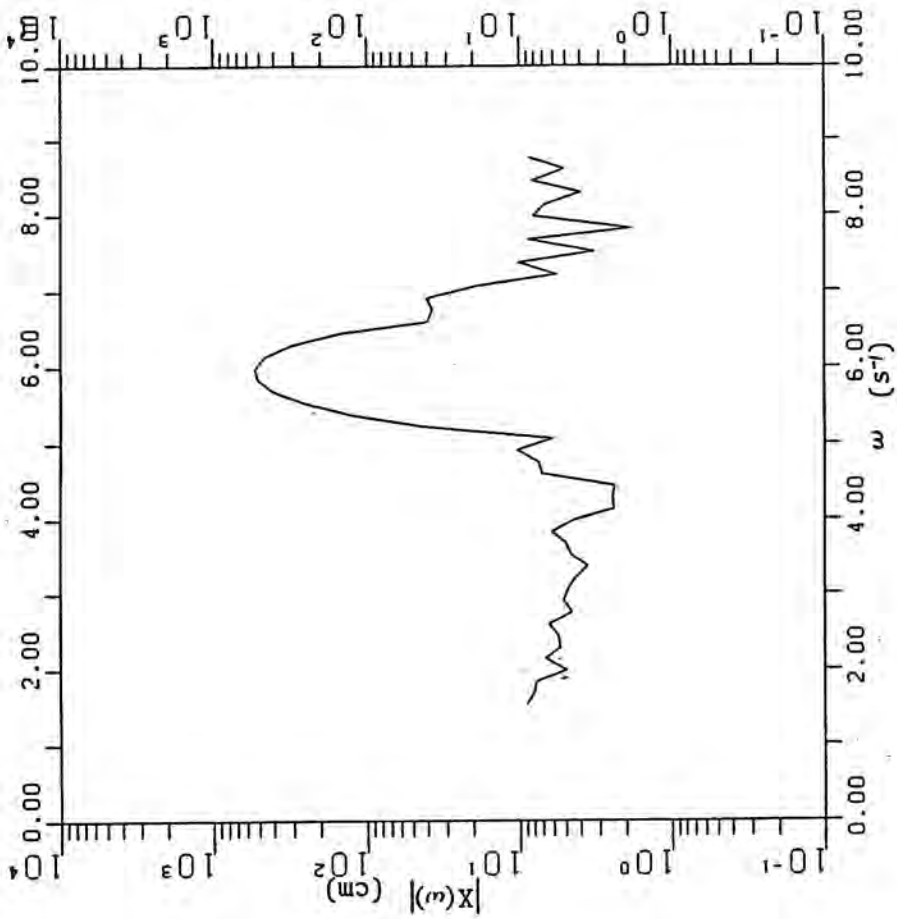


Fig. 3-10(d) Fourier Transform of First Order Incident Wave Record at $x=0$ (Case 5-4)

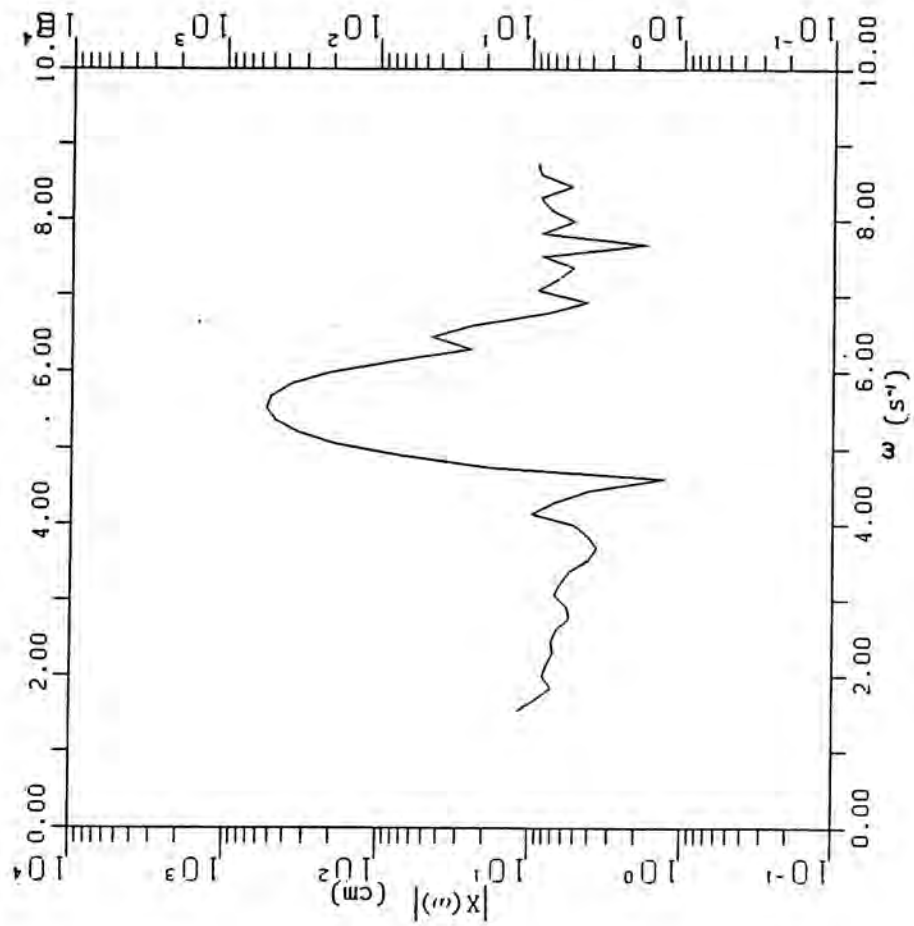


Fig. 3-10(e) Fourier Transform of First Order Incident Wave Record at $x=0$ (Case 5-5)

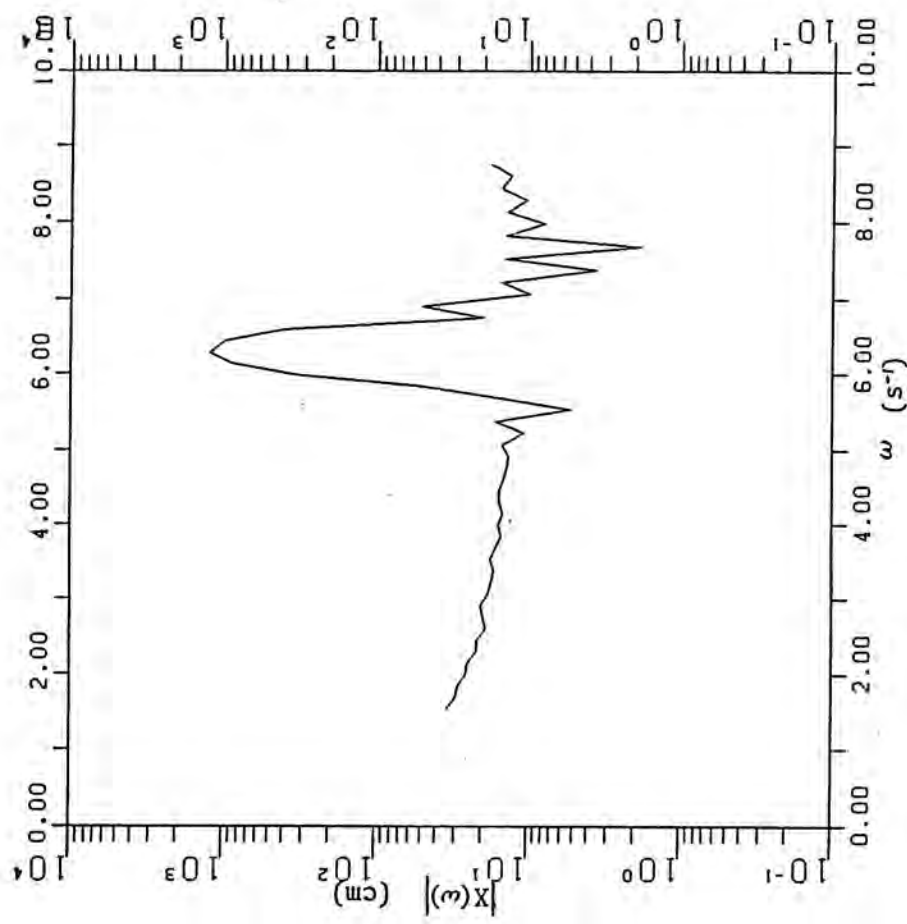


Fig. 3-10(f) Fourier Transform of First Order Incident Wave Record at $x=0$ (Case 5-6)

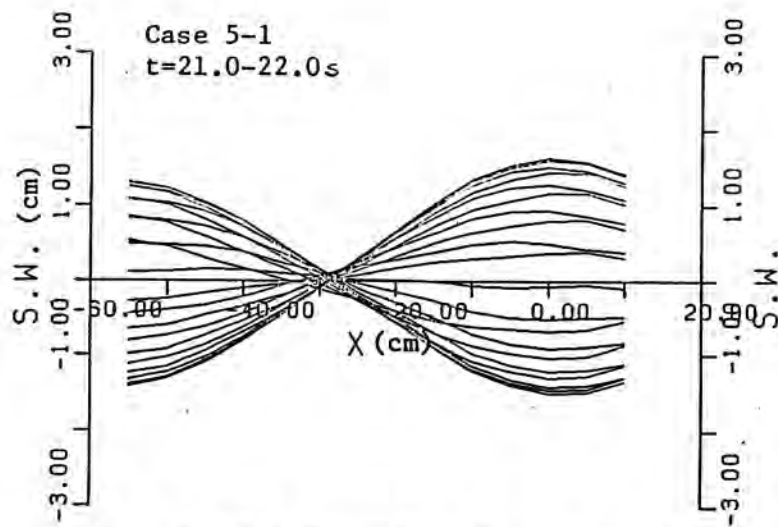


Fig. 3-11(a) Short Wave Profile during One Wave Period

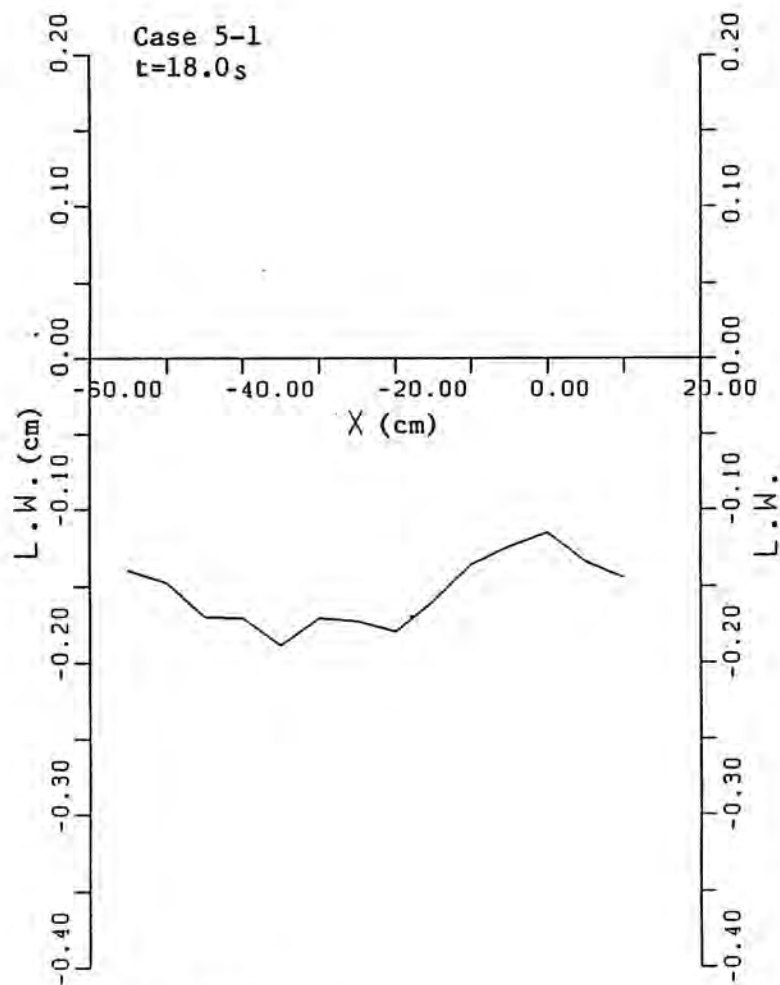


Fig. 3-12(a)
Measured ζ_{20} versus x

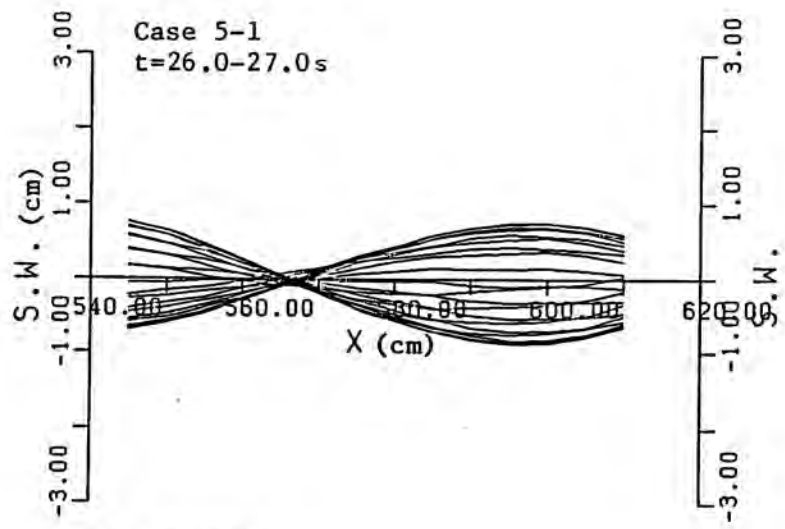


Fig. 3-11(b)

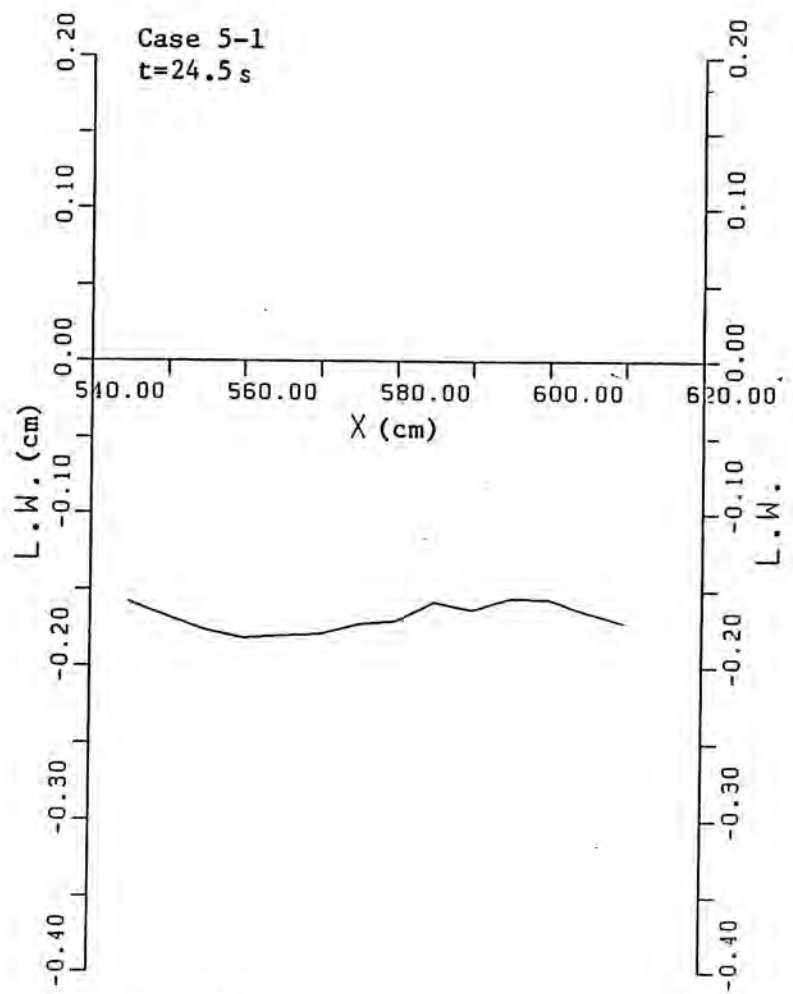


Fig. 3-12(b)

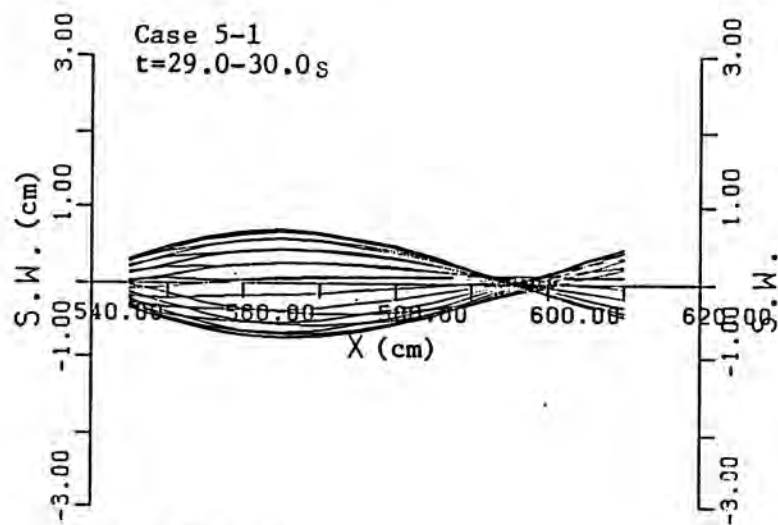


Fig. 3-11(c)

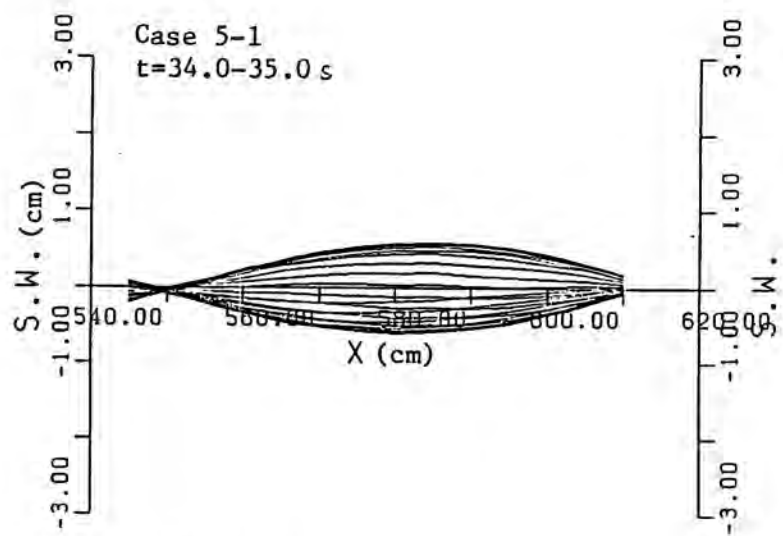


Fig. 3-11(d)

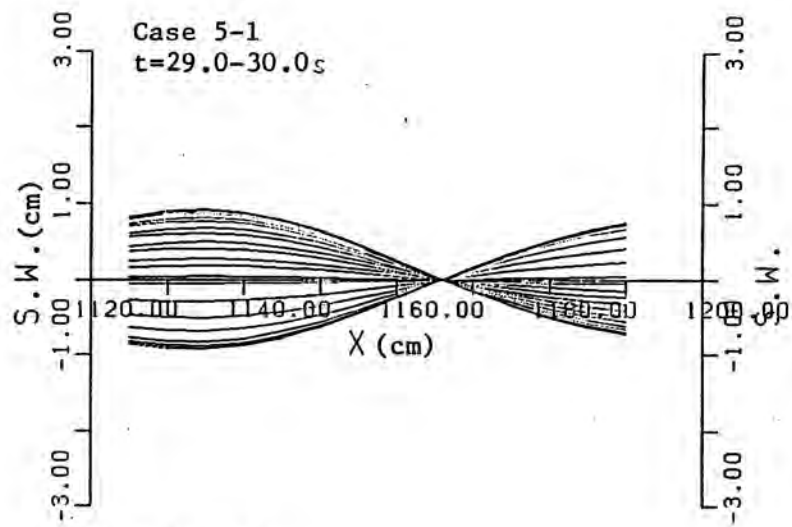


Fig. 3-11(e)

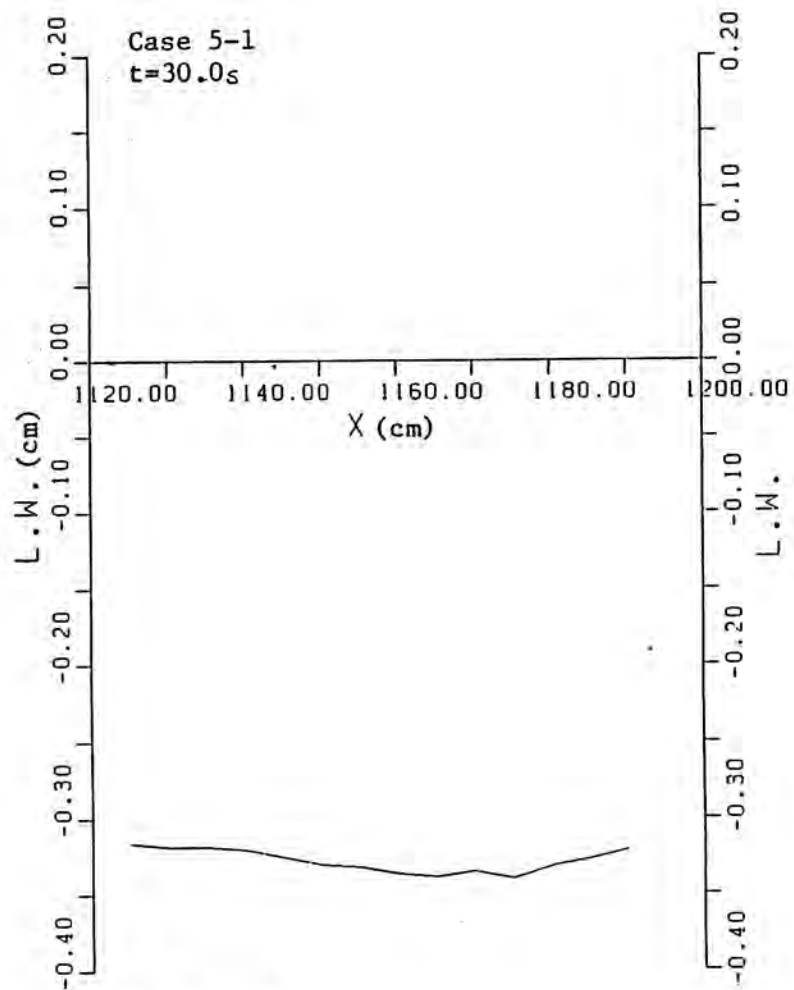


Fig. 3-12(e)

