The Dissipation Range of Wind-Wave Spectra Observed on a Lake

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ABSTRACT

Based on an interpretation of a field experiment it is argued that, due to breaking of wind waves in deep water, the dissipation of energy is restricted to a range of frequencies $\omega > \omega_g$ much higher than the frequency ω_m of the dominant waves. In this dissipation range the spectrum has the form $S(\omega) = \beta g^2\omega^{-5}$ where g is the acceleration due to gravity and $\beta = 0.025$. For spectral wave components at $\omega \leqslant \omega_g$, only a local balance between energy input from the wind and the weak, third-order, nonlinear interaction is important. Asymptotically as $\omega \gg \omega_m$ the wind input becomes unimportant, and the wave spectrum has the Kitaigorodskii form of a Kolgomorov analog $S(\omega) = 2a\epsilon_0{}^{1/3}g^{4/3}\omega^{-4}$ where ϵ_0 is a constant flow of mean energy per unit surface area through the spectrum dissipated at high frequencies (when multiplied by g and water density ρ_w). From a method of M. S. Longuet-Higgins we estimate the magnitude of the dissipation (due to wave breaking) and find the Kolmogorov constant to be $a \approx 0.6$. When a model, explained by Phillips, for wind energy input to the wave spectrum is applied to a simplified spectral model prescribing the scales of dissipation and growth of spectral wave components, good agreement is found with measurements by Donelan et al. of the coefficient $2a\epsilon_0{}^{1/3}$ and its dependence on the frequency ω_m of the dominant waves at the spectral peak.

1. Introduction

a. Development of the theory

Since the measurements in the late 1950s it has been known from experiments that the frequency spectra of water elevation due to wind waves at different stages of development practically lie on one line. Phillips (1958) proposed that this universal line is due to the statistical limitation of the random wave field caused by wave breaking, yielding, on similarity grounds, a spectral form in the frequency domain

$$S(\omega) = \beta g^2\omega^{-5} \quad (1.1)$$

called Phillips' saturation form where g is the acceleration due to gravity, ω (angular) frequency, and β is a universal constant.

Since then especially the work of K. Hasselmann and co-workers has shown, beginning with the derivation of the wave kinetic equation (1962) through the JONSWAP report (1973) to recent numerical calculations (e.g. Komen et al. 1984), that weak nonlinear interactions between wave components is an important mechanism for the development of the wind-wave spectrum with fetch or time. However, until recently wind input to the spectral wave energy and dissipation of this energy have both been understood only in a qualitative sense, their quantitative description being very speculative. A general similarity form of wave energy input from the wind has now been empirically established by Plant (1982) and explained in more general terms by Phillips (1985).

Concerning the dissipation by wave breaking, Kitaigorodskii (1983) suggested that wave breaking is important only at frequencies higher than some frequency ω_g of gravitational instability, which is much higher than the frequency of dominant waves ω_m. The weak nonlinear interactions then serve to redistribute energy from the range $\omega_m \leqslant \omega \leqslant \omega_g$ to new waves at $\omega \leqslant \omega_m$ and to dissipation at $\omega \geqslant \omega_g$ in such a way that the nonlinear divergence of energy in the range $\omega_m \leqslant \omega$

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$\leqslant \omega_g$ is balanced by the wind energy input in a stationary wave field. Several experiments, of which Kahma (1981), Forristall (1981) and Donelan et al. (1985) are the most convincing, have shown that the high-frequency part of the spectrum is adequately described by the form $S(\omega) = \alpha_u g u_{10} \omega^{-4}$ where u_{10} is the mean wind speed 10 meters above the water and α_u is nearly independent of wind speed and fetch and has the mean value $\alpha_u \approx 4.5 \times 10^{-3}$ (Kahma 1981).

This paper provides a theoretical general explanation of the development of wind-wave spectra. The idea may be outlined as follows.

After some time of wind action (or some fetch over which wind has acted on the waves) the wave field has developed through initial wave generation processes to a state where the wind acts mainly on the dominant spectral wave components at a wavenumber k_m in wavenumber space. According to the hypothesis of Kitaigorodskii (1983), dissipation due to gravitational instability takes place only at very high wavenumbers, $k_g \gg k_m$. It was further suggested by Kitaigorodskii that wind energy input becomes asymptotically negligible at high wavenumbers, and that a $k_m \ll k < k_g$ range exists where local weak nonlinear interactions are the only forces acting on individual wave components, with possibly a sharp transition to the range of breaking waves at some $k > k_g$.

The energy loss from waves may generally be due to either gravitational instability of the potential motion (i.e., wave breaking), wave–turbulence interaction (e.g., see Kitaigorodskii and Lumley 1983), or viscous decay of the wave motion. Any such physical process that leads to energy loss from the wave potential motion to mean kinetic energy, turbulent energy or heat, we will term *dissipation* and its characteristic wavenumber range, the *dissipation range.*

b. *The spectral asymptotes of weak nonlinear energy transfer*

The weak nonlinear interactions are governed by the Hasselmann (1962) nonlinear energy transfer equation which is written in terms of spectral wave action density $N_{\mathbf{k}} = F_{\mathbf{k}}/\omega_{\mathbf{k}}$ where $F_{\mathbf{k}}$ is the energy density in wavenumber space and $\omega_{\mathbf{k}}$ is given by the dispersion relation.

When surface currents are negligible, the energy transfer equation for the action spectrum $N_{\mathbf{k}}$ is

$$\frac{\partial N_{\mathbf{k}}}{\partial t} + \mathbf{C}_g \cdot \nabla_x N_{\mathbf{k}} = S_{\mathbf{k}}^{nl} + N_{\mathbf{k}}^{+} - D_{\mathbf{k}} \quad (1.2)$$

where $\mathbf{C}_g$ is the group velocity of the waves of wavenumber $\mathbf{k}$, $S_{\mathbf{k}}^{nl}$ the directional spectrum of the rate of convergence of weak nonlinear interactions, $N_{\mathbf{k}}^{+}$ is the spectrum of action input from the wind to the waves, and $D_{\mathbf{k}}$ the spectrum of the rate of action loss from waves.

We wish to describe the spectral ranges which are horizontally homogeneous and in equilibrium. This means that the left-hand side of Eq. (1.2) is zero. Due to weak, nonlinear interactions the spectral flux convergence is (Hasselmann 1963)

$$S_{\mathbf{k}}^{nl} = \iiint g^2 |Q|^2 \times \{(N_{\mathbf{k}1} + N_{\mathbf{k}2}) N_{\mathbf{k}3} N_{\mathbf{k}} - (N_{\mathbf{k}3} + N_{\mathbf{k}}) N_{\mathbf{k}1} N_{\mathbf{k}2}\} \times \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\delta(\omega_{\mathbf{k}} + \omega_{\mathbf{k}1} - \omega_{\mathbf{k}2} - \omega_{\mathbf{k}2}) \times d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3. \quad (1.3)$$

Integration is taken over all wavenumber triplets ($\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$) which interact resonantly with wavenumber $\mathbf{k}$ according to the resonance condition

$$\mathbf{k} + \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3$$

$$\omega_{\mathbf{k}} + \omega_{\mathbf{k}1} = \omega_{\mathbf{k}2} + \omega_{\mathbf{k}3}.$$

For infinitely deep water the interaction coefficient Q is a third-order homogeneous polynomial in the four variables ($\mathbf{k}$, $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$).

In Eq. (1.3) the spectral flux convergence scales to the third order in the action density $N_{\mathbf{k}}$, and assuming a scalar dispersion relation $\omega^2 = gk$ it scales to the power 19/2 in wavenumber modulus k. This scaling consideration, which was used by Kitaigorodskii (1983) and Phillips (1985), leads to the similarity form

$$S_{\mathbf{k}}^{nl} \propto g^{3/2} k^{19/2} N_{\mathbf{k}}^{3}. \quad (1.4)$$

The existence of two equilibrium ranges of the Kolmogorov type in atmospheric turbulence, for which both the energy source and dissipation terms in Eq. (1.2) are zero, was proposed by Zakharov and Zaslavskii (1982) and Kitaigorodskii (1983). The energy transfer equation then reduces to a so-called kinetic equation

$$S_{\mathbf{k}}^{nl} = 0. \quad (1.5)$$

With angularly integrated action, N_k, defined by $N_k = \int_{-\pi}^{\pi} N_{\mathbf{k}} d\theta$, an exact solution of Eq. (1.5) was found by Zakharov and Filonenko (1968) in the form of a power function $N_k \propto k^{-4}$.

