# Acoustic-gravity waves in atmospheric and oceanic waveguides

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A theory of guided propagation of sound in layered, moving fluids is extended to include acoustic-gravity waves (AGWs) in waveguides with piecewise continuous parameters. The orthogonality of AGW normal modes is established in moving and motionless media. A perturbation theory is developed to quantify the relative significance of the gravity and fluid compressibility as well as sensitivity of the normal modes to variations in sound speed, flow velocity, and density profiles and in boundary conditions. Phase and group speeds of the normal modes are found to have certain universal properties which are valid for waveguides with arbitrary stratification. The Lamb wave is shown to be the only AGW normal mode that can propagate without dispersion in a layered medium. © *2012 Acoustical Society of America*.
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## I. INTRODUCTION

Acoustic-gravity waves (AGWs) are mechanical waves in compressible fluids in a gravity field.[1–3] At frequencies much larger than the buoyancy frequency, AGWs reduce to acoustic waves (infrasound). At sufficiently low frequencies, where the fluid can be treated as incompressible, AGWs reduce to surface and internal gravity waves. The term "acoustic-gravity waves" is usually employed when restoring forces due to both gravity and compressibility are significant.

AGWs are known to propagate hundreds and thousands of kilometers in atmospheric and coupled atmospheric-oceanic waveguides.[4–8] In the past, theoretical investigations of long-range propagation of AGWs primarily aimed to explain signals generated by volcanic eruptions and nuclear explosions in the atmosphere.[2,4–7,9–11] Recently, interest in AGW propagation modeling has been renewed by a vast expansion of the observation network,[8] evidence of ionospheric[8,12–15] and tropospheric[16–18] manifestations of earthquakes and tsunamis, as well as by possible application of these manifestations for tsunami early detection and warning.[12–18]

Any detailed description of AGW fields in the atmosphere and ocean undoubtedly requires numerical modeling for specific environmental conditions. However, theoretical investigations[19–24] of general properties and the resulting qualitative understanding of normal modes have proven very useful in acoustics, and are likely to play a similar role for AGWs. While earlier AGW studies typically assumed an ideal gas half-space overlying a rigid boundary,[1,4,9–11,25] investigations of AGW propagation above the ocean surface and, more generally, of coupling of physical processes in the ocean and atmosphere, require an environmental model that allows for a general equation of state of the fluid as well as the presence of fluid-fluid interfaces and compliant boundaries. To study properties of AGW normal modes in such media, in this paper we extend to AGWs a theory[22,24] previously developed for acoustic waveguides in stratified, moving fluids.

The paper is organized as follows. In Sec. II, linearized equations of motion in stratified, moving, compressible fluids in a gravity field are cast in a form convenient for the analysis of waves in media with piecewise continuous parameters. Orthogonality relations for AGW normal modes in the same or in distinct waveguides are derived in Sec. III. A close relationship between the AGW mode orthogonality and wave energy conservation is demonstrated in the Appendix. Variations of the mode wave number resulting from perturbations in various environmental parameters are quantified in Sec. IV. General properties of phase and group velocity of normal modes in generic stratified waveguides are established in Sec. V. Section VI summarizes our findings.

## II. EQUATIONS OF MOTION

Consider continuous linear waves of frequency $\omega$ in a fluid with background (i.e., unperturbed by waves) pressure $p_0$, density $\rho$, sound speed $c$, and flow velocity $\mathbf{u}$ in a uniform gravity field with acceleration $\mathbf{g}$. The fluid is stationary (i.e., its parameters are independent of time $t$) in the absence of waves. Time dependence $\exp(-i\omega t)$ of the wave field is assumed and suppressed. Linearization of the Euler, continuity, and state equations with respect to wave amplitude leads to the following set of equations[24,26] governing wave fields:

$$\nabla p + \rho \frac{d^2\mathbf{w}}{dt^2} + (\mathbf{w}\cdot\nabla)\nabla p_0 - (p + \mathbf{w}\cdot\nabla p_0)\frac{\nabla p_0}{\rho c^2} = 0, \quad (1)$$

$$\nabla\cdot\mathbf{w} + (p + \mathbf{w}\cdot\nabla p_0)/\rho c^2 = 0, \quad (2)$$

where $p$ and $\mathbf{w}$ are the pressure perturbation and oscillatory displacement of fluid particles due to the wave and $d/dt = -i\omega + \mathbf{u}\cdot\nabla$ is the convective time derivative. Wave-induced fluid velocity perturbation $\mathbf{v}$ is related to the oscillatory displacement by the equation[24,26] $\mathbf{v} = d\mathbf{w}/dt - (\mathbf{w}\cdot\nabla)\mathbf{u}$. In Eqs. (1) and (2), we assume wave propagation to be an adiabatic thermodynamic process and disregard irreversible

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processes associated with viscosity, thermal conductivity, and diffusion of admixtures such as salt in seawater and water vapor in atmospheric air.