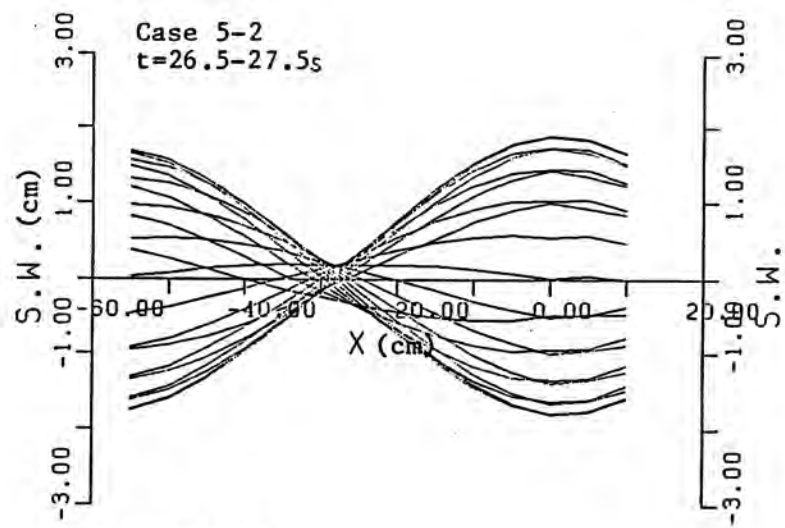


Fig. 3-11(f)

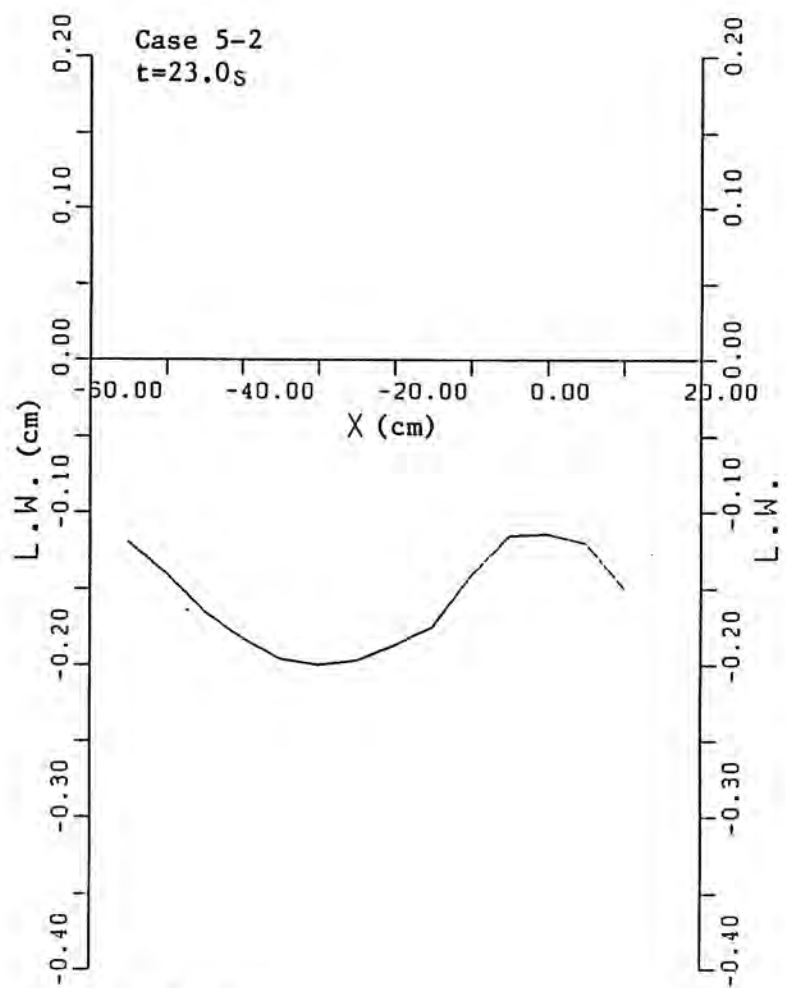


Fig. 3-12(f)

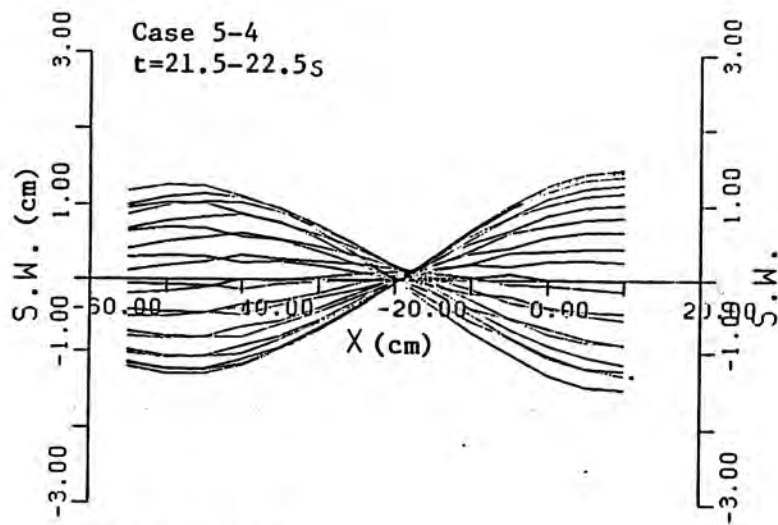


Fig. 3-11(g)

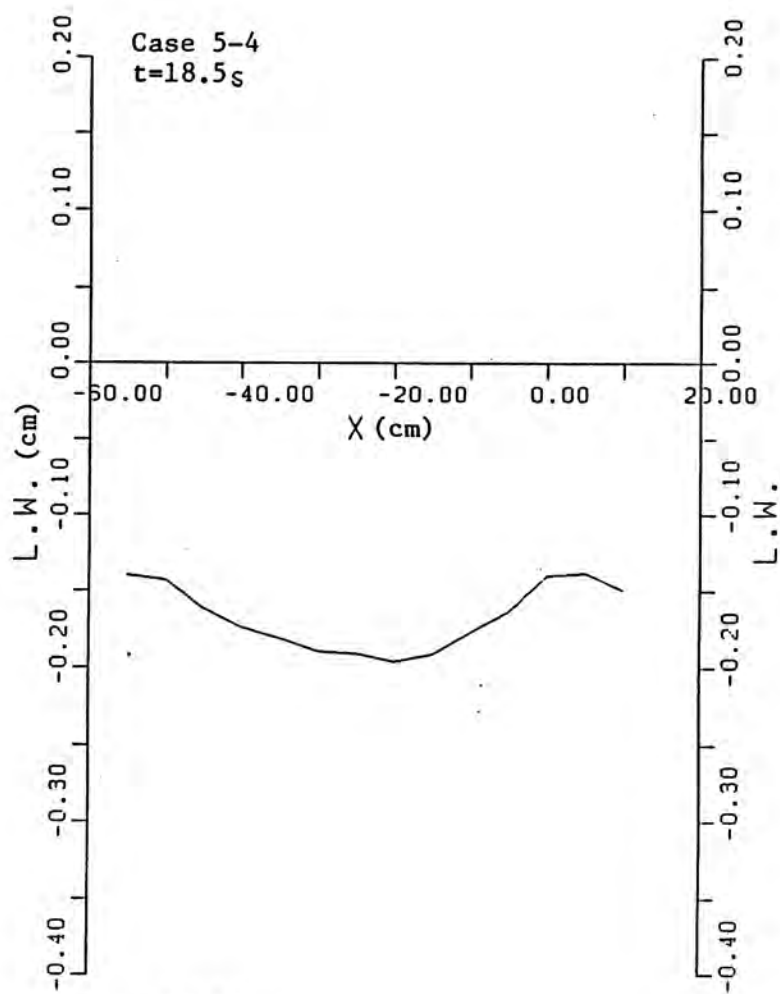


Fig. 3-12(g)

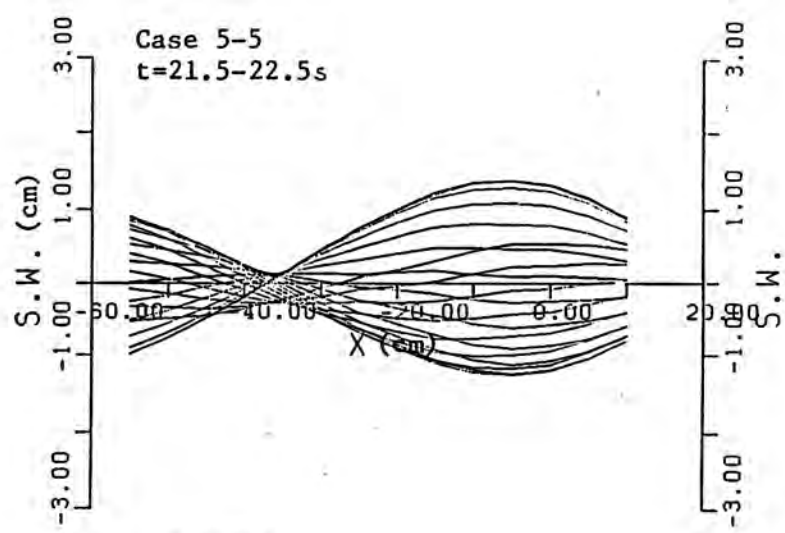


Fig. 3-11(h)

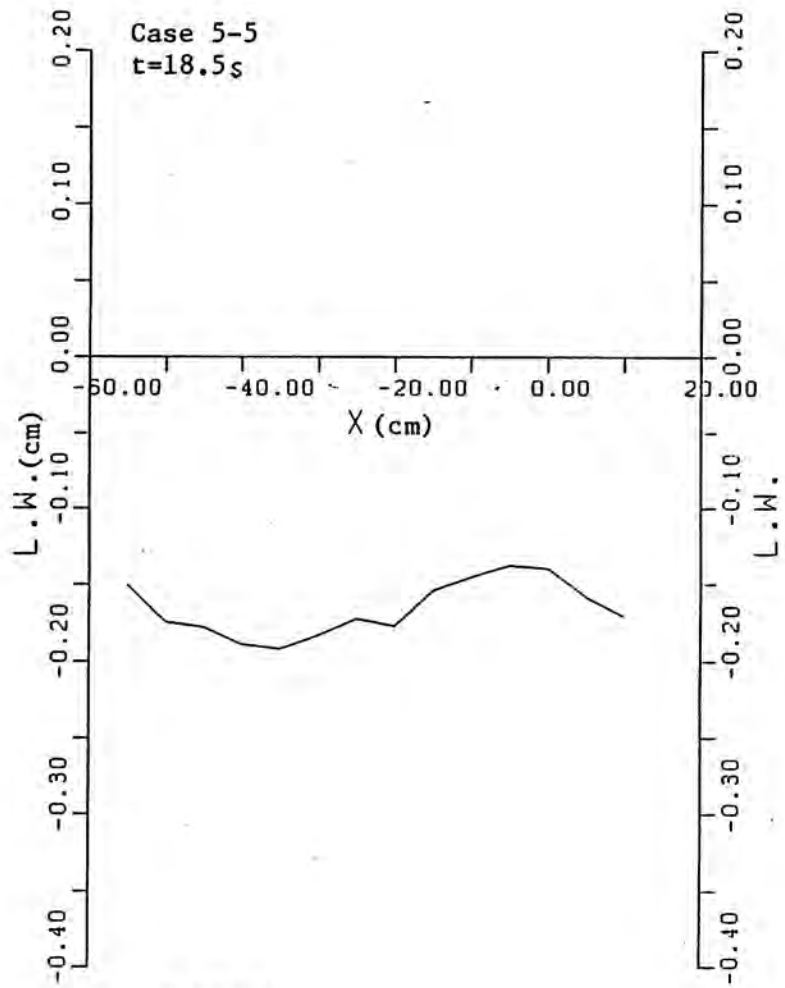


Fig. 3-12(h)

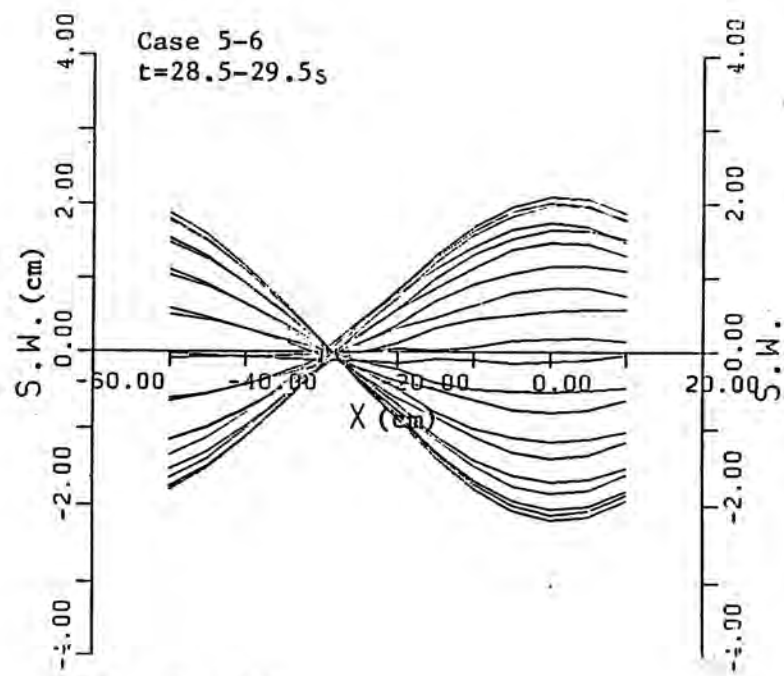


Fig. 3-11(i)

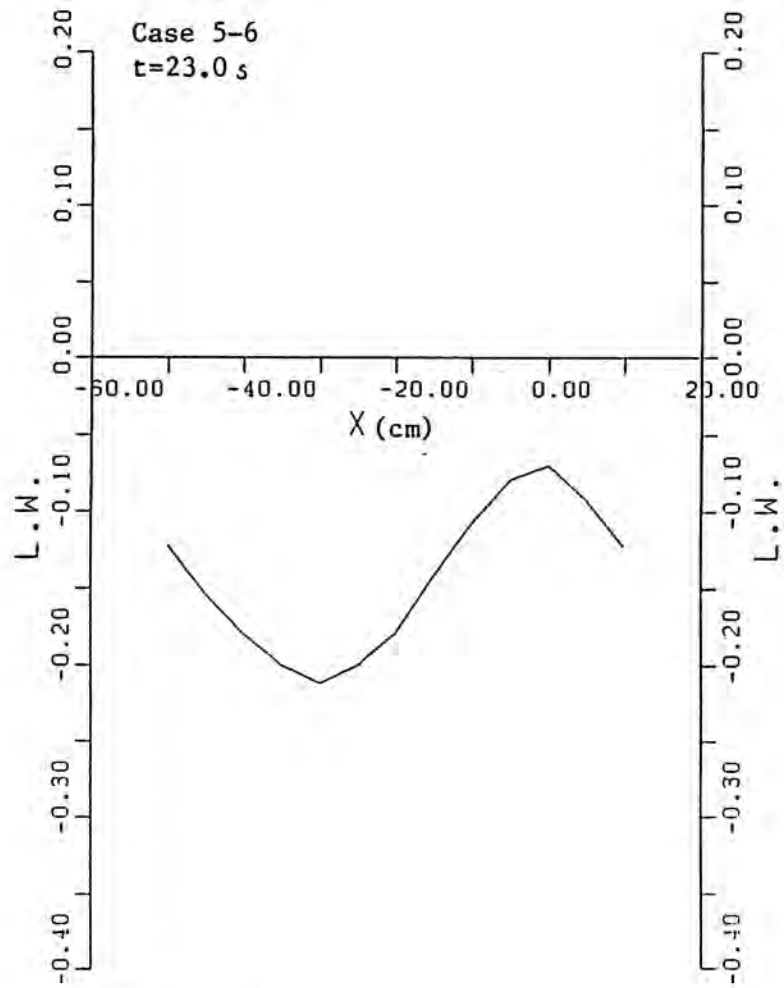


Fig. 3-12(i)

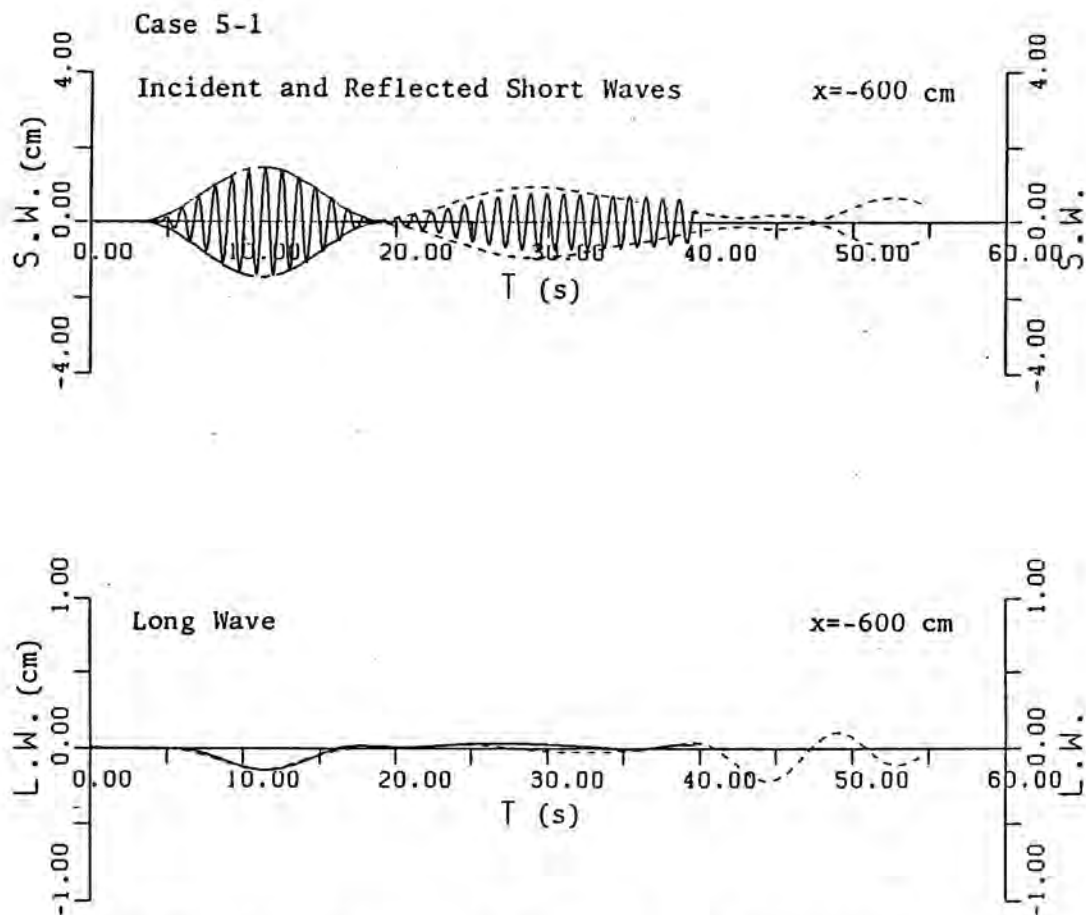
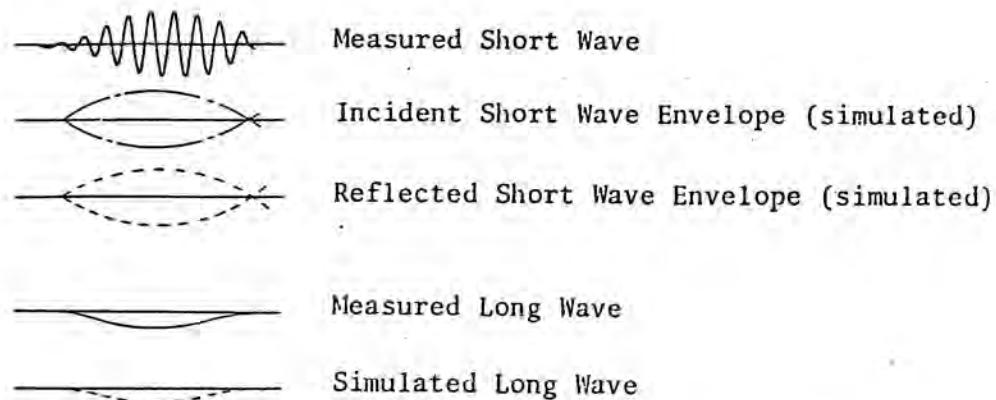


Fig. 3-13(a) Experimental and Simulated Results of Short Wave Envelope and Long Wave



