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WAVE ACTION AND WAVE-MEAN FLOW INTERACTION, WITH APPLICATION TO STRATIFIED SHEAR FLOWS

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1. GENERAL THEORY

Over the last two decades wave action principles and the associated wave-mean flow interaction theorems have become increasingly important for the study of the various kinds of waves that occur in fluid mechanics. Action density is a well-established entity in classical mechanics and plays a central role in the Lagrangian and Hamiltonian development of that subject. However, only relatively recently have the corresponding entities in fluid mechanics been identified and exploited. This is partly because fluid mechanics has traditionally been developed in an Eulerian framework and wave action principles are more obvious in a Lagrangian framework, and partly because the classes of waves for which wave action principles are particularly useful (e.g. internal gravity waves in stratified shear flows) have only recently received much attention.

The current interest in wave action began with the pioneering work of Whitham (1965, 1970), who introduced the wave action equation through the averaged variational principle. Although the initial motivation was the study of finite-amplitude waves, it was soon recognized that the wave action equation was also useful for the study of linearized waves on a mean flow (Bretherton & Garrett 1968). However, because these early theories were Lagrangian in concept, it became necessary to develop Lagrangian equations of motion in contrast to the more familiar Eulerian equations of motion. The key concept here is a correct definition of the Lagrangian-mean flow with respect to which particle displacements can be defined;

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these particle displacements then serve as the appropriate field variables in a Lagrangian formulation. The preliminary ideas were developed by Dewar (1970) and Bretherton (1971) and culminated in the generalized Lagrangian-mean formulation of Andrews & McIntyre (1978a,b).

In this review, we describe wave action principles and wave-mean flow interaction theorems in three stages. In this section, we present a general theory based on a Lagrangian formulation of the equations of motion. Our treatment is in the spirit of Whitham's approach (Whitham 1965, 1970), but follows the development by Hayes (1970) more closely. Because wave action and wave-mean flow interaction are intertwined concepts, we complement the discussion on wave action by introducing the radiation stress tensor and describing its role in the wave energy equation and the mean flow equation. Our treatment is based on the ideas of Dewar (1970) and Bretherton (1971), but goes beyond their results in that there is no restriction to slowly varying linearized waves.

In order to apply this general theory to fluids, we turn in Section 2 to a description of the generalized Lagrangian-mean formulation of Andrews & McIntyre. Although the results of this section are complete, they are obtained in a form where their application to specific problems generally requires more discussion. Rather than give a catalog of all the contexts in fluid mechanics where wave action principles have been invoked, we instead give in Section 3 a brief account of internal gravity waves in a stratified shear flow. Our purpose here is didactic; that is, our concern is not so much to present some specific results as to illustrate how the general theory is adapted in a specific case.

Formulation

We begin by supposing that the physical system is specified by the vector-valued field ϕ (x_i), where x_i $(i = 0, 1, 2, 3)$ are the independent variables. Subsequently it will be useful to distinguish between $x_0 = t$, a timelike coordinate, and x_α $(\alpha = 1, 2, 3)$, which are spacelike coordinates. Throughout we employ the dual summation convention that Latin indices are summed over the range 0 to 3, but Greek indices are summed over the range 1 to 3. In the absence of dissipation, we suppose that the physical system obeys a variational principle with a Lagrangian density $L(\phi_i, \phi; x_i)$, where ϕ_i denotes the partial derivative $\partial\phi/\partial x_i$. Then the equations of motion are

$$\frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial \phi_i}\right) - \frac{\partial L}{\partial \phi} = Q. \tag{1.1}$$

Here the generalized force Q represents the effects of dissipation. A useful consequence of this formulation (see Hayes 1970) is that if ψ is any field with the same dimension as ϕ, then

$$\frac{\partial}{\partial x_i}\left(\psi\frac{\partial L}{\partial \phi_i}\right) = \psi_i\frac{\partial L}{\partial \phi_i} + \psi\frac{\partial L}{\partial \phi} + \psi Q. \tag{1.2}$$

For instance, putting $\psi = \phi_j$ in (1.2), it follows that

$$\frac{\partial T_{ji}}{\partial x_i} = -\frac{\partial L}{\partial x_j} + \phi_j Q, \tag{1.3a}$$

where

$$T_{ji} = \phi_j\frac{\partial L}{\partial \phi_i} - L\,\delta_{ji}. \tag{1.3b}$$

Here $\partial L/\partial x_j$ on the right-hand side of (1.3a) is the explicit derivative of L with respect to x_j, and T_{ji} can be identified as the energy-momentum tensor of classical theoretical physics (see, for example, Landau & Lifshitz 1962). Although the precise physical interpretation of the components of T_{ji} will depend *inter alia* on the choice of Lagrangian density, we shall find it useful to identify T_{00} as the energy density and $T_{0\alpha}$ as its flux, and $T_{\alpha 0}$ as the momentum density and $T_{\alpha\beta}$ as the corresponding fluxes.

WAVE ACTION In order to define wave action density and its flux, we must first introduce the notion of an ensemble average $\langle\ \rangle$. We base our discussion on the ideas of Sturrock (1962) and Hayes (1970), which were further developed by Andrews & McIntyre (1978b). The relationship with the more specialized notions of Whitham (1965, 1970), Dewar (1970), Dougherty (1970), and Bretherton (1971) are developed below. We suppose that $\phi(x_i, \theta)$ depends smoothly on the ensemble parameter θ, such that

$$\phi(x_i, \theta+2\pi) = \phi(x_i, \theta). \tag{1.4}$$

We then define the averaging operator

$$\langle\ \rangle = \frac{1}{2\pi}\int_0^{2\pi}(\qquad)\,d\theta. \tag{1.5}$$

For simplicity, we sometimes denote the mean field $\langle\phi\rangle$ by $\bar{\phi}$, and we note the important observation that all mean quantities are independent of θ. The averaging operator commutes with $\partial/\partial x_i$, and has other simple and obvious properties [see Andrews & McIntyre (1978a,b) for an extensive discussion].

We next define the wave perturbation or disturbance field $\hat{\phi}(x_i, \theta)$ of ϕ by

$$\phi = \bar{\phi} + \hat{\phi}. \tag{1.6}$$

Clearly $\hat{\phi}$ has a zero mean ($\langle\hat{\phi}\rangle = 0$). Although we have called $\hat{\phi}$ the wave perturbation of ϕ, there is at present no restriction on the magnitude of $\hat{\phi}$ vis-à-vis that of $\bar{\phi}$, nor on their relative scales. Next put $\psi = \hat{\phi}_\theta$ in (1.2), where $\hat{\phi}_\theta = \partial\hat{\phi}/\partial\theta$, and apply the averaging operator $\langle\ \rangle$. It follows that (see Hayes 1970)

$$\frac{\partial}{\partial x_i}\left\langle \hat{\phi}_\theta \frac{\partial L}{\partial \phi_i} \right\rangle = \langle \hat{\phi}_\theta Q \rangle. \tag{1.7}$$

This is the wave action equation. In the absence of dissipation ($Q = 0$), it gives a local conservation law. It is a consequence of the invariance of the mean Lagrangian $\langle L \rangle$ to changes in θ. We shall find it useful to identify

$$\mathbf{A} = \left\langle \hat{\phi}_\theta \frac{\partial L}{\partial \phi_t} \right\rangle, \qquad \mathbf{B}_\alpha = \left\langle \hat{\phi}_\theta \frac{\partial L}{\partial \phi_\alpha} \right\rangle \tag{1.8}$$

as the wave action density and flux, respectively. Here and subsequently, we write ϕ_t in place of ϕ_0, which denotes the partial derivative $\partial\phi/\partial t$. From (1.8) it is clear that both $\mathbf{A}$ and $\mathbf{B}_\alpha$ are $O(a^2)$ in the limit of small wave amplitude a. Here the wave amplitude parameter a is a measure of the magnitude of $\hat{\phi}$ and its derivatives. Consequently, $\mathbf{A}$ and $\mathbf{B}_\alpha$ are wave properties and are the appropriate general measures of wave activity. The analogue of (1.7) in classical mechanics is obtained by restricting the independent variables to t alone. The action density $\mathbf{A}$ can then be recognized as $\oint p\, d\phi$, where p is the momentum conjugate to ϕ, and the integral is over one cycle. The action equation (1.7) is then the basis for the study of adiabatic invariants (for a lucid discussion, see Landau & Lifshitz 1960).

It may be shown that both the density and flux are unaffected by smooth, monotonic transformations of the ensemble parameter θ, and by the addition of flux terms to the Lagrangian density L that leave the equations of motion (1.1) unchanged. Further, under a Galilean transformation $x'_\alpha = x_\alpha - U_\alpha t, t' = t$, where U_α is a constant velocity, $\mathbf{A}$ is invariant, and $\mathbf{B}_\alpha$ transforms to $\mathbf{B}_\alpha - U_\alpha \mathbf{A}$; thus (1.7) is left invariant. Under a Lorentz transformation, the four-vector $(\mathbf{A}, \mathbf{B}_\alpha)$ transforms according to the usual laws for a relativistic four-vector. These properties are in strong contrast to the corresponding properties of the energy-momentum tensor and Equation (1.3a).

The application of (1.7) depends upon the delineation of the family $\phi(x_i, \theta)$. One possibility that has received much attention is to identify θ as a phase shift in the phase of the wave; the averaging operator (1.5) is then an average over this phase. Thus we put

$$\hat{\phi} = \hat{\phi}(x_i, s(x_i) - \theta), \tag{1.9a}$$

and

$$\kappa_i = \frac{\partial s}{\partial x_i}. \tag{1.9b}$$

Here $s(x_i)$ is the phase, and ϕ is periodic in s. Let κ_α be the wavenumber components, and $\omega = -\kappa_0$ the frequency. It follows that

$$\mathbf{A} = \frac{\partial}{\partial \omega}\langle L \rangle, \qquad \mathbf{B}_\alpha = -\frac{\partial}{\partial \kappa_\alpha}\langle L \rangle, \tag{1.10}$$

and the wave action equation (1.7) takes the form obtained by Whitham (1970). In this form, the wave action equation can be recognized as the Euler equation that is obtained when the mean Lagrangian $\langle L \rangle$ is subjected to variations in the phase $s(x_i)$, and is an application of Whitham's averaged variational principle (Whitham 1965, 1970), here extended to include dissipative effects (Ostrovsky & Pelinovsky 1972).

It is a remarkable fact that the wave action equation (1.7) is formally exact. It is valid without any restriction in wave amplitude, or without any assumption that the mean field is slowly varying with respect to the waves. However, the utility of this is reduced in practice by the presence of two kinds of error. The first of these has been called by Hayes (1970) the *identification* error, and occurs whenever the ensemble parameter θ is interpreted as a phase shift. It arises as a result of the identification of the family (1.9a) with a particular solution of interest, and can in principle be made arbitrarily small with respect to a small parameter characterizing the difference in scale between the rapidly varying phase of the waves and other variations, such as those in the mean field. The second kind of error is due to the implicit hypothesis that only a single wave is present, and is particularly severe in strongly nonlinear systems. To some extent it can be removed by allowing θ to be vector-valued (see Hayes 1970), so that the family $\phi(x_i, \theta)$ describes a multiple wave system. This aspect has been largely neglected in the wave action literature, although this is compensated by the extensive literature on wave interactions. In the extreme case of strongly nonlinear random wave interactions, the action density obeys a diffusion equation in phase space (see, for instance, Abarbanel 1981).