In their comprehensive review of the theory of the kinetic wave equation, Zakharov and Zaslavskii (1982) pointed to another power-form solution of Eq. (1.5), $N_k \propto k^{-23/6}$. The solutions scale with a constant spectral flux of wave energy density $\rho_w g \epsilon_0$ and of wave action density $\rho_w g \epsilon_N$, respectively,

$$N_k = a \epsilon_0^{1/3} g^{-2/3} k^{-4} \quad (1.6)$$

or

$$N_k = a_1 \epsilon_N^{1/3} g^{-1/2} k^{-23/6}. \quad (1.7)$$

The factors a and a_1 are assumed to be universal constants of order unity in analogy with Kolmogorov's theory of isotropic turbulence. Kitaigorodskii (1983), using some estimates for the overall momentum balance, gave plausible arguments for the constant a actually being of order unity.

c. On the existence of a quasi-universal form of the spectral tail

It is convenient to express spectral density and frequency in dimensionless form. The scaling parameters are wind velocity at a certain height, e.g. 10 meters, u_{10}, and gravity, g. The dimensionless frequency is for linear deep-water waves with phase velocity $c = g/\omega$,

$$\tilde{\omega} = \frac{\omega u_{10}}{g} = \frac{u_{10}}{c}, \qquad (1.8)$$

and the dimensionless spectral density

$$\tilde{S}(\tilde{\omega}) = \frac{S(\omega)g^3}{u_{10}^5}. \qquad (1.9)$$

In terms of the dimensionless energy flux

$$\tilde{\epsilon}_0 = \frac{\epsilon_0 g}{u_{10}^3} \qquad (1.10)$$

the asymptotic spectrum corresponding to Eq. (1.6) as $\omega \gg \omega_c$, where $\omega_c = g/c_c$ is a characteristic frequency of energy input from the wind, becomes

$$\tilde{S}(\tilde{\omega}) = 2a\tilde{\epsilon}_0{}^{1/3}\tilde{\omega}^{-4}. \qquad (1.11)$$

Correspondingly, in terms of the dimensionless action flux

$$\tilde{\epsilon}_N = \frac{\epsilon_N g^2}{u_{10}^4}, \qquad (1.12)$$

the frequency spectrum corresponding to Eq. (1.7) becomes

$$\tilde{S}(\tilde{\omega}) = 2a_1\tilde{\epsilon}_N{}^{1/3}\tilde{\omega}^{-11/3}. \qquad (1.13)$$

Besides momentum and energy density, also the total mean action per unit surface area of a system of waves is conserved by weakly nonlinear interactions (Zakharov and Zaslavskii 1982). Energy and thereby action are created by wind at low frequencies near the spectral peak. Some fraction of the energy input is dissipated at much higher frequencies. A much smaller fraction of action (= energy/frequency) is dissipated together with the energy (Zakharov and Zaslavskii 1983). A large fraction of the action input is thus transferred to generation of new waves near the spectral peak. Here, fully developed waves show $\tilde{\omega}_m \approx 0.7$ [see Eq. (1.8)], while experiments show that energy input to the waves takes place for $\tilde{\omega} > \tilde{\omega}_c \approx 2$. Therefore, Zakharov and Zaslavskii (1983) suggested that the spectral form in Eq. (1.7) may be found in the frequency range $0.7 \leqslant \tilde{\omega} \leqslant 2$. The action flux to new waves is advected away, carried by the waves and gives rise to the growth of total action (or of energy) density of the wave spectrum with fetch. From measurements of this growth Zakharov and Zaslavskii (1983) estimated the action flux and by comparing with the magnitude of the well-known Pierson–Moskowitz model spectrum, they estimated the universal constant of Eq. (1.7) to be $a_1 \approx 0.8$.

Based on the measurements reported in the present paper we shall argue (section 4c) that with energy cascading from low to high frequencies the spectral form [Eq. (1.6)] applies to young, growing waves at relatively short fetches or time with the Kolmogorov type constant a estimated to be $a \approx 0.6$. It is worth pointing out that the energy flux spectral form (Eq. (1.6)) applies to young waves while the action flux form [Eq. (1.7)] applies to well developed waves, but yet the spectral tail has the overall quasi-universal form of Eq. (1.1). This means that at some stage of development with the dimensionless peak frequency $\tilde{\omega}_m = \tilde{\omega}_{mt}$ say, there is transition from the spectral form [Eq. (1.11)] to the spectral form [Eq. (1.13)] in the sense that both equations equally represent the spectral value at some frequency ω_t on the spectral tail. Equating (1.11) and (1.13) at ω_t yields

$$2a_1\tilde{\epsilon}_N{}^{1/3} = 2a\tilde{\epsilon}_0{}^{1/3}\tilde{\omega}_t{}^{-1/3}. \qquad (1.14)$$

Zaslavskii and Lobysheva (1983) applied empirical data to the power-form relation $\tilde{\epsilon}_N = b_N\tilde{\omega}_m{}^{\alpha_N}$, and found for developed waves with $\tilde{\omega}_m \leqslant 2$ the values $b_N \approx 1.4 \times 10^{-8}$; $\alpha_N \approx 1.0$. In Fig. 1 are shown selected well-developed wave spectra of Forristall (1981) some of which seem indeed to be of the form $S(\omega) \propto \omega^{-11/3}$ rather than ω^{-4} with the straight line representing $2a_1\tilde{\epsilon}_N{}^{1/3} = 3.6 \times 10^{-3}$. Later we shall show (see Fig. 6) that approximately $2a\tilde{\epsilon}_0{}^{1/3}\tilde{\omega}_m{}^{1/3} = b_0 = 5 \times 10^{-3}$ in the neighborhood of $\tilde{\omega}_m = 2$. Thus, for transition in the neighborhood of $\tilde{\omega}_t \approx \tilde{\omega}_{m_t} = 2$ we have

$$a_1\tilde{\omega}_{m_t}\left(\frac{\omega_t}{\omega_{m_t}}\right)^{1/3} = b_0/(2b_N{}^{1/3}) \approx 1.0. \qquad (1.15)$$

An empirical estimate of the transition peak frequency $\tilde{\omega}_{m_t}$ may be derived from the assumed continuity of the growth de/dt across the transition. For ideal fetch-limited situations empirical relations have been proposed in the dimensionless form $d\tilde{e}/d\tilde{t} = b_e\tilde{\omega}_m{}^{\alpha_e}$ for very developed waves (Zaslavskii and Lobysheva 1983, see above, with $\tilde{e} = \tilde{\omega}_m\tilde{\epsilon}_N$) and for growing waves (JONSWAP: Hasselman et al. 1973, $b_e \approx 0.8 \times 10^{-3}$, $\alpha_e = -1$). Equating the two expressions for $d\tilde{e}/d\tilde{t}$ yields $\tilde{\omega}_{m_t} \approx 1.8$. In connection with Eq. (1.15) this yields $a_1(\omega_t/\omega_{m_t})^{1/3} \approx 0.6$. Although these estimates show fairly good consistency with our hypothesis, an $\omega^{-11/3}$ spectral form has been identified only for the case of Fig. 1a while it is completely absent in the spectra of Fig. 1b where the dimensionless angular peak frequency $\tilde{\omega}_m$ is only slightly larger than one.

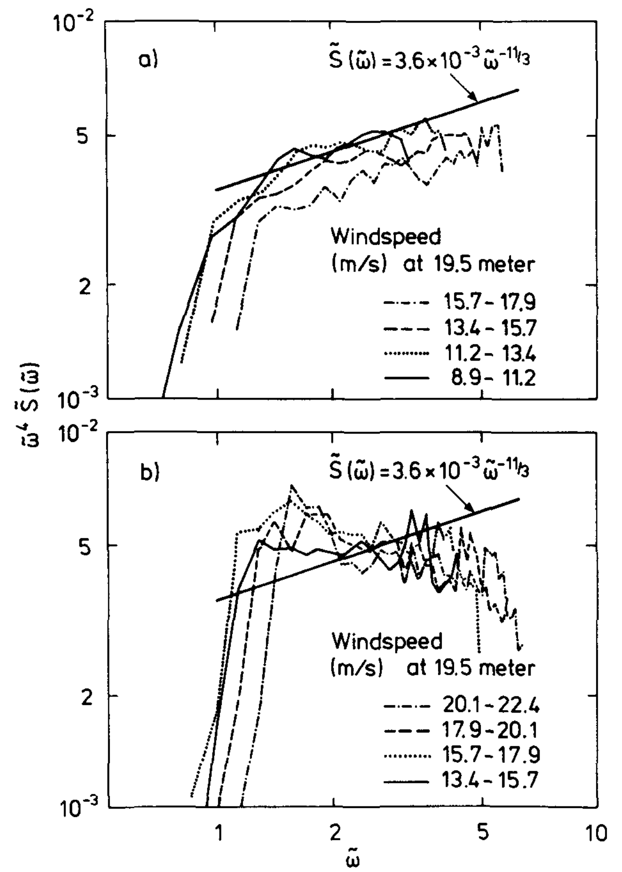

FIG. 1. Spectra of well-developed waves selected from Forristall (1981) using the criteria that 1) the spectral tail extends to at least three times the peak frequency and 2) the apparent 90% confidence limits should be less than ±30% (a): Hurricane Anita, (b) Hurricane Eloise.

2. The high-frequency spectral tail

a. Spectral cascade and dissipation of wave energy

Zakharov and Zaslavskii estimated the universal constant a_1 of the asymptotic spectrum in Eq. (1.7) using empirical data for the flux of wave action to the spectral peak at low frequencies. We shall now similarly estimate the universal constant a in the asymptotic spectrum in Eq. (1.6) from indirect estimates of the dissipation ϵ_0 at high frequencies. In general scaling terms, this was done by Kitaigorodskii (1983) who showed that a is of order unity. The calculation presented here uses the method of Longuet-Higgins (1969) whose aim was to estimate the universal constant β of Phillips' saturation spectrum [Eq. (1.1)].