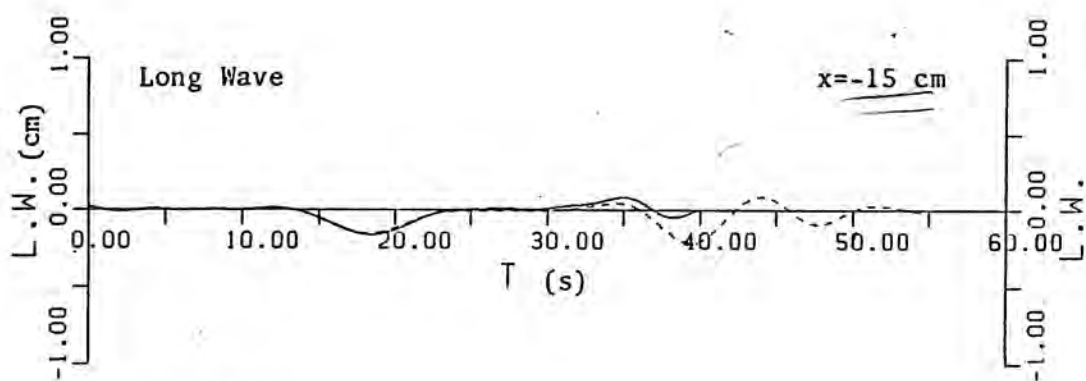
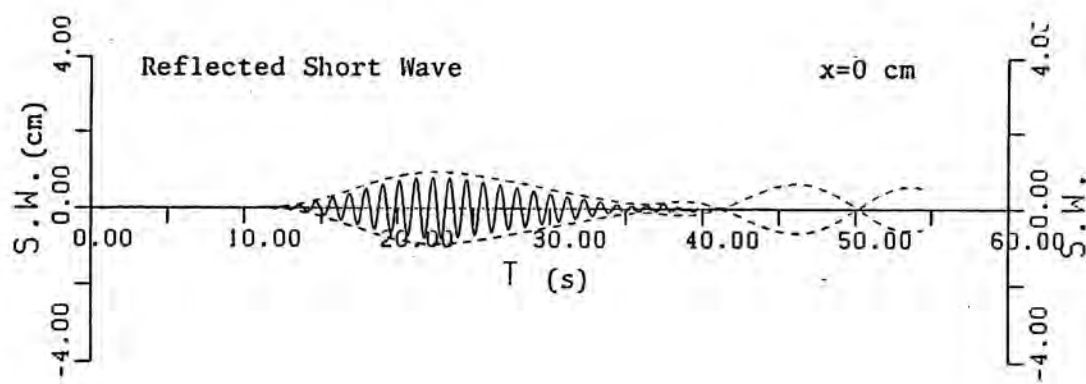
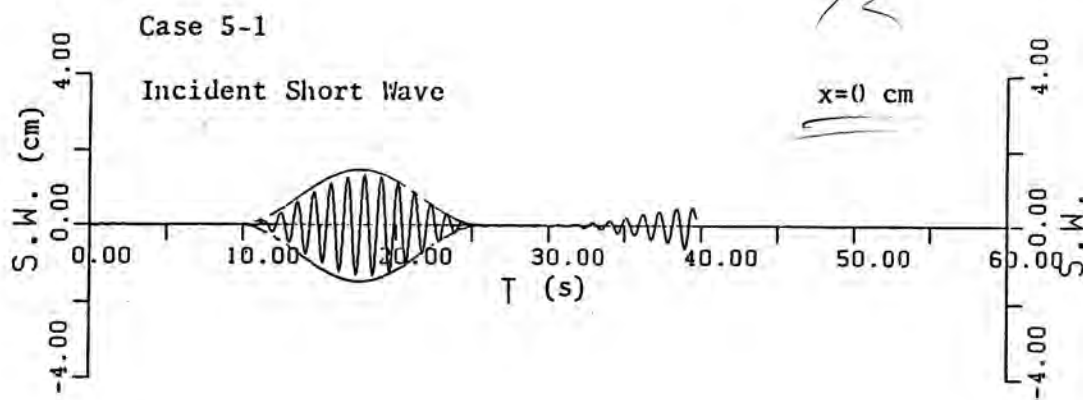


Fig. 3-13(b)

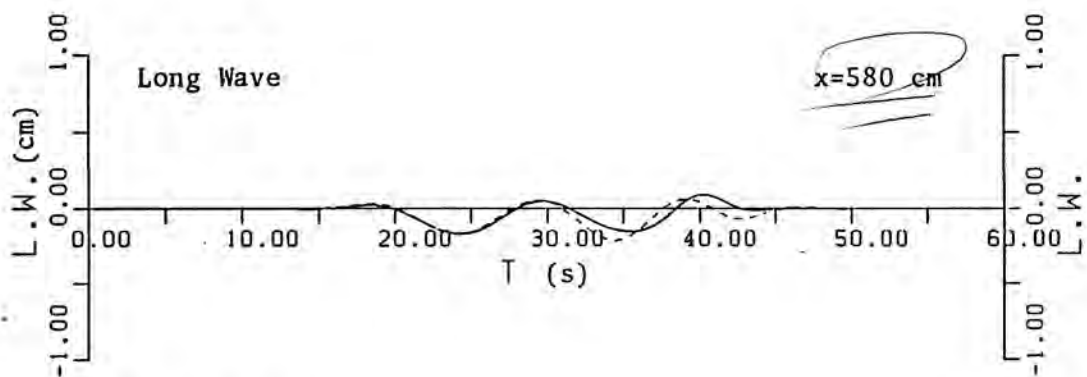
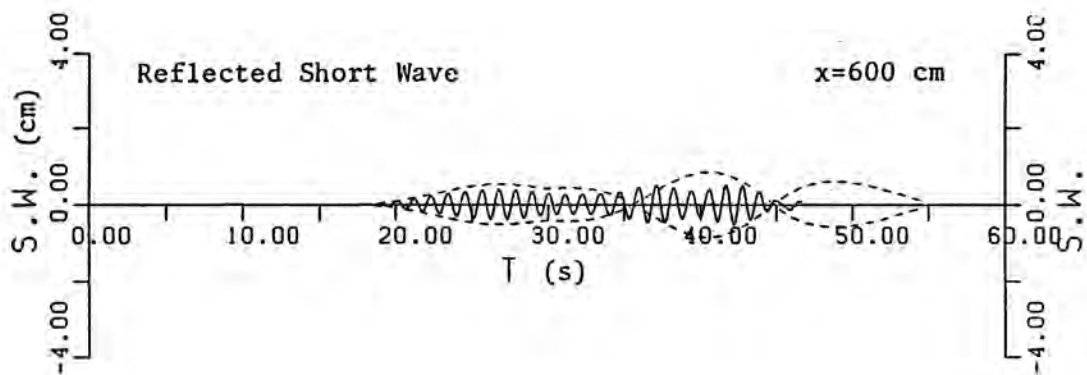
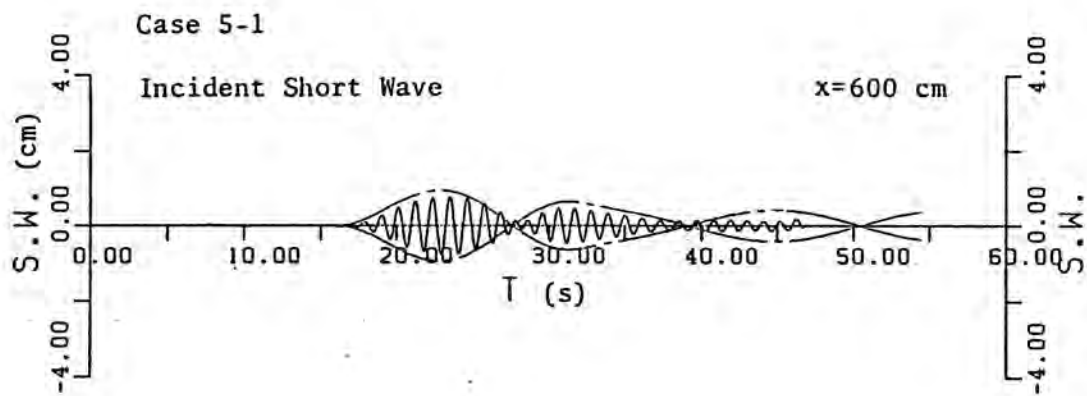


Fig. 3-13(c)

Case 5-1

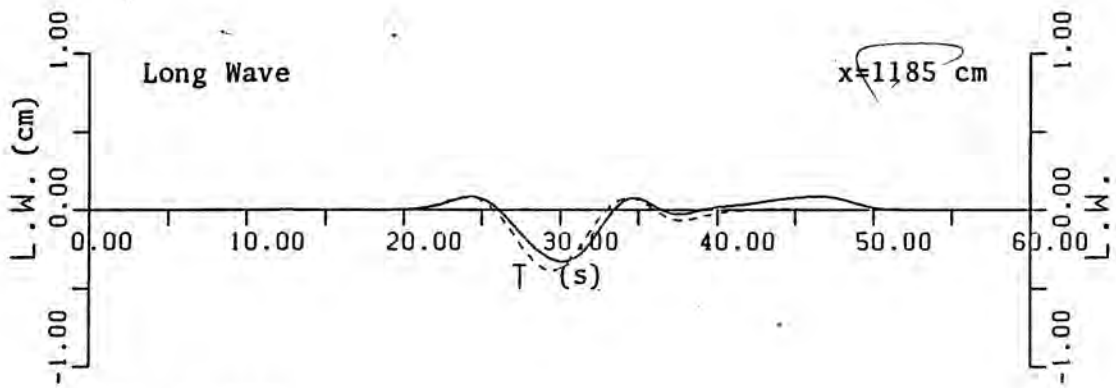
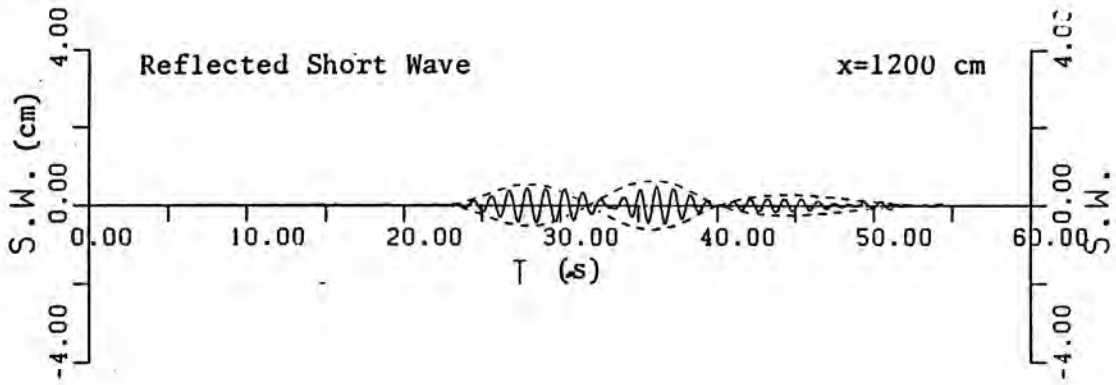
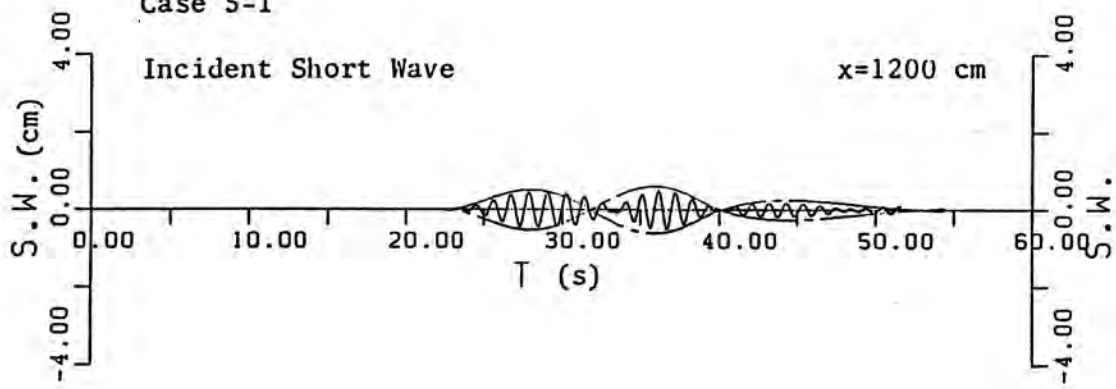


Fig. 3-13(d)

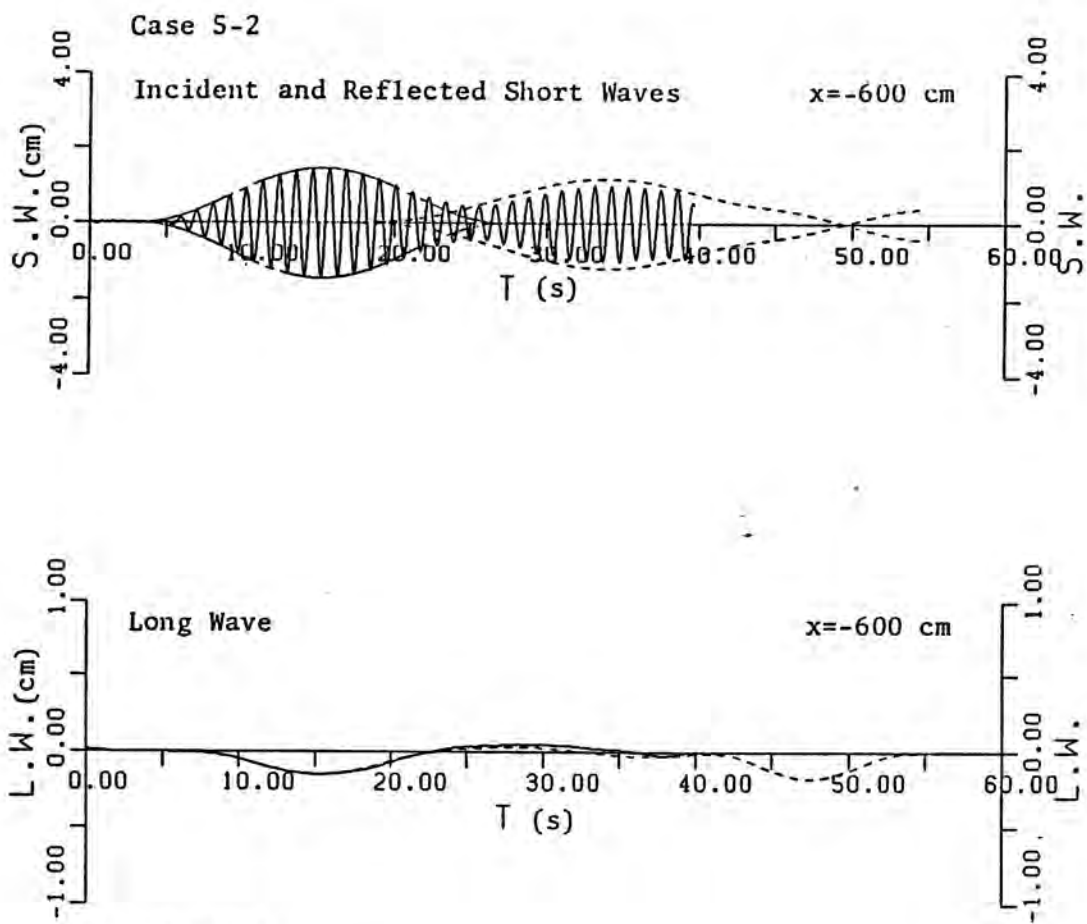


Fig. 3-14(a)

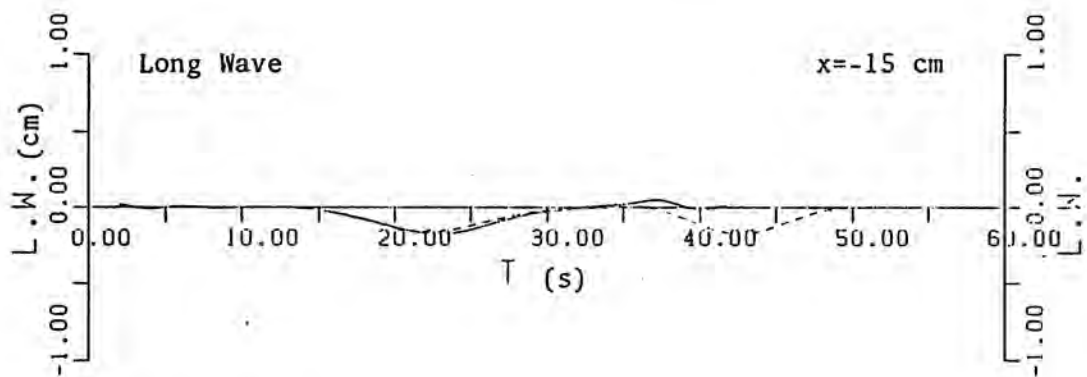
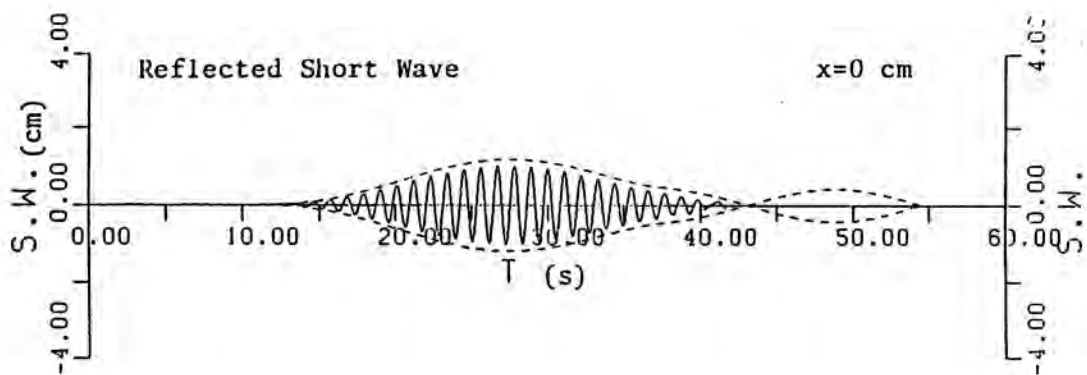
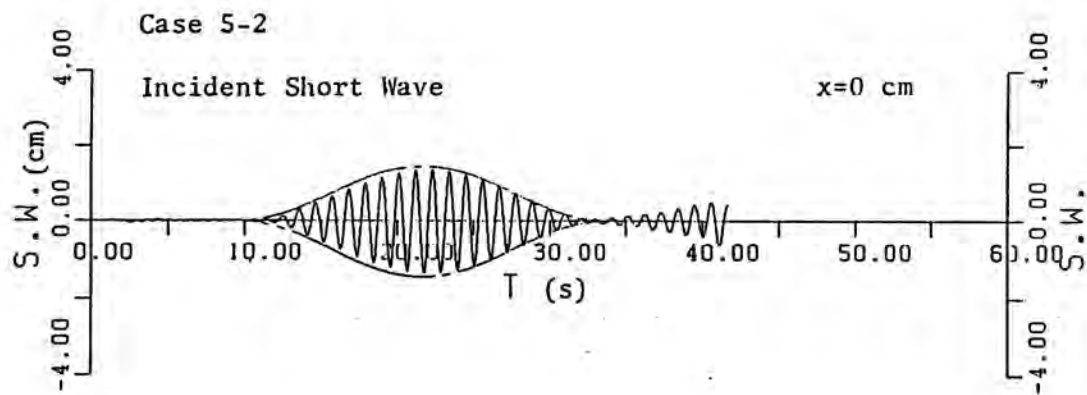


Fig. 3-14(b)

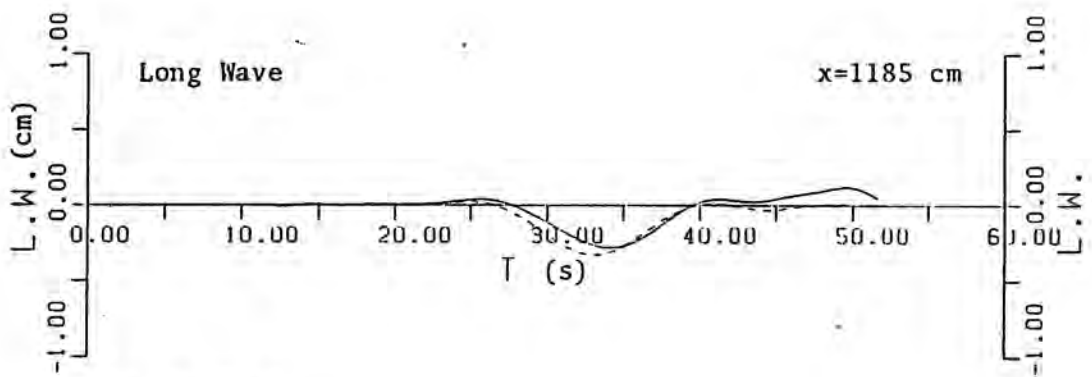
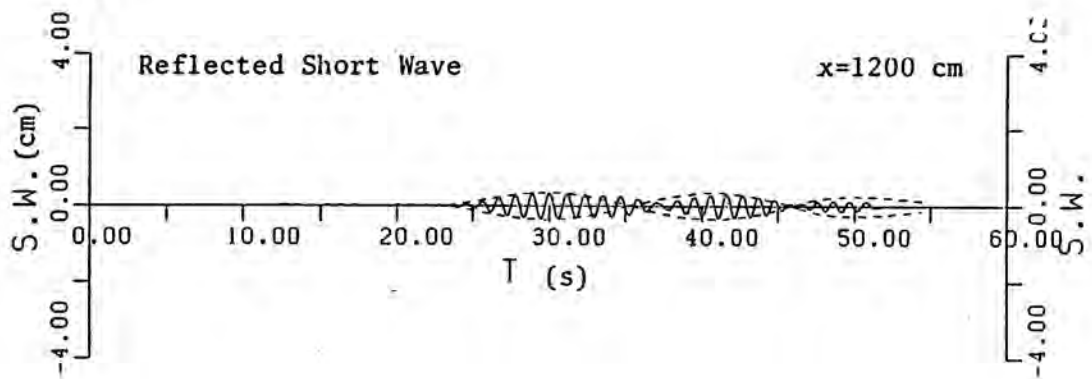
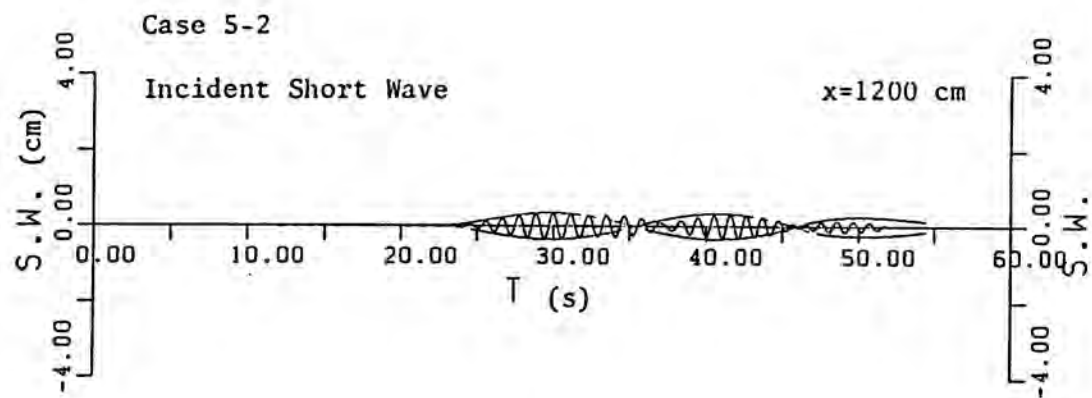