Subsequent theoretical analysis is greatly simplified by having only one scalar and one vector dependent variables, $p$ and $\mathbf{w}$, in Eqs. (1) and (2). This form of equations of motion is obtained by eliminating unknown wave-induced perturbations in mass density, entropy density, and concentrations of admixtures from linearized Euler, continuity, and state equations as well as equations expressing conservation of entropy and mass of each admixture in fluid particles.[24,26] Remarkably, of all the thermodynamic partial derivatives entering the linearized equation of state only one characteristic of the fluid, the sound speed, is present in Eqs. (1) and (2). No specific form of the equation of state is assumed in derivation of Eqs. (1) and (2).[24,26] While relation between background pressure, density, and sound speed may be different in fluids with different equations of state, wave fields in fluids with any equation of state (or in a medium comprised of fluids with distinct equations of state) can be investigated using Eqs. (1) and (2) when the environmental parameters $p_0$, $\rho$, $c$, and $\mathbf{u}$ are given as functions of position.

Introduce a Cartesian coordinate system with horizontal coordinates $x$ and $y$ and vertical coordinate $z$ increasing upward (Fig. 1). Then $\mathbf{g} = (0, 0, -g)$. Let the fluid be horizontally stratified, with the background flow being horizontal: $\mathbf{u} = (u_x, u_y, 0)$ and the parameters $p_0$, $\rho$, $c$, and $\mathbf{u}$ depending on the vertical coordinate $z$ only. Then $p_0$ and $\rho$ are related by the hydrostatic equilibrium equation $dp_0/dz = -\rho g$, while $c(z)$ and $\mathbf{u}(z)$ can be arbitrary. We will assume that the background stratification is hydrodynamically stable.

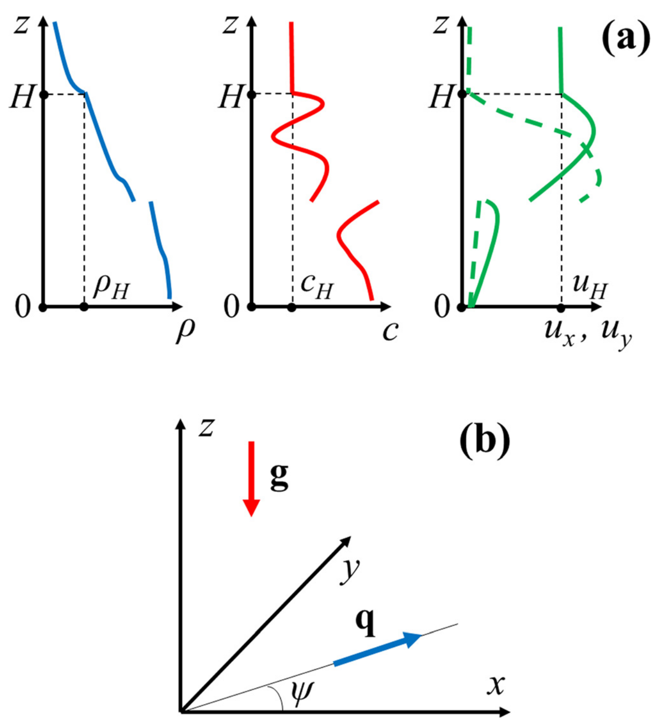


FIG. 1. (Color online) Geometry of the problem. Background density $\rho$, sound speed $c$, and flow velocity $\mathbf{u} = (u_x, u_y, 0)$ are functions of the vertical coordinate $z$ and may be discontinuous at interfaces within the fluid (a). Variation of the field of a normal mode in the $xy$ plane is characterized by a horizontal wave vector $\mathbf{q}$, direction of which is determined by azimuthal angle $\psi$ (b).