PSEUDOMOMENTUM If the averaging operator (1.5) is applied to (1.3a) we obtain equations for the total energy $\langle T_{00} \rangle$ and total momentum $\langle T_{\alpha 0} \rangle$. However, these equations are not generally as useful as the wave action equation in determining wave properties, as they contain both wave and mean flow expressions; in particular, the mean flow expressions will contain $O(a^2)$ contributions due to the waves. To obtain a mathematical analogue of $\langle T_{ij} \rangle$ for the waves, we follow the procedure of Andrews & McIntyre (1978b). First define the "undisturbed" Lagrangian by

$$L_0 = L(\bar{\phi}_i, \bar{\phi}; x_i) \tag{1.11}$$

and then put

$$L_1(\hat{\phi}_i, \hat{\phi}; x_i) = L - L_0. \tag{1.12}$$

Note that L_1 is an $O(a^2)$ wave property, and that the explicit dependence of L_1 on x_i includes the dependence of L_1 on the mean field $\bar{\phi}$ and its derivatives $\bar{\phi}_i$. Then put $\psi = \hat{\phi}_j$ in (1.2) and apply the averaging operator (1.5). The result is

$$\frac{\partial}{\partial x_i}\mathbf{T}_{ji} = -\left\langle \frac{\partial L_1}{\partial x_j} \right\rangle + \langle \hat{\phi}_j Q \rangle, \tag{1.13a}$$

where

$$\mathbf{T}_{ji} = \left\langle \hat{\phi}_j \frac{\partial L_1}{\partial \phi_i} - L_1\ \delta_{ji} \right\rangle. \tag{1.13b}$$

$\mathbf{T}_{ji}$ is an $O(a^2)$ wave property. We shall call $\mathbf{T}_{00}$ the pseudoenergy, $\mathbf{T}_{0\alpha}$ its flux, $-\mathbf{T}_{\alpha 0}$ the pseudomomentum, and $-\mathbf{T}_{\alpha\beta}$ the corresponding flux. The sign conventions have been chosen to agree with historical convention (see Andrews & McIntyre 1978b). Note that $\mathbf{T}_{ji}$ can be identified as the averaged energy-momentum tensor for the disturbance Lagrangian L_1. Unlike the wave action density, $\mathbf{T}_{00}$ is not Galilean invariant and transforms to $\mathbf{T}_{00} + U_\alpha \mathbf{T}_{\alpha 0}$; however, the pseudomomentum $-\mathbf{T}_{\alpha 0}$ is Galilean invariant. Equation (1.13a) is a conservation equation only in the absence of dissipation ($Q = 0$) and when L_1 is explicitly independent of x_j; in particular, this latter condition requires the mean field $\bar{\phi}$ to be independent of x_j.

The relationship between (1.13a) and the wave action equation is obtained by observing that when the mean field $\bar{\phi}$ is independent of a particular coordinate x_j, then, by invoking a suitable ergodic principle, we may replace the averaging operator (1.5) with averaging over that coordinate and so identify θ with $-x_j$. The wave action equation then reduces to (1.13a), the wave action density to $-\mathbf{T}_{j0}$ for $j \neq 0$, and the flux $\mathbf{B}_\alpha$ to $-\mathbf{T}_{j\alpha}$ for $\alpha \neq j$; note that the diagonal term $\mathbf{T}_{jj}$ is now absent from (1.13a), being independent of x_j by definition. This application of the wave action equation is quite common, although in the literature it has not always been recognized as such. It is important to note the distinction between energy and momentum on the one hand, and pseudoenergy and pseudomomentum on the other. The former quantities are conserved, in the absence of dissipation, whenever the total system, represented by the full

Lagrangian L, is independent of t or x_α respectively. The latter quantities are conserved, again in the absence of dissipation, whenever the mean field, represented by $\langle L_1 \rangle$, is independent of t or x_α, respectively.

WAVE ENERGY Wave energy is a constantly recurring theme in the literature, although as we show below it is generally not as useful a concept as wave action. This arises, in part, from ambiguities in its definition. Bretherton & Garrett (1968) have given a comprehensive discussion of an appropriate definition for wave energy in the context of linearized waves. Adapting their conclusions, we define the wave energy density $\mathbf{E}$ as the pseudoenergy in a frame with respect to which the mean state is locally at rest. Specifically, let us now postulate that the mean field $\bar{\phi}$ consists of a mean velocity $\bar{u}_\alpha$ and a vector-valued mean field $\bar{\lambda}$. In applications, $\bar{\lambda}$ will incorporate the mean density, mean magnetic field, mean fluid depth, etc. Thus we define

$$\mathbf{E} = \mathbf{T}_{00} + \bar{u}_\alpha \mathbf{T}_{\alpha 0}, \tag{1.14a}$$

or

$$\mathbf{E} = \left\langle \frac{d\hat{\phi}}{dt} \frac{\partial L_1}{\partial \phi_t} - L_1 \right\rangle, \tag{1.14b}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \bar{u}_\alpha \frac{\partial}{\partial x_\alpha}. \tag{1.14c}$$

Note that d/dt is the time derivative following the mean motion. The corresponding definition of wave energy flux is

$$\mathbf{F}_\alpha = \left\langle \frac{d\hat{\phi}}{dt} \frac{\partial L_1}{\partial \phi_\alpha} - \bar{u}_\alpha (L_1 + \mathbf{E}) \right\rangle. \tag{1.15}$$

Then from (1.13a), or more directly by putting $\psi = d\hat{\phi}/dt$ in (1.2), it follows that the wave energy equation is

$$\frac{\partial \mathbf{E}}{\partial t} + \frac{\partial}{\partial x_\alpha}(\bar{u}_\alpha \mathbf{E} + \mathbf{F}_\alpha) = -\left\langle \frac{\partial L_1}{\partial t} + \bar{u}_\alpha \frac{\partial L_1}{\partial x_\alpha} \right\rangle + \frac{\partial \bar{u}_\alpha}{\partial x_i} \mathbf{T}_{\alpha i} + \left\langle \frac{d\hat{\phi}}{dt} Q \right\rangle. \tag{1.16}$$

Note that here, as in (1.13a), $\partial L_1/\partial x_i$ denotes the explicit derivative of L_1 with respect to x_i, including the dependence of L_1 on x_i through $\bar{u}_\alpha$ and $\bar{\lambda}$. Equation (1.16) demonstrates that whenever the mean state is varying, wave energy is not conserved, and is generally exchanged with the mean flow. This contrasts unfavorably with the wave action equation, which states that wave action is conserved in the absence of dissipation, regardless of the variability of the mean state.

RADIATION STRESS At this point is is useful to make some further hypotheses concerning the Lagrangian L. These are motivated by the generalized Lagrangian-mean description of fluid flow introduced by Andrews & McIntyre (1978a) (see also Dewar 1970, Bretherton 1971), which we describe in more detail in Section 2. We suppose that L_1 depends on $\hat{\phi}_t$ only through its dependence on $d\hat{\phi}/dt$; further, apart from the dependence of L_1 on $\bar{u}_\alpha$ through $d\hat{\phi}/dt$, any other explicit dependence of L_1 on $\bar{u}_\alpha$ is bilinear in $\bar{u}_\alpha$ and the disturbance variables $\hat{\phi}$ and $\hat{\phi}_i$. This last property arises from the fact that the full Lagrangian is usually at most quadratic in the velocity field. It then follows that

$$\left\langle \frac{\partial L_1}{\partial \bar{u}_\alpha} \right\rangle = \mathbf{T}_{\alpha 0}. \tag{1.17}$$

Next, following Garrett (1968) and Dewar (1970), we observe that in many cases of interest λ will satisfy an equation of the form

$$\frac{d\lambda}{dt} + \Lambda_{\alpha\beta}\lambda \frac{\partial \bar{u}_\alpha}{\partial x_\beta} = \sigma. \tag{1.18}$$

Here σ represents dissipative effects. When σ is zero, λ is a mean quantity ($\lambda = \bar{\lambda}$); however, it is useful for the applications to be discussed later to allow λ to have a fluctuating component $\hat{\lambda}$ when σ is nonzero. These dissipative components are not included in the disturbance components $\hat{\phi}$, but are included in the radiation stress tensor defined below. The wave energy equation then becomes

$$\frac{\partial \mathbf{E}}{\partial t} + \frac{\partial}{\partial x_\alpha}(\bar{u}_\alpha \mathbf{E} + \mathbf{F}_\alpha) = -R_{\alpha\beta}\frac{\partial \bar{u}_\alpha}{\partial x_\beta} - \left\langle \frac{dL_1}{dt} \right\rangle_{\mathrm{e}} + \left\langle \frac{d\hat{\phi}}{dt} Q \right\rangle - \left\langle \frac{\partial L_1}{\partial \lambda} \sigma \right\rangle, \tag{1.19a}$$

where

$$R_{\alpha\beta} = -\mathbf{T}_{\alpha\beta} + \bar{u}_\beta \mathbf{T}_{\alpha 0} - \left\langle \Lambda_{\alpha\beta}\lambda \frac{\partial L_1}{\partial \lambda} \right\rangle. \tag{1.19b}$$

Although the nomenclature is not universal, we shall call $R_{\alpha\beta}$ the radiation stress tensor. Here $\langle \partial L_1/\partial x_i \rangle_{\mathrm{e}}$ denotes the explicit derivative of L_1 with respect to x_i when the disturbance variables $\hat{\phi}$, $\hat{\phi}_i$ and the mean variables $\bar{u}_\alpha$ and λ are all held constant. Also note the useful results in this context that

$$\mathbf{B}_\alpha = \bar{u}_\alpha \mathbf{A} + \left\langle \hat{\phi}_\theta \left(\frac{\partial L_1}{\partial \hat{\phi}_\alpha} \right)_{\mathrm{d}} \right\rangle, \tag{1.20a}$$

$$\mathbf{F}_\alpha = \left\langle \frac{d\hat{\phi}}{dt}\left(\frac{\partial L_1}{\partial \hat{\phi}_\alpha}\right)_{\mathrm{d}} \right\rangle, \tag{1.20b}$$

$$\mathbf{T}_{\alpha\beta} = \bar{u}_\beta \mathbf{T}_{\alpha 0} + \left\langle \hat{\phi}_\alpha \left(\frac{\partial L_1}{\partial \hat{\phi}_\beta}\right)_{\mathrm{d}} - L_1\,\delta_{\alpha\beta} \right\rangle. \tag{1.20c}$$

Here $(\partial L_1/\partial \hat{\phi}_\alpha)_{\mathrm{d}}$ denotes the derivative with respect to $\hat{\phi}_\alpha$ when $d\hat{\phi}/dt$ is held constant. An important consequence of these equations is that when the averaging operator (1.5) can be interpreted as an average over the coordinate x_γ (i.e. θ can be identified with $-x_\gamma$), then the off-diagonal components of $R_{\gamma\beta}$ differ from $\mathbf{B}_\beta - \bar{u}_\beta \mathbf{A}$ only by the terms involving $\Lambda_{\gamma\beta}$; in particular, if $\Lambda_{\gamma\beta}$ is itself diagonal, then $R_{\gamma\beta}$ for $\gamma \neq \beta$ is exactly equal to $\mathbf{B}_\beta - \bar{u}_\beta \mathbf{A}$.

MEAN FLOW To complete the description of the interaction between the waves and the mean flow, equations describing the forcing of the mean flow by the waves are needed. These can be obtained by applying the averaging operator (1.5) to (1.1) or, alternatively, to (1.3). In the present context, when the mean field consists only of the mean velocity $\bar{u}_\alpha$ and the vector-valued mean field $\bar{\lambda}$ that satisfies (1.18), the most convenient result is

$$\frac{\partial}{\partial t}\left(\frac{\partial L_0}{\partial \bar{u}_\alpha}\right) + \frac{\partial}{\partial x_\beta}\left(\bar{u}_\beta \frac{\partial L_0}{\partial \bar{u}_\alpha} - \Lambda_{\alpha\beta}\bar{\lambda}\frac{\partial L_0}{\partial \bar{\lambda}} + L_0\,\delta_{\beta\alpha}\right) - \left\langle \frac{\partial L}{\partial x_\alpha}\right\rangle_{\mathrm{e}} = -\frac{\partial R_{\alpha\beta}}{\partial x_\beta} + \langle Q_\alpha \rangle. \tag{1.21}$$

Since L_0 is quadratic in $\bar{u}_\alpha$, this is just the mean momentum equation and can be obtained by averaging the full-momentum equation [i.e. those equations in (1.1) corresponding to the components of ϕ for which $d\phi/dt$ is the total velocity u_α].