Fig. 3-14(c)

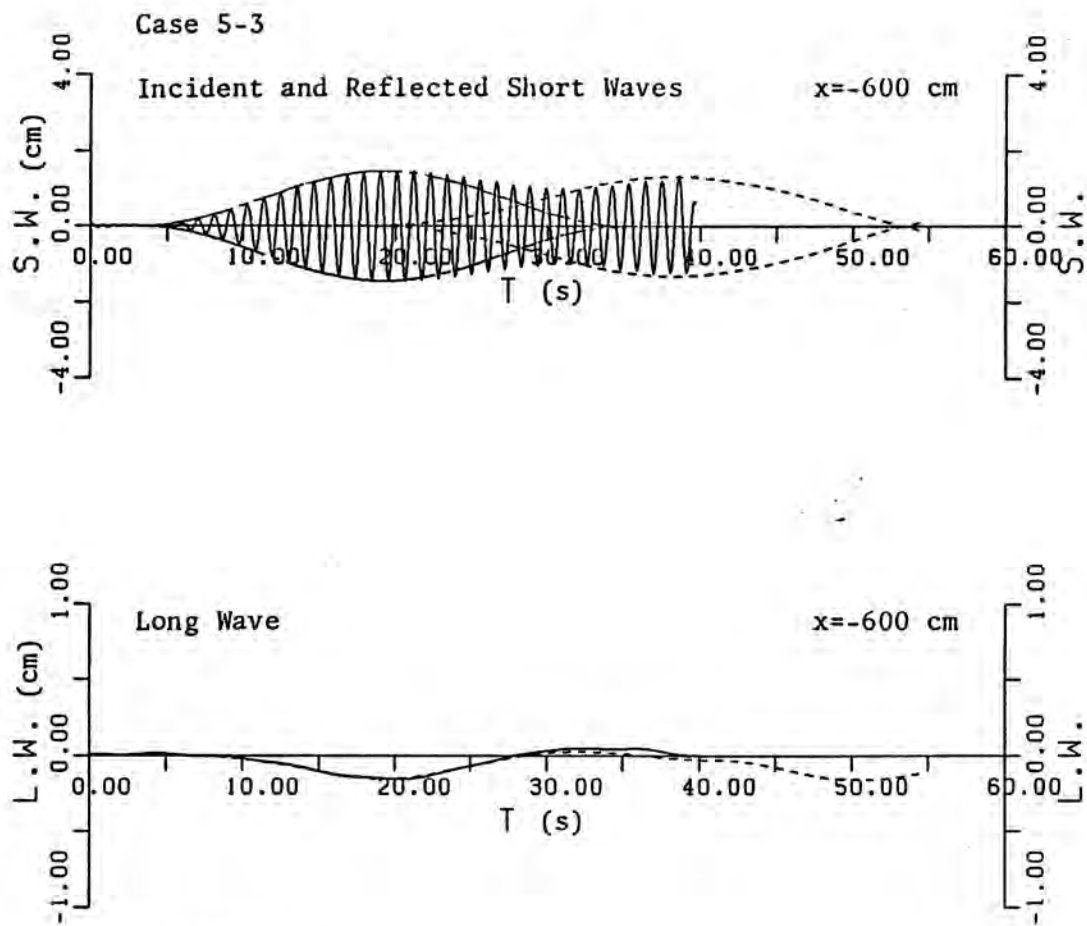


Fig. 3-15(a)

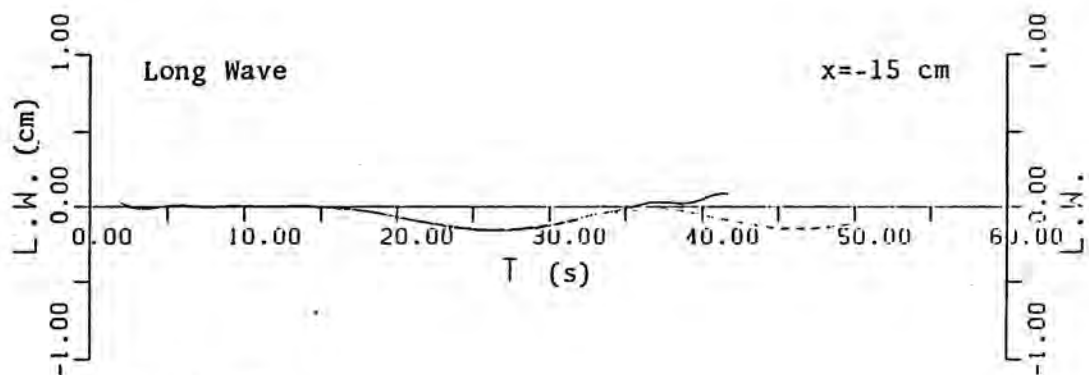
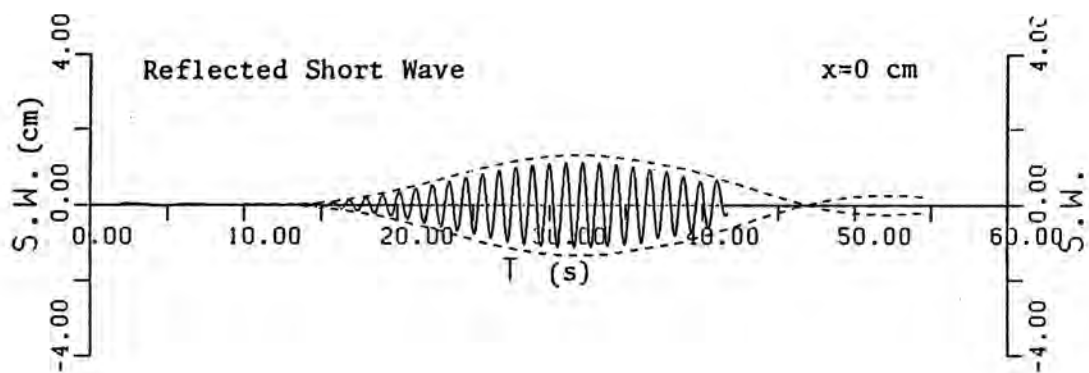
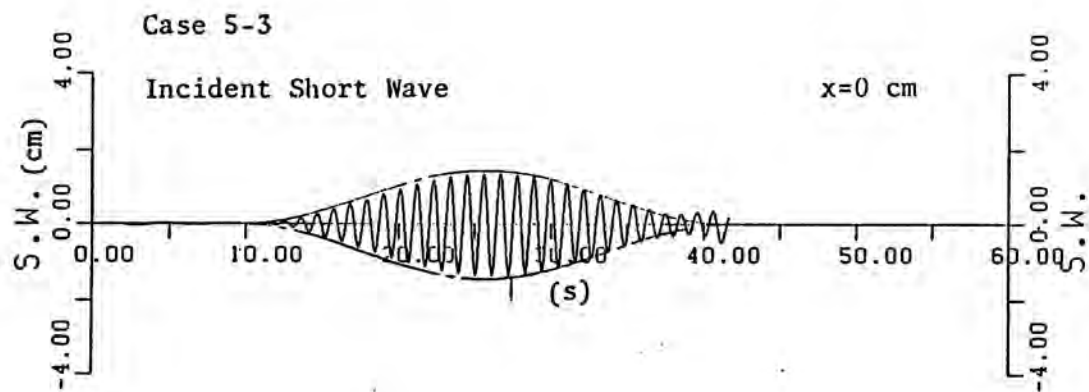


Fig. 3-15(b)

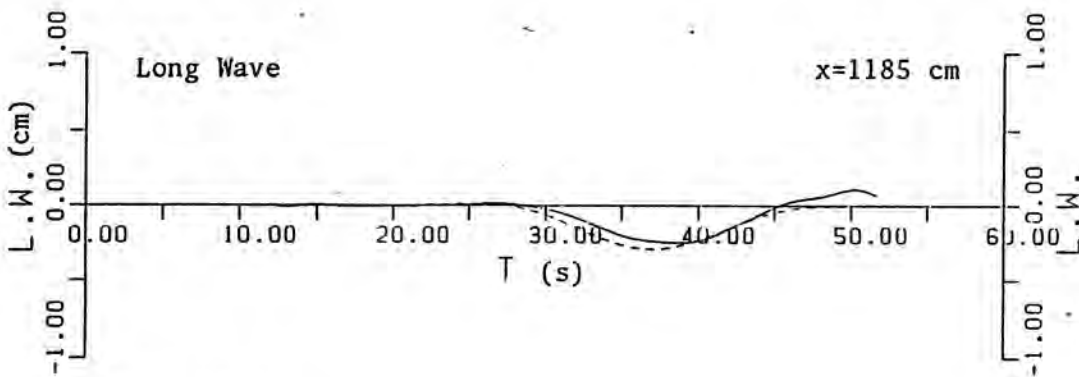
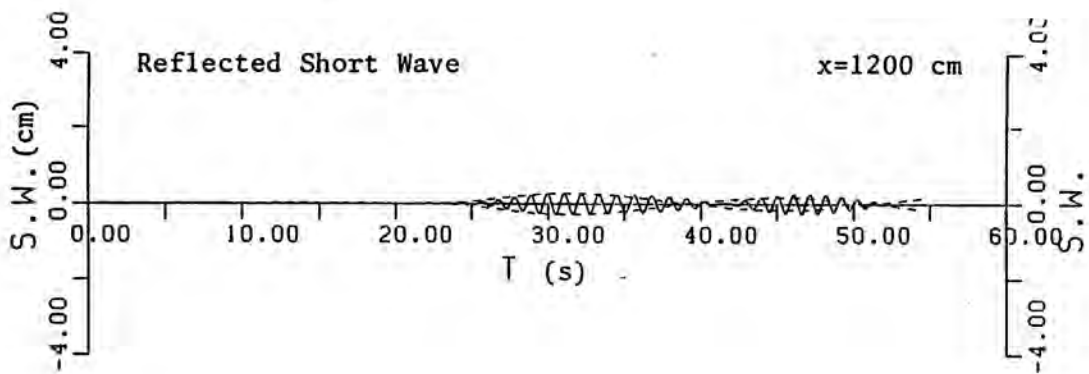
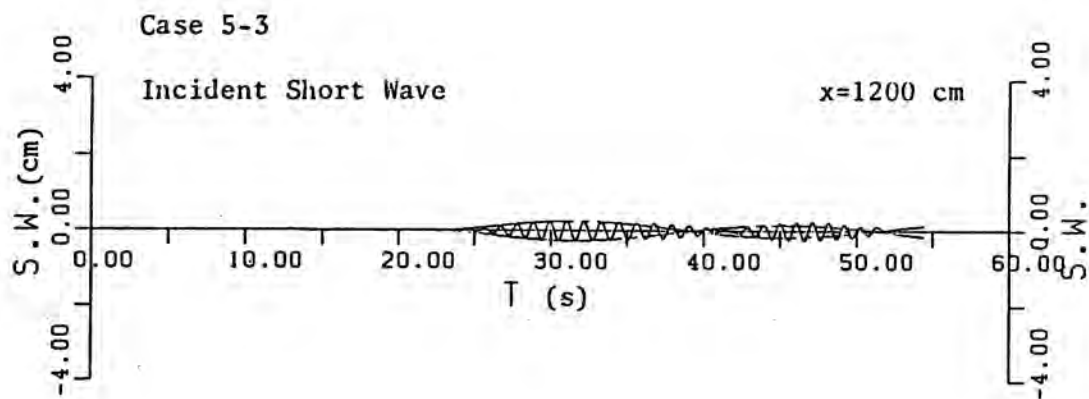


Fig. 3-15(c)

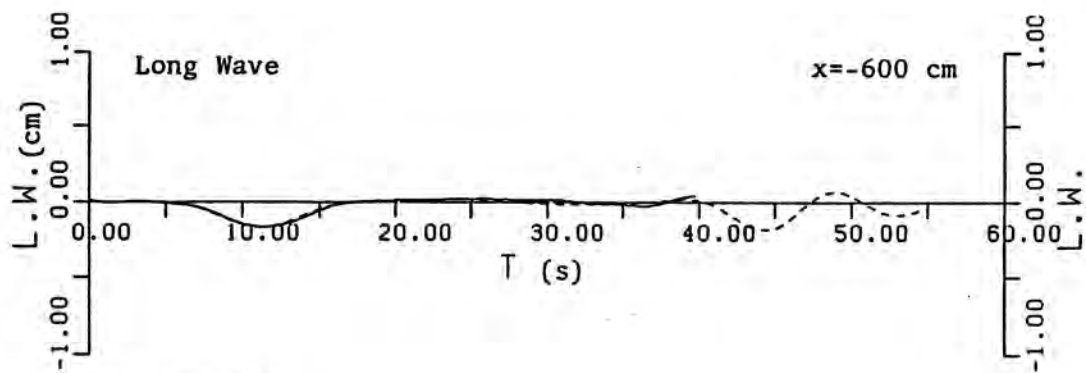
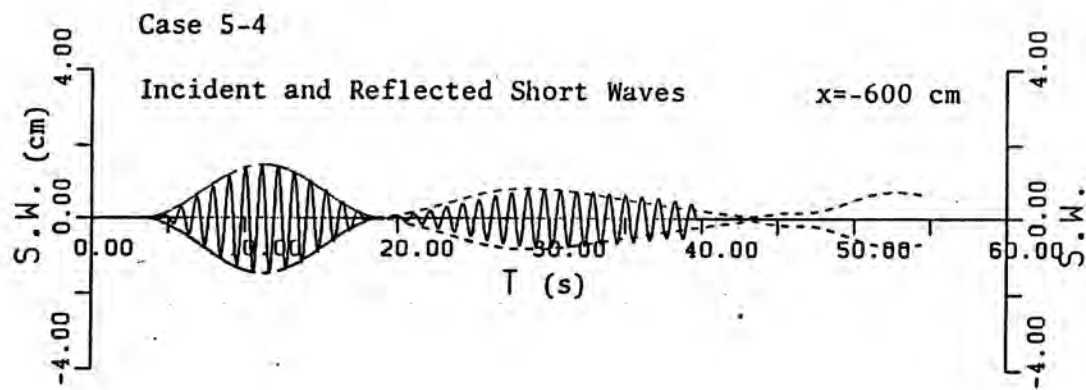


Fig. 3-16(a)

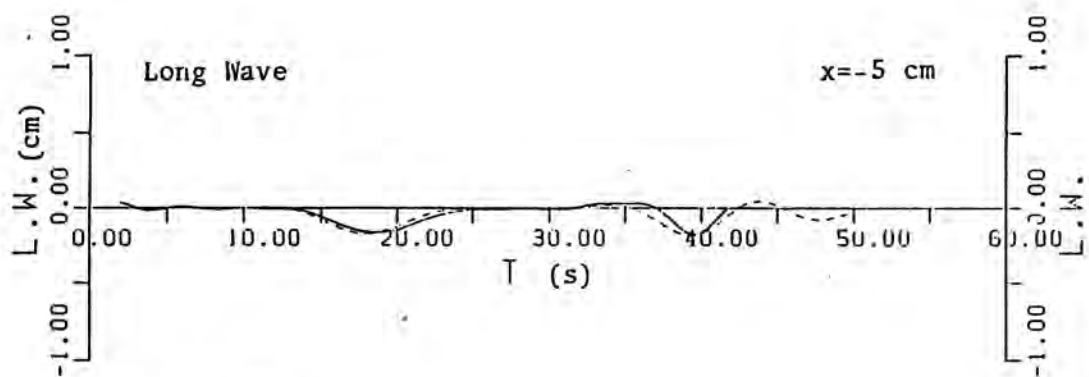
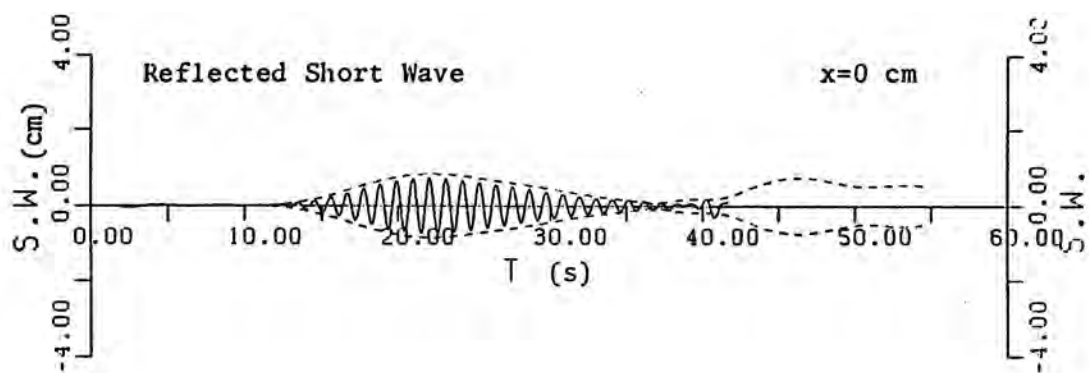
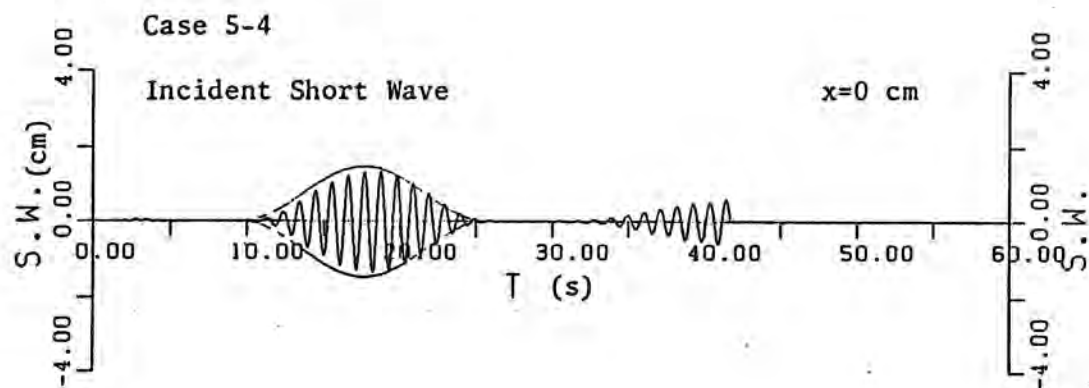


Fig. 3-16(b)

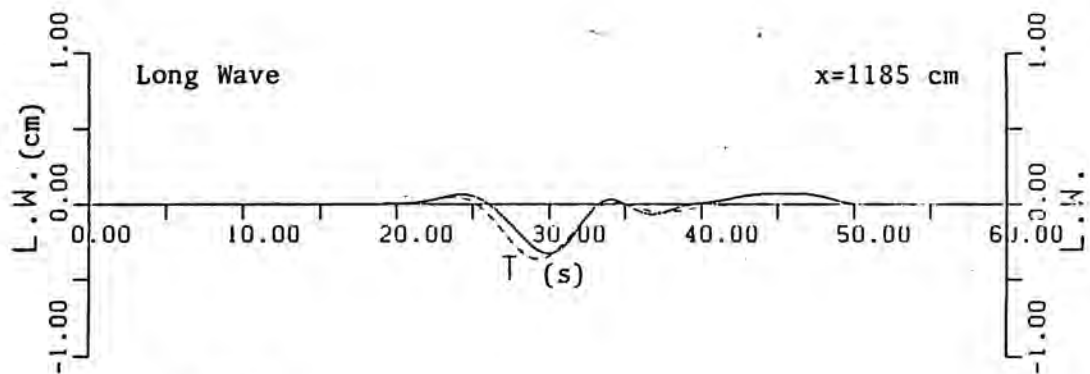
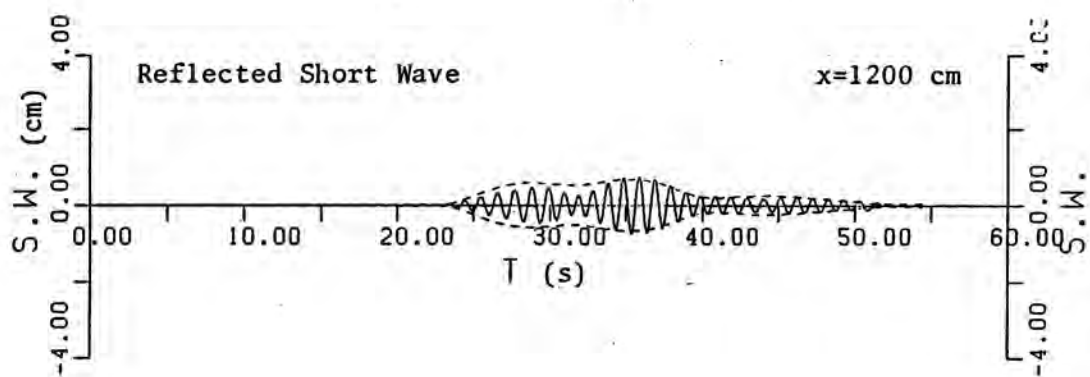
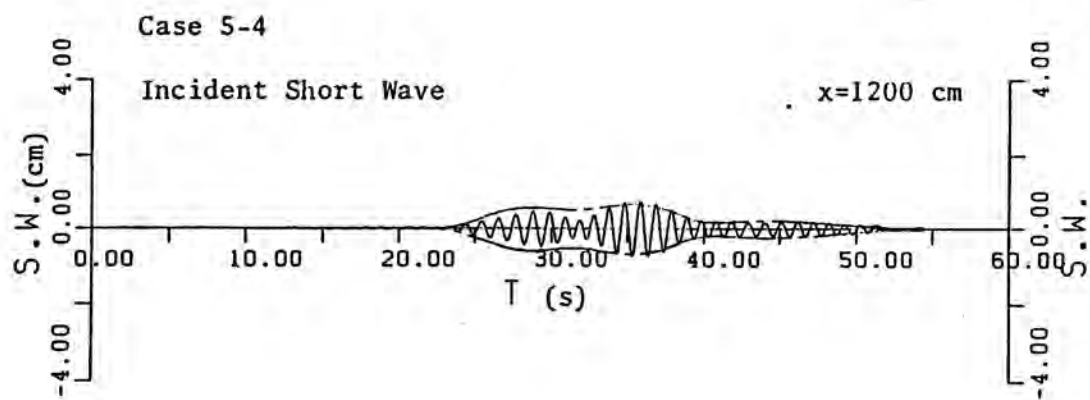