Following the arguments in section 1, we will adopt the assumption of Kitaigorodskii (1983) that a frequency range exists below a frequency ω_g where the spectrum has the form of Eq. (1.6) and is unaffected by wave breaking. Above ω_g the spectrum is limited by wave breaking. This assumption of a sharp transition at ω_g implies that only wave components in a narrow band of frequencies higher than ω_g combine into breaking-wave fronts. The energy transfer equations on either side of the transition are

$$S_{\mathbf{k}}^{nl} = 0 \quad \text{for} \quad \omega < \omega_g \tag{2.1}$$

$$S_{\mathbf{k}}^{nl} - D_{\mathbf{k}} = 0 \quad \text{for} \quad \omega \geq \omega_g \tag{2.2}$$

where we have assumed that the wind energy input is negligible near the transitional frequency ω_g. Using the similarity expression Eq. (1.4) for $S_{\mathbf{k}}^{nl}$, we find

$$D_{\mathbf{k}} \propto g^{3/2}k^{19/2}N_{\mathbf{k}}^{3}. \tag{2.3}$$

For the fully saturated spectrum above ω_g limited by wave breaking, we postulate that the wave-action density dissipated per wave period $2\pi/\omega$ is a constant fraction, r, of the spectral wave action density

$$(2\pi/\omega)\frac{D_{\mathbf{k}}}{N_{\mathbf{k}}} = r. \tag{2.4}$$

Inserting Eq. (2.4) in Eq. (2.3) we find a similarity form of the wave-action density expressed in terms of the wavenumber modulus k,

$$N_k = Bg^{-1/2}k^{-9/2}, \tag{2.5}$$

where B is a numerical constant. The energy spectrum in frequency representation is readily found to be the well-known Phillips' (1958) form [Eq. (1.1) with $\beta = 2B$] which in dimensionless terms reads

$$\tilde{S}(\tilde{\omega}) = 2B\tilde{\omega}^{-5}. \tag{2.6}$$

We assume that B is a wind-independent constant characterizing the spectral range of dissipation due to wave breaking at frequencies $\omega \geq \omega_g$. In the literature earlier estimations of B have been found by fitting an ω^{-5} spectral form very close to the spectral peak. For the JONSWAP spectrum for example, B was found to decrease with increasing dimensionless fetch (Hasselmann et al. 1973), and Geernaert et al. (1986) found B to increase as a function of both ω and dimensionless peak frequency $\tilde{\omega}_m$. We believe this observed variation of B to be due to the spectral tail actually being of the form (1.11) near the spectral peak rather than the assumed power ω^{-5}. Following this idea, Mitsuyasu et al. (1980) demonstrated that the mean JONSWAP spectrum is approximated very well by the form (1.11) in the range $1.2 \leq \omega/\omega_m \leq 2$, with $2a\tilde{\epsilon}_0^{1/3} = 1.0B\tilde{\omega}_m^{-1}$, where B is the wind-dependent JONSWAP estimate.

The action-dissipation density is found from Eqs. (2.4) and (2.5)

$$D_k = r\frac{\omega}{2\pi}N_k = \frac{1}{2\pi}rBk^{-4}, \tag{2.7}$$

and the spectral density of *energy* dissipation rate is found to be

$$\epsilon_k = \omega D_k = \frac{1}{2\pi}rBg^{1/2}k^{-7/2}. \tag{2.8}$$

We are now able to calculate the total energy dissipation in the saturation range $[k_g, \infty]$,

$$\epsilon_0 = \int_{k_g}^{\infty} \epsilon_k k dk. \quad (2.9)$$

Using as dispersion relation

$$\omega^2 = gk, \quad (2.10)$$

we find

$$\epsilon_0 = \frac{1}{3\pi} r B g^2 \omega_g^{-3} \quad (2.11)$$

where $\omega_g = (gk_g)^{1/2}$, and in dimensionless terms:

$$\tilde{\epsilon}_0 = \frac{1}{3\pi} r B \tilde{\omega}_g^{-3}. \quad (2.12)$$

Similar to the definition of energy dissipation, Eq. (2.9), we may define the total action dissipation

$$D_0 = \int_{k_g}^{\infty} D_k k dk, \quad (2.13)$$

and if the dispersion relation [Eq. (2.10)] is valid to sufficient accuracy, the characteristic frequency scale of dissipation, ω_d, may be defined as

$$\omega_d = \frac{\epsilon_0}{D_0} = \frac{4}{3} \omega_g. \quad (2.14)$$

We apply the assumption of a sharp transition at ω_g where the expressions in Eqs. (1.11) and (2.6) are equal and find by use of Eq. (2.12)

$$2a\tilde{\epsilon}_0^{1/3} = 2B\tilde{\omega}_g^{-1}, \quad (2.15)$$

from which an expression for the universal constant a is derived

$$a = \left(\frac{3\pi B^2}{r}\right)^{1/3}. \quad (2.16)$$

There is no obvious way of determining theoretically the ratio r, but as we will see in section 4, a fine agreement is found with observations of the spectral form $S(\omega) = 2a\tilde{\epsilon}_0^{1/3} u_{10} g \omega^{-4}$ reported in literature, if we use the value

$$r = e^{-1/(16B)}. \quad (2.17)$$

This value obtained by Longuet-Higgins (e.g., see Phillips, 1977, p. 196) is the ratio of energy lost by wave breaking in one overall mean wave period defined as $\bar{T} = 2\pi/\bar{\sigma}$, where

$$\bar{\sigma}^2 = \frac{\int_0^{\infty} \omega^2 S(\omega) d\omega}{\int_0^{\infty} S(\omega) d\omega},$$

to the total wave energy

$$\int_0^{\infty} S(\omega) d\omega$$

of a spectrum $S(\omega) = 2Bg^2\omega^{-5}$ for $\omega \geqslant \omega_m$, $S(\omega) = 0$ for $\omega < \omega_m$, where it was assumed that the wave maxima are Rayleigh distributed.

b. Energy input from the wind to waves

According to Phillips' theory (1985) the frequency range appears to be very wide where energy input to waves is important. Phillips assumes that the pertubation of wind stress over a wave is proportional to $\rho_a u_*^2 ak$, a is the amplitude, k the wavenumber, and the mean wind stress is $\tau = \rho_a u_*^2$. The energy flux to the wave is the mean scalar product of the stress variations and the orbital velocity akc of the water surface and is thus proportional to $\rho_a u_*^2 (ak)^2 c$. The rate of energy density transfer $\rho_w g F^+(k, \theta)$ to a directional spectrum $F(k, \theta)$ may thus be written as

$$\rho_w g F^+(k, \theta) = m \rho_w g \phi^2(\theta) \left(\frac{u_*}{c}\right)^2 \omega F(k, \theta), \quad (2.18)$$

where m is a proportionality constant. The transfer rate of action density $N(k, \theta) = F(k, \theta)/\omega$ is similarly

$$N^+(k, \theta) = m\phi^2(\theta)(u_*/c)^2 \omega N(k, \theta). \quad (2.19)$$

In Eqs. (2.18) and (2.19) the term $\phi^2(\theta)$ indicates the influence on the wave component moving in a direction θ relative to the mean wind direction. The parameter m and the form of $\phi^2(\theta)$ must be deduced from experiments.

Phillips assumes that an equilibrium range of the wave spectrum exists where the wind energy input is locally balanced both by the weak nonlinear flux divergence *and* by dissipation due to breaking waves. In the context of the present paper we assume that the dissipation is negligible over the range where energy input from wind is important so that the energy transfer equation (1.2) is reduced to

$$S_{\mathbf{k}}^{nl} + N_{\mathbf{k}}^{+} = 0 \quad \text{for} \quad \omega_c \leqslant \omega \ll \omega_g, \quad (2.20)$$

where ω_c is a lower limit to wind energy input. The solution of Eq. (2.20) must be an asymptotic solution of the wave-kinetic equation (1.5) as $N_{\mathbf{k}}^{+}$ tends to zero.

Applying the similarity expression [Eq. (1.4)] and the source function [Eq. (2.19)] to Eq. (2.20) we find

$$N(k, \theta) = \gamma\phi(\theta) \frac{u_*}{c} \omega^{1/2} g^{-5/4} k^{-19/4}, \quad (2.21)$$

where γ is a proportionality constant. Assuming that the dispersion relation $\omega^2 = gk$ is valid independent of direction of wave propagation, we have $c = (g/k)^{1/2}$

and

$$N(k,\theta) = \begin{cases} \gamma\phi(\theta)u_*g^{-1}k^{-1}, & \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases} \quad (2.22)$$

which is similar to Eq. (1.6) if we replace $g\epsilon_0$ with the scaling parameter u_*^3.

The directionally averaged spectrum is

$$N_k = \int_{-\pi}^{\pi} N(k,\theta)d\theta = \gamma' u_* g^{-1} k^{-4},$$

where

$$\gamma' = \gamma \int_{-\pi/2}^{\pi/2} \phi(\theta)d\theta.$$

The corresponding frequency spectrum is

$$S(\omega) = 2\gamma' u_* g \omega^{-4}. \quad (2.23)$$

Substituting Eq. (2.22) into Eq. (2.19), taking the directional average and transforming to the frequency domain, we find the rate of energy input from the wind

$$S^+(\omega) = 2mu_*^2 \frac{\omega^3}{g^2}\gamma\left(\int_{-\pi/2}^{\pi/2}\phi^3(\theta)d\theta\right)u_*g\omega^{-4} \quad (2.24)$$

which by use of Eq. (2.23) becomes

$$S^+(\omega) = m'u_*^2 \frac{\omega^3}{g^2} S(\omega) \quad (2.25)$$

where

$$m' = m \frac{\int_{-\pi/2}^{\pi/2}\phi^3(\theta)d\theta}{\int_{-\pi/2}^{\pi/2}\phi(\theta)d\theta}.$$

As shown in a recent work by Donelan et al. (1985), several generally known experiments favour a directional distribution proportional to $\phi(\theta) = \text{sech}^2(\beta\theta)$ with $\beta \approx 1$ except very close to the spectral peak.