However, a simpler and more revealing derivation of (1.21) is to apply Whitham's averaged variational principle (Whitham 1965, 1970, Ostrovsky & Pelinovsky 1972)

$$\delta \int \langle L \rangle \, dx_\alpha \, dt = -\int \langle Q\delta\phi \rangle \, dx_\alpha \, dt \tag{1.22}$$

with respect to variations in $\bar{u}_\alpha$ and λ. This is the procedure used by Dewar (1970) and Bretherton (1971) in the context of small-amplitude waves. The variations in $\bar{u}_\alpha$ and λ are obtained by considering variations Δx_α in x_α, where Δx_α is a Lagrangian variation, or a variation incurred on a given fluid particle moving with the mean velocity. The corresponding Lagrangian variations $\Delta\bar{\phi}$ in any mean quantity $\bar{\phi}$ must be distinguished from the Eulerian variation $\delta\bar{\phi}$, the variation incurred at a given point x_α [see

Bretherton (1970) for a lucid discussion of this point]. They are related by the expression

$$\delta\bar{\phi} = \Delta\bar{\phi} - \Delta x_\alpha \frac{\partial \bar{\phi}}{\partial x_\alpha}. \tag{1.23}$$

Thus the variations in $\bar{u}_\alpha$ and λ are given by

$$\delta\bar{u}_\alpha = \frac{d}{dt}(\Delta x_\alpha) - \Delta x_\beta \frac{\partial \bar{u}_\alpha}{\partial x_\beta}, \tag{1.24a}$$

$$\delta\lambda = -\Lambda_{\alpha\beta}\lambda \frac{\partial}{\partial x_\beta}(\Delta x_\alpha) - \Delta x_\beta \frac{\partial \lambda}{\partial x_\beta}. \tag{1.24b}$$

The expression (1.24b) is valid only for a restricted class of tensors $\Lambda_{\alpha\beta}$ (see Dewar 1970), which, however, includes all cases so far encountered in the literature. Applying the principle (1.22), it now follows that

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \bar{u}_\alpha}\langle L\rangle\right) + \frac{\partial}{\partial x_\beta}\left(\bar{u}_\beta \frac{\partial}{\partial \bar{u}_\alpha}\langle L\rangle - \left\langle \Lambda_{\alpha\beta}\lambda \frac{\partial L}{\partial \lambda}\right\rangle\right)$$
$$+ \frac{\partial \bar{u}_\beta}{\partial x_\alpha}\frac{\partial}{\partial \bar{u}_\beta}\langle L\rangle + \left\langle \frac{\partial \lambda}{\partial x_\beta}\frac{\partial L}{\partial \lambda}\right\rangle = \langle Q_\alpha\rangle + \langle \hat{\phi}_\alpha Q\rangle. \tag{1.25}$$

Decomposing L into L_1 and L_0 (1.12), we can derive (1.21) by using (1.13a), (1.17), and (1.19b). An alternative to (1.21) that does not involve the radiation stress tensor $R_{\alpha\beta}$ but instead contains the pseudomomentum $-\mathbf{T}_{\alpha 0}$ can be derived from (1.25) by decomposing L into L_1 and L_0 (1.12) and then using only (1.13a). This latter form is the one preferred by Andrews & McIntyre (1978a) and is often more useful, particularly for irrotational flow and for situations where one component of the divergence of the radiation stress tensor is larger, by an order of magnitude in some small parameter, than all the other terms in (1.21). For examples of this, see Andrews & McIntyre (1978a) or Grimshaw (1979). An important application of this alternative procedure arises when the averaging operator (1.5) can be interpreted as an average over the coordinate x_γ (i.e. θ can be identified with $-x_\gamma$). Then if $\Lambda_{\alpha\beta}$ is diagonal, the off-diagonal components of $R_{\gamma\beta}$ are just $\mathbf{B}_\beta - \bar{u}_\beta \mathbf{A}$, and so in the γ-component of (1.21) the divergence of the radiation stress tensor is given by

$$-\frac{\partial R_{\gamma\beta}}{\partial x_\beta} = \frac{\partial \mathbf{A}}{\partial t} + \frac{\partial}{\partial x_\beta}(\bar{u}_\beta \mathbf{A}) - \langle \hat{\phi}_\gamma Q\rangle. \tag{1.26}$$

If (1.21) is multiplied by $\bar{u}_\alpha$ and the result is added to (1.19a), we obtain the mean-energy equation

$$\frac{\partial \mathbf{E}^{\mathrm{T}}}{\partial t}+\frac{\partial}{\partial x_\beta}\left\{\bar{u}_\beta \mathbf{E}^{\mathrm{T}}+\bar{u}_\alpha\left(-\Lambda_{\alpha\beta}\lambda\frac{\partial L_0}{\partial \lambda}+L_0\,\delta_{\beta\alpha}\right)+\mathbf{F}_\beta+\bar{u}_\alpha R_{\alpha\beta}\right\}$$
$$+\left\langle\frac{\partial L}{\partial t}\right\rangle_{\mathrm{e}}=\left\langle \bar{u}_\alpha Q_\alpha+\frac{d\hat{\phi}}{dt}Q-\frac{\partial L}{\partial \lambda}\sigma\right\rangle, \quad (1.27\mathrm{a})$$

where

$$\mathbf{E}^{\mathrm{T}}=\bar{u}_\alpha\frac{\partial \bar{L}_0}{\partial \bar{u}_\alpha}-L_0+\mathbf{E}. \quad (1.27\mathrm{b})$$

Here $\mathbf{E}^{\mathrm{T}}$ is the total mean energy. In some applications this equation is more useful than the wave energy equation (1.19a), as it is in conservation form in the absence of dissipation and explicit time dependence.

The mean field equations are thus (1.18) and (1.21), and apart from the dissipative terms and the term representing the effects associated with external forces $(\langle \partial L/\partial x_\alpha\rangle)_{\mathrm{e}}$, the radiation stress tensor $R_{\alpha\beta}$ represents the sole effect due to the waves. Note that $R_{\alpha\beta}$ is generally asymmetric and has no simple relationship to the Reynolds stresses and buoyancy fluxes encountered in Eulerian formulations of the mean flow equations. These mean-field equations are complemented by the wave action equation (1.7), which is generally the most useful way of describing the effect of the mean field on the waves. For finite-amplitude waves the two sets of equations are coupled. However, for linearized wave motion the action density and flux can be evaluated correct to $O(a^2)$, with the mean field fixed at the basic state values. The mean field changes due to the waves can then be calculated from (1.21), where $R_{\alpha\beta}$ can be evaluated correct to $O(a^2)$ independently of the mean field changes. Much of the literature on wave action and wave-mean flow interaction has considered only this special case of linearized wave motion.

SLOWLY VARYING WAVES Slowly varying, almost-plane waves have the representation (1.9a) where the dependence on the phase $s(x_i)$ is rapidly varying relative to the explicit dependence on x_i (although Rossby waves on a β-plane are an important exception). From (1.9a) it follows that

$$\hat{\phi}_i=-\kappa_i\hat{\phi}_\theta+\frac{\partial\hat{\phi}}{\partial x_i}, \quad (1.28)$$

where the explicit derivative $\partial\hat{\phi}/\partial x_i$ can be neglected compared to $\kappa_i\hat{\phi}_\theta$. The following approximate expressions can then be derived from (1.8), (1.13b), (1.14a), (1.15), and (1.19b):

$$\mathbf{T}_{00} \approx \omega \mathbf{A} - \bar{L}_1, \qquad \mathbf{T}_{0\alpha} \approx \omega \mathbf{B}_\alpha, \tag{1.29a}$$

$$\mathbf{T}_{\alpha 0} \approx -\kappa_\alpha \mathbf{A}, \qquad \mathbf{T}_{\alpha\beta} \approx -\kappa_\alpha \mathbf{B}_\beta - \bar{L}_1\, \delta_{\alpha\beta}, \tag{1.29b}$$

$$\mathbf{E} \approx \omega^* \mathbf{A} - \bar{L}_1, \qquad \mathbf{F}_\alpha \approx \omega^*(\mathbf{B}_\alpha - \bar{u}_\alpha \mathbf{A}), \tag{1.29c}$$

$$R_{\alpha\beta} \approx \kappa_\alpha(\mathbf{B}_\beta - \bar{u}_\beta \mathbf{A}) + \bar{L}_1\, \delta_{\alpha\beta} - \left\langle \Lambda_{\alpha\beta} \lambda \frac{\partial L_1}{\partial \lambda} \right\rangle, \tag{1.29d}$$

where

$$\omega^* = \omega - \kappa_\alpha \bar{u}_\alpha. \tag{1.29e}$$

Here ω^* is the intrinsic wave frequency.

SLOWLY VARYING LINEARIZED WAVES For small-amplitude waves further simplifications are possible. First note that if we put $\psi = \hat{\phi}$ in (1.2), then it follows that

$$\frac{\partial}{\partial x_i} \left\langle \hat{\phi} \frac{\partial L_1}{\partial \phi_i} \right\rangle = \left\langle \hat{\phi}_i \frac{\partial L_1}{\partial \hat{\phi}_i} + \hat{\phi} \frac{\partial L_1}{\partial \hat{\phi}} \right\rangle + \langle \hat{\phi} Q \rangle. \tag{1.30}$$

This can be regarded as a virial theorem (Hayes 1974, Andrews & McIntyre 1978b). For linearized wave motion, L_1 is at most quadratic in the disturbance quantities $\hat{\phi}$ and $\hat{\phi}_i$. Hence the first term on the right-hand side is just $2\bar{L}_1$. For slowly varying waves, the left-hand side can be neglected and, assuming that the dissipative term can likewise be neglected, it follows that $\bar{L}_1 \approx 0$. This in turn implies equipartition of energy in nonrotating systems. Thus for slowly varying linearized waves, (1.29c) shows that the action density $\mathbf{A}$ is given by the classical result $\mathbf{E}/\omega^*$. In the context of fluid mechanics, this result was first derived by Bretherton & Garrett (1968) using the averaged variational principle (Whitham 1965), although the identification of action density in terms of an energy density divided by a local frequency has antecedents in the classical theory of adiabatic invariants (Landau & Lifshitz 1960). Analogous results in the context of plasma physics were developed by Dewar (1970) and Dougherty (1970).