Fig. 3-16(c)

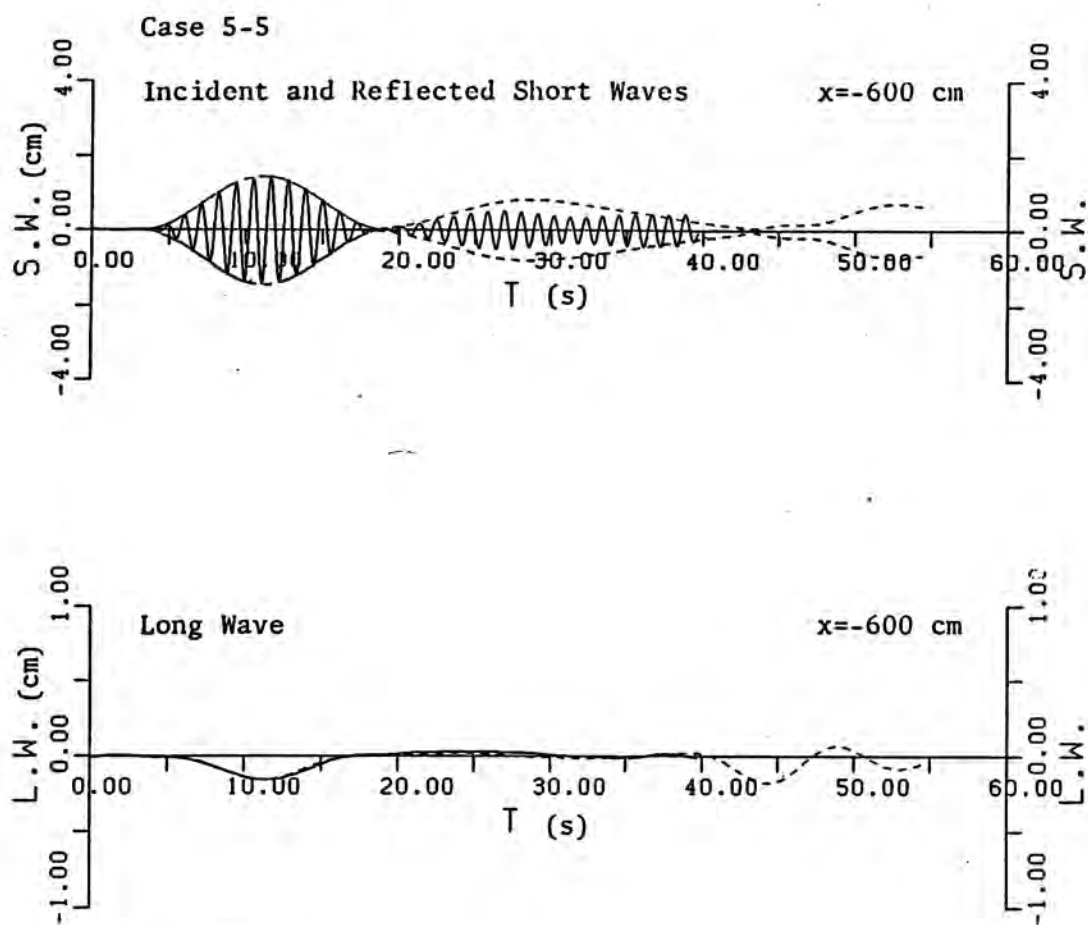


Fig. 3-17(a)

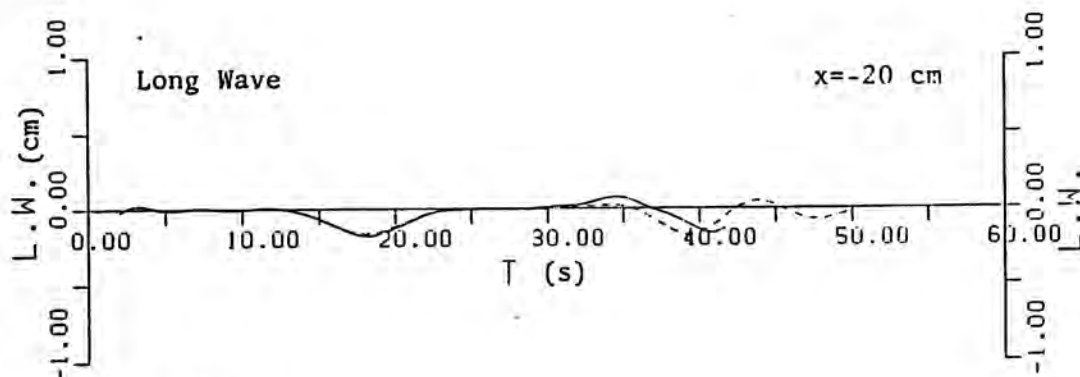
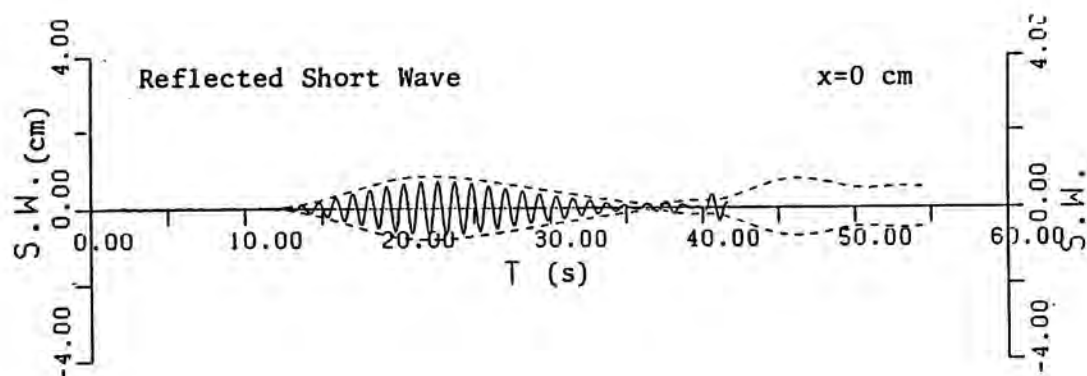
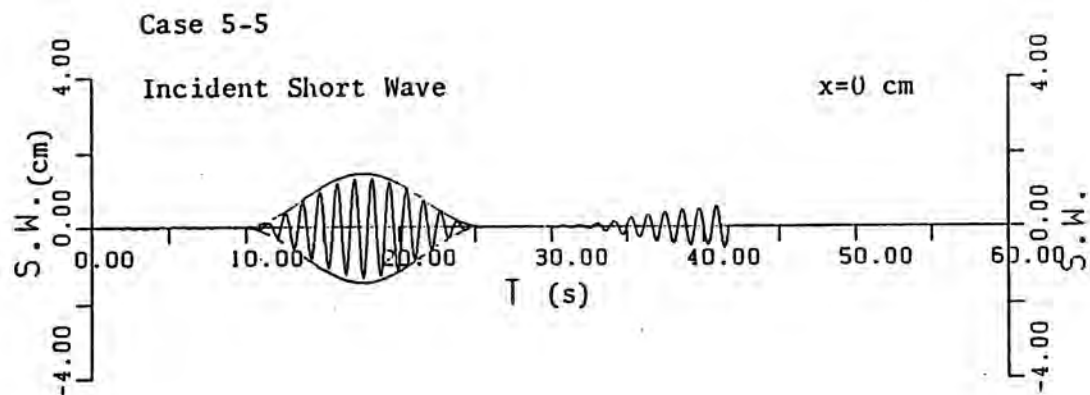


Fig. 3-17(b)

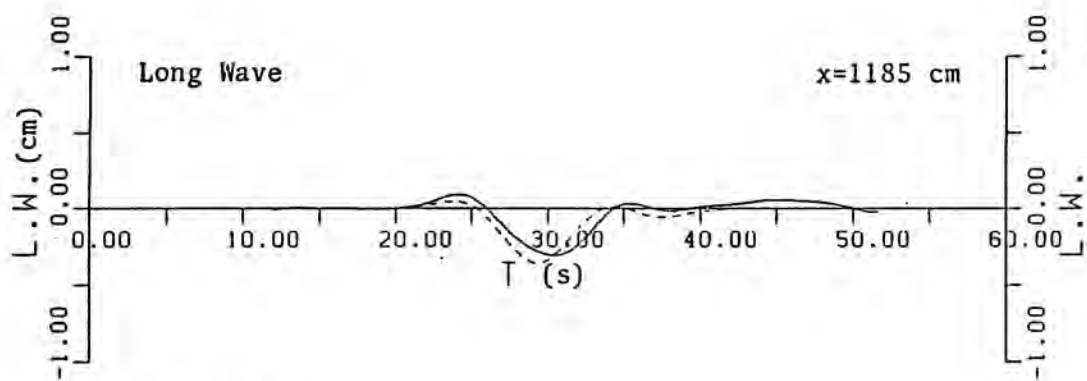
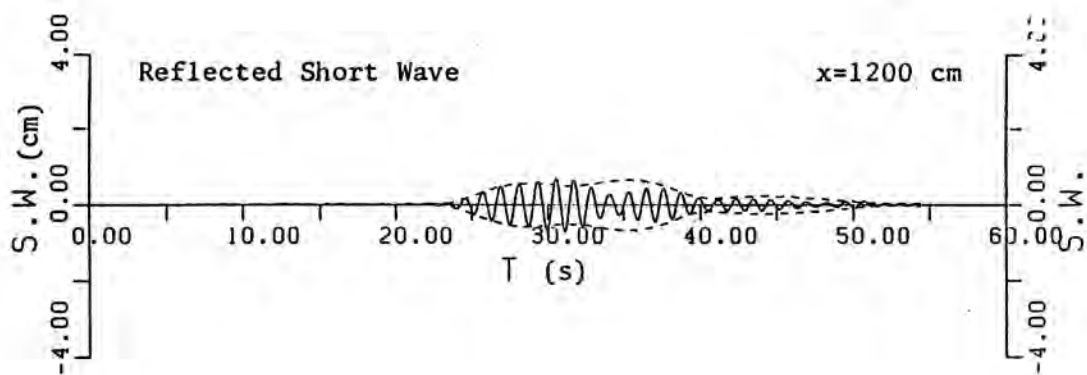
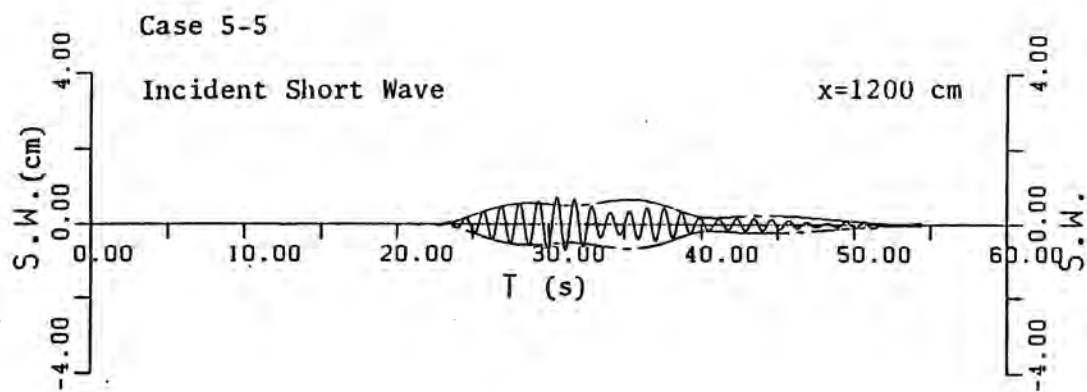


Fig. 3-17(c)

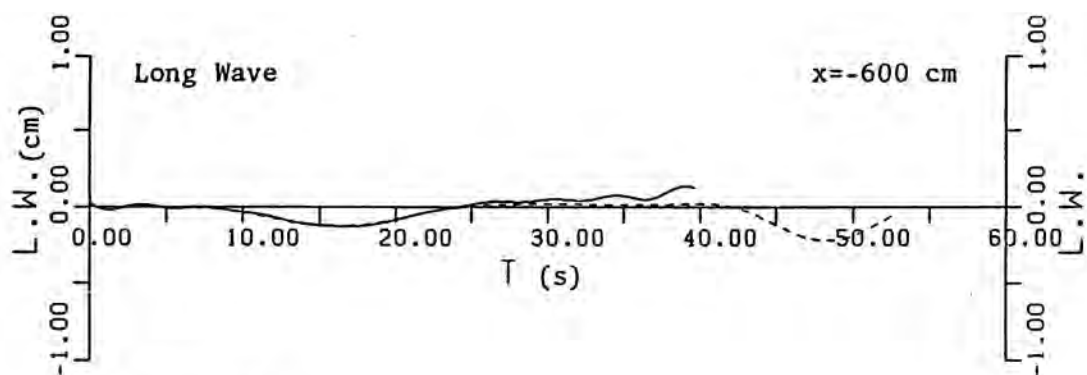
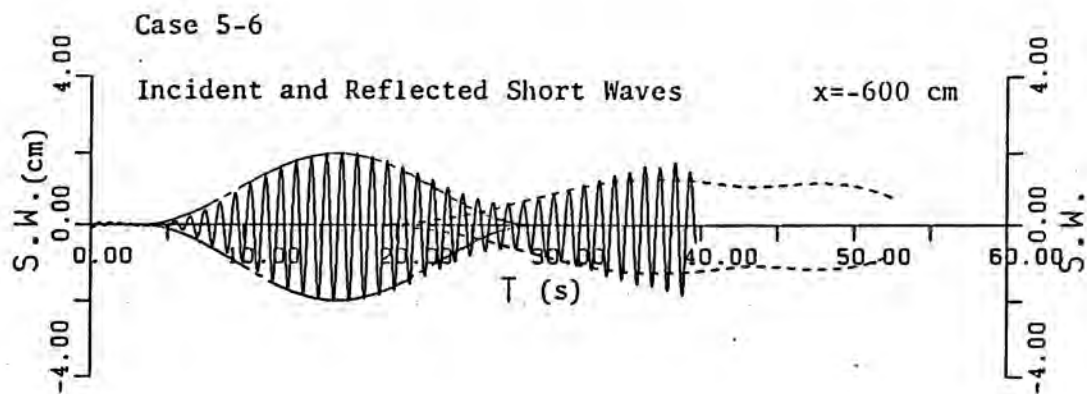


Fig. 3-18(a)

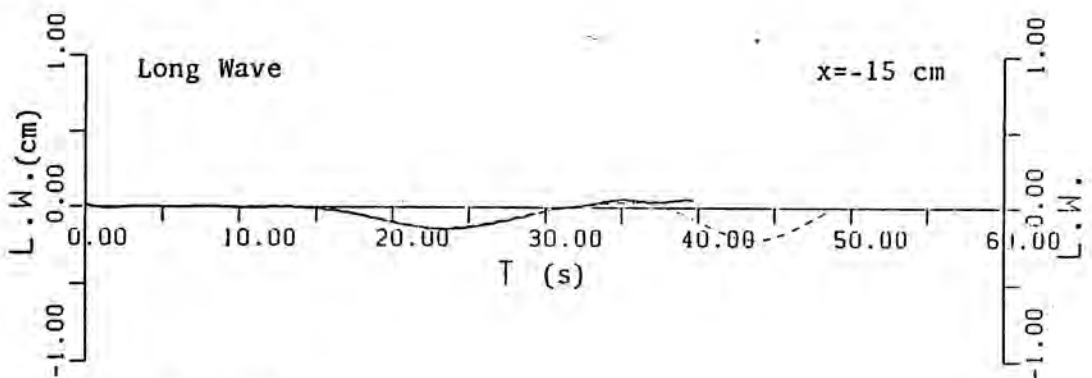
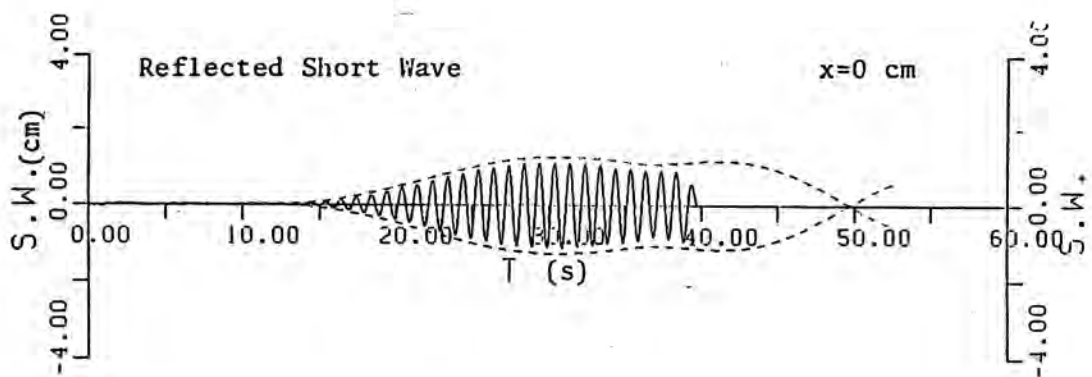
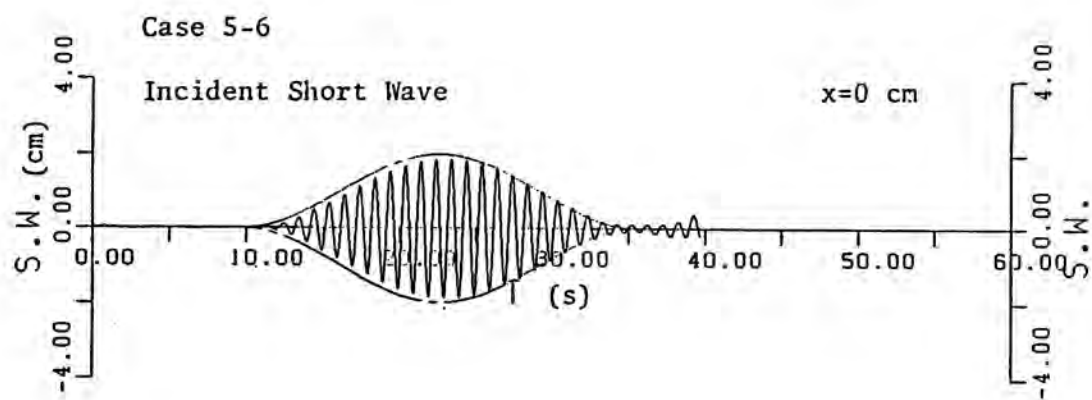


Fig. 3-18(b)

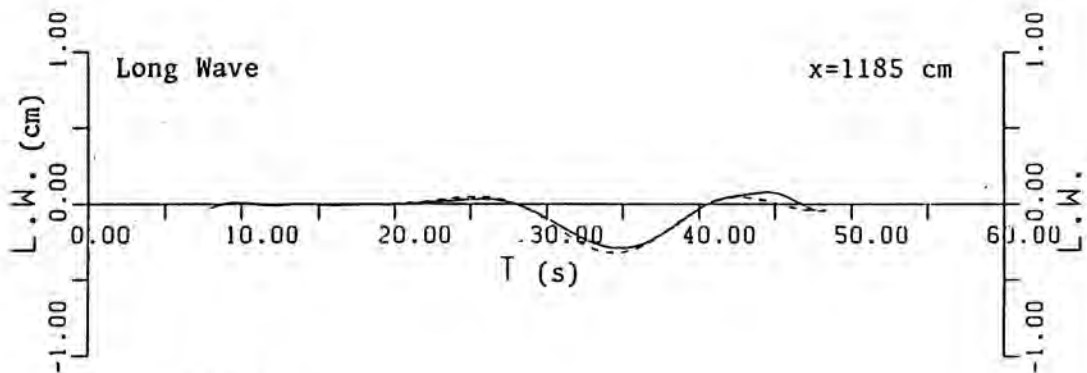
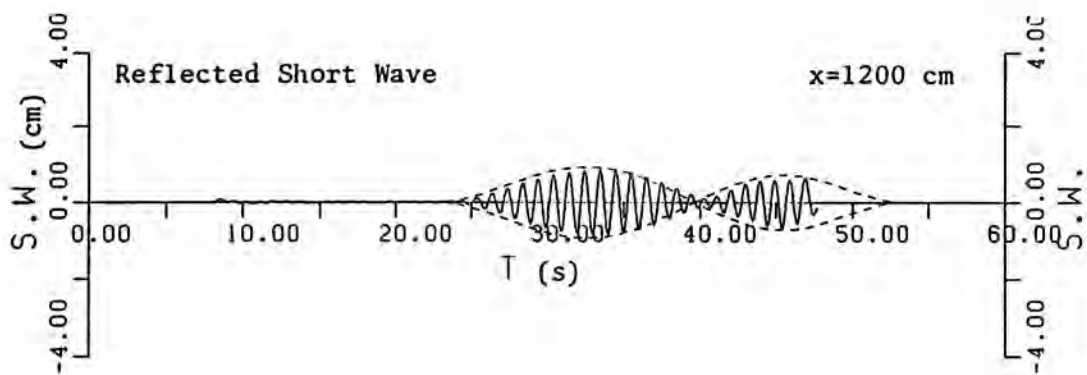
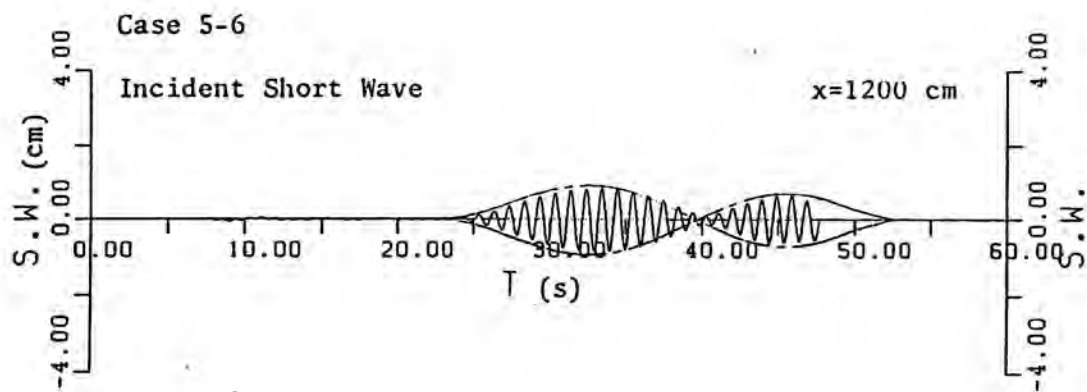