From Eq. (2.25) we find

$$m' = \frac{4}{15}\left(2 + \text{sech}\,\frac{\pi}{2}\beta\right)m \approx \frac{8}{15}m.$$

The value of m was determined by Plant (1982) who compared several experiments where θ was close to 0 with Eq. (2.19) and found

$$m = 0.04 \pm 0.02.$$

We then have an estimate for the coefficient m' of Eq. (2.25)

$$m' = 0.02 \pm 0.01. \quad (2.26)$$

In dimensionless terms using u_{10} and g as scaling parameters, the rate of energy input, Eq. (2.25), becomes

$$\tilde{S}^+(\tilde{\omega}) = h\tilde{\omega}^3\tilde{S}(\tilde{\omega}), \quad (2.27)$$

where $h = m'C_D$ and the drag coefficient, $C_D = u_*^2/u_{10}^2$. The rate of action input similarily becomes

$$\tilde{N}^+(\tilde{\omega}) = h\tilde{\omega}^2\tilde{S}(\tilde{\omega}). \quad (2.28)$$

In dimensionless terms the spectrum Eq. (2.23) reads

$$\tilde{S}(\tilde{\omega}) = 2\gamma' C_D{}^{1/2}\tilde{\omega}^{-4}, \quad (2.29)$$

and attains by hypothesis the asymptotic form in Eq. (1.11) as $\tilde{\omega} \to \infty$. Kitaigorodskii (1983) denoted the numerical constant of the $\tilde{\omega}^{-4}$ law by α_u, so we have

$$\tilde{S}(\tilde{\omega}) = \alpha_u\tilde{\omega}^{-4},$$

$$\alpha_u \equiv 2a\tilde{\epsilon}_0{}^{1/3}. \quad (2.30)$$

We assume that the numerical constant of Eq. (2.29) asymptotically adjusts to match the form in Eq. (2.30) so that $2\gamma' C_D{}^{1/2} = \alpha_u$. For a spectrum of the form in Eq. (2.30) the rate of energy input from the wind [Eq. (2.27)] is

$$\tilde{S}^+(\tilde{\omega}) = h\alpha_u\tilde{\omega}^{-1}. \quad (2.31)$$

Many wave-spectra measurements show that they do indeed have the form of [Eq. (2.30)] except in a narrow range around the frequency ω_m of the spectral peak, e.g., Donelan (1985) and Kahma (1981). Because of the weak decrease with increasing frequency of the form in Eq. (2.31), only a small fraction of the total energy (or action) input to the wave spectrum occurs in the frequency range close to the peak. Thus, the slight deviation from the form in Eq. (2.31) near the peak may be neglected in a model of the total energy input. We may assume a model spectrum of the form

$$\tilde{S}(\tilde{\omega}) = \begin{cases} 0, & \text{for } \tilde{\omega} < \tilde{\omega}_m \\ \alpha_u\tilde{\omega}^{-4}, & \text{for } \tilde{\omega}_m \leqslant \tilde{\omega} < \tilde{\omega}_g \\ \beta\tilde{\omega}^{-5}, & \text{for } \tilde{\omega} \geqslant \tilde{\omega}_g \end{cases} \quad (2.32)$$

where $\beta = 2B$, α_u is given by Eq. (2.30) and $\tilde{\omega}_g$ is the dimensionless critical frequency of gravitational instability. Inserting Eq. (2.32) into Eqs. (2.27) and (2.28), we may derive model expressions for the energy and action input to the wave field.

3. The experiment

The field measurements were made on Lake Washington during one morning in August 1977. The fetch was approximately 5 km in the wind direction over deep water, and at the measuring site the water depth was about 4 m.

The instruments were mounted on a mast located 20 m from the shore. All instrumental outputs were sampled at a rate of 25.55 per second. Three-dimensional fluctuating components of the wind together with temperature and humidity were measured 67 cm above the water surface, using hot wires mounted on a wind vane. The measurements were calibrated against a Gill-propeller system and thermocouples on the mast 3.6 m above the water surface. The atmospheric stratification was slightly unstable with a Monin–Obukhov length of the order of 100 m so a neutral mean wind at 10 m was obtained from the logarithmic profile $u(10\text{ m}) - u(67\text{ cm}) = (u_*/\kappa)\ \ln 10/0.67$, κ is von Kármán's constant. The value of the friction velocity, u_*, was estimated from the hot-wire measurements with a standard error of about 10 percent (Hansen 1985).

Using a 0.4-mm diameter nichrome resistance wire, the fluctuating water height was measured. The response of this wire was examined in a wave tank study by Liu et al. (1982) with special emphasis on the response to waves in the capillary gravity range. In their study the resistance gauge (RG) was compared to a laser displacement gauge (LDG). In Fig. 2 we have plotted their measured ratios of mean square surface displacement $\zeta^2_{\text{RG}}/\zeta^2_{\text{LDG}}$ against the peak frequencies of the rather narrow spectra. The straight line in the figure is of the form

$$H^2_{\text{RG}}(f) = \left(\frac{f}{f_0}\right)^{-n}, \quad f_0 = 1\text{ Hz}, \quad n = 0.5 \qquad (3.1)$$

and works well with the data of Liu et al. around 5 Hz. In Fig. 3 we have fitted by eye to the rear face of the LDG spectrum a spectral model of the form

$$\hat{S}(f) = \alpha u_s g^* f^{-4}, \qquad (3.2)$$

where $g^* = g + \gamma k^2$, k is related to f by the wave dispersion relation $(2\pi f)^2 = gk + \gamma k^3$ and αu_s is just a best fit coefficient with dimension velocity. We see that the functions $S(f) = H^2_{\text{RG}}(f) \cdot \hat{S}(f)$ agree well with the measured RG-spectra over the whole range 4 Hz $\leqslant f \leqslant$ 30 Hz and Eq. (3.1) is thus chosen as a valid spectral response function for all frequencies $f \geqslant f_0$. Although the spectral response function given by Eq. (3.1) is a good approximation to the data of Liu et al. (1982) over the broad frequency range 4 Hz $\leqslant f \leqslant$ 30 Hz, the application to the range 1 Hz $\leqslant f \leqslant$ 4 Hz is a pure extrapolation. As a support to consider $f_0 = 1$ Hz as a fair representation of the point where the response is 100%, we may consider a reference to an earlier experiment by Liu et al. (1982), using a stainless steel wire with presumably a better wetting characteristic than the 0.4 mm nichrome wire used on Lake Washington. This measurement yielded a 100% response at 1.6 Hz.

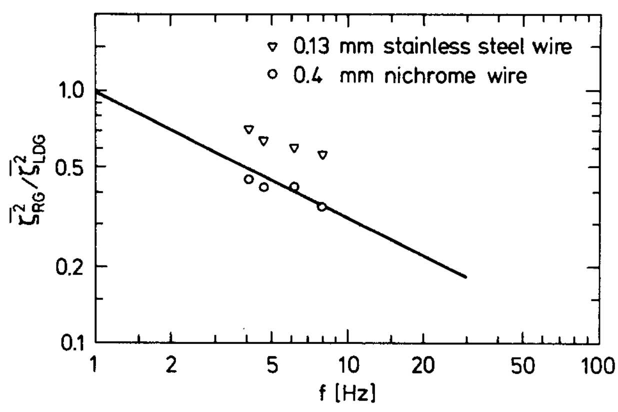

FIG. 2. Response function for the wave gauge. The points are the ratios between mean square surface elevation of wind-tunnel water waves measured by resistance wire (RG) and laser beam (LDG) as a function of frequency of the spectral peak from the data of Liu et al. (1982). The line is our "eyeball" fit to the response $H^2_{\text{RG}}(f)$ at all the measured frequencies given by Eq. (3.1) and extrapolated to 1 Hz.

In our Lake Washington measurements the voltage output from the RG was recorded in a digital form by a FSK (frequency shift keying) system. In the FSK a low-pass filter $H^2_{\text{FSK}}(\omega)$ was applied to avoid aliasing.

The output of the digitizer is an integer $l(t)$. Assume that the digitized signal $y(t) = l(t) \cdot \Delta$ is within $\pm\Delta/2$ from the true signal where Δ is the digitization step, then for a steep wave spectrum the discretization noise is approximately white and attains the value $N_D(\omega) = N = \Delta^2/(12\omega_N)$, where ω_N is the Nyquist frequency (Kristensen and Kirkegaard 1987).

At the high-frequency tail of the measured spectrum we identified an apparent white noise, which was about twice the digitization noise $N(\omega) = N \approx 2N_D$. We estimated a minimum value of the noise N individually for each spectrum.

The construction of the wave spectrum $S(\omega)$ from the spectrum $S_{\text{DIG}}(\omega)$ of the digitally recorded spectrum involved three steps as shown in Fig. 4. In this figure the spectrum is multiplied by ω^4, and the spectrum and frequency are scaled by use of $\alpha_u u_{10}$ and g. The Kitaigorodskii constant α_u is calculated from the spectrum $S(\omega)$ as the average of $S(\omega) \cdot \omega^4$ over the frequency interval from $1.5\omega_m$ to $3.0\omega_m$ where ω_m is the spectral frequency at the peak of the spectrum or of the dominant waves. The three steps in constructing $S(\omega)$ from $S_{\text{DIG}}(\omega)$ are

1) Subtraction of the noise yielding the spectrum before digitizing

$$S_{\text{FSK}}(\omega) = S_{\text{DIG}}(\omega) - N.$$

A slight underestimation of the noise results in a small residual noise appearing at the highest frequencies.