Next, for linearized waves, $\bar{L}_1$ will be quadratic in the wave amplitude, and hence given by an expression of the form

$$\bar{L}_1 \approx D(\omega^*, \kappa_\alpha; \lambda) a^2, \tag{1.31}$$

where the explicit dependence on ω^*, rather than just ω, is a consequence of the hypothesis that the mean velocity $\bar{u}_\alpha$ is slowly varying. Hence, assuming Galilean invariance, the averaged Lagrangian $\bar{L}_1$ can be evaluated approximately in a frame with respect to which the mean state is locally at rest. But $\bar{L}_1 \approx 0$ and so $D \approx 0$; this must be equivalent to the local dispersion relation

$$\omega^* = W^*(\kappa_\alpha; \lambda). \tag{1.32}$$

But from (1.10) it now follows that

$$\mathbf{B}_\alpha \approx c_\alpha \mathbf{A}, \tag{1.33a}$$

where

$$c_\alpha = \bar{u}_\alpha + c_\alpha^*, \tag{1.33b}$$

and

$$c_\alpha^* = \frac{\partial W^*}{\partial \kappa_\alpha}. \tag{1.33c}$$

Here c_α^* is the intrinsic group velocity. The wave action equation (1.7) now reduces to the form proposed by Bretherton & Garrett (1968). In the absence of dissipation this is

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{E}}{\omega^*}\right) + \frac{\partial}{\partial x_\alpha}\left([\bar{u}_\alpha + c_\alpha^*]\frac{\mathbf{E}}{\omega^*}\right) \approx 0. \tag{1.34}$$

With the same approximations, the energy flux $\mathbf{F}_\alpha \approx c_\alpha^* \mathbf{E}$, and the wave energy equation (1.19a) becomes

$$\frac{\partial \mathbf{E}}{\partial t} + \frac{\partial}{\partial x_\alpha}([\bar{u}_\alpha + c_\alpha^*]\mathbf{E}) \approx -R_{\alpha\beta}\frac{\partial \bar{u}_\alpha}{\partial x_\beta}, \tag{1.35a}$$

where

$$R_{\alpha\beta} \approx \frac{\mathbf{E}}{\omega^*}\left\{\kappa_\alpha c_\beta^* + \Lambda_{\alpha\beta}\lambda\frac{\partial W^*}{\partial \lambda}\right\}. \tag{1.35b}$$

It is readily verified (see Garrett 1968) that (1.34) and (1.35a) are equivalent. Finally, we note that the pseudomomentum $-\mathbf{T}_{\alpha 0}$ is approximately given by $\kappa_\alpha \mathbf{E}/\omega^*$.

MODAL WAVES In many applications the waves are confined to a waveguide by the presence of boundaries. Consequently, the waves possess a propagating character only with respect to coordinates that vary along the waveguide, and have a modal character across the waveguide. Following the notions of Hayes (1970) and Andrews & McIntyre (1978b), we suppose that a boundary Σ to the waveguide is undisturbed and impermeable to the fluid. The appropriate boundary condition on Σ is then

$$\text{either} \quad \hat{\phi} = 0 \quad \text{on} \quad \Sigma, \tag{1.36a}$$

$$\text{or} \quad n_i \frac{\partial L}{\partial \hat{\phi}_i} = 0 \quad \text{on} \quad \Sigma. \tag{1.36b}$$

Here n_i are the components of the normal to Σ; for instance, if Σ is given by $F(x_i) = 0$, then $n_i \propto \partial F/\partial x_i$. We also allow for the possibility that (1.36a) holds for some components of $\hat{\phi}$, and (1.36b) for the remaining components. It follows that

$$\hat{\phi}_\theta n_i \frac{\partial L}{\partial \hat{\phi}_i} = 0 \quad \text{on} \quad \Sigma. \tag{1.37}$$

Thus, the wave action flux normal to Σ vanishes on Σ. For simplicity, we are considering only nondissipative boundary conditions on Σ; for cases where dissipative boundary conditions are discussed, see Grimshaw (1981, 1982).

Let us now suppose, for simplicity, that the x_3-coordinate varies across the waveguide, which is bounded above and below by the surfaces $x_3 = F_\pm(t, x_1, x_2)$, respectively. The coordinates t, x_1, and x_2 thus characterize the propagation space. The wave action equation (1.7) continues to hold locally. However, the x_3-derivative in this equation will generally be the dominant term, and it is useful to remove it by integrating across the waveguide. Using the boundary condition (1.37), it follows that

$$\frac{\partial \mathscr{A}}{\partial t} + \frac{\partial \mathscr{B}_1}{\partial x_1} + \frac{\partial \mathscr{B}_2}{\partial x_2} = \int_{F-}^{F+} \langle \hat{\phi}_\theta Q \rangle \, dx_3. \tag{1.38a}$$

where

$$\mathscr{A} = \int_{F-}^{F+} \mathbf{A} \, dx_3, \tag{1.38b}$$

$$\mathscr{B}_\alpha = \int_{F-}^{F+} \mathbf{B}_\alpha \, dx_3. \tag{1.38c}$$

Equation (1.38a) is a global form of the wave action equation appropriate for modal waves; $\mathscr{A}$ and $\mathscr{B}_\alpha$ are the global wave action and flux, respectively. The analogue of (1.9a) for modal waves is obtained by restricting the phase s to be a function of only the variables t, x_1, and x_2. Since integration across the waveguide commutes with the averaging operator (1.5), it follows from (1.10) that

$$\mathscr{A} = \frac{\partial \mathscr{L}}{\partial \omega}, \qquad \mathscr{B}_\alpha = -\frac{\partial \mathscr{L}}{\partial \kappa_\alpha}, \tag{1.39a}$$

where

$$\mathscr{L} = \int_{F-}^{F+} \langle L \rangle \, dx_3. \tag{1.39b}$$

Thus, the wave action equation (1.38a) can be obtained from Whitham's averaged variational principle applied directly to $\mathscr{L}$.

The analogous global results for other quantities, such as the wave energy, the pseudomomentum, and the mean field, can also be obtained by integration across the waveguide. However, simple results analogous to (1.38a) are not generally obtained. For slowly varying waves, $\partial\hat{\phi}/\partial x_i$ can be neglected compared with $\kappa_i\hat{\phi}_\theta$ for $i = 0, 1, 2$, but $\partial\hat{\phi}/\partial x_3$ cannot be neglected. Nevertheless, the relations (1.29a–d) will continue to hold, provided the indices α, β are restricted to the values 1 and 2. Quantities such as the global wave energy, etc., can then be obtained by integrating (1.29a–d) across the waveguide. In this context, it is useful to note that the virial theorem (1.30) holds locally. By considering linearized waves, integrating across the waveguide, and using the boundary conditions (1.36a, b), it may be shown that the integral across the waveguide of $\bar{L}_1$ (i.e. $\mathscr{L}_1$) is approximately equal to zero. Thus for slowly varying, linearized modal waves whose dispersion relation is $\omega = W(\kappa_\alpha; t, x_1, x_2)$, we have $\mathscr{B}_\alpha = c_\alpha\mathscr{A}$, where c_α is the total group velocity $\partial W/\partial\kappa_\alpha$. This is the counterpart of (1.33a) for modal waves. Also, if we define

$$\mathscr{E} = (\omega - \kappa_\alpha v_\alpha)\mathscr{A} = \int_{F-}^{F+} \omega^* \mathbf{A}\, dx_3, \tag{1.39c}$$

where these equations also act as the definition of the mean velocity v_α, then we obtain the counterparts of (1.34) and (1.35a) for modal waves, i.e. replace E with $\mathscr{E}$ and ω^* with $\omega - \kappa_\alpha v_\alpha$, etc. The terms involving the mean field $\bar{\lambda}$ must of course be interpreted to apply to a different quantity that obeys a relation analogous to (1.18), with $\bar{u}_\alpha$ replaced with v_α and α, β restricted to the values 1 and 2. Also, the dispersion relation is assumed to take the form (1.32), with ω^* replaced with $\omega - \kappa_\alpha v_\alpha$.

2. FLUIDS AND THE GENERALIZED LAGRANGIAN-MEAN FORMULATION

An important feature of the general theory of Section 1 is that a Lagrangian formulation of the problem is an essential preliminary step to the efficient derivation of the wave action equation and the mean flow equations. Thus, in order to apply the results of Section 1 to specific cases involving fluids, the following points should be noted:

1. The problem should be formulated in terms of particle displacements ξ_α from a mean position that moves with the mean velocity $\bar{u}_\alpha$.
2. The wave action equation (1.7) is obtained by scalar multiplication of the momentum equation with $\partial\xi_\alpha/\partial\theta$, and averaging.

3. The wave energy equation (1.19a) is obtained by scalar multiplication of the momentum equation with $d\xi_\alpha/dt$, and averaging.
4. The mean flow equations should take the form (1.18) and (1.21), where the latter is obtained from averaging the momentum equation.

If these procedures are followed, it is often not necessary to identify the Lagrangian specifically, although its existence underlies the general theory. In particular, the radiation stress tensor is often most conveniently obtained by deriving the wave energy equation (1.19a) and the mean-flow equation (1.21) and consequently identifying $R_{\alpha\beta}$.

Lagrangian-Mean Formulation

The equations of motion for a conducting, compressible fluid in the nonrelativistic case are

$$\rho\frac{du_\alpha}{dt}+2\rho\varepsilon_{\alpha\beta\gamma}\Omega_\beta u_\gamma+\rho\frac{\partial\Phi}{\partial x'_\alpha}+\frac{\partial q}{\partial x'_\alpha}-\frac{B_\beta}{\mu}\frac{\partial B_\alpha}{\partial x'_\beta}=\rho X_\alpha, \tag{2.1a}$$

$$\frac{d\rho}{dt}+\rho\frac{\partial u_\alpha}{\partial x'_\alpha}=0, \tag{2.1b}$$

$$\frac{dS}{dt}=h, \tag{2.1c}$$

$$\frac{dB_\alpha}{dt}-B_\beta\frac{\partial u_\alpha}{\partial x'_\beta}+B_\alpha\frac{\partial u_\beta}{\partial x'_\beta}=j_\alpha, \tag{2.1d}$$

where

$$q=p+\frac{1}{2\mu}B_\alpha B_\alpha \tag{2.1e}$$

and

$$\frac{d}{dt}=\frac{\partial}{\partial t}+u_\alpha\frac{\partial}{\partial x'_\alpha}. \tag{2.1f}$$

Here x'_α is the Eulerian coordinate such that a fluid particle at x'_α has velocity u_α. The notation is standard; in particular, Ω_α is the constant angular velocity of the frame of reference, $\Phi(x'_\alpha)$ is the potential for both the gravitational and centrifugal forces, $p(\rho, S)$ is the thermodynamic pressure, S is the entropy, and B_α is the magnetic field. The terms X_α, h, and j_α represent, respectively, the effects of nonconservative and dissipative forces, nonadiabatic motion, and finite magnetic conductivity. In particular, note that $\partial j_\alpha/\partial x'_\alpha=0$, so that $\partial B_\alpha/\partial x'_\alpha=0$ is a consequence of (2.1d).

The appropriate Lagrangian formulation of these equations is the generalized Lagrangian-mean formulation of Andrews & McIntyre (1978a) (see also Dewar 1970, Bretherton 1971). For a comprehensive account and justification of this theory in the absence of a magnetic field, the reader is referred to Andrews & McIntyre (1978a) [see also McIntyre (1977, 1980), Grimshaw (1979), or Dunkerton (1980) for simplified versions], as here we give only a brief outline. Let x_α be generalized Lagrangian coordinates and let $\xi_\alpha(t, x_\beta)$ be the particle displacements, defined so that

$$x'_\alpha = x_\alpha + \xi_\alpha. \tag{2.2}$$

Then, for any given u_α there is a unique "reference" velocity $\bar{u}_\alpha(t, x_\beta)$, such that when the point x_α moves with velocity $\bar{u}_\alpha$ the point x'_α moves with velocity u_α. It follows that the material time derivative (2.1f) is also given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \bar{u}_\alpha \frac{\partial}{\partial x_\alpha}. \tag{2.3}$$

The generalized Lagrangian-mean formulation is now obtained by letting $\bar{u}_\alpha$ be the mean velocity, precisely that introduced in Section 1, and requiring that

$$\langle \xi_\alpha(t, x_\beta) \rangle = 0. \tag{2.4}$$

Note, in particular, that (2.3) agrees with our previous definition (1.14c). The reader should also note that our notation differs in one important respect from that of Andrews & McIntyre (1978a,b); here $\bar{u}_\alpha$ denotes the Lagrangian-mean velocity, rather than $\bar{u}^L_\alpha$ used in Andrews & McIntyre (1978a,b), who use the single overbar to denote Eulerian means. No confusion should arise, as Eulerian means are not discussed in this article.