Fig. 3-18(c)

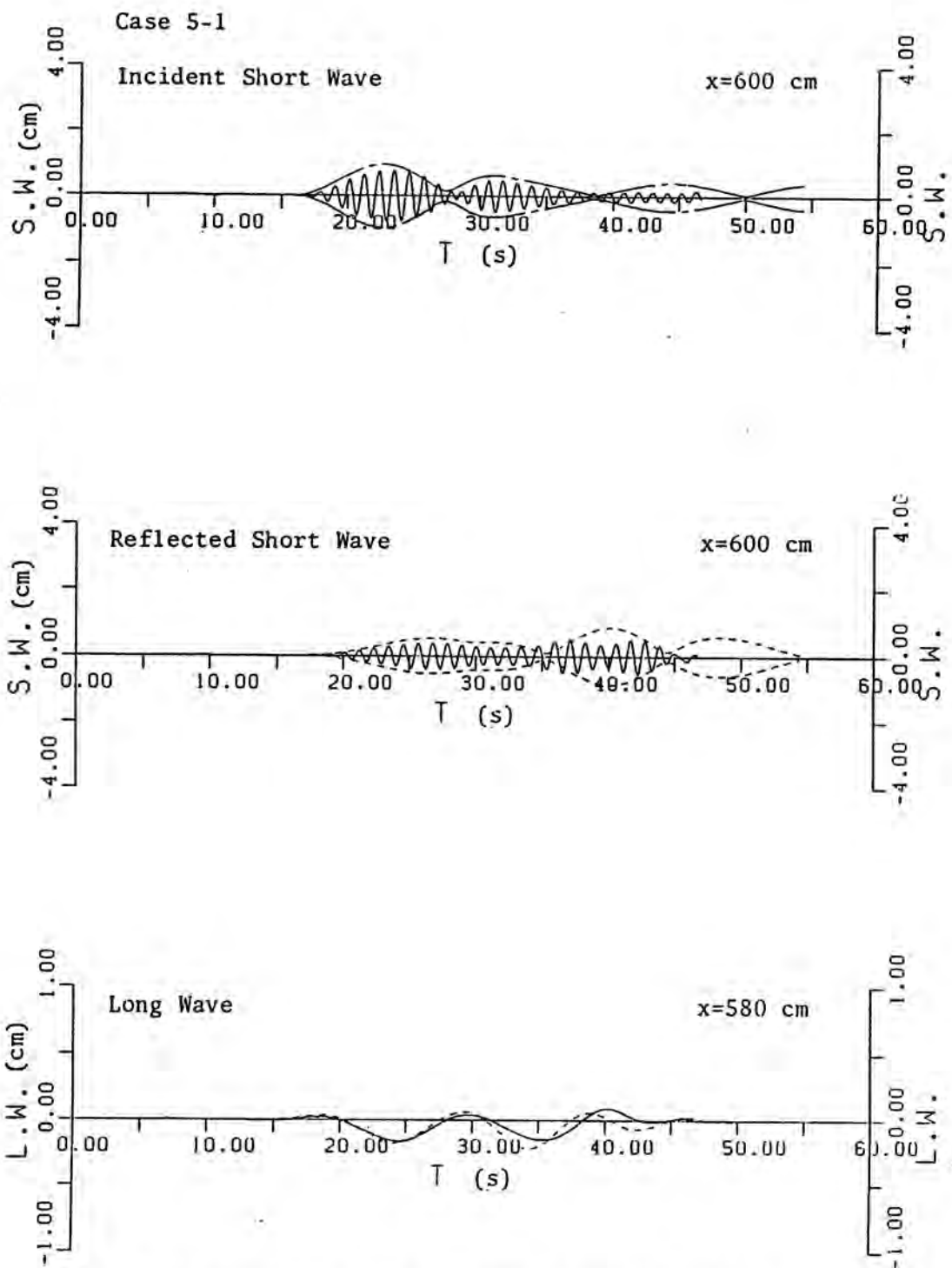


Fig. 3-19(a) Experimental and Simulated Results of Short Wave Envelope and Long Wave (corrected h and D)

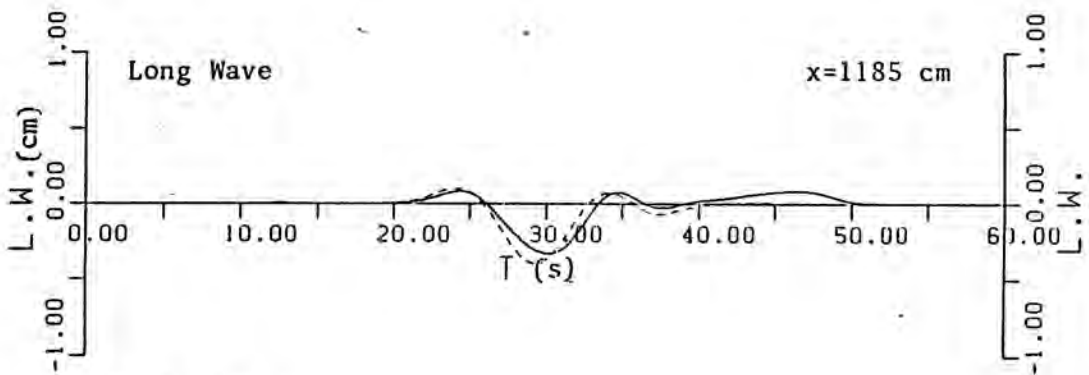
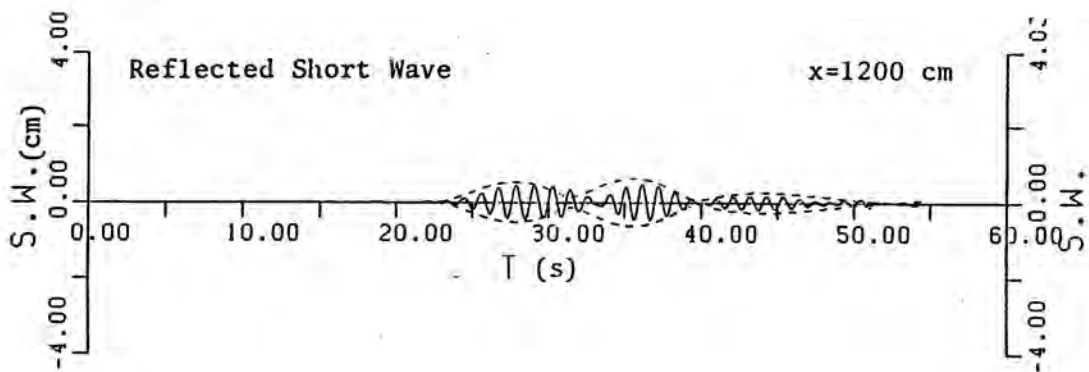
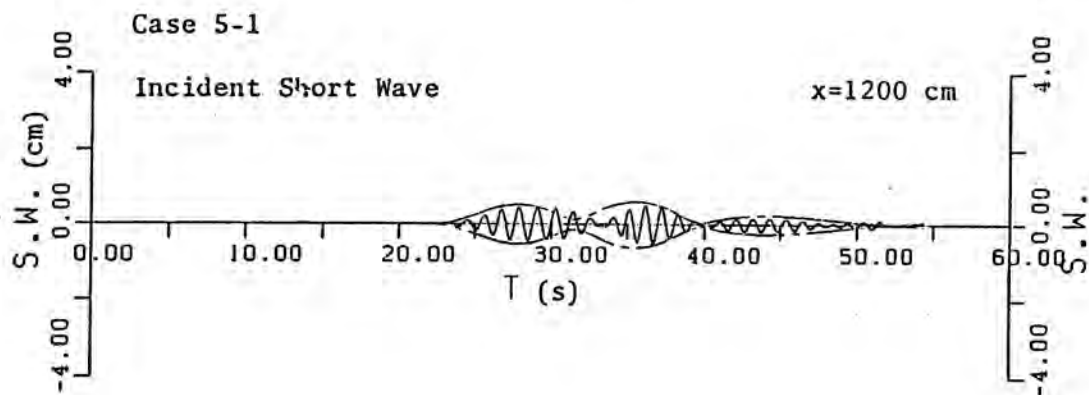


Fig. 3-19(b)

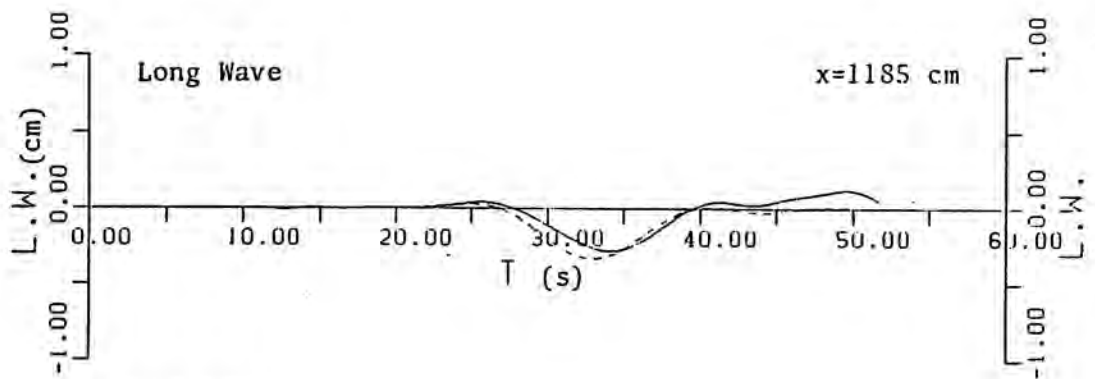
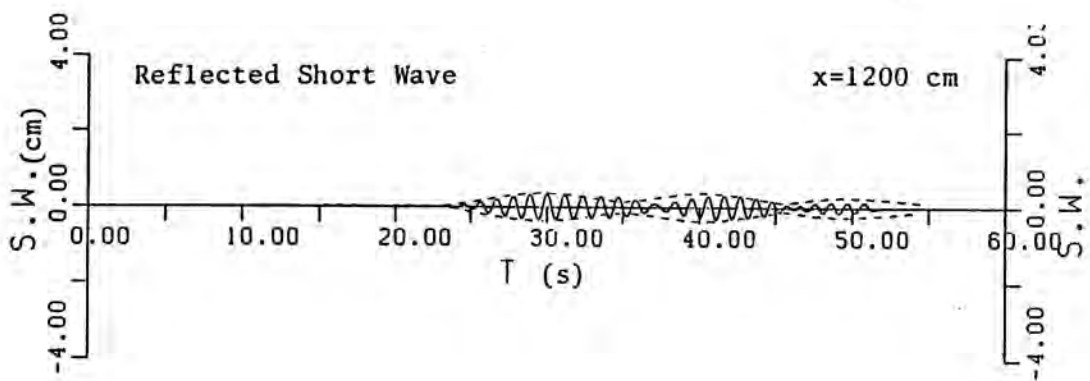
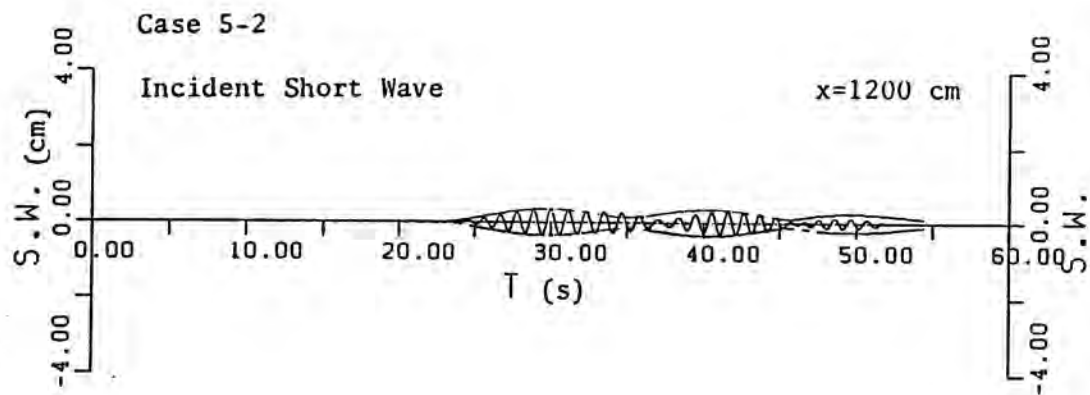


Fig. 3-19(c)

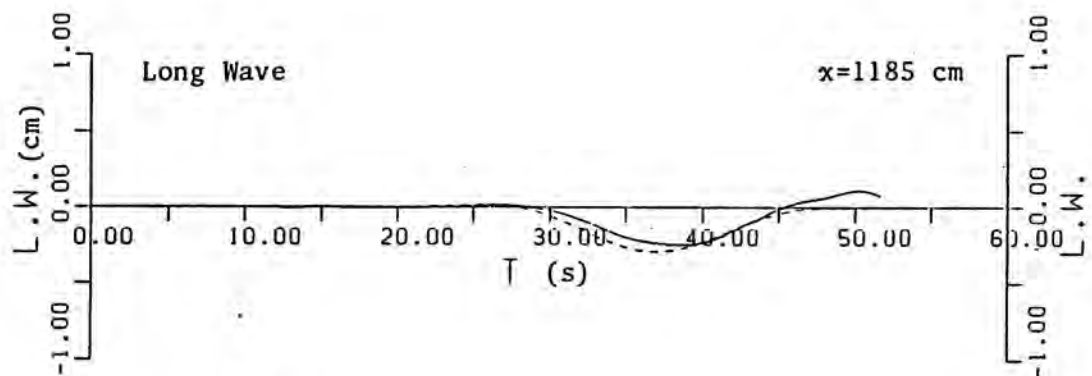
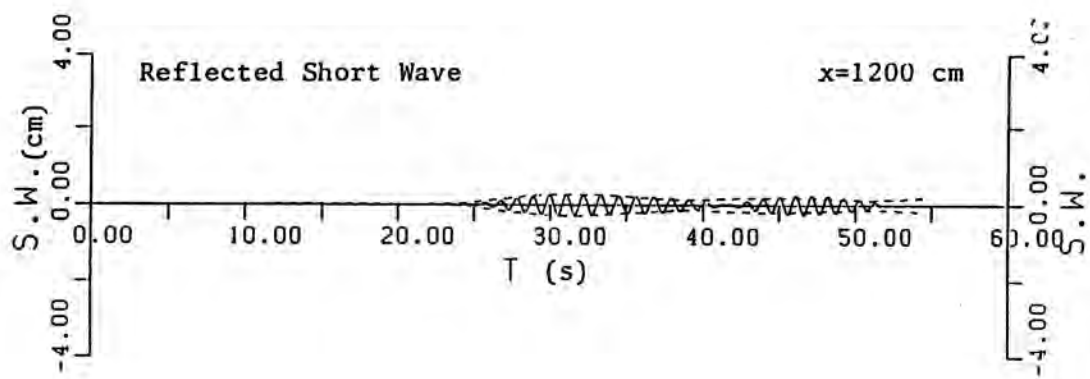
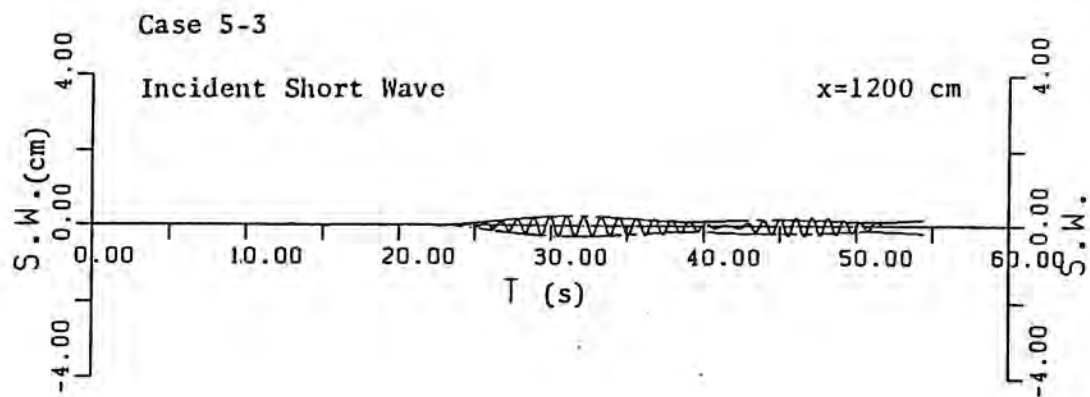


Fig. 3-19(d)

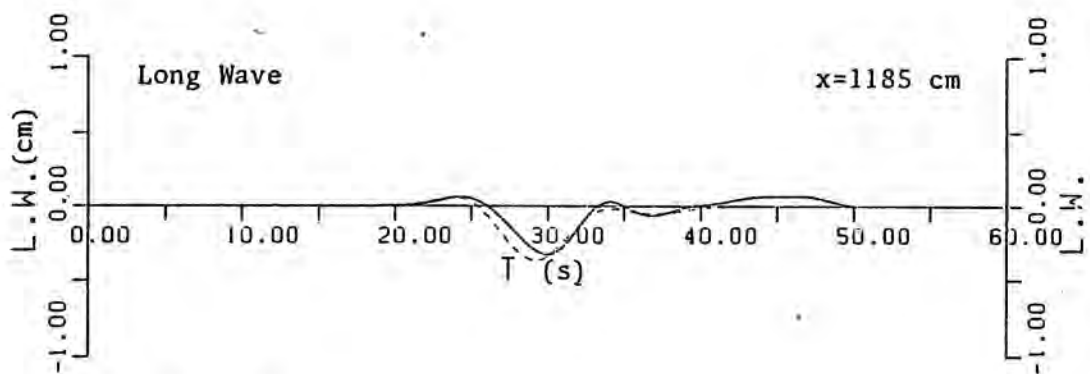
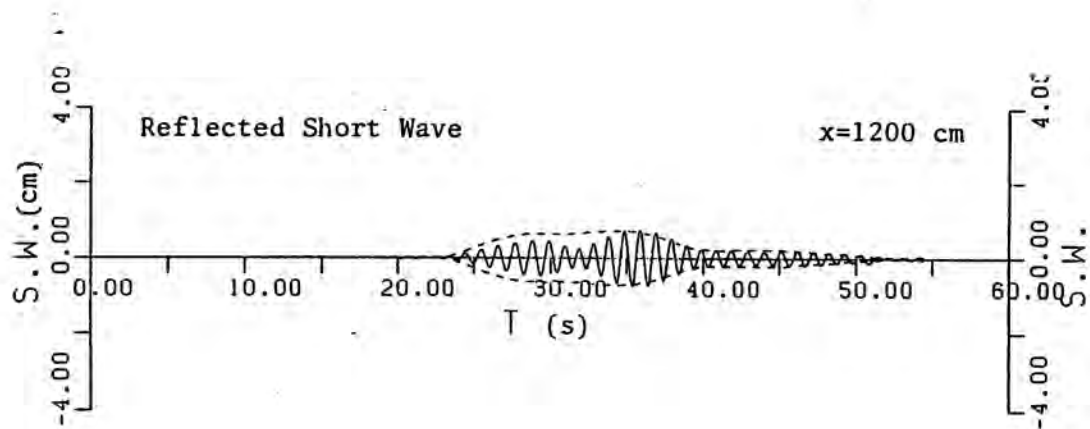
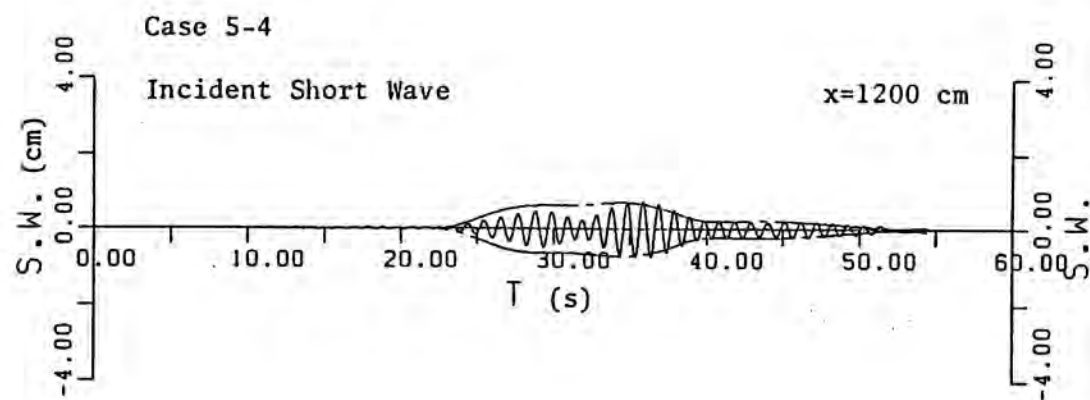


Fig. 3-19(e)