2) Correction for the low-pass filter of the FSK-system yielding the output from the resistance gauge system

$$S_{\text{RG}}(\omega) = S_{\text{FSK}}(\omega) \cdot \frac{1}{H^2_{\text{FSK}}(\omega)}.$$

3) Correction for the wave gauge response yielding the estimated true spectrum

$$S(\omega) = S_{\text{RG}}(\omega) \cdot \frac{1}{H^2_{\text{RG}}(\omega)}.$$

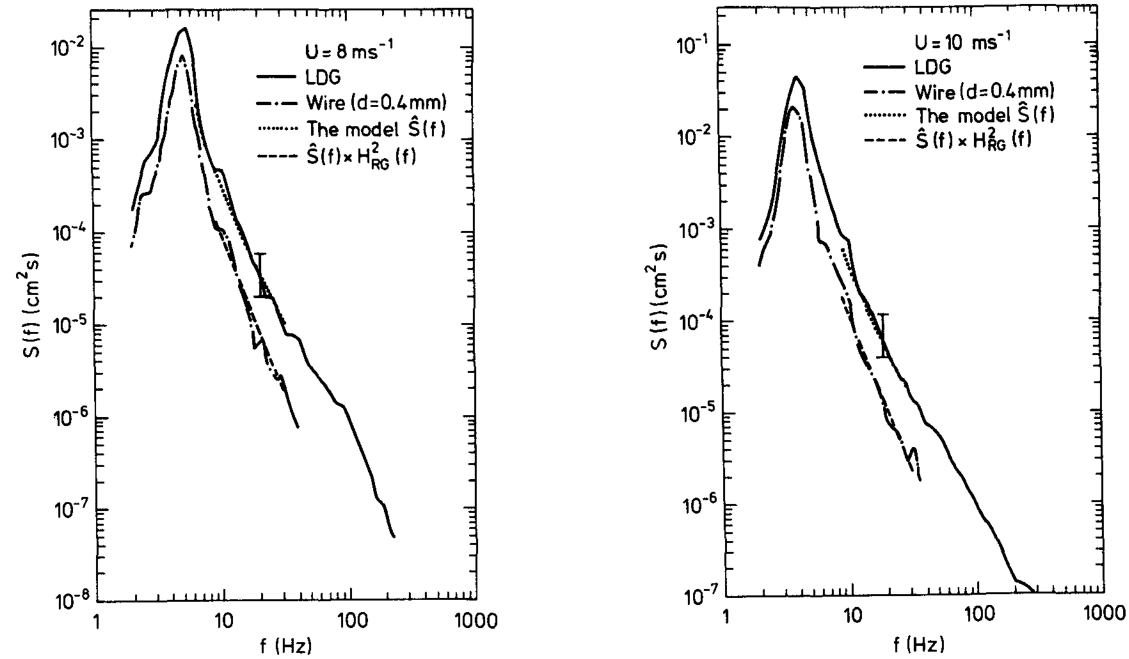

FIG. 3. Wind-tunnel wave spectra by Liu et al. (1982) measured by laser (LDG) and resistance wire. It is seen that the frequency response function $H^2_{RG}(f)$ given in Eq. (3.1) applied to the high-frequency model of the true (LDG) spectrum $\hat{S}(f)$ in Eq. (3.2) yields a function which is in good agreement with the measured resistance wire spectrum. Two wind speed cases are shown: (a) U = 8 m s^{-1} and (b) U = 10 m s^{-1}. The error bars represent the 95 percent confidence limits of the spectra.

In the Lake Washington experiment the wind was calm in the early morning, but around 0600 local time the wind increased from approximately 2 m s^{-1} to approximately 6 m s^{-1} in 20 minutes. From 0635 the wave spectra calculated from segments of the time series of 11 min each seemed to have a high-frequency tail in equilibrium with the wind. From the following four hours of measurement 14 such 11-min segments with a rather steady wind were chosen. Listed in Table 1 are 10-m mean wind, u_{10}, the Kitaigorodskii constant α_u estimated from the range [$1.5\omega_m$, $3.0\omega_m$], the phase speed of the spectral peak and the stability parameter z/L at z = 0.67 m.

During the first two hours of measurement the wind direction was almost parallel to the shore of the lake permitting waves only from directions ~0° to ~90° from the shore to arrive at the measuring tower. This might be the reason for the rather low values of α_u for runs 246 and 248 due to the truncated direction distribution.

The wave spectra from runs 246 and 248 are shown in composite forms in Fig. 5a.

From the value of the group velocity $c_m/2$ of the wave spectral peak, which is about 1.5 m s^{-1} for all runs, we find that the travelling time over the 5-km fetch is about 3 × 10^3 s or 50 minutes. Thus the wave spectral peaks for runs 252 and 254 may be strongly influenced by higher winds at earlier times of their development, and the level α_u of the spectral tail may be

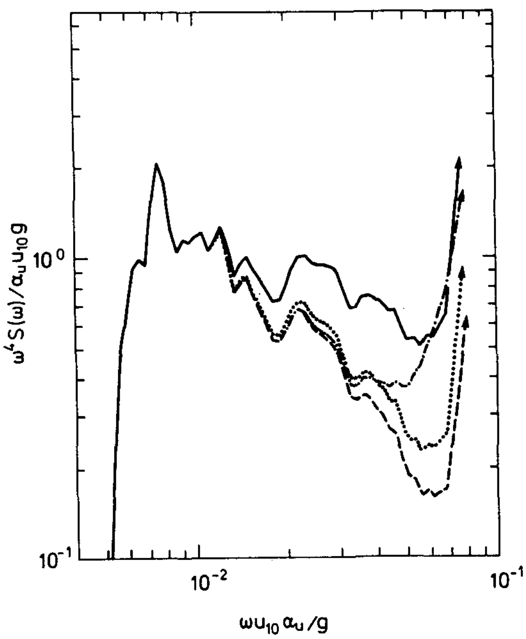

FIG. 4. Construction of the wave spectrum from the raw data. The figure shows the spectrum multiplied by ω^4. The spectrum and angular frequency are scaled with the parameters $\alpha_u u_{10}$ and g where α_u is the mean of $\omega^4 S(\omega)/u_{10}g$ over the frequency range $1.5\omega_m$ to $3.0\omega_m$. Dot-dash: raw spectrum; dash: after subtracting white noise, residual noise indicated by the arrows; dotted: after correction for low-pass filter of the recording system; solid: the true spectrum after correction for the wave-gauge response function defined by Eq. (3.1).

TABLE 1. Ten-meter wind speed, Kitaigorodskii factor of $S(\omega) = a_u u_{10} g \omega^{-4}$, phase speed c_m of the spectral peak and stability parameter for the selected time segments.

Run	Time (approx.) (local time)	u_{10} (m s^{-1})	$a_u \times 10^{-3}$	c_m (m s^{-1})	z/L
246					
b)	0635	6.5	3.6	2.6	−0.03
c)	0645	7.1	3.3	2.7	−0.02
d)	0655	6.1	3.3	2.4	−0.04
248					
a)	0715	6.3	3.5	3.0	−0.04
b)	0725	6.0	3.6	3.2	−0.05
c)	0735	5.9	3.7	3.3	−0.05
d)	0745	6.2	3.4	3.3	−0.07
252					
a)	0905	4.9	4.3	2.7	−0.06
c)	0925	5.0	4.2	2.9	−0.05
d)	0935	4.8	4.6	2.8	−0.08
254					
a)	1020	3.6	5.8	2.8	−0.11
b)	1030	3.3	6.1	3.0	−0.20
c)	1040	3.4	6.2	2.8	−0.19
d)	1050	3.4	6.6	2.6	−0.29
Mean		5.0	4.4	2.9	−0.09

higher than will be found in steady wind conditions. The spectra from runs 252 and 254 are shown in composite form in Fig. 5b.

It caused some trouble to find the voltage-to-water height calibrations for the resistance gauge because some unknown gain was introduced by mistake in the recording end. The results were therefore calibrated against some spectra measured with the same wave gauge on the same site four months earlier. However, we estimate this calibration to be accurate within ±20% for the measure α_u, or β, of the spectral level. The rms deviation of our values for α_u (Table 1) from the relation empirically obtained by Donelan et al. (1985), $\alpha_u = 0.006(u_{10}/c_m)^{-0.45}$, is 17% of the mean value, $\alpha_u = 4.4 \times 10^{-3}$. This compares very well with the 20% accuracy of our calibration and coincidentially the mean $\alpha_u = 4.4 \times 10^{-3}$ is very close to the mean values of Donelan et al. (1985) and Kahma (1981).

4. Interpretation of results

a. The transition from ω^{-4} to ω^{-5}

The composite spectra in Fig. 5 clearly have a transition at a high frequency ω_g (approximately at $\omega_g \approx 3\omega_m$) from an ω^{-4} range below ω_g to roughly the form depicted by the lower of the broken curves. These curves are derived from some plausibility arguments given in section 4b and represent the enhanced spectra due to a vertical profile of steady wind-drift velocity below the water surface. The straight line of slope −1 in Fig. 5 is the corresponding intrinsic spectrum of the form $S(\omega) = \beta g^2 \omega^{-5}$.

The value $\beta = 0.025 \pm 20\%$ is an overall best fit (iteratively found) to the spectra in the range $0.03 < \omega u_{10} \alpha_u / g < 0.05$ of the spectral model given by Eqs. (4.11) and (4.4). Here, the surface drift velocity is chosen among various possibilities as $U_d(z=0) = u_*/2$

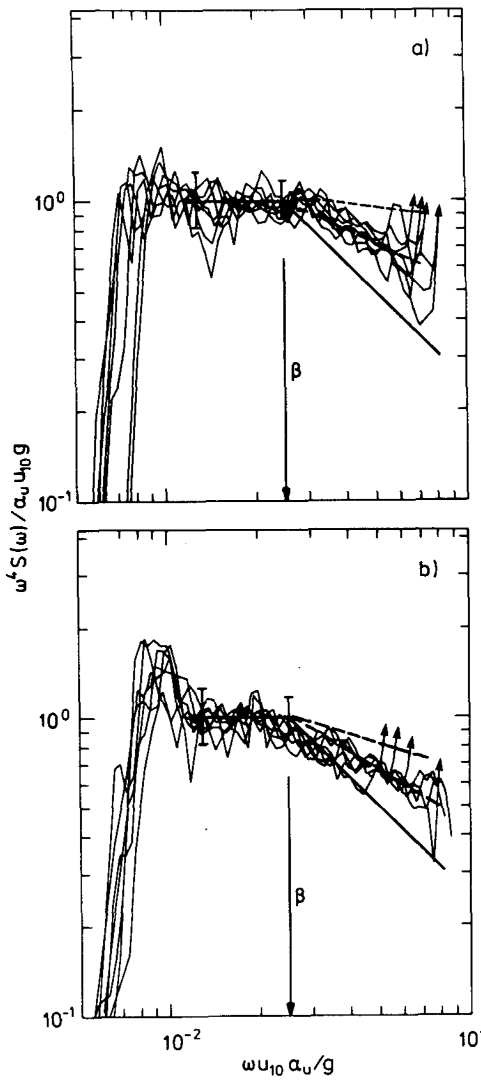

FIG. 5. Composite spectra of 11-min time series with (a) steady wind between 5.9 and 7.1 m s^{-1} and (b) steady wind between 3.3 and 5.0 m s^{-1}. The spectra in this figure are scaled as in Fig. 4 and are presented in two parts by mean wind speed (a) u_{10} = 5.9 to 7.1 m s^{-1}, (b) u_{10} = 3.3–5.0 m s^{-1} (see Table 1). The vertical bars at approx. $1.5\omega_m$ and $3\omega_m$ are here 90% confidence intervals (the spectral peaks lie at $\omega_m u_{10} \alpha_u / g$ slightly less than 10^{-2}. The solid lines are the models (2.30) and (2.6) for the intrinsic spectrum intersecting at $\omega_g u_{10} \alpha_u / g = \beta = 0.025$. The broken curves are the enhanced spectrum due to a wind-drift velocity profile in the water as explained in section 4b. The lower of the broken curves was chosen as the best fit to the observed spectra and corresponds in both (a) and (b) to a surface drift velocity $u_d = 0.015\bar{u}_{10}$ while the upper broken curves describes the effect of $u_d = 0.03\bar{u}_{10}$.