Next we define a mean density $\tilde{\rho}$, so that

$$\frac{d\tilde{\rho}}{dt} + \tilde{\rho} \frac{\partial \bar{u}_\alpha}{\partial x_\alpha} = 0. \tag{2.5}$$

It is an immediate consequence of (2.1b) and (2.5) that

$$\rho J = \tilde{\rho}, \tag{2.6a}$$

where

$$J = \det\,[\partial x'_\alpha/\partial x_\beta]. \tag{2.6b}$$

For the magnetic field, we define a new variable H_α by

$$JB_\alpha = H_\beta \frac{\partial x'_\alpha}{\partial x_\beta}, \quad \text{or} \quad H_\alpha = B_\beta K_{\beta\alpha}. \tag{2.7}$$

Here $K_{\alpha\beta}$ is the α, β-cofactor of J, and so

$$K_{\alpha\beta}\frac{\partial x'_\alpha}{\partial x_\gamma} = \delta_{\beta\gamma}J = K_{\beta\alpha}\frac{\partial x'_\gamma}{\partial x_\alpha}. \tag{2.8}$$

It is useful to note that $K_{\alpha\beta}$ is the derivative of J with respect to $\partial x'_\alpha/\partial x_\beta$ and that $\partial K_{\alpha\beta}/\partial x_\beta = 0$. With the definitions (2.7), it can now be shown that (2.1d) becomes

$$\frac{dH_\alpha}{dt} - H_\beta\frac{\partial \bar{u}_\alpha}{\partial x_\beta} + H_\alpha\frac{\partial \bar{u}_\beta}{\partial x_\beta} = k_\alpha = j_\beta K_{\beta\alpha}. \tag{2.9}$$

Also, $\partial k_\alpha/\partial x_\alpha = 0$, so that $\partial H_\alpha/\partial x_\alpha = 0$ is a consequence of (2.9). The entropy equation (2.1c) is left unchanged, and the final step is the converting of the momentum equation (2.1a) to its Lagrangian form. The result is

$$\tilde{\rho}\frac{du_\alpha}{dt} + 2\tilde{\rho}\varepsilon_{\alpha\beta\gamma}\Omega_\beta u_\gamma + \tilde{\rho}\frac{\partial\Phi}{\partial x'_\alpha} + \frac{\partial}{\partial x_\beta}\left(qK_{\alpha\beta} - \frac{B_\alpha H_\beta}{\mu}\right) = \tilde{\rho}X_\alpha. \tag{2.10}$$

Here the velocity u_α is given by

$$u_\alpha = \bar{u}_\alpha + \frac{d\xi_\alpha}{dt}. \tag{2.11}$$

In summary, the generalized Lagrangian-mean equations are (2.10), the entropy equation (2.1c), the magnetic equation (2.9), and the mean density equation (2.5). They can be identified as the Euler equations (1.1) for the Lagrangian

$$L(u_\alpha, x'_\alpha, \partial x'_\alpha/\partial x_\beta, \tilde{\rho}, S, H_\alpha)$$
$$= \tilde{\rho}\{\tfrac{1}{2}u_\alpha u_\alpha + \varepsilon_{\alpha\beta\gamma}\Omega_\alpha x'_\beta u_\gamma - \Phi(x'_\alpha) - E(\rho, S)\} - \frac{J}{2\mu}B_\alpha B_\alpha. \tag{2.12}$$

Here we recall that ρ and B_α are defined in terms of $\tilde{\rho}$ and H_α by (2.6a) and (2.7), respectively. Also, $E(\rho, S)$ is the internal energy per unit mass, and

$$\frac{\partial E}{\partial \rho} = \frac{p}{\rho^2}, \qquad \frac{\partial E}{\partial S} = T, \tag{2.13}$$

where T is the temperature. Variations in x'_α (or equivalently ξ_α) then give (2.10), with $Q_\alpha = \tilde{\rho}X_\alpha$. Equation (2.5) for $\tilde{\rho}$ involves only mean quantities, and so acts as a constraint on the Lagrangian variations Δx_α, which determine the mean flow equation (1.21). Since S and H_α are mean quantities when the dissipative terms h and k_α vanish, Equations (2.1c) and (2.9) are in the same category. However, in order to keep the correspondence with the general theory of Section 1 as close as possible, we define the generalized forces Q_S and Q_{H_α} so that the corresponding Euler equation is an identity:

$$Q_S = \tilde{\rho} T, \qquad Q_{H_\alpha} = \frac{1}{\mu} B_\beta \frac{\partial x'_\beta}{\partial x_\alpha}. \tag{2.14}$$

Finally, we identify λ as the 5-vector whose components are $\tilde{\rho}$, S, and H_α. Then it may be verified that each of the equations (2.1c), (2.5), and (2.9) leads to an equation of the form (1.18) for λ (Dewar 1970), and that the variations in λ then satisfy (1.24b) as required.

WAVE ACTION This can now be obtained directly from (1.7), or by following the procedure of Andrews & McIntyre (1978b) and multiplying (2.10) by $\partial\xi_\alpha/\partial\theta$ and averaging. The result is

$$\frac{\partial \mathbf{A}}{\partial t} + \frac{\partial \mathbf{B}_\alpha}{\partial x_\alpha} = \mathbf{D}, \tag{2.15a}$$

where

$$\mathbf{A} = \left\langle \frac{\partial \xi_\alpha}{\partial \theta}\left(\tilde{\rho}\frac{d\xi_\alpha}{dt} + \tilde{\rho}\varepsilon_{\alpha\beta\gamma}\Omega_\beta \xi_\gamma\right)\right\rangle, \tag{2.15b}$$

$$\mathbf{B}_\alpha = \bar{u}_\alpha \mathbf{A} + \left\langle \frac{\partial \xi_\beta}{\partial \theta}\left(qK_{\beta\alpha} - \frac{B_\beta H_\alpha}{\mu}\right)\right\rangle, \tag{2.15c}$$

$$\mathbf{D} = \left\langle \frac{\partial \xi_\alpha}{\partial \theta}\tilde{\rho} X_\alpha + \frac{\partial S}{\partial \theta}\tilde{\rho} T + \frac{\partial H_\alpha}{\partial \theta}\frac{B_\beta}{\mu}\frac{\partial x'_\beta}{\partial x_\alpha}\right\rangle. \tag{2.15d}$$

That $\mathbf{D}$ represents the effects of dissipation follows from the identification of X_α as representing nonconservative and dissipative forces, and from the fact that S and H_α are disturbance quantities only when the dissipative terms h and k_α are nonzero. The expressions (2.15b–d) agree with those obtained by Andrews & McIntyre (1978b) in the absence of a magnetic field, although the dissipative term has been written in a different form here.

For linearized wave motion, it is useful to introduce the Eulerian pressure perturbation

$$q' = \hat{q} - \xi_\alpha \frac{\partial \bar{q}}{\partial x_\alpha} + O(a^2). \tag{2.16}$$

Then it may be shown that [see Andrews & McIntyre (1978b) or Grimshaw (1980)]

$$\mathbf{B}_\alpha - \bar{u}_\alpha \mathbf{A} = \left\langle q' \frac{\partial \xi_\alpha}{\partial \theta}\right\rangle + \frac{\partial}{\partial x_\beta}\left\langle \bar{q}\xi_\beta \frac{\partial \xi_\alpha}{\partial \theta}\right\rangle - \frac{1}{\mu}\left\langle H_\alpha H_\beta \frac{\partial \xi_\gamma}{\partial \theta}\left\{\frac{\partial \xi_\gamma}{\partial x_\beta} + \delta_{\gamma\beta}\left(1 - \frac{\partial \xi_\sigma}{\partial x_\sigma}\right)\right\}\right\rangle + O(a^3). \tag{2.17}$$

Note that the second term here is identically nondivergent and can be omitted from (2.15a). In many applications only the first term of (2.17) is significant; for instance, if all mean quantities depend only on a single coordinate, say x_3, and H_α is normal to this direction ($H_3 = 0$), then the only relevant component of $\mathbf{B}_\alpha$ is $\mathbf{B}_3$ and this is just $\langle q'\partial\xi_3/\partial\theta\rangle$. Equation (2.15a) can be rederived correct to $O(a^2)$ without invoking the generalized Lagrangian-mean formulation; the linearized momentum equation is multiplied by $\partial\xi_\alpha/\partial\theta$ and then averaged. If one invokes the basic flow equations (i.e. the mean flow equations to zeroth order in a), Equation (2.15a) follows with $\mathbf{A}$ given by (2.15b) and $\mathbf{B}_\alpha$ by (2.17) (McIntyre 1977, 1980, or Grimshaw 1980). This derivation also applies the useful result that the Eulerian velocity perturbation is given by

$$u'_\alpha = \frac{d\xi_\alpha}{dt} - \xi_\beta \frac{\partial \bar{u}_\alpha}{\partial x_\beta} + O(a^2). \tag{2.18}$$

LAGRANGIAN-MEAN FLOW To conform with the definitions of Section 1, we define

$$L_0 = L(\bar{u}_\alpha, x_\alpha, \delta_{\alpha\beta}, \tilde{\rho}, \bar{S}, \bar{H}_\alpha), \tag{2.19}$$

where L is given by (2.12). The mean flow equation is then obtained from (1.21), or more directly by averaging (2.10) (Andrews & McIntyre 1978a). The result is

$$\tilde{\rho}\frac{d\bar{u}_\alpha}{dt} + 2\tilde{\rho}\varepsilon_{\alpha\beta\gamma}\Omega_\beta\bar{u}_\gamma + \tilde{\rho}\left\langle\frac{\partial\Phi}{\partial x'_\alpha}\right\rangle + \frac{\partial\tilde{q}}{\partial x_\alpha} - \frac{1}{\mu}\frac{\partial}{\partial x_\beta}(\bar{H}_\alpha\bar{H}_\beta)$$
$$= -\frac{\partial R_{\alpha\beta}}{\partial x_\beta} + \langle\tilde{\rho}X_\alpha\rangle, \tag{2.20a}$$

where

$$\tilde{q} = p(\tilde{\rho}, \bar{S}) + \frac{1}{2\mu}\bar{H}_\alpha\bar{H}_\alpha, \tag{2.20b}$$

and

$$R_{\alpha\beta} = \delta_{\alpha\beta}\langle qJ - \tilde{q}\rangle - \left\langle q\frac{\partial\xi_\gamma}{\partial x_\alpha}K_{\gamma\beta}\right\rangle - \frac{1}{\mu}\langle B_\alpha H_\beta - \bar{H}_\alpha\bar{H}_\beta\rangle. \tag{2.20c}$$

It can be verified that $R_{\alpha\beta}$ is the radiation stress tensor defined by (1.19b). An alternative form of (2.20a), involving the pseudomomentum $-\mathbf{T}_{\alpha 0}$, can be obtained by first multiplying (2.10) by $\partial x'_\alpha/\partial x_\beta$ and then averaging (see Andrews & McIntyre 1978a). Here, from (1.13b),

$$\mathbf{T}_{\alpha 0} = \left\langle\frac{\partial\xi_\gamma}{\partial x_\alpha}\left(\tilde{\rho}\frac{d\xi_\gamma}{dt} + \tilde{\rho}\varepsilon_{\gamma\beta\delta}\Omega_\beta\xi_\beta\right)\right\rangle, \tag{2.21a}$$

and

$$\mathbf{T}_{\alpha\beta}=\bar{u}_\beta\mathbf{T}_{\alpha 0}+\left\langle\frac{\partial\xi_\gamma}{\partial x_\alpha}\left(qK_{\gamma\beta}-\frac{B_\gamma H_\beta}{\mu}\right)\right\rangle-\langle L_1\rangle\,\delta_{\alpha\beta}, \tag{2.21b}$$

where we recall that L_1 is $L-L_0$ [see (2.12) and (2.19)]. The wave energy density $\mathbf{E}$ (1.14b) and flux $\mathbf{F}_\alpha$ (1.15) are given by

$$\mathbf{E}=\left\langle\tilde{\rho}\left\{\frac{1}{2}\frac{d\xi_\alpha}{dt}\frac{d\xi_\alpha}{dt}+\Phi(x_\alpha+\xi_\alpha)-\Phi(x_\alpha)+E(\tilde{\rho}J^{-1},S)-E(\tilde{\rho},\bar{S})\right\}\right.$$
$$\left.+\frac{J}{2\mu}B_\alpha B_\alpha-\frac{1}{2\mu}\bar{H}_\alpha\bar{H}_\alpha\right\rangle, \tag{2.22a}$$

$$\mathbf{F}_\alpha=\left\langle\frac{d\xi_\beta}{dt}\left(qK_{\beta\alpha}-\frac{B_\beta H_\alpha}{\mu}\right)\right\rangle. \tag{2.22b}$$