In stratified fluids, arbitrary AGW fields can be represented by superpositions of waves with harmonic dependence on horizontal coordinates:

$$p(\mathbf{r}) = P(z)\exp(i\mathbf{q}\cdot\mathbf{r}),$$
$$\mathbf{w}(\mathbf{r}) = [\mathbf{h}(z) + W(z)\hat{\mathbf{z}}]\exp(i\mathbf{q}\cdot\mathbf{r}). \quad (3)$$

Here $\mathbf{q}$ and $\mathbf{h}$ are 2-D horizontal vectors: $\mathbf{q} = (q_x, q_y, 0)$, $\mathbf{h} = (h_x, h_y, 0)$. From Eqs. (1)–(3) we obtain

$$\mathbf{h} = \frac{i\mathbf{q}P}{\omega^2\rho\beta^2}, \quad \mathbf{v} = \left(\frac{\mathbf{q}P}{\omega\rho\beta} - W\frac{d\mathbf{u}}{dz} - i\omega\beta W\hat{\mathbf{z}}\right)\exp(i\mathbf{q}\cdot\mathbf{r}),$$
$$\beta = 1 - \frac{\mathbf{q}\cdot\mathbf{u}}{\omega} \quad (4)$$

and a set of first-order, ordinary differential equations

$$\frac{dP}{dz} + \frac{g}{c^2}P = \rho(\omega^2\beta^2 - N^2)W, \quad (5)$$

$$\frac{dW}{dz} - \frac{g}{c^2}W = \left(\frac{q^2}{\omega^2\beta^2} - \frac{1}{c^2}\right)\frac{P}{\rho}, \quad (6)$$

for the unknown functions $P$ and $W$. Similar equations can be found in the literature.[9,10,25,27] Here $N^2 = -g\rho^{-1}d\rho/dz - g^2/c^2$, and $N$ is the buoyancy frequency.

The quantity $p + \mathbf{w}\cdot\nabla p_0 = \tilde{P}(z)\exp(i\mathbf{q}\cdot\mathbf{r})$, where

$$\tilde{P} = P - \rho g W, \quad (7)$$

has the meaning of the Lagrangian pressure perturbation, i.e., wave-induced pressure perturbation in a moving fluid particle[1,24] as opposed to the (Eulerian) pressure perturbation $p$ at a fixed point in space. Using $\tilde{P}$ as an unknown function instead of $P$, Eqs. (5) and (6) become

$$\frac{d\tilde{P}}{dz} + \frac{gq^2}{\omega^2\beta^2}\tilde{P} = \rho\left(\omega^2\beta^2 - \frac{g^2q^2}{\omega^2\beta^2}\right)W, \quad (8)$$

$$\frac{dW}{dz} - \frac{gq^2}{\omega^2\beta^2}W = \left(\frac{q^2}{\omega^2\beta^2} - \frac{1}{c^2}\right)\frac{\tilde{P}}{\rho}. \quad (9)$$

While $d\rho/dz$ enters Eq. (5) through $N^2$, Eqs. (8) and (9) do not contain spatial derivatives of environmental parameters. In Eqs. (5)–(6) and (8)–(9), all the effects of background flows are described through the quantity $\beta(z)$. $\beta$ has the meaning of the ratio of the wave frequency in the reference frame, which follows the local background flow, to the wave frequency in the original reference frame; $\beta \equiv 1$ in quiescent fluids. Governing equations for AGWs in moving fluids differ from the equations in quiescent fluids by substitution of $\omega\beta$ for $\omega$.

The governing equations (5)–(6) and (8)–(9) are supplemented by boundary conditions. On a horizontal fluid-fluid interface, the linearized boundary conditions[24,26] consist in the continuity of $\tilde{P}$ and $W$. On a horizontal locally reacting (impedance) boundary with impedance $Z$, the boundary condition[24,26] is

$$\tilde{P} = -i\omega Z W. \quad (10)$$



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In particular, $\tilde{P}=0$ on a free surface, where $Z=0$, and $W=0$ on a rigid surface, where $Z=\infty$. Note that generally $p\neq 0$ on a free surface as long as $g\neq 0$.

In investigations of AGW propagation, the atmosphere is usually modeled[1,2,9] as a half-space with density vanishing at $z\to+\infty$. We will assume that at $z>H$ sound speed and flow velocity are constant and density decreases with height exponentially (Fig. 1):

$$c=c_H,\quad \mathbf{u}=\mathbf{u}_H,\quad \rho=\rho_H\exp\Big(2\mu(H-z)\Big),\quad z>H. \tag{11}$$

For the medium to be stably stratified, it is necessary that $N^2>0$ and, hence, $\mu>g/2c_H^2$. At $z>H$, linearly independent solutions to Eqs. (5) and (6) are

$$P=\exp[(-\mu\mp s)z],$$
$$W=\frac{(gc_H^{-2}-\mu\mp s)\exp[(\mu\mp s)z-2\mu H]}{\rho_H(\omega^2\beta_H^2-2g\mu+g^2c_H^{-2})}, \tag{12}$$

where $\beta_H=1-\mathbf{q}\cdot\mathbf{u}_H/\omega$ and

$$s=\sqrt{\mu^2-\frac{\omega^2\beta_H^2}{c_H^2}+\left(1-g\frac{2\mu-gc_H^{-2}}{\omega^2\beta_H^2}\right)q^2},\quad \mathrm{Re}\, s\geq 0. \tag{13}$$

For normal modes, there should be $\mathrm{Re}\, s>0$ and the solution with upper signs should be chosen in Eq. (12).[9,24] Note that the amplitude of the vertical displacement in a normal mode may increase exponentially with height, but the pressure and the product $PW$ tend to zero at $z\to+\infty$.