where the air friction velocity u_* is the measured mean over the runs represented in Figs. 5a and 5b, respectively. The upper broken curve corresponds to $U_d(z = 0) = u_*$ and is merely shown to illustrate the dependence of the model result given by Eq. (4.8) on the choice for U_d. The drift profile in the highest fractions of mm of the water where the turbulent eddy viscosity is weaker than molecular viscosity has little influence on the waves so we may consider the depth $z = 0$ to lie just below this layer.

b. The correction for influence by drift motion and low-frequency orbital motion on the high-frequency part of the spectrum

The high-frequency part of the wind-wave spectrum is Doppler-shifted by both the surface drift current and the orbital motion of waves of lower frequency. In this section we will argue that the influence of the orbital motion is negligible compared to that of the drift current as long as we consider only frequencies just above the transition from the ω^{-4} to the ω^{-5} spectral forms in Fig. 5. The situation in this range is different from what it is for waves with a wavelength of a few centimeters which are intensively studied by microwave backscatter experiments. Using wire data, the Doppler shift of centrimetric waves due to orbital motion of the dominant waves was found to have the same order of magnitude as the influence of drift current (Ataktürk and Katsaros 1986). Their dataset is also from Lake Washington under similar conditions as the August 1977 dataset discussed here.

Following Kitaigorodskii et al. (1975) we assume that in the presence of low-frequency wave motion the spectral form in the two-dimensional wavenumber plane is

$$F(k, \theta) = Bk^{-4}\phi(\theta) \tag{4.1}$$

where k is the wavenumber modulus and $\phi(\theta)$ the directional distribution of wave energy propagation such that the energy transport is $Bk^{-4}\phi(\theta)kd\theta$ at wavenumber k in the interval dk moving in the direction $d\theta$. The frequency spectrum measured at a fixed point is

$$S(\omega) = \int_{-\pi/2}^{\pi/2} (kF(k, \theta)/c_g)\phi(\theta)d\theta \tag{4.2}$$

where c_g is the group velocity, $c_g = d\omega/dk$. $S(\omega)$ thus depends on the dispersion relation which has the form

$$\omega(k) = [(g + a^\sigma)k]^{1/2} + \mathbf{u}\cdot\mathbf{k}, \tag{4.3}$$

where a^σ is the local vertical component of surface acceleration due to the orbital particle motion in the low-frequency waves and $\mathbf{u} = \mathbf{u}^\sigma + \mathbf{u}_d$ is the sum of orbital and surface-drift velocity.

First, we consider only the drift component, u_d of u and assume it is aligned with the mean wave direction. Kitaigorodskii et al. (1975) solved Eq. (4.3) for k. By applying the transformation Eq. (4.2) to the spectral form in Eq. (4.1), they found for the case where low-frequency orbital motion is zero

$$S(\omega) = 2Bg^2\omega^{-5}J(\omega_d). \tag{4.4}$$

Here, $J(\omega_d)$ has the form

$$J(\omega_d) = 1 + 3\mu\omega_d + \nu\omega_d^2 \tag{4.5}$$

to the second order in the dimensionless frequency $\omega_d = \omega u_d/g$ where μ and ν depend on the directional distribution of energy. For the directional distribution $\phi(\theta) = 2/\pi \cos^2\theta$ for $|\theta| \leq \pi/2$ and $\phi(\theta) = 0$ elsewhere, they derive exact values

$$\mu = \frac{8}{3\pi}, \quad \nu = \frac{3}{4}. \tag{4.6}$$

The wave components of wavenumber k ride on a surface-drift current profile with a scaling depth of order $1/k$. Thus, the drift velocity is to be taken at a depth $z_d = 1/k = g/\omega^2$, and the dimensionless frequency ω_d of Eq. (4.5) must be evaluated as

$$\omega_d = \frac{\omega u_d(z_d)}{g}. \tag{4.7}$$

To estimate $u_d(z_d)$ we shall below seek to relate it to the formation of turbulence by wave breaking at frequencies above ω_g, the transition frequency between the forms ω^{-4} and ω^{-5}.

According to the observations by Donelan (1978) breaking waves generate a turbulent patch extending to the depth of approximately the wave height, which means that the characteristic scale of the surface perturbation is $H \sim 2a \sim 1/k_g$. We assume that the wavenumber k_g of the breaking wave shall be taken as $k_g = \omega_g^2/g$ where ω_g is the transitional frequency defined by applying the model in Eq. (2.32). Assuming a simple linear profile for the drift velocity and pointing the z-axis *downward*, we can write

$$\frac{du_d}{dz} = \text{const} = -\frac{u_d(0)}{H} \quad \text{for} \quad 0 \leq z \leq H. \tag{4.8}$$

This assumption agrees quantitatively with theories of a constant turbulent exchange coefficient A_z,

$$-A_z\frac{du_d}{dz} = w_*^2 \tag{4.9}$$

where w_*^2 is the vertical momentum transfer τ normalized with the water density;

$$w_*^2 = \frac{\tau}{\rho_w},$$

or w_* is the friction velocity in the water.

Dimensional arguments (Long 1979) yields $A_z = \lambda w_* H$. Using the results of the measurements by Robertson (1959), Long estimates $\lambda = 0.12$. Inserting Long's expression in Eq. (4.9), we find

$$\frac{du_d}{dz} = -\frac{w_*}{\lambda H},$$

and for our velocity profile [Eq. (4.9)] to hold, we must have

$$\frac{w_*}{\lambda} = u_d(0). \tag{4.10}$$

Normalizing the Lake Washington momentum transfer τ above the surface with air density and 10-m wind speed, we obtain:

$$\frac{\tau}{\rho_a u_{10}^2} = u_*^2/u_{10}^2 = 1.1 \times 10^{-3}.$$

As only a negligible fraction of the momentum transfer is retained in waves, we have $w_*^2/u_*^2 = (\rho_a/\rho_w) \approx 1.2 \times 10^{-3}$, and thus for our experiment $w_*/u_{10} \approx 1.2 \times 10^{-3}$. Using Eq. (4.10) with $\lambda \approx 0.12$, we find

$$\frac{u_d(0)}{u_{10}} \approx 0.01.$$

For simplicity we assume the linear dispersion relation $\omega^2 = gk$ for all frequencies of interest here. From Eq. (4.8) we find the drift current at depth $z = 1/k$

$$u_d\left(z = \frac{1}{k}\right) = \left(\frac{\omega^2 - \omega_g{}^2}{\omega^2}\right) u_d(0).$$

The dimensionless frequency ω_d to be used in the formula of Kitaigorodskii et al. (Eq. (4.5)) is thus by Eq. (4.7)

$$\omega_d = \frac{\omega}{g} u_d\left(z = \frac{1}{k}\right) = \frac{\omega}{g} u_d(0)\left(1 - \frac{\omega_g{}^2}{\omega^2}\right)$$
$$= \tilde{\omega} \cdot \left(1 - \frac{\omega_g{}^2}{\omega^2}\right) \cdot \frac{u_d(0)}{u_{10}} \tag{4.11}$$

where $\tilde{\omega}$ is the dimensionless frequency using g and u_{10} as scaling parameters, i.e. $\tilde{\omega} = \omega u_{10}/g$.

The assumption is here that in Lake Washington the current is negligible at depths larger than $1/k_g$. On larger lakes, though, and especially under open-ocean conditions, deeper current systems like the Ekman profile are generated by turbulent transport of momentum, and the function $J(\omega_d)$ may be much larger under such conditions. The lower stipled curves in Figs. 5a and 5b, which appear to correspond well with the composite spectra, are given by Eq. (4.4) where $2B = \beta = 2.5 \times 10^{-2}$ and $J(\omega_d)$ is calculated from Eq. (4.5) by use of our model, Eq. (4.11). ω_d is an increasing function of $\omega u_{10}\alpha_u/g$, and we consider the approximate formula, Eq. (4.5), to be accurate only up to $\omega u_{10}\alpha_u/g = 5 \times 10^{-2}$ where $\omega_d = 0.14$.