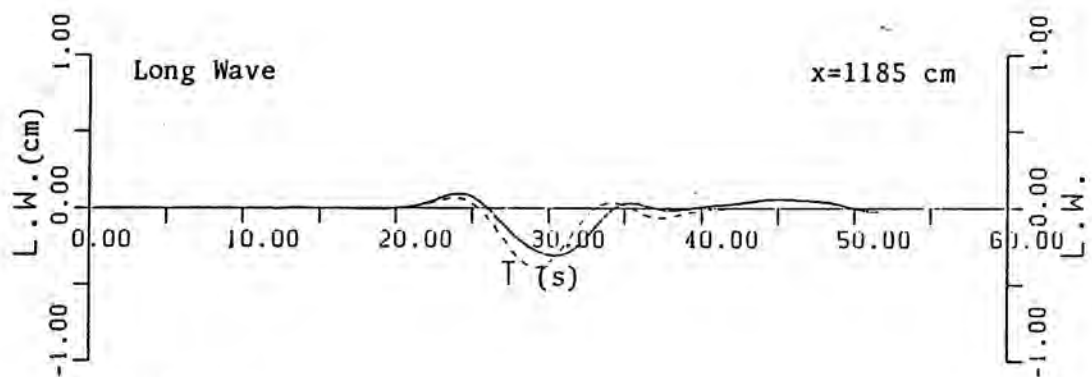
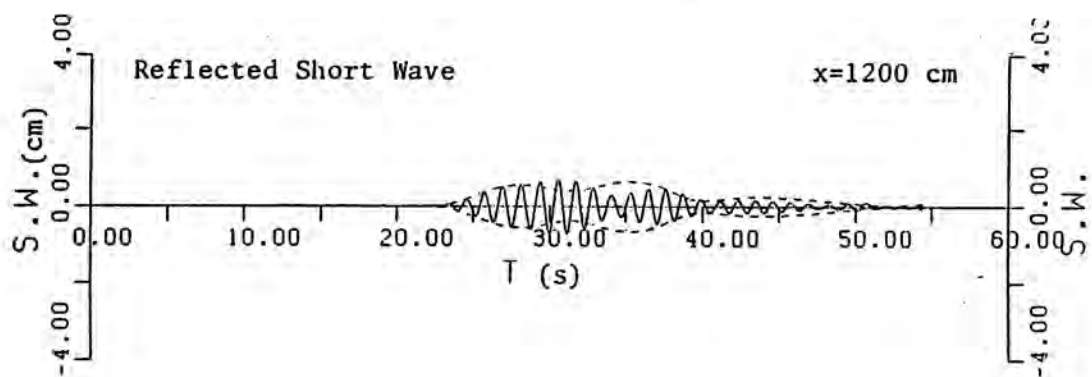
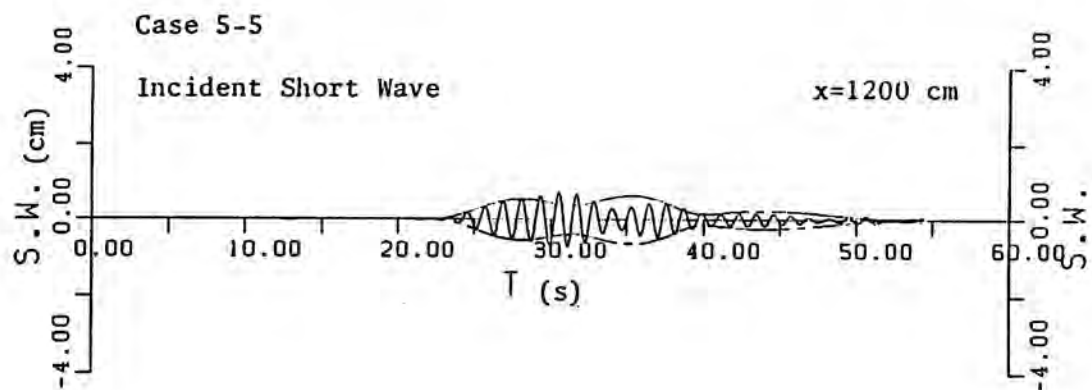


Fig. 3-19(f)

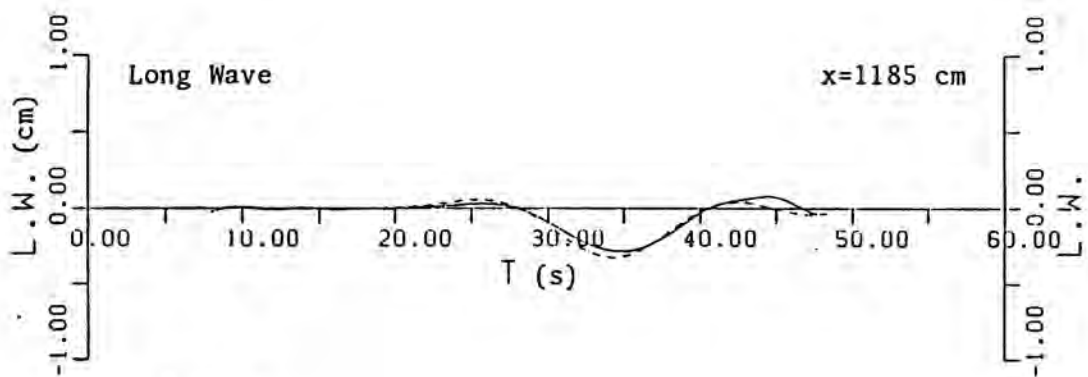
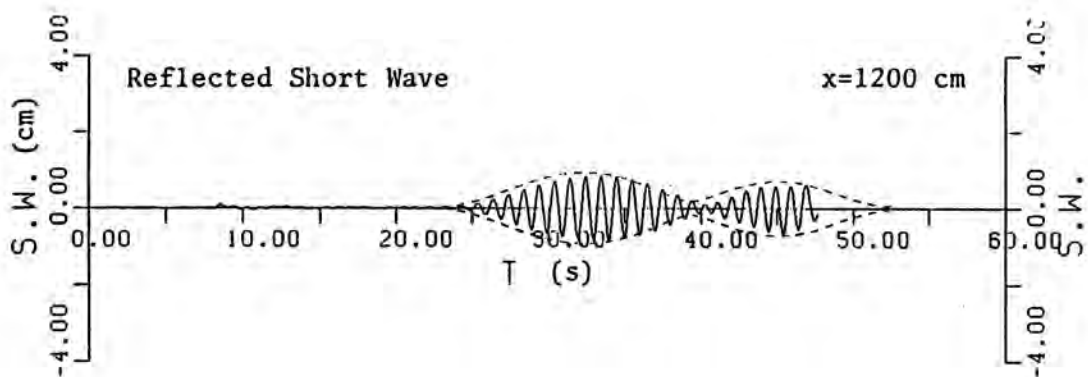
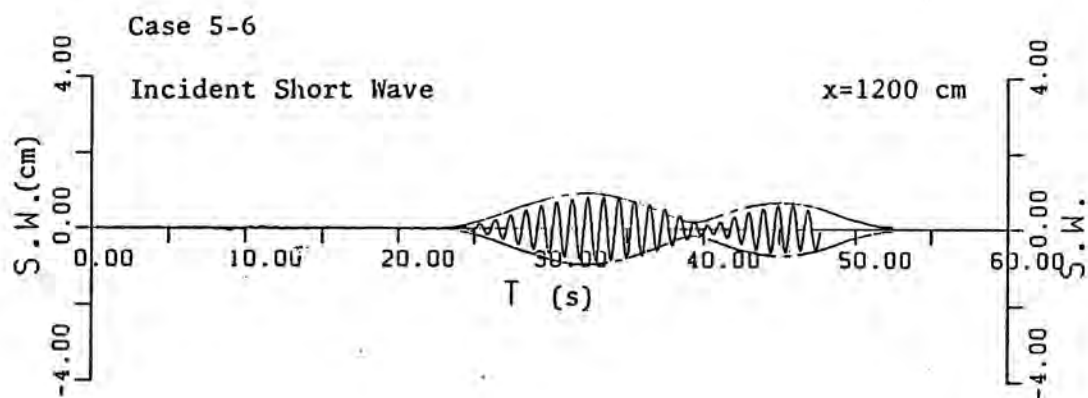


Fig. 3-19(g)

Chapter 4 Conclusion

We obtained the approximate equations for first order short wave envelopes and second order long waves over spatially periodical sandbars, and their solutions were obtained in various conditions (Chapter 2). An experiment was carried out to confirm the theoretical results (Chapter 3).

In Chapter 2, we found the following things,

- 1) As in the case of a constant depth (Mei 1985), above an inclined bottom topography, strong reflection can be induced when the Bragg condition is met, and the scattering process depends on whether the modulational frequency is above or below a threshold frequency. A sandbar region can work as a kind of filter for incident waves with continuous spectrum.
- 2) A wave packet is strongly reflected and becomes dispersive once it enters a sandbar region.
- 3) For periodically modulated wave groups above sandbars, unlocked free long waves appear besides the wave set-down associated with the wave envelope. In particular, when the free waves have the same wavelength as of wave groups in the sandbar region, their amplitude becomes unbounded.
- 4) When a wave packet enters a sandbar region, the wave set-down originally locked to the incident packet, is unlocked and propagates with the speed of long waves in both directions. At the same time the steady set-downs for the deformed incident and reflected wave packets appear.

In Chapter 3, the following things were shown.

- 1) For detuned wave trains, the wave height distributions over sandbars showed distinctive properties for various modulational frequencies above or below the threshold frequency. They agreed with the numerical results very well in the extent of this experiment.
- 2) When kD was rather large compared with the other small parameters, the modified governing equations considering the second order in kD gave better numerical results.
- 3) The reflection and the dispersion of a wave packet, the appearance and the propagation of unlocked free long waves were clearly observed, and agreed with the simulation very well, as far as the main part of the spectrum of incident waves were not very apart from the resonant frequency.

Together with the study by Mei (1985), the strong resonant reflection of water waves by periodic sandbars were shown both in theoretically and experimentally. This result is useful not only

in understanding the mechanism of the formation of longshore sandbars, but also in considering the wave field in coastal zone, which is the most important for engineering purposes.

It is to be noticed that in the area where longshore sandbars are formed on natural coasts, the relative wavelength kh is usually small ($kh = O(0.1)$). So the theory derived here is valid only when the ratio of wave height to water depth A/h is very small ($kh \gg A/h$). When nonlinearity is important, a more precise approach is necessary.

It is important to know by field investigations whether this kind of reflection actually takes place on natural beaches. An experiment on a sloping movable bed with a few initial sandbars may be worth conducting, so as to see whether the sandbars develop further offshore by the effect of reflected waves caused by the sandbars themselves.

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```

00000010 C*****
00000020 C
00000030 C          CALCULATION OF REFLECTION COEFFICIENT
00000040 C
00000050 C          - DETUNED WAVE, NORMAL INCIDENCE -
00000060 C
00000070 C          - PLOT OF A,B VERSUS X
00000080 C          FOR OME/OM(X=0)
00000090 C          PARAMETER K*XH(X=0),LO/L
00000100 C          OM*L/CG(X=0)
00000110 C*****
00000120 C
00000130 C          REAL      KH,KDHO,KDH,KOHO
00000140 C          REAL      X(101),CG(101),OM(101),P(101),
00000150 C          +          F(101),G(101),GX(101),
00000160 C          +          AA(10,101),BB(10,101),
00000170 C          +          GO(10)
00000180 C          COMPLEX   A(100),B(100),C(100),D(100),L(100),U(100),W(100),
00000190 C          +          Z(100),BBB(101),A11(100),
00000200 C          +          A12(100),A22(100),Q(101),DBB(101),CI
00000210 C
00000230 C          DATA N/100/
00000240 C          CI=(0.,1.)
00000250 C          WRITE(6,*) 'KH(X=0)= '
00000260 C          READ(5,*) KOHO
00000270 C          WRITE(6,*) 'LO/L= '
00000280 C          READ(5,*) XLO
00000290 C          WRITE(6,*) 'OM*L/CG(X=0)= '
00000300 C          READ(5,*) F(1)
00000310 C
00000320 C          WRITE(6,*) 'NUMBER OF PLOT (NG)= '
00000330 C          READ(5,*) NG
00000340 C          WRITE(6,*) 'OME/OM(X=0)= '
00000350 C          READ(5,*) (GO(I),I=1,NG)
00000360 C
00000370 C          DX=-1./N
00000380 C          KDHO=KOHO*TANH(KOHO)
00000390 C          CGO=1./KOHO*(1.+2.*KOHO/SINH(2.*KOHO))
00000400 C
00000540 C          DO 10 I=1,N+1
00000550 C          X(I)=DX*(I-1)
00000560 C          H=(XLO-X(I))/XLO
00000570 C          KDH=KDHO*H
00000580 C          KH=FK(KDH)
00000590 C          CG(I)=1./KH*(1.+2.*KH/SINH(2.*KH))*H/CGO
00000600 C          OM(I)=SINH(2.*KOHO)/SINH(2.*KH)
00000610 C          OM(I)=SINH(2.*KOHO)/SINH(2.*KH)*KH/KOHO/H
00000620 C          F(I)=F(1)*OM(I)/CG(I)
00000640 C          10 CONTINUE
00000650 C
00000660 C          DO 90 JJJ=1,NG
00000670 C          G(1)=GO(JJJ)
00000680 C
00000690 C          DO 13 I=1,N+1
00000780 C          G(I)=G(1)/OM(I)
00000790 C          13 CONTINUE
00000800 C
00000801 C          GX(1)=(-3.*G(1)+4.*G(2)-G(3))/2./DX
00000802 C          DO 11 I=2,N

```

```

00000803      GX(I)=(G(I+1)-G(I-1))/2./DX
00000804      11  CONTINUE
00000805      GX(N+1)=(3.*G(N+1)-4.*G(N)+G(N-1))/2./DX
00000806 C
00000807      DO 12 I=1,N+1
00000808          P(I)=-1./F(I)
00000809          Q(I)=(0.,1.)*GX(I)+F(I)*(G(I)**2-1.)
00000810      12  CONTINUE
00000811 C
00000812      DO 20 I=1,N
00000813          A11(I)=1./DX*(P(I)+P(I+1))+DX*(1./2.*Q(I)+1./6.*Q(I+1))
00000814          A12(I)=-1./DX*(P(I)+P(I+1))+DX/6.*(1.*Q(I)+1.*Q(I+1))
00000815          A22(I)=1./DX*(P(I)+P(I+1))+DX*(1./6.*Q(I)+1./2.*Q(I+1))
00000816      20  CONTINUE
00000820 C
00000830          A(1)=A22(1)+A11(2)
00000830          C(1)=A12(2)
00000840          DO 30 I=2,N-1
00000860              A(I)=A22(I)+A11(I+1)
00000870              B(I)=A12(I)
00000880              C(I)=A12(I+1)
00000890      30  CONTINUE
00000900          A(N)=A22(N)-2.*(0.,1.)*G(N+1)
00000910          B(N)=A12(N)
00000920 C
00000930          DO 40 I=1,N-1
00000940              Z(I)=(0.,0.)
00000950      40  CONTINUE
00000960          Z(N)=-2.*(0.,1.)*SQRT(CG(N+1))
00000970 C
00000980          CALL LUDCMP(A,B,C,D,L,U,N)
00000990          CALL FBSOLV(D,L,U,BBB,Z,W,N)
00001000 C
00001010          DO 45 I=2,N+1
00001020              I2=N+3-I
00001030              BBB(I2)=BBB(I2-1)
00001040              BB(JJJ,I2)=CABS(BBB(I2)/SQRT(CG(I2)))
00001050      45  CONTINUE
00001060              BB(JJJ,1)=0.
00001070              BBB(1)=(0.,0.)
00001080 C
00001090              DBB(1)=(-3.*BBB(1)+4.*BBB(2)-BBB(3))/2./DX
00001100              DO 50 I=2,N
00001110                  DBB(I)=(BBB(I+1)-BBB(I-1))/2./DX
00001120      50  CONTINUE
00001130              DBB(N+1)=(3.*BBB(N+1)-4.*BBB(N)+BBB(N-1))/2./DX
00001140 C
00001150              DO 55 I=1,N+1
00001152                  AA(JJJ,I)=CABS(BBB(I)*G(I)+DBB(I)/F(I)/CI)
00001154                  + /SQRT(CG(I))
00001180      55  CONTINUE
00001190 C
00001320          WRITE(6,*) JJJ
00001330      90  CONTINUE
00001340 C
00001341 C      WRITE(7,*) 'CG,OM,F,G,BBB'
00001342 C      DO 1000 I=1,N+1
00001343 C      WRITE(7,*) CG(I),OM(I),F(I),G(I),BBB(I)
00001344 C1000 CONTINUE
00001360          CALL FIG(X,AA,BB,N,NG,CG)
00001370          STOP
00001380          END
00001385 *STARTC DDXYPT1
00001390 C*****
00001400 C
00001410          SUBROUTINE LUDCMP(A,B,C,D,L,U,N)

```

```

00001420 C
00001430      COMPLEX L(N),A(N),B(N),C(N),D(N),U(N)
00001440 C
00001450      D(1)=A(1)
00001460      U(1)=C(1)
00001470      DO 10 I=2,N-1
00001480      L(I)=B(I)/D(I-1)
00001490      D(I)=A(I)-L(I)*U(I-1)
00001500      U(I)=C(I)
00001510      10 CONTINUE
00001520      L(N)=B(N)/D(N-1)
00001530      D(N)=A(N)-L(N)*U(N-1)
00001540 C
00001550      RETURN
00001560      END
00001570 C*****
00001580 C
00001590      SUBROUTINE FBSOLV(D,L,U,Y,Z,W,N)
00001600 C
00001610      COMPLEX L(N),D(N),U(N),Y(N),Z(N),W(N)
00001620 C
00001630      W(1)=Z(1)
00001640      DO 10 I=2,N
00001650      W(I)=Z(I)-W(I-1)*L(I)
00001660      10 CONTINUE
00001670 C
00001680      Y(N)=W(N)/D(N)
00001690      DO 20 I=1,N-1
00001700      II=N-I
00001710      Y(II)=(W(II)-Y(II+1)*U(II))/D(II)
00001720      20 CONTINUE
00001730 C
00001740      RETURN
00001750      END
00001760 C*****
00001770 C
00001780      FUNCTION FK(FKDH)
00001790 C
00001800      FKHO=FKDH
00001810      DO 10 I=1,1000
00001820      FKH=FKHO-(FKHO-FKDH/TANH(FKHO))/(1.+FKDH/(SINH(FKHO))**2)
00001830      IF(ABS((FKH-FKHO)/FKHO).LT.1.0E-4) GO TO 20
00001840      FKHO=FKH
00001850      10 CONTINUE
00001860      WRITE(6,*) 'FK:NO CONVERGENCE'
00001870      STOP
00001880 C
00001890      20 FK=FKH
00001900      RETURN
00001910      END
00001920 C*****
00001930 C
00001940      SUBROUTINE FIG(X,AA,BB,N,NG,CG)
00001950 C
00001960      REAL X(101),AA(10,101),BB(10,101),Y(101),CG(101)
00001970 C
00001980      WRITE(6,*) 'XMN,XX,XC,XAX,YMN,XX,YC,YAX'
00001990      READ(5,*) XMN,XX,XC,XAX,YMN,XX,YC,YAX
00002000      CALL SPACE(XMN,XX,XC,XAX,0,YMN,XX,YC,YAX,0,3)
00002010      DO 10 I=1,NG
00002020      DO 20 J=1,N+1
00002030      Y(J)=AA(I,J)
00002040      20 CONTINUE
00002050      CALL ILINE(X,Y,N+1,1,0,1)
00002060      10 CONTINUE
00002070      DO 11 I=1,NG

```

```

00002080      DO 21 J=1,N+1
00002090      Y(J)=BB(I,J)
00002100      21 CONTINUE
00002110      CALL ILINE(X,Y,N+1,2,0,1)
00002120      11 CONTINUE
00002140      DO 22 J=1,N+1
00002150      Y(J)=SQRT(CG(N+1)/CG(J))
00002160      22 CONTINUE
00002170      CALL ILINE(X,Y,N+1,4,0,1)
00002181      CALL VSTERM(ID,ID)
00002182      CALL GPSLTM
00002183      RETURN
00002184      END

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OF B7368.HARA.FORT LIST END

MEMBER CHAR1 OF B7368.HARA.FORT

```

00000040 C*****
00000050 C
00000060 C      CALCULATION OF REFLECTION COEFFICIENT AND LONG WAVE
00000070 C
00000080 C      UNSTEADY WAVE, NORMAL INCIDENCE, CONSTANT DEPTH
00000090 C
00000100 C      3-RD A-B
00000110 C
00000130 C      LINK WITH LONG1,AL40
00000140 C*****
00000150 C
00000160 C      REAL      KH
00000170 C      REAL      F1(601)
00000180 C      COMPLEX AL
00000190 C      COMMON      OMEGA
00000200 C      +          /BLK2/N,NDIV,NN,F,KH,CGC,GHCC,DX,DT,MO,MMM,NT
00000210 C      DATA      PI/3.14159/
00000220 C
00000230 C      WRITE(6,*) 'OMEGA,N,NDIV,F,KH='
00000240 C      READ(5,*) OMEGA,N,NDIV,F,KH
00000250 C      WRITE(6,*) 'OMEGA=',OMEGA
00000260 C      WRITE(6,*) 'N,NDIV=',N,NDIV
00000270 C      WRITE(6,*) 'F,KH=',F,KH
00000280 C      NN=N/NDIV
00000290 C
00000300 C      CGC=0.5+KH/SINH(2.*KH)
00000310 C      GHCC=KH/TANH(KH)
00000320 C      WRITE(6,*) 'CG/CO=',CGC/SQRT(GHCC)
00000330 C
00000340 C      DX=-1./NN
00000350 C      DT=-DX*F/SQRT(GHCC)*CGC/2.
00000360 C      MO=1
00000370 C      RMMM=PI/DT
00000380 C      WRITE(6,*) 'PI/DT=',RMMM
00000390 C      WRITE(6,*) 'PI/DT?,TIME/PI?'
00000400 C      READ(5,*) MMM,NT
00000410 C      WRITE(6,*) 'MMM,NT=',MMM,NT
00000420 C
00000430 C      DO 10 I=N/2+1,N+1
00000440 C      TAU=(I-(N/2+1))*(-DX)*F
00000450 C      F1(I)=(CABS(AL(TAU)))*2*(CGC-0.25)/(CGC**2*TANH(KH)-KH)
00000460 C      10 CONTINUE
00000470 C      DO 20 I=1,N/2
00000480 C      F1(I)=0.
00000490 C      20 CONTINUE
00000500 C
00000540 C      WRITE(11) NT+1,N/(N/600)+1
00000580 C      WRITE(12) NT+1,N/(N/600)+1
00000590 C
00000600 C      CALL XXX(N,NN,NDIV,DX)
00000610 C      CALL TIMEST(F1)
00000620 C
00000650 C
00000660 C      STOP
00000670 C      END
00000671 C *STARTC LONG1
00000672 C *STARTC AL40
00000680 C*****
00000690 C