The wave energy equation can now be obtained from (1.19a) or by multiplying (2.10) by $d\xi_\alpha/dt$ and averaging. The result is

$$\frac{\partial\mathbf{E}}{\partial t}+\frac{\partial}{\partial x_\alpha}(\bar{u}_\alpha\mathbf{E}+\mathbf{F}_\alpha)=-R_{\alpha\beta}\frac{\partial\bar{u}_\alpha}{\partial x_\beta}$$
$$+\left\langle\tilde{\rho}\bar{u}_\alpha\left(\frac{\partial\Phi}{\partial x_\alpha}(x_\beta+\xi_\beta)-\frac{\partial\Phi}{\partial x_\alpha}(x_\beta)\right)\right\rangle+\mathbf{D}^{\mathrm{E}}, \tag{2.23a}$$

where

$$\mathbf{D}^{\mathrm{E}}=\left\langle\frac{d\xi_\alpha}{dt}\tilde{\rho}X_\alpha+h\tilde{\rho}\{T(\rho,S)-T(\tilde{\rho},\bar{S})\}+\frac{1}{\mu}k_\alpha\left\{B_\beta\frac{\partial x'_\beta}{\partial x_\alpha}-\bar{H}_\alpha\right\}\right\rangle. \tag{2.23b}$$

Here $\mathbf{D}^{\mathrm{E}}$ represents the effects of dissipation. Finally, the total energy equation is (1.27a), which here becomes

$$\frac{\partial\mathbf{E}^{\mathrm{T}}}{\partial t}+\frac{\partial}{\partial x_\alpha}\left\{\bar{u}_\alpha(\mathbf{E}^{\mathrm{T}}+\bar{q})-\bar{u}_\beta\frac{\bar{H}_\beta\bar{H}_\alpha}{\mu}+F_\alpha+\bar{u}_\beta R_{\beta\alpha}\right\}$$
$$=\left\langle u_\alpha\hat{\rho}X_\alpha+h\tilde{\rho}T(\rho,S)+\frac{1}{\mu}JB_\alpha j_\alpha\right\rangle, \tag{2.24a}$$

where

$$\mathbf{E}^{\mathrm{T}}=\left\langle\tilde{\rho}\left\{\frac{1}{2}\bar{u}_\alpha\bar{u}_\alpha+\frac{1}{2}\frac{d\xi_\alpha}{dt}\frac{d\xi_\alpha}{dt}+\bar{\Phi}(x_\alpha+\xi_\alpha)+E(\tilde{\rho}J^{-1},S)\right\}\right.$$
$$\left.+\frac{J}{2\mu}B_\alpha B_\alpha\right\rangle. \tag{2.24b}$$

INCOMPRESSIBLE FLOW The corresponding results for an incompressible flow may be obtained by taking a limit in which the local sound speed becomes infinite, although some care should be taken when the Boussinesq approximation is also made due to the presence of a large hydrostatic component in the pressure field [see Grimshaw (1975a) or McIntyre (1977, 1980)]. Alternatively, we may proceed directly from the equations of motion for incompressible flow. These are just (2.1a) and (2.1d), with (2.1b) and (2.1c) replaced with

$$\frac{\partial u_\alpha}{\partial x'_\alpha} = 0, \tag{2.25a}$$

$$\frac{1}{\rho}\frac{d\rho}{dt} = m, \tag{2.25b}$$

where m is the counterpart of h in (2.1c) and represents the effects of nonadiabatic motion. In (2.1e) the pressure p is no longer the thermodynamic pressure, and is instead an independent variable in its own right. In the generalized Lagrangian-mean formulation we again define $\tilde{\rho}$ and J by (2.6a) and (2.6b), respectively. In place of (2.5) we now have

$$\frac{1}{\tilde{\rho}}\frac{d\tilde{\rho}}{dt} + \frac{\partial \bar{u}_\alpha}{\partial x_\alpha} = m. \tag{2.26}$$

Also, J is a mean quanity $\bar{J}$, which satisfies the equation

$$\frac{d\bar{J}}{dt} + \bar{J}\frac{\partial \bar{u}_\alpha}{\partial x_\alpha} = 0. \tag{2.27}$$

Note that because of the dissipative term m in (2.25), $\tilde{\rho}$ will have a fluctuating component. The Lagrangian-mean equation is then (2.10), and we identify λ with the 5-vector $\tilde{\rho}$, $\bar{J}$, and H_α. A suitable Lagrangian is

$$L(u_\alpha, x'_\alpha, \partial x'_\alpha/\partial x_\beta, p, \tilde{\rho}, \bar{J}, H_\alpha)$$
$$= \tilde{\rho}\{\tfrac{1}{2}u_\alpha u_\alpha + \varepsilon_{\alpha\beta\gamma}\Omega_\alpha x'_\beta u_\gamma - \Phi(x'_\alpha)\} + p(J-\bar{J}) - J\frac{B_\alpha B_\alpha}{2\mu}. \tag{2.28}$$

Here the disturbance fields to be varied are x'_α (or equivalently ξ_α) and p. However, Q_α is now given by $\tilde{\rho}X_\alpha + \tilde{\rho}m(u_\alpha + \varepsilon_{\alpha\beta\gamma}\Omega_\beta x'_\gamma)$. We also define $Q_{\tilde{\rho}}$ and Q_{H_α} so that the corresponding Euler equations are identities.

The wave action equation is again (2.15a), with $\mathbf{A}$ and $\mathbf{B}_\alpha$ again given by (2.15b) and (2.15c), respectively. However, the dissipative term $\mathbf{D}$ is now

$$\mathbf{D} = \left\langle \frac{\partial \xi_\alpha}{\partial \theta}\{\tilde{\rho}X_\alpha + \tilde{\rho}m(u_\alpha + \varepsilon_{\alpha\beta\gamma}\Omega_\beta x'_\gamma\}\right\rangle - \left\langle \frac{\partial \tilde{\rho}}{\partial \theta}\{\tfrac{1}{2}u_\alpha u_\alpha + \varepsilon_{\alpha\beta\gamma}\Omega_\beta x'_\gamma - \bar{\Phi}(x'_\alpha)\}\right\rangle$$
$$+ \left\langle \frac{\partial H_\alpha}{\partial \theta}\frac{B_\beta}{\mu}\frac{\partial x'_\beta}{\partial x_\alpha}\right\rangle. \tag{2.29}$$

If we assume for simplicity that m is zero, the mean flow equation is again (2.20a), with the proviso that in the expression (2.20b) for $\tilde{q}$, $p(\tilde{\rho}, \bar{S})$ is replaced with $\bar{p}$. The radiation stress tensor is again given by (2.20c). The pseudomomentum and its flux are still given by (2.21a) and (2.21b), with the proviso that L_0 is now $L(\bar{u}_\alpha, x_\alpha, \delta_{\alpha\beta}, \bar{p}, \tilde{\rho}, \bar{J}, \bar{H}_\alpha)$, where L is given by (2.28). The wave energy density is again given by (2.22a), with the proviso that the terms involving the internal energy E are replaced by $\bar{p}(1-\bar{J})$; the wave energy flux is again given by (2.22b). The wave energy equation is again (2.23a), but the dissipative term now takes a different form from (2.23b), and an extra term $(1-\bar{J})\, d\bar{p}/dt$ must be included on the right-hand side.

CURVILINEAR COORDINATES So far, our results in this section have been expressed in Cartesian coordinates. However, in the general theory of Section 1 we may allow the coordinates x_α to be any set of spacelike coordinates. By way of illustration, let us now consider the case when x_α are the cylindrical polar coordinates r, λ, and z. Analogous results using spherical polar coordinates have been obtained by F. P. Bretherton (personal communication). Thus, we let

$$x_1 = r, \qquad x_2 = \lambda, \qquad x_3 = z, \tag{2.30}$$

be generalized Lagrangian coordinates, whose Eulerian counterparts are r', λ', and z'. The particle displacements are then defined by [see (2.2)]

$$\xi_1 = r' - r, \qquad \xi_2 = \lambda' - \lambda, \qquad \xi_3 = z' - z. \tag{2.31}$$

The velocity components in the Eulerian coordinate directions are

$$u_1 = \frac{dr'}{dt}, \qquad u_2 = r'\frac{d\lambda'}{dt}, \qquad u_3 = \frac{dz'}{dt}, \tag{2.32a}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \bar{u}_1 \frac{\partial}{\partial r} + \frac{\bar{u}_2}{r}\frac{\partial}{\partial \lambda} + \bar{u}_3 \frac{\partial}{\partial z}. \tag{2.32b}$$

Here $\bar{u}_\alpha$ are the mean velocity components in the Lagrangian coordinate directions, which must be carefully distinguished from the Eulerian coordinate directions. The Lagrangian is again given by (2.12); for simplicity, we suppose that the axis of rotation is in the z-direction, that the potential Φ is axisymmetric, and that there is no magnetic field ($B_\alpha = 0$). Thus the Lagrangian is

$$L\left(u_\alpha, x'_\alpha, \frac{\partial x'_\alpha}{\partial x_\beta}, r\tilde{\rho}, S\right) = r\tilde{\rho}\{\tfrac{1}{2}u_\alpha u_\alpha + \Omega r' u_2 - \Phi(r', z') - E(\rho, S)\}, \tag{2.33a}$$

where

$$r'\rho J = r\tilde{\rho}, \tag{2.33b}$$

and J is again defined by (2.6b), but now x'_α and x_α are the cylindrical polar coordinates. Variations in x'_α (or equivalently ξ_α) then give the equations of motion:

$$\tilde{\rho}\left(\frac{du_1}{dt} - \frac{u_2^2}{r'} - 2\Omega u_2 + \frac{\partial \Phi}{\partial r'}\right) + \frac{r'}{r}\frac{\partial}{\partial x_\beta}(pK_{r\beta}) = \tilde{\rho}X_1, \tag{2.34a}$$

$$\tilde{\rho}\left(\frac{du_2}{dt} + \frac{u_1 u_2}{r'} + 2\Omega u_1\right) + \frac{1}{r}\frac{\partial}{\partial x_\beta}(pK_{\lambda\beta}) = \tilde{\rho}X_2, \tag{2.34b}$$

$$\tilde{\rho}\left(\frac{du_3}{dt} + \frac{\partial \Phi}{\partial z'}\right) + \frac{r'}{r}\frac{\partial}{\partial x_\beta}(pK_{z\beta}) = \tilde{\rho}X_3. \tag{2.34c}$$

Here the generalized forces are given by $Q_1 = \tilde{\rho}rX_1$, $Q_2 = \tilde{\rho}rr'X_2$, and $Q_3 = \tilde{\rho}rX_3$. The counterpart of (2.5) is

$$\frac{d}{dt}(r\tilde{\rho}) + r\tilde{\rho}\left\{\frac{\partial \bar{u}_1}{\partial r} + \frac{1}{r}\frac{\partial \bar{u}_2}{\partial \lambda} + \frac{\partial \bar{u}_3}{\partial z}\right\} = 0, \tag{2.35}$$

while S again satisfies (2.1c), where d/dt is now given by (2.32b).