We now turn toward the orbital component $\mathbf{u}^\sigma$ of $\mathbf{u}$ in Eq. (4.3). A wave solution for the surface elevation, ζ^σ, of the linearized equations of motion valid at the water surface is

$$\zeta^\sigma = A^\sigma e^{i(\mathbf{k}^\sigma \cdot x - \omega^\sigma t)}.$$

The corresponding surface orbital velocity u^σ and acceleration a^σ in the direction of $\mathbf{k}$ are

$$\begin{cases} u^\sigma = \omega^\sigma \zeta^\sigma \\ a^\sigma = -(\omega^\sigma)^2 \zeta^\sigma = gk^\sigma \zeta^\sigma. \end{cases} \tag{4.12}$$

In the case of non-negligible orbital motion but zero drift velocity, Kitaigorodskii et al. (1975) find that the ω^{-5}-spectrum is augmented by the factor $J^\sigma(\bar{\omega})$

$$S(\omega) = 2Bg^2\omega^{-5}J^\sigma(\bar{\omega})$$

where $J^\sigma(\bar{\omega})$ is the time average $J^\sigma(\bar{\omega}) = \langle (1 + (a^\sigma/g))^2 J(\bar{\omega}) \rangle$; J is the function in Eq. (4.5), and the variable $\bar{\omega}$ is defined as $\bar{\omega} = \omega u^\sigma/(g + a^\sigma)$. In terms of k^σ and ω^σ using Eq. (4.12), $\bar{\omega}$ may be written

$$\bar{\omega} = \frac{\omega}{\omega^\sigma} \frac{1}{1/k^\sigma\zeta^\sigma - 1}.$$

When averaging the function J in Eq. (4.5) it yields a new function in terms of second moment in $k^\sigma\zeta^\sigma$,

$$J^\sigma\left(\frac{\omega}{\omega^\sigma}, k^\sigma\zeta^\sigma\right) = 1 + q^*\left(\frac{\omega}{\omega^\sigma}\right)\langle (k^\sigma)^2\zeta^2 \rangle, \tag{4.13}$$

with

$$q^*\left(\frac{\omega}{\omega^\sigma}\right) = 1 - 3r\frac{\omega}{\omega^\sigma} + q\left(\frac{\omega}{\omega^\sigma}\right)^2, \tag{4.14}$$

where r and q are both of the order of 1.

The first two terms of q^* arise from the acceleration a^σ while the term $q(\omega/\omega^\sigma)^2$ comes from the Doppler shift. It is seen that the effect of acceleration is negligible compared with that of Doppler shift only for very high ratios of ω/ω^σ. For $\omega/\omega^\sigma \approx 3$, the two parts are of equal magnitude but of opposite sign, yielding $q^* \approx 0$. For $\omega/\omega^\sigma \leqslant 3$ the acceleration effect dominates yielding negative but small values of the q^*-function.

In case of a wave spectrum $S(\omega^\sigma)$ the augmentation function $J^\sigma(\omega)$ is $1 + q^*(\omega/\omega^\sigma)(k^\sigma)^2 S(\omega^\sigma)d\omega^\sigma$ arising from the interval $d\omega^\sigma$ around ω^σ. For simplicity we will assume that the total augmentation function $J^s(\omega/\omega_m)$ is a product of the contributions $J^\sigma(\omega)$ from all spectral components in the range $[\omega_m, \omega[$. To a sufficient degree of accuracy we can put $(k^\sigma) = (\omega^\sigma)^2/g$, and we find

$$J^s\left(\frac{\omega}{\omega_m}\right) = 1 + \int_{\omega_m}^{\omega} q^*\left(\frac{\omega}{\omega^\sigma}\right)\frac{(\omega^\sigma)^4}{g^2} S(\omega^\sigma)d\omega^\sigma. \tag{4.15}$$

Using the spectrum $S(\omega) = \alpha_u u_{10} g \omega^{-4}$ we find that the ratio of the function $J^s(\omega/\omega_m)$ to the drift motion augmentation function $J(\omega_d)$ increases with increasing ω. At the upper end of the definition of $J(\omega_d)$ ($\omega u_{10}\alpha_u/g \approx 5 \times 10^{-2}$), we find from Eq. (4.15) for the typical value $\omega/\omega_m = 8$ that $J^s(\omega/\omega_m) = 1.08$ while the effect of a constant drift motion is found from Eq. (4.4). Using $\omega_d = 0.14$, $J(\omega_d) = 1.37$. Therefore, we neglect the effect of orbital motion for the range considered, $\omega u \alpha_u / g \leqslant 5 \times 10^{-2}$. (For a discussion of the effect of orbital velocity of the long waves on the point measurement of wave frequency spectra, see Ataktürk and Katsaros 1987).

c. *The Kolmogorov-type universal constant a*

Here, the experimental results shall be compared with the theoretical discussion of sections 1 and 2. The universal constant a of the wave analog [Eq. (1.6)] to the Kolmogorov spectrum of isotropic turbulence may be estimated from the theoretical result in Eq. (2.17), using the value of Phillips constant $2B = \beta = 0.025$ found from Fig. 5 and the ratio of energy loss $r = \exp[-1/(8\beta)]$. This yields

$$a = 0.6.$$

As the estimate [Eq. (2.17)] for the dissipation rate is not a theoretical one but merely a postulate, our results must be checked against other results before we are able to accept the value $a = 0.6$. The key for doing this is the conservation of overall energy and action.

d. *The energy and action budget for fetch-limited spectra*

A simple model for the overall energy and action budget per unit surface area will be formulated using the spectral model [Eq. (2.32)] for the energy and action source Eqs. (2.27) and (2.28). This model will be used to estimate the uncertainty of our estimate of the Kitaigorodskii constant α_u. We assume that the energy and action dissipation take place at the scale ω_d given by Eq. (2.14), and the growth of wave energy and action takes place from the peak frequency ω_m and is advected to larger fetches. We will further simplify the model by assuming the energy/action source to be zero for $\omega \geqslant \omega_g$ and for $\omega \leqslant \omega_c$ where ω_c is given by $\omega_c = 2g/u_{10}$ on the basis of the discussion in section 1a. The model equations are

conservation of energy:

$$q = q^+ - \epsilon_0$$

conservation of action:

$$p = p^+ - \epsilon_0/\omega_d$$

advection at ω_m:

$$p = q/\omega_m,$$

where $\omega_d = \frac{4}{3}\omega_g$. We assume that input from the wind occurs only to wave components in the interval $[\omega_c, \omega_g]$, and find from Eqs. (2.27) and (2.28) in dimensionless terms

energy input:

$$\tilde{q}^+ = h\alpha_u \ln \omega_g/\omega_c$$

action input:

$$\tilde{p}^+ = h\alpha_u(\tilde{\omega}_c^{-1} - \tilde{\omega}_g^{-1}),$$

where

$$\omega_c = \begin{cases} \omega_m, & \text{for } \omega_m \geqslant 2g/u_{10} \quad \text{young waves} \\ 2g/u_{10}, & \text{for } \omega_m \leqslant 2g/u_{10} \quad \text{developed waves} \end{cases}$$

and

$$\alpha_u = 2a\tilde{\epsilon}_0^{1/3}. \qquad (4.16)$$

The assumption of a sharp transition at ω_g implies for the model spectrum [Eq. (2.32)]

$$\tilde{\omega}_g = \beta/\alpha_u. \qquad (4.17)$$

From the model equations we directly derive the forms of $\tilde{q}$ and $\tilde{\epsilon}_0$ as explicit functions of the ratio ω_g/ω_m, with the parameters a and h,

$$\frac{\tilde{\epsilon}_0}{h\alpha_u} = \frac{\ln(\omega_g/\omega_c) - (\omega_m/\omega_c - \omega_m/\omega_g)}{1 - \omega_m/\omega_d} \qquad (4.18)$$

where $\alpha_u = 2a\tilde{\epsilon}_0^{1/3}$ and $\omega_d = (4/3)\omega_g$, and

$$\frac{\tilde{q}}{h\alpha_u} = \frac{\ln(\omega_g/\omega_c) - (\omega_d/\omega_c - \omega_d/\omega_g)}{1 - \omega_d/\omega_m}. \qquad (4.19)$$

By use of Eq. (4.17) we find implicitly from Eq. (4.18) α_u as a function of $\tilde{\omega}_m$ with the parameters a, h, b. This function is shown in Fig. 6 calculated with the values $\beta = 0.025$, $a = 0.6$ from our results, and $h = 2 \times 10^{-5}$ from Eq. (2.26), assuming $c_D = 1 \times 10^{-3}$. Donelan et al. (1985) found empirically the relation

$$\alpha_u = 0.006\tilde{\omega}_m^{-0.45}$$

also shown in Fig. 6 and valid for the range $1 \leqslant \tilde{\omega}_m \leqslant 4$.

It is interesting to compare the result with the general considerations by Hasselmann et al. (1976) who found that the spectral form in the vicinity of the peak was practically invariant for various measured spectra. They

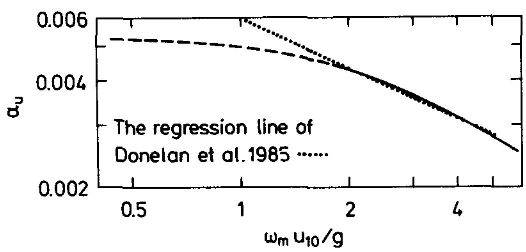

FIG. 6. The coefficient α_u of the spectrum $S(\omega) = \alpha_u u_{10} g \omega^{-4}$ calculated from the model for energy and action balance, Eqs. (4.16), (4.17) and (4.18) as a function of dimensionless spectral peak frequency $\omega_m u_{10}/g$ for $\omega_m u_{10}/g \geqslant 2$ (solid curve) and assuming the energy input from wind to be zero for $\omega_m u_{10}/g < 2$ (broken curve). The dotted line is a regression fit from Donelan et al. (1985) to their experimental results.

defined a form parameter [for the spectral tail: $S(\omega) = \beta g^2 \omega^{-5}$]

$$\lambda_\beta \equiv \tilde{e}\tilde{\omega}_m{}^4/\beta, \qquad (4.20)$$

where $\tilde{e} \equiv \int_0^\infty \tilde{S}(\tilde{\omega})d\tilde{\omega}$ is the dimensionless wave energy per unit surface area.