```

```

00000700      FUNCTION AOO(T,DT,AOOR,AOOI)
00000710 C
00000720      COMPLEX AOO
00000730      REAL      AOOR(1500),AOOI(1500)
00000740 C
00000750      TN=T/DT
00000760      N=TN
00000770      DTN=TN-N
00000780      AOORR=(1.-DTN)*AOOR(1+N)+DTN*AOOR(2+N)
00000790      AOOII=(1.-DTN)*AOOI(1+N)+DTN*AOOI(2+N)
00000800      AOO=CMPLX(AOORR,AOOII)
00000810 C
00000820      RETURN
00000830      END
00000840 C*****
00000850 C
00000860      FUNCTION BL(T,DT,BLR,BLI)
00000870 C
00000880      COMPLEX BL
00000890      REAL      BLR(1500),BLI(1500)
00000900 C
00000910      TN=T/DT
00000920      N=TN
00000930      DTN=TN-N
00000940      BLRR=(1.-DTN)*BLR(1+N)+DTN*BLR(2+N)
00000950      BLII=(1.-DTN)*BLI(1+N)+DTN*BLI(2+N)
00000960      BL=CMPLX(BLRR,BLII)
00000970 C
00000980      RETURN
00000990      END
00001070 C*****
00001080 C
00001090      SUBROUTINE XXX(N,NN,NDIV,DX)
00001100 C
00001110      REAL      X(601)
00001120 C
00001130      DO 10 I=1,N+1
00001140      X(I)=DX*(I-(NDIV/2-1)*NN-1)
00001150 10  CONTINUE
00001160      WRITE(11) (X(N+2-I),I=1,N+1,N/600)
00001170      WRITE(12) (X(N+2-I),I=1,N+1,N/600)
00001180 C
00001190      RETURN
00001200      END
00001210 C*****
00001220 C
00001230      SUBROUTINE TIMEST(F1)
00001240 C
00001250      REAL      KH
00001260      REAL      AOR(101),BOR(101),AOI(101),BOI(101),
00001270      +          A1R(601),B1R(601),A1I(601),B1I(601),
00001280      +          A8R(101),B8R(101),A8I(101),B8I(101),
00001290      +          A9R(101),B9R(101),A9I(101),B9I(101),
00001300      +          AOOR(1500),AOOI(1500),BLR(1500),BLI(1500),
00001310      +          H(601),F1(601),HLO(601),HRO(601)
00001320      COMPLEX    AO,BO,AL,DAO,DBO,A1AO,B1BO,A1,B1,
00001330      +          A8,A9,B8,B9,AOO,BL
00001340      COMMON     /BLK2/N,NDIV,NN,F,KH,CGC,GHCC,DX,DT,MO,MMM,NT
00001350 C
00001360      DO 10 I=1,NN+1
00001370      AOR(I)=0.
00001380      BOR(I)=0.
00001390      AOI(I)=0.
00001400      BOI(I)=0.
00001410      A8R(I)=0.
00001420      B8R(I)=0.

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00001430      A8I(I)=0.
00001440      B8I(I)=0.
00001450      A9R(I)=0.
00001460      B9R(I)=0.
00001470      A9I(I)=0.
00001480      B9I(I)=0.
00001490 10  CONTINUE
00001500      DO 11 I=1,N+1
00001510      A1R(I)=0.
00001520      B1R(I)=0.
00001530      A1I(I)=0.
00001540      B1I(I)=0.
00001550 11  CONTINUE
00001560      DO 12 I=1,1500
00001570      A00R(I)=0.
00001580      A00I(I)=0.
00001590      B1R(I)=0.
00001600      B1I(I)=0.
00001610 12  CONTINUE
00001620 C
00001630      DO 40 III=1,MMM*NT+1
00001640 C
00001650      IF(III.LE.MO) GO TO 130
00001660 C
00001670      DO 60 I=1,NN
00001680      A8=CMPLX(A8R(I),A8I(I))
00001690      A9=CMPLX(A9R(I),A9I(I))
00001700      IF(I.EQ.1) DAO=-3.*CMPLX(AOR(I),AOI(I))
00001710      +      +4.*CMPLX(AOR(I+1),AOI(I+1))
00001720      +      -CMPLX(AOR(I+2),AOI(I+2))
00001730      IF(I.NE.1) DAO=CMPLX(AOR(I+1),AOI(I+1))
00001740      +      -CMPLX(AOR(I-1),AOI(I-1))
00001750      A1A0=-CMPLX(BOR(I),BOI(I))*(0.,1.)*DT-DAO/2./DX/F*DT
00001760      IF(III.EQ.MO+1) A1=A1A0
00001770      IF(III.EQ.MO+2) A1=(3.*A1A0-A9)/2.
00001780      IF(III.GE.MO+3) A1=(23.*A1A0-16.*A9+5.*A8)/12.
00001790      A1=A1+CMPLX(AOR(I),AOI(I))
00001800      A8=A9
00001810      A9=A1A0
00001820      A1R(I)=REAL(A1)
00001830      A1I(I)=AIMAG(A1)
00001840      A8R(I)=REAL(A8)
00001850      A8I(I)=AIMAG(A8)
00001860      A9R(I)=REAL(A9)
00001870      A9I(I)=AIMAG(A9)
00001880 60  CONTINUE
00001890      A1R(NN+1)=REAL(AL((III-MO)*DT))
00001900      A1I(NN+1)=AIMAG(AL((III-MO)*DT))
00001910      DO 62 JJJ=2,1500
00001920      JJJJ=1502-JJJ
00001930      A00R(JJJJ)=A00R(JJJJ-1)
00001940      A00I(JJJJ)=A00I(JJJJ-1)
00001950 62  CONTINUE
00001960      A00R(1)=A1R(1)
00001970      A00I(1)=A1I(1)
00001980 C
00001990      DO 65 I=2,NN+1
00002000      B8=CMPLX(B8R(I),B8I(I))
00002010      B9=CMPLX(B9R(I),B9I(I))
00002020      IF(I.EQ.NN+1) DBO=3.*CMPLX(BOR(I),BOI(I))
00002030      +      -4.*CMPLX(BOR(I-1),BOI(I-1))
00002040      +      +CMPLX(BOR(I-2),BOI(I-2))
00002050      IF(I.NE.NN+1) DBO=CMPLX(BOR(I+1),BOI(I+1))
00002060      +      -CMPLX(BOR(I-1),BOI(I-1))
00002070      B1B0=-CMPLX(AOR(I),AOI(I))*(0.,1.)*DT+DBO/2./DX/F*DT
00002080      IF(III.EQ.MO+1) B1=B1B0

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00002090      IF(III.EQ.MO+2) B1=(3.*B1B0-B9)/2.
00002100      IF(III.GE.MO+3) B1=(23.*B1B0-16.*B9+5.*B8)/12.
00002110      B1=B1+CMPLX(B0R(I),B0I(I))
00002120      B8=B9
00002130      B9=B1B0
00002140      B1R(I)=REAL(B1)
00002150      B1I(I)=AIMAG(B1)
00002160      B8R(I)=REAL(B8)
00002170      B8I(I)=AIMAG(B8)
00002180      B9R(I)=REAL(B9)
00002190      B9I(I)=AIMAG(B9)
00002200 65 CONTINUE
00002210      B1R(1)=0.
00002220      B1I(1)=0.
00002230      DO 67 JJJ=2,1500
00002240      JJJJ=1502-JJJ
00002250      BLR(JJJJ)=BLR(JJJJ-1)
00002260      BLI(JJJJ)=BLI(JJJJ-1)
00002270 67 CONTINUE
00002280      BLR(1)=B1R(NN+1)
00002290      BLI(1)=B1I(NN+1)
00002300 C
00002310      DO 110 I=1,NN+1
00002320      A0R(I)=A1R(I)
00002330      A0I(I)=A1I(I)
00002340      B0R(I)=B1R(I)
00002350      B0I(I)=B1I(I)
00002360 110 CONTINUE
00002370 C
00002380      IF((III-1).NE.(III-1)/(N/600*2)*(N/600*2)) GO TO 40
00002390 C
00002400      DO 120 I=1,NN+1
00002410      II=NN+2-I
00002420      A1R(II+(NDIV/2-1)*NN)=A1R(II)
00002430      B1R(II+(NDIV/2-1)*NN)=B1R(II)
00002440      B1R(II)=0.
00002450      A1I(II+(NDIV/2-1)*NN)=A1I(II)
00002460      B1I(II+(NDIV/2-1)*NN)=B1I(II)
00002470      B1I(II)=0.
00002480 120 CONTINUE
00002490 C
00002500      DO 70 I=1,(NDIV/2-1)*NN
00002510      TAU2=(I-(NDIV/2-1)*NN-1)*DX*F
00002520      A1=A00(TAU2,DT,A00R,A00I)
00002530      A1R(I)=REAL(A1)
00002540      A1I(I)=AIMAG(A1)
00002550 70 CONTINUE
00002560 C
00002570      DO 80 I=(NDIV/2*NN)+1+1,NDIV*NN+1
00002580      TAU=(NDIV/2*NN+1-I)*DX*F
00002590      B1=BL(TAU,DT,BLR,BLI)
00002600      B1R(I)=REAL(B1)
00002610      B1I(I)=AIMAG(B1)
00002620 80 CONTINUE
00002630 C
00002640 130 IF((III-1).NE.(III-1)/(N/600*2)*(N/600*2)) GO TO 40
00002650 C
00002660      DO 50 I=(NDIV/2*NN)+1+1,NDIV*NN+1
00002670      TAU=(NDIV/2*NN+1-I)*DX*F
00002680      A1=AL((III-MO)*DT+TAU)
00002690      A1R(I)=REAL(A1)
00002700      A1I(I)=AIMAG(A1)
00002710 50 CONTINUE
00002720 C
00002730      CALL LONG(III,A1R,A1I,B1R,B1I,F1,HLO,HRO,P1,P2)
00002740 C

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00002750      IF((III-1)/MMM*MMM.EQ.(III-1))
00002760      + WRITE(12) (CABS(CMPLX(A1R(N+2-I),A1I(N+2-I))),I=1,N+1,N/600)
00002770      IF((III-1)/MMM*MMM.EQ.(III-1))
00002780      + WRITE(12) (CABS(CMPLX(B1R(N+2-I),B1I(N+2-I))),I=1,N+1,N/600)
00002790      IF((III-1)/MMM*MMM.EQ.(III-1)) WRITE(6,*) III
00002800 C
00002810      40  CONTINUE
00002820 C
00002830      RETURN
00002840      END
00002850 C*****
00002860 C
00002870      SUBROUTINE LONG(III,A1R,A1I,B1R,B1I,F1,HLO,HRO,P1,P2)
00002880 C
00002890      REAL      KH
00002900      REAL      A1R(601),B1R(601),A1I(601),B1I(601),
00002910      +          H(601),F1(601),HLO(601),HRO(601)
00002920      COMMON     /BLK2/N,NDIV,NN,F,KH,CGC,GHCC,DX,DT,MO,MMM,NT
00002930 C
00002940      H(N+1)=0.
00002950      DO 150 I=NDIV/2*NN+2,N
00002960      AAB= ((A1R(I+1)**2+A1I(I+1)**2+B1R(I+1)**2+B1I(I+1)**2) *
00002970      +      -(A1R(I-1)**2+A1I(I-1)**2+B1R(I-1)**2+B1I(I-1)**2))
00002980      +      /2./DX
00002990      H(I)=1./F**2*(CGC-0.25)/KH*GHCC/CGC**2*AAB
00003000 150  CONTINUE
00003010      I=NDIV/2*NN+1
00003020      AAB= (-3.*(A1R(I)**2+A1I(I)**2+B1R(I)**2+B1I(I)**2)
00003030      +      +4.*(A1R(I+1)**2+A1I(I+1)**2+B1R(I+1)**2+B1I(I+1)**2)
00003040      +      -(A1R(I+2)**2+A1I(I+2)**2+B1R(I+2)**2+B1I(I+2)**2))
00003050      +      /2./DX
00003060      AAB2= (3.*(A1R(I)**2+A1I(I)**2+B1R(I)**2+B1I(I)**2)
00003070      +      -4.*(A1R(I-1)**2+A1I(I-1)**2+B1R(I-1)**2+B1I(I-1)**2)
00003080      +      +(A1R(I-2)**2+A1I(I-2)**2+B1R(I-2)**2+B1I(I-2)**2))
00003090      +      /2./DX
00003100      H(I)=1./F**2*(CGC-0.25)/KH*GHCC/CGC**2*AAB
00003110      P2=AAB-AAB2
00003120      ABAB=2.*(A1I(I)*B1R(I)-A1R(I)*B1I(I))
00003130      HHH=1./F**2*(CGC-0.25)/KH*GHCC/CGC**2*AAB2
00003140      +      +1./F*GHCC/CGC/KH*ABAB
00003150      HDIF=HHH-H(I)
00003160      DO 152 I=(NDIV/2-1)*NN+2,NDIV/2*NN
00003170      AAB= ((A1R(I+1)**2+A1I(I+1)**2+B1R(I+1)**2+B1I(I+1)**2)
00003180      +      -(A1R(I-1)**2+A1I(I-1)**2+B1R(I-1)**2+B1I(I-1)**2))
00003190      +      /2./DX
00003200      ABAB=2.*(A1I(I)*B1R(I)-A1R(I)*B1I(I))
00003210      H(I)=1./F**2*(CGC-0.25)/KH*GHCC/CGC**2*AAB
00003220      +      +1./F*GHCC/CGC/KH*ABAB-HDIF
00003230 152  CONTINUE
00003240      I=(NDIV/2-1)*NN+1
00003250      AAB= (-3.*(A1R(I)**2+A1I(I)**2+B1R(I)**2+B1I(I)**2)
00003260      +      +4.*(A1R(I+1)**2+A1I(I+1)**2+B1R(I+1)**2+B1I(I+1)**2)
00003270      +      -(A1R(I+2)**2+A1I(I+2)**2+B1R(I+2)**2+B1I(I+2)**2))
00003280      +      /2./DX
00003290      AAB2= (3.*(A1R(I)**2+A1I(I)**2+B1R(I)**2+B1I(I)**2)
00003300      +      -4.*(A1R(I-1)**2+A1I(I-1)**2+B1R(I-1)**2+B1I(I-1)**2)
00003310      +      +(A1R(I-2)**2+A1I(I-2)**2+B1R(I-2)**2+B1I(I-2)**2))
00003320      +      /2./DX
00003330      ABAB=2.*(A1I(I)*B1R(I)-A1R(I)*B1I(I))
00003340      H(I)=1./F**2*(CGC-0.25)/KH*GHCC/CGC**2*AAB
00003350      +      +1./F*GHCC/CGC/KH*ABAB-HDIF
00003360      HHH=1./F**2*(CGC-0.25)/KH*GHCC/CGC**2*AAB2-HDIF
00003370      P1=AAB-AAB2
00003380      HDIF=HDIF+HHH-H(I)
00003390      DO 154 I=2,(NDIV/2-1)*NN
00003400      AAB= ((A1R(I+1)**2+A1I(I+1)**2+B1R(I+1)**2+B1I(I+1)**2)

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00003410      +      -(A1R(I-1)**2+A1I(I-1)**2+B1R(I-1)**2+B1I(I-1)**2))
00003420      +      /2./DX
00003430      H(I)=1./F**2*(CGC-0.25)/KH*GHCC/CGC**2*AABB-HDIF
00003440 154      CONTINUE
00003450      H(1)=-HDIF
00003460 C
00003470      CALL LONG2(III,H,F1,HLO,HRO,P1,P2)
00003480 C
00003490      RETURN
00003500      END

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MEMBER CHAR1 OF B7368.HARA.FORT LIST END

MEMBER LONG1 OF B7368.HARA.FORT

```
00000010 C*****
00000020 C
00000030 C      CALCULATION OF LONG WAVE (SUBROUTINE)
00000040 C
00000050 C*****
00000060 C
00000070 C      SUBROUTINE LONG2(III,H,F1,HLO,HRO,P1,P2)
00000080 C
00000090 C      REAL      KH
00000100 C      REAL      H(601),HLO(601),HRO(601)
00000110 C      +          ,F1(601),Y(601)
00000120 C      COMPLEX AL
00000130 C      COMMON    OMEGA
00000140 C      +          /BLK2/N,NDIV,NN,F,KH,CGC,GHCC,DX,DT,MO,MMM,NT
00000150 C
00000160 C      CO=1./F*SQRT(GHCC)/CGC
00000170 C      DDT=DT*2.
00000180 C
00000190 C      P1=-CO**2/2./SINH(2.*KH)*P1
00000200 C      P2=-CO**2/2./SINH(2.*KH)*P2
00000210 C
00000220 C      IF(III.GE.2) GO TO 45
00000230 C      DO 40 I=1,N+1
00000240 C      Y(I)=F1(I)
00000250 C      Y(I)=-Y(I)/(CGC-0.25)*(CGC**2*TANH(KH)-KH)
00000260 C      40 CONTINUE
00000270 C      DO 41 I=1,(NDIV/2-1)*NN
00000280 C      HLO(I)=(H(I)+P1+P2)*DDT/2.
00000290 C      HRO(I)=(H(I)+P1+P2)*DDT/2.
00000300 C      41 CONTINUE
00000310 C      I=(NDIV/2-1)*NN+1
00000320 C      HLO(I)=(H(I)+P2)*DDT/2.
00000330 C      HRO(I)=(H(I)+P2)*DDT/2.
00000340 C      DO 42 I=(NDIV/2-1)*NN+2,NDIV/2*NN
00000350 C      HLO(I)=(H(I)+P2)*DDT/2.
00000360 C      HRO(I)=(H(I)+P2)*DDT/2.
00000370 C      42 CONTINUE
00000380 C      I=NDIV/2*NN+1
00000390 C      HLO(I)=H(I)*DDT/2.
00000400 C      HRO(I)=(H(I)+P2)*DDT/2.
00000410 C      DO 43 I=NDIV/2*NN+2,N+1
00000420 C      HLO(I)=H(I)*DDT/2.
00000430 C      HRO(I)=H(I)*DDT/2.
00000440 C      43 CONTINUE
00000450 C
00000460 C      GO TO 55
00000470 C
00000480 C      45 DO 60 I=2,N+1
00000490 C      I2=3+N-I
00000500 C      HLO(I2)=HLO(I2-1)+H(I2)*DDT
00000510 C      IF(I2.EQ.(NDIV/2*NN+1)) HLO(I2)=HLO(I2)+P2*DDT/2.
00000520 C      IF(I2.LE.(NDIV/2*NN)) HLO(I2)=HLO(I2)+P2*DDT
00000530 C      IF(I2.EQ.((NDIV/2-1)*NN+1)) HLO(I2)=HLO(I2)+P1*DDT/2.
00000540 C      IF(I2.LE.((NDIV/2-1)*NN)) HLO(I2)=HLO(I2)+P1*DDT
00000550 C      60 CONTINUE
00000560 C      HLO(1)=HLO(2)
00000570 C      DO 70 I=1,N
00000580 C      HRO(I)=HRO(I+1)+H(I)*DDT
00000590 C      IF(I.EQ.(NDIV/2*NN+1)) HRO(I)=HRO(I)+P2*DDT/2.
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00000600      IF(I.LE.(NDIV/2*NN)) HRO(I)=HRO(I)+P2*DDT
00000610      IF(I.EQ.((NDIV/2-1)*NN+1)) HRO(I)=HRO(I)+P1*DDT/2.
00000620      IF(I.LE.((NDIV/2-1)*NN)) HRO(I)=HRO(I)+P1*DDT
00000630 70    CONTINUE
00000640      HRO(N+1)=0.
00000650 C
00000660      IF((III-1)/MMM*MMM.NE.(III-1)) GO TO 50
00000670 C
00000680      DO 100 I=1,N+1
00000690      IPLUS=I+(III-1)/2
00000700      IMINUS=I-(III-1)/2
00000710      IF(IPLUS.GT.N+1) IPLUS=N+1
00000720      IF(IMINUS.LT.1) IMINUS=1
00000730      Y(I)=HRO(I)-HLO(I)
00000740      IF(I.EQ.((NDIV/2-1)*NN+1)) Y(I)=Y(I)-P1*DDT/2.
00000750      IF(I.EQ.(NDIV/2*NN+1)) Y(I)=Y(I)-P2*DDT/2.
00000760      Y(I)=-0.5/CO*Y(I)+0.5*(F1(IPLUS)+F1(IMINUS))
00000770      +0.5/CO/F*(F1(IPLUS)-F1(IMINUS))
00000780      Y(I)=-Y(I)/(CGC-0.25)*(CGC**2*TANH(KH)-KH)
00000790 100  CONTINUE
00000800 C
00000810 55  WRITE(11) (Y(N+2-I),I=1,N+1)
00000820 C
00000830 50  RETURN
00000840      END

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MEMBER LONG1 OF B7368.HARA.FORT LIST END

MEMBER AL40 OF B7368.HARA.FORT

```
00000010 C*****
00000020 C
00000030 C      INPUT A(T)
00000040 C
00000050 C*****
00000060 C
00000070 C      FUNCTION AL(T)
00000080 C
00000090 C      COMPLEX AL
00000100 C      COMMON OMEGA
00000110 C
00000120 C      AL=SIN(OMEGA*T)**2
00000130 C      IF(T.GT.3.14159/OMEGA) AL=0.
00000140 C      IF(T.LT.0.) AL=0.
00000150 C
00000160 C      RETURN
00000170 C      END
```

MEMBER AL40 OF B7368.HARA.FORT LIST END