The wave action equation can now be obtained from (1.7), or by multiplying (2.34a), (2.34b), and (2.34c) by $\partial\xi_\alpha/\partial\theta$ and averaging. The result is (2.15a), where now

$$\mathbf{A} = r\tilde{\rho}\left\langle \frac{\partial \xi_\alpha}{\partial \theta} h'_\alpha u_\alpha + \Omega r'^2 \frac{\partial \xi_2}{\partial \theta} \right\rangle, \tag{2.36a}$$

$$\mathbf{B}_\alpha = h_\alpha^{-1}\bar{u}_\alpha \mathbf{A} + \left\langle \frac{\partial \xi_\beta}{\partial \theta} r' p K_{\beta\alpha} \right\rangle, \tag{2.36b}$$

$$\mathbf{D} = r\tilde{\rho}\left\langle \frac{\partial \xi_\alpha}{\partial \theta} h'_\alpha X_\alpha + \frac{\partial S}{\partial \theta} T \right\rangle, \tag{2.36c}$$

where

$$h'_1 = h_1 = 1; \qquad h'_2 = r',\ h_2 = r; \qquad h'_3 = h_3 = 1. \tag{2.36d}$$

For linearized wave motion, we introduce the Eulerian pressure perturbation p' by (2.16), with q replaced by p, and x_α and ξ_α defined by (2.30) and (2.37), respectively. It may then be shown that [compare (2.17)]

$$\mathbf{B}_\alpha - \bar{u}_\alpha \mathbf{A} = \left\langle rp' \frac{\partial \xi_\alpha}{\partial \theta} \right\rangle + \frac{\partial}{\partial x_\beta}\left\langle r\bar{p}\xi_\beta \frac{\partial \xi_\alpha}{\partial \theta} \right\rangle + O(a^3). \tag{2.37}$$

The second term is identically nondivergent and can be omitted from (2.15a). When the basic flow is zonal [i.e. $\bar{u}_1$ and $\bar{u}_3$ are $O(a^2)$] and the averaging operator is interpreted as a zonal average (i.e. θ is identified with $-\lambda$), the wave action equation (2.15a) reduces to the generalized Eliassen-Palm relation derived by Andrews & McIntyre (1978c), and $r^{-1}\mathbf{A}$ is the angular pseudomomentum.

The mean flow equations can now be obtained from (1.21) [after allowing for the presence of geometrical factors involving r in L, (2.33a) and (1.18)], or more directly, by averaging (2.34a–c). For instance, from the azimuthal equation (2.34b), we obtain

$$\tilde{\rho}\frac{dM}{dt}+\frac{\partial\tilde{p}}{\partial\lambda}=-\frac{1}{r}\frac{\partial}{\partial x_\beta}R_{\lambda\beta}+\langle\tilde{\rho}r'X_2\rangle, \tag{2.38a}$$

where

$$R_{\lambda\beta}=\delta_{\lambda\beta}\langle r'pJ-r\tilde{p}\rangle-\left\langle r'p\frac{\partial\xi_\gamma}{\partial\lambda}K_{\gamma\beta}\right\rangle, \tag{2.38b}$$

and

$$M=\langle r'u_2+\Omega r'^2\rangle. \tag{2.38c}$$

Here $\tilde{p}$ is $p(\tilde{\rho},\bar{S})$, M is the mean specific angular momentum about the z-axis, and $R_{\lambda\beta}$ is the azimuthal component of the radiation stress tensor. In particular, when the averaging operator is interpreted as a zonal average (i.e. θ is identified with $-\lambda$), the off-diagonal components of $R_{\lambda\beta}$ are identical with $\mathbf{B}_\beta-\bar{u}_\beta\mathbf{A}$; note that the diagonal components will not now appear in (2.38a). With the further restriction to linearized waves on a zonal basic flow, (2.38a) reduces to a generalized Charney-Drazin theorem (see the similar results obtained by Andrews & McIntyre 1978c). Since the divergence of the radiation stress tensor in (2.38a) is here given by (1.26a), it follows that the M will change only in response to wave transience or dissipative effects (for an explicit demonstration of this and the relationship between M and the zonal mean flow, see Dunkerton 1980).

These results and their counterparts in spherical polar coordinates are now finding extensive application in stratospheric meteorology. In this context the literature abounds with results on conservation equations for wave activity, derived usually for linearized waves and using various approximations (e.g. quasi-geostrophy, hydrostatic, slowly varying mean flows). These results are now generally called Eliassen-Palm relations after the pioneering work of Eliassen & Palm (1961), and can be recognized as special cases of the wave action equation. The corresponding results for the mean flow, such as (2.38a), are known variously as nonacceleration theorems, or Charney-Drazin theorems after the initial work by Charney &

Drazin (1961). For recent and comprehensive reviews of this now extensive and rapidly growing subject, see Andrews & McIntyre (1978c), McIntyre (1980), Dunkerton (1980), and Uryu (1980). The significant feature of the Eliassen-Palm relations on the one hand and the Charney-Drazin theorems on the other is the equality between the flux terms of the wave action equation and the wave forcing terms in the mean flow equation. The general theory of Section 1 shows that this duality is not a peculiarity of the equations governing stratospheric circulation, but is instead a general property of wave-mean flow interactions.

3. APPLICATIONS TO STRATIFIED SHEAR FLOWS

It is not possible in a single article to cover all instances where the wave action equation has proved a useful tool in elucidating wave-mean flow interaction. Instead, we discuss a specific case that is relatively familiar and sufficiently simple to permit a compact description. In applying the general theory, the reader is reminded that it is preferable to use the principles enunciated at the beginning of Section 2, rather than a slavish use of the subsequent formulae. This is particularly relevant when additional approximations, such as small wave amplitude or slowly varying waves, are being invoked.

Internal Gravity Waves

We consider internal gravity waves propagating on a basic state consisting of a horizontal shear flow $u_0(z)$ in the x-direction and the density profile $\rho_0(z)$. Here z is a coordinate in the vertical direction. We assume that the flow is incompressible and ignore the effects of rotation, magnetic fields, and dissipation. Then the linearized, two-dimensional equations of motion for the particle displacements $\xi(t, x, z)$ and $\zeta(t, x, z)$ in the horizontal and vertical directions, respectively, are

$$\rho_0 \frac{d^2\xi}{dt^2} + \frac{\partial p'}{\partial x} = 0, \tag{3.1a}$$

$$\rho_0 \frac{d^2\zeta}{dt^2} + \frac{\partial p'}{\partial z} + \rho_0 N^2 \zeta = 0, \tag{3.1b}$$

$$\frac{\partial \xi}{\partial x} + \frac{\partial \zeta}{\partial z} = 0, \tag{3.1c}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x}. \tag{3.1d}$$

Here d/dt is the linearized approximation to the material time derivative (2.3), and N^2 is the Brunt-Väisälä frequency $-g\rho_0^{-1}\,d\rho_0/dz$. Note that the equations have been formulated using the Eulerian pressure perturbation p' [see (2.16)], rather than its Lagrangian counterpart $\hat{p}$. For incompressible flow, p' is generally found to be a more convenient entity than $\hat{p}$, which is dominated by a large hydrostatic component [see Grimshaw (1975b) or McIntyre (1977, 1980)]. Also note that the Eulerian velocity perturbations are given by (2.18):

$$u' = \frac{d\xi}{dt} - \zeta\frac{\partial u_0}{\partial z}, \qquad w' = \frac{d\zeta}{dt}. \tag{3.2}$$

The derivation of (3.1a–c) is either from (2.10) (Grimshaw 1979), or from the linearized Eulerian equations after using (3.2).

Next we seek solutions of (3.1a–c) for which

$$\zeta = \psi\ \exp(ikx - i\omega t - i\theta) + \text{c.c.}, \tag{3.3}$$

with similar expressions for the other variables. At first, suppose that ψ is a function of z alone. Then $\psi(z)$ satisfies the equation

$$\frac{\partial}{\partial z}\left(\rho_0\omega^{*2}\frac{\partial\psi}{\partial z}\right) + \rho_0 k^2(N^2 - \omega^{*2})\psi = 0, \tag{3.4a}$$

where

$$\omega^* = \omega - ku_0. \tag{3.4b}$$

Here ω^* is the intrinsic frequency (1.29e). Equation (3.4a) is transformed into the Taylor-Goldstein equation when ψ is replaced by $\phi = \omega^*\psi$. As such, its properties are well known (see, for instance, Booker & Bretherton 1967). Equation (3.4a) has the wave invariant

$$\mathbf{B} = \left\langle p'\frac{\partial\zeta}{\partial\theta}\right\rangle, \tag{3.5a}$$

or

$$k^2\mathbf{B} = -2\ \mathrm{Im}\left\{\rho_0\omega^{*2}\frac{\partial\psi}{\partial z}\psi^*\right\} \tag{3.5b}$$

Here $\langle\ \rangle$ is an average over the phase-shift parameter θ [see (1.5)], and from (3.3) is equivalent to an average over a wavelength in the x-direction. From (2.17), $\mathbf{B}$ can be recognized as the vertical component of the wave action flux, correct to $O(a^2)$, where for simplicity we have omitted the subscript 3. It is a constant of the motion except at critical levels where $\omega^* = 0$. Of course, this result is a consequence of (2.15a), where only the z-derivative term survives; it can also be derived directly from (3.4a) or by using (3.5b). Using (3.1a) and (3.2), we can show that

$$k\mathbf{B} = \langle \rho_0 u'w' \rangle, \tag{3.6}$$

which is the vertical flux of horizontal momentum, or the xz-component of Reynolds stress. Historically it was in this form that **B** was first identified, but the general theory of the previous sections shows that (3.5a) is the more fundamental expression.

CRITICAL LEVELS AND OVER-REFLECTION Let us now suppose that $u_0(z) \to U_{1,2}$ and $N \to N_{1,2}$ as $z \to \pm\infty$, respectively. Then

$$\rho_0^{1/2}\psi \sim I \exp(im_2 z) + R \exp(-im_2 z) \quad \text{as} \quad z \to -\infty, \tag{3.7a}$$

$$\rho_0^{1/2}\psi \sim T \exp(im_1 z) \quad \text{as} \quad z \to \infty, \tag{3.7b}$$

where

$$m_{1,2}^2 + \frac{N^4}{4g^2} = \left(\frac{N^2}{(v-U_{1,2})^2} - k^2\right). \tag{3.7c}$$

Here v is the phase speed ωk^{-1} in the x-direction. We suppose that the wave frequency ω and wave number k are such that $m_{1,2}$ are both real, and that the signs of $m_{1,2}$ are chosen so that I, R, and T correspond to incident, reflected, and transmitted waves, respectively. From (3.5b),

$$\mathbf{B} = 2m_2(v-U_2)^2\{|R|^2 - |I|^2\} \quad \text{as} \quad z \to -\infty, \tag{3.8a}$$

and

$$\mathbf{B} = -2m_1(v-U_1)^2|T|^2 \quad \text{as} \quad z \to \infty. \tag{3.8b}$$

Since, from (3.5a), $\mathbf{B} = \omega^*\langle p'w' \rangle$ and $\langle p'w' \rangle$ is the vertical flux of wave energy, it follows that

$$-km_{1,2}(v-U_{1,2}) > 0. \tag{3.9}$$

If there are no critical levels in the flow, then **B** is constant throughout and we can equate (3.8a) with (3.8b). The result is an expression for the conservation of wave action and implies that $|R|^2 < |I|^2$.