Hasselmann et al. (1976) found that λ_β has the almost constant value 0.25 for the JONSWAP spectra. According to Mitsuyasu et al. (1980), a fit of the form $S(\omega) = \alpha_u u_{10} g \omega^{-4}$ to the near-peak spectral tail of the mean JONSWAP spectrum yields $\alpha_u = 0.50\beta\tilde{\omega}_m{}^{-1}$. A form parameter defined for a spectral tail $S(\omega) = \alpha_u u_{10} g \omega^{-4}$ is

$$\lambda_\alpha = \tilde{e}\tilde{\omega}_m{}^3/\alpha_u. \qquad (4.21)$$

The argument of Mitsuyasu et al. yields $\lambda_\alpha = 2.0\lambda_\beta$ for the mean JONSWAP spectrum.

We may approximate the function $\alpha_u(\tilde{\omega}_m)$ from Fig. 6 by the power law

$$\alpha_u = 0.0052\tilde{\omega}_m{}^{-1/3} \qquad (4.22)$$

in the vicinity of $\tilde{\omega}_m = 2$ which represents the center of most reported fetch-limited field data. Taking the constant value $\lambda_\alpha = 0.50$ for the JONSWAP data, we find from Eq. (4.21) that $\tilde{e} = 0.0026\tilde{\omega}_m{}^{-10/3}$. This approximation for $\tilde{e}$ is valid only in the neighbourhood of $\tilde{\omega}_m = 2$. Hasselmann et al. (1976) found for the JONSWAP data $\tilde{e} = 0.0024\tilde{\omega}_m{}^{-10/3}$. Donelan et al. (1985) found $\tilde{e} = 0.0027\tilde{\omega}_m{}^{-3.3}$.

Equation (4.22) corresponds to the function $\tilde{\epsilon}_0/(h\alpha_u) = 0.16\omega_g/\omega_m$, from which we see that $\alpha_u \equiv 2a\tilde{\epsilon}_0{}^{1/3}$ scales like

$$\alpha_u = b_\alpha a h^{1/3}\beta^{1/3}\tilde{\omega}_m{}^{-1/3}, \quad b_\alpha = 1.1. \qquad (4.23)$$

If we insert Eq. (4.23) in (4.21), we see that $\tilde{e}$ scales like

$$\tilde{e} = b_e a h^{1/3}\beta^{1/3}\tilde{\omega}_m{}^{-10/3}, \quad b_e = \lambda_\alpha b_\alpha = 0.56. \qquad (4.24)$$

We note again that Eqs. (4.23) and (4.24) shall be considered as approximations in the vicinity of $\tilde{\omega}_m = 2$.

Figure 6 suggests that our value for α_u obtained from the form $\alpha_u \propto a h^{1/3}\beta^{1/3}$ is correct. From Eq. (2.26) we expect that h is correct within ±50 percent, and in section 3 we estimated the uncertainty of β to be roughly 20 percent. Accordingly, the universal constant a is correctly estimated within $\pm\frac{1}{3}(\Delta h/h) \pm \frac{1}{3}(\Delta\beta/\beta) = \pm 23$ percent or roughly

$$a = 0.6 \pm 0.1. \qquad (4.25)$$

5. Conclusions

We have demonstrated that for an intermediate range of high frequencies, i.e. in the high-frequency tail of the wind-wave spectrum in equilibrium, $S(\omega) = \alpha_u u_{10} g \omega^{-4}$ for $\omega < \omega_g$ and $S(\omega) = \beta g^2 \omega^{-5}$ for $\omega \geq \omega_g$, a simple model describes our experimental data when account has been taken of Doppler shift due to the wind-drift velocity near the water surface. The coefficient α_u is parameterized as $\alpha_u = 2a\tilde{\epsilon}_0{}^{1/3}$ where through the spectrum from low to high frequencies by nonlinear interactions the dimensionless constant flux of energy $\tilde{\epsilon}_0$ basically is a function of the ratio between wind speed u_{10} and wave phase speed c_p of the spectral peak, $\tilde{\epsilon}_0 = f(u_{10}/c_p)$. The Kolmogorov-type universal constant a has been determined, using a simple model for the overall balance of energy and action; the calculations based on the measurements suggest that $a = 0.6 \pm 0.1$.

For similarity reasons involving the equation for nonlinear spectral flux convergence, we have suggested a spectral model for the dissipation of wave energy due to wave breaking. This implies that the wave spectrum in the dissipation range has the form $S(\omega) = \beta g^2 \omega^{-5}$.

This model is attractive though perhaps simplistic because it conserves energy and action and balances the empirically known energy input from the wind in the form suggested by Phillips (1985) in such a way that the resulting growth of the total wave energy agrees well with empirical data reported in literature.

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REFERENCES

Ataktürk, S. A., and K. B. Katsaros, 1987: Intrinsic frequency spectra of short gravity-capillary waves obtained from temporal measurements of wave height on a lake. *J. Geophys. Res.*, **92**, 5131–5141.

Donelan, M. A., 1978: Whitecaps and momentum transfer. *Turbulent Fluxes through the Sea Surface, Wave Dynamics, and Prediction*, A. Favre and K. Hasselmann, Eds. Plenum Press, 15 pp.

——, J. Hamilton and W. H. Hui, 1985: Directional spectra of wind-generated waves. *Phil. Trans. Roy. Soc. London*, **A315**, 509–562.

Forristall, G. Z., 1981: Measurements of a saturated range in ocean wave spectra. *J. Geophys. Res.*, **86**, 8075–8084.

Geernaert G., K. B. Katsaros and K. Richter, 1986: Variation of the drag coefficient and its dependence on sea state. *J. Geophys. Res.*, **91**, 7667–7679.

Hansen, C., 1985: Theoretical and experimental investigations of the spectra of wind-generated waves. M.S. thesis, Dept. of Physical Oceanography, University of Copenhagen.

Hasselmann, K., 1962: On the nonlinear energy transfer in a gravity wave spectrum, Part 1. *J. Fluid Mech.*, **12**, 481–500.

——, 1963: On the nonlinear energy transfer in a gravity spectrum, Part 2. *J. Fluid Mech.*, **15**, 273–281.

——, et al., 1973: Measurements of wind wave growth and swell decay during the Joint North Sea Wave Project (JONSWAP). *Ergänzungsheft zur Deutschen Hydrographischen Zeits.* Reihe A, **12**.

——, D. B. Ross, P. Müller and W. Sell, 1976: A parametric wave prediction model. *J. Phys. Oceanogr.*, **6**, 200–228.

Kahma, K. K., 1981: A study of the growth of the wave spectrum with fetch. *J. Phys. Oceanogr.*, **11**, 1503–1515.

Kitaigorodskii, S. A., 1983: On the theory of equilibrium range in the spectrum of wind-generated gravity waves. *J. Phys. Oceanogr.*, **13**, 816–827.

——, and J. L. Lumley, 1983: Wave turbulence in the upper ocean. Part I: The energy balance of the interacting fields of surface wind waves and wind-induced three-dimensional turbulence. *J. Phys. Oceanogr.*, **13**, 1977–1987.

——, V. P. Krasitskii and M. M. Zaslavskii, 1975: On Phillips theory of equilibrium range in the spectra of wind-generated gravity waves. *J. Phys. Oceanogr.*, **5**, 410–420.

Komen, G. J., S. Hasselmann and K. Hasselmann, 1984: On the existence of a fully developed wind-sea spectrum, *J. Phys. Oceanogr.*, **14**, 1271–1285.

Kristensen, L., and P. Kirkegaard, 1987: Digitization noise in power spectral analysis. *J. Atmos. Oceanic Technol.*, **4**, 328–335.

Liu, H.-T., K. B. Katsaros and M. A. Weissman, 1982: Dynamic response of thin-wire wave gauges. *J. Geophys. Res.*, **87**, 5686–5698.

Long, R. S., 1977: Some aspects of turbulence in geophysical systems. *Advances in Applied Mechanics*, Academic.

Longuet-Higgins, M. S., 1969: On wave breaking and the equilibrium spectrum of wind-generated waves. *Proc. Roy. Soc. London*, **310**, 151–159.

Mitsuyasu, H., F. Tasai, T. Suhara, S. Mizuno, M. Ohkusu, T. Honda and K. Rikiishi, 1980: Observation of the power spectrum of ocean waves using a cloverleaf buoy. *J. Phys. Oceanogr.*, **10**, 286.

Phillips, O. M., 1958: The equilibrium range of the spectrum of wind-generated ocean waves. *J. Fluid Mech.*, **4**, 426–434.

——, 1977; *The Dynamics of the Upper Ocean*, Cambridge University Press, 196 pp.

——, 1985: Spectral and statistical properties of the equilibrium range in wind-generated gravity waves. *J. Fluid Mech.*, **156**, 505–531.

Plant, W. J., 1982: A relationship between wind stress and wave slope. *J. Geophys. Res.*, **87**, 1961–1967.

Robertson, J. W., 1959: On turbulent plane-cuette flow. *Proc. Sixth Annual Conf. on Fluid Mechanics*, 169–182.

Zakharov, V. E., and N. N. Filonenko, 1966: The energy spectrum for random surface waves. *Dokl. Acad. Sci. USSR*, **170**, 1291–1295.

——, and M. M. Zaslavskii, 1982: The kinetic equation and Kolmogorov spectra in the weak turbulence theory of wind waves. *Izv., Atmos. Oceanic Phys.*, **18**, 747–753.

——, and ——, 1983: Shape of the spectrum of energy-carrying components of a water surface in the weak-turbulence theory of wind waves. *Izv., Atmos. Oceanic Phys.*, **19**, 207–212.

Zaslavskii, M. M., and L. B. Lobysheva, 1983: Estimate of wave fluxes of momentum, energy and action from field data. *Oceanology*, **23**, 574–579.