However, if there is a critical level, then **B** is constant throughout except at the critical level. Suppose there is a single critical level at $z = 0$, where $\omega^* = 0$. Then, near $z = 0$,

$$\rho_0^{1/2}\psi \approx A(v-u_0)^{-1/2+i\mu} + B(v-u_0)^{-1/2-i\mu}, \tag{3.10a}$$

where

$$\mu^2 = \left\{N^2\left(\frac{\partial u_0}{\partial z}\right)^{-2} - \frac{1}{4}\right\} \quad \text{at} \quad z = 0. \tag{3.10b}$$

Following Booker & Bretherton (1967), we determine the branch of $v-u_0$ as z passes through zero by assuming that the critical level is viscosity dominated. To be explicit, suppose that $k > 0$ and $\partial u_0/\partial z$ is positive at $z = 0$; then $v-u_0$ is real and positive for $z < 0$, and is given by $|v-u_0|\, e^{i\pi}$ for $z > 0$. Suppose first that μ is real and positive (i.e. the local Richardson number is greater than 1/4). Then, from (3.5b),

$$\mathbf{B} = 2\mu\left(\frac{\partial u_0}{\partial z}\right)_0 \{|A|^2-|B|^2\} \quad \text{for} \quad z<0 \tag{3.11a}$$

and

$$\mathbf{B} = -2\mu\left(\frac{\partial u_0}{\partial z}\right)_0 \{|A|^2 \exp(-2\mu\pi)-|B|^2 \exp(2\mu\pi)\} \quad \text{for} \quad z>0. \tag{3.11b}$$

Recalling the sign conventions, it follows that the "A"-wave is upgoing and the "B"-wave is downgoing. In either case a wave passing through the critical level is absorbed (Booker & Bretherton 1967). Further, since $\mathbf{B}$ is constant throughout $z \gtrless 0$, respectively, we may equate (3.8a) with (3.11a), and (3.8b) with (3.11b). It then follows that $|R|^2 < |I|^2$.

However, if $\mu = i\nu$, where $0 < \nu < 1/2$ (i.e. the local Richardson number is less than 1/4), then

$$\mathbf{B} = 2i\nu\left(\frac{\partial u_0}{\partial z}\right)_0 \{AB^*-A^*B\} \quad \text{for} \quad z<0, \tag{3.12a}$$

$$\mathbf{B} = -2i\nu\left(\frac{\partial u_0}{\partial z}\right)_0 \{AB^* \exp(-2i\nu\pi)-A^*B \exp(2i\nu\pi)\} \quad \text{for} \quad z>0. \tag{3.12b}$$

Again, we may equate (3.8a) with (3.12a), and (3.8b) with (3.12b). As $\nu \to 0$, the critical level looks more like a vortex sheet and $\mathbf{B}$ is continuous at $z = 0$. It follows that $|R|^2 > |I|^2$ in this limit, and the incident wave is over-reflected (Acheson 1976). For small but nonzero ν, this argument suggests that there may be over-reflection, since, from (3.12a,b), the jump in $\mathbf{B}$ across the critical level is $O(\nu)$.

The importance of this discussion in relation to the wave action equation is that it illustrates how a knowledge of local solutions [i.e. (3.7a,b) or (3.10a)], together with a wave invariant (3.5b), enables a number of significant conclusions to be made without necessarily solving the wave equation (3.4a). For a similar account of critical levels when compressibility, rotation, and magnetic effects are included, see Grimshaw (1980), which also contains a review of the extensive literature in linearized wave motion near critical levels.

WAVE ACTION AND ENERGY As a preliminary to discussing the mean flow, we first allow ψ in (3.3) to depend on both z and t. Then the wave action equation (2.15a) is

$$\frac{\partial \mathbf{A}}{\partial t}+\frac{\partial \mathbf{B}}{\partial z}=0, \tag{3.13a}$$

where

$$\mathbf{A}=\left\langle\rho_0\left(\frac{\partial \xi}{\partial \theta}\frac{d\xi}{dt}+\frac{\partial \zeta}{\partial \theta}\frac{d\zeta}{dt}\right)\right\rangle, \tag{3.13b}$$

and $\mathbf{B}$ is given by (3.5a). The expressions (3.5b) and (3.6) for $\mathbf{B}$ are no longer valid, although they are first approximations when ψ is a slowly varying function of t. Equation (3.13a) can be derived directly from (3.1a,b) by multiplying with $\partial\xi/\partial\theta$ and $\partial\zeta/\partial\theta$, respectively. Note here that since θ is a phase-shift parameter in the x-direction, we can identify $k\mathbf{A}$ as the x-component of pseudomomentum $-\mathbf{T}_{10}$, and $k\mathbf{B}$ is the corresponding vertical flux.

As a comparison, the wave energy equation (2.23a) is

$$\frac{\partial \mathbf{E}}{\partial t}+\frac{\partial \mathbf{F}}{\partial z}=-R_{13}\frac{\partial u_0}{\partial z}, \tag{3.14a}$$

where

$$\mathbf{E}=\left\langle\tfrac{1}{2}\rho_0\left\{\left(\frac{d\xi}{dt}\right)^2+\left(\frac{d\zeta}{dt}\right)^2+N^2\zeta^2\right\}\right\rangle, \tag{3.14b}$$

$$\mathbf{F}=\langle p'w'\rangle, \tag{3.14c}$$

$$R_{13}=-\langle p'\zeta_x\rangle=k\mathbf{B}. \tag{3.14d}$$

Here R_{13} is the xz-component of the radiation stress tensor (2.20c). Equation (3.14a) is most simply derived from (3.1a,b) by multiplying with $d\xi/dt$ and $d\zeta/dt$, respectively. It can also be obtained from the counterpart of (2.22a,b) for incompressible flow. However, if this approach is followed, (2.22a) yields an expression for $\mathbf{E}$ that differs from (3.14b) by

$$\frac{\partial}{\partial z}\left\{\langle g\rho_0\zeta\rangle+\frac{1}{2}\frac{\partial}{\partial z}\langle\rho_0\zeta^2\rangle\right\}. \tag{3.15}$$

Note, however, that a corresponding term $(\partial/\partial t)\{\text{——}\}$ occurs in $\mathbf{F}$ (2.22b), and consequently can be omitted in (3.14b,c). This illustrates the fact that expressions such as (2.22a,b) derived from a Lagrangian may not always yield familiar expressions, and emphasizes the desirability of a direct derivation from the particular equation of motion being considered.

When ψ is slowly varying in both z and t, it is readily shown that $\mathbf{E} \approx \omega^* \mathbf{A}$ and $\mathbf{F} \approx \omega^* \mathbf{B}$, in agreement with the general results (1.29c) for slowly varying waves, recalling that for linearized waves $\bar{L}_1 \approx 0$. Further, slowly varying waves have a slowly varying vertical wave number $m(t, z)$, determined from the dispersion relation

$$\omega^{*^2} = N^2 k^2 (k^2 + m^2)^{-1}. \tag{3.16}$$

The vertical group velocity is $c_3 = \partial\omega/\partial m$, and $\mathbf{B} \approx c_3 \mathbf{A}$. In this form the wave action equation (3.13a) holds without restriction on wave amplitude, provided that u_0 is replaced with $\bar{u}$ in ω^* (3.4b), since, for slowly varying waves in incompressible flow, the waves are transverse and expressions such as (3.3) hold without restriction in amplitude (see Grimshaw 1975a).

MEAN FLOW The horizontal component of the Lagrangian-mean flow equation (2.20a) is, correct to $O(a^2)$,

$$\rho_0 \frac{\partial \bar{u}}{\partial t} + \rho_0 \bar{w} \frac{\partial u_0}{\partial z} = -\frac{\partial R_{13}}{\partial z}. \tag{3.17}$$

Also, the vertical component $\bar{w}$ is determined from (2.27), which, correct to $O(a^2)$, is

$$\frac{\partial \bar{J}}{\partial t} + \frac{\partial \bar{w}}{\partial z} = 0, \tag{3.18a}$$

and

$$\bar{J} - 1 = -\frac{\partial^2}{\partial z^2} \langle \tfrac{1}{2}\zeta^2 \rangle. \tag{3.18b}$$

Assuming a state of no disturbance before the arrival of the waves, and using the wave action equation (3.13a), it follows that

$$\rho_0(\bar{u} - u_0) = k\mathbf{A} - \rho_0 \frac{\partial u_0}{\partial z} \frac{\partial}{\partial z} \langle \tfrac{1}{2}\zeta^2 \rangle. \tag{3.19}$$

Remarkably, this result is exact without any restriction to slowly varying waves. If the slowly varying hypothesis is invoked, the second term on the right-hand side of (3.19) is omitted.

The total energy equation (2.24a) is here most simply obtained by multiplying (3.17) with u_0 and adding the result to (3.14a). We find that

$$\frac{\partial}{\partial t}(\mathbf{E} + \rho_0 u_0 k\mathbf{A}) + \frac{\partial}{\partial z}(\mathbf{F} + u_0 k\mathbf{B}) = 0. \tag{3.20}$$

Here the wave-induced total energy density is $(\mathbf{E} + \rho_0 u_0 k\mathbf{A})$, and (3.20) gives a succinct description of how the wave-induced mean flow term $k\mathbf{A}$

combines with the wave energy **E** to ensure conservation of total energy. In particular, note that for slowly varying waves the total energy density is $\mathbf{E}v(v-u_0)^{-1}$, and the total energy flux is just this quantity multiplied by the vertical group velocity c_3. Acheson (1976) has shown how these expressions provide an energetic explantation of the phenomenon of over-reflection where the wave energy flux is directed away from the critical level in both $z > 0$ and $z < 0$, but the total energy flux is one-signed.

For slowly varying waves, both the wave action equation (3.13a) and the mean flow equation (3.17) are valid without any restriction on wave amplitude (Grimshaw 1975a). Combined with the dispersion relation (3.16), in which ω^* is $\omega - k\bar{u}$, they form a set of three coupled equations for the wave action density **A**, the mean flow $\bar{u}$, and the vertical wave number m. Numerical solutions of this set are described by Grimshaw (1975b), and Dunkerton (1981) has obtained analytic solutions by invoking the hydrostatic approximation in the dispersion relation (3.16). This is one of the rare instances where finite-amplitude wave-mean flow interaction can be analyzed in a simple analytic manner.

Conclusion

This brief account of wave action and wave-mean flow interaction for internal gravity waves is intended as a didactic illustration of the general theory. Although this particular example can also be analyzed using Eulerian means and wave energy arguments, it should be clear that wave action and Lagrangian-mean concepts lead simply and directly to the main conclusions. The advantages that ensue when the wave action equation and Lagrangian means are employed are particularly clear when Coriolis forces are included (Grimshaw 1975a, McIntyre 1980, Andrews 1980).

Dissipative and nonconservative effects are readily incorporated in the above discussion in the manner described in Section 2. However, some caution is needed when the basic state is maintained by nonconservative or diabatic terms that may not appear explicitly in the linearized wave equations [i.e. (3.1a–c) or their counterparts]. In this situation, wave action is not generally conserved. An example of this occurs when the Brunt-Väisälä frequency varies with time, but there is no corresponding basic vertical velocity; the time variation in the basic density profile must then be maintained by diabatic terms and so m in (2.26) is nonzero, and consequently the dissipative term **D** (2.29) in the wave action equation (2.15a) is nonzero (see Rotunno 1977). An analogous situation occurs for Rossby waves on a nonzonal flow (Young & Rhines 1980).

The wave action equation occurs in a variety of other physical systems. The extension of the theory described in this section to include Coriolis forces and its application to stratospheric meteorology has already been referred to at the end of Section 2. Completely contrasting physical systems are sound waves and surface gravity waves, as in both cases the Eulerian flow is irrotational. For a summary of wave action conservation in acoustic waveguides, the reader is referred to Andrews & McIntyre (1978b). The development of wave action concepts in water waves can be found in the pioneering work of Whitham (1965, 1970) and Bretherton & Garrett (1968); applications to finite-amplitude water waves began with Lighthill (1965) and have been extensively developed by Peregrine & Thomas (1979) and Stiassnie & Peregrine (1979). Finally, although it is beyond the scope of this review to delve into applications to plasma physics, the reader may like to consult Dewar (1970, 1972) or Dougherty (1970, 1974) for the development of Lagrangian concepts in that context.

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