## III. NORMAL MODE ORTHOGONALITY RELATIONS

Consider AGWs in a half-space $z>0$ with an impedance boundary at $z=0$. We want to compare wave fields in two different media ($j=1, 2$). Both media are stratified fluids, but these have generally different sound speed, flow velocity, and density profiles $c_j(z)$, $\mathbf{u}_j(z)$, $\rho_j(z)$ and may even have different gravity accelerations $g_j$ and boundary impedances $Z_j$. In each medium, we will consider its own normal mode with horizontal wave vector $\mathbf{q}_j=q_j(\cos\psi_j, \sin\psi_j, 0)$, frequency $\omega_j$, and vertical dependencies $P_j(z)$, $\tilde{P}_j(z)$, $W_j(z)$ of the Eulerian and Lagrangian pressure perturbations and vertical displacement. The functions $\tilde{P}_j(z)$ and $W_j(z)$ satisfy corresponding governing Eqs. (8) and (9) as well as boundary conditions at $z=0$ and $z\to+\infty$, which are discussed above in Sec. II.

Let parameters of both fluids be continuous in a layer $z_1<z<z_2$. By multiplying Eq. (8) for $\tilde{P}_1$ by $-W_2$, Eq. (8) for $\tilde{P}_2$ by $W_1$, Eq. (9) for $W_2$ by $-\tilde{P}_1$, and Eq. (9) for $W_1$ by $\tilde{P}_2$, summing up the results, and integrating the sum over $z$, we obtain the identity

$$(\tilde{P}_2W_1-\tilde{P}_1W_2)\big|_{z=z_1}^{z=z_2}=\int_{z_1}^{z_2}dz\Bigg\{\left(\frac{q_1^2c_1^2-\omega_1^2\beta_1^2}{\omega_1^2\beta_1^2\rho_1c_1^2}-\frac{q_2^2c_2^2-\omega_2^2\beta_2^2}{\omega_2^2\beta_2^2\rho_2c_2^2}\right)\tilde{P}_1\tilde{P}_2+\left(\frac{g_1q_1^2}{\omega_1^2\beta_1^2}-\frac{g_2q_2^2}{\omega_2^2\beta_2^2}\right)$$
$$\times(\tilde{P}_2W_1+\tilde{P}_1W_2)-\left[\rho_1\left(\omega_1^2\beta_1^2-\frac{g_1^2q_1^2}{\omega_1^2\beta_1^2}\right)-\rho_2\left(\omega_2^2\beta_2^2-\frac{g_2^2q_2^2}{\omega_2^2\beta_2^2}\right)\right]W_1W_2\Bigg\}, \tag{14}$$

where $\beta_j=1-\mathbf{q}_j\cdot\mathbf{u}_j/\omega_j$.We assume that $\beta_j(z)\neq 0$.[28] The half-space $z>0$ is divided into a set of layers, where parameters of both media are continuous, by the boundary $z=0$ and fluid-fluid interfaces in each of the media. Applying Eq. (14) to individual layers, summing up the results, and taking into account continuity of $\tilde{P}_j(z)$ and $W_j(z)$ on fluid-fluid interfaces, one finds that Eq. (14) remains valid when media parameters are piecewise continuous at $z_1<z<z_2$. Moreover, it follows from Eq. (12) (with upper signs) that the integral in the right-hand side of Eq. (14) converges at $z_2\to+\infty$, provided $\tilde{P}_2W_1\to 0$ and $\tilde{P}_1W_2\to 0$ at $z_2\to+\infty$. Then, we obtain

$$(\tilde{P}_1W_2-\tilde{P}_2W_1)\big|_{z=0}=\int_0^{+\infty}dz\Bigg\{\left(\frac{q_1^2c_1^2-\omega_1^2\beta_1^2}{\omega_1^2\beta_1^2\rho_1c_1^2}-\frac{q_2^2c_2^2-\omega_2^2\beta_2^2}{\omega_2^2\beta_2^2\rho_2c_2^2}\right)\tilde{P}_1\tilde{P}_2+\left(\frac{g_1q_1^2}{\omega_1^2\beta_1^2}-\frac{g_2q_2^2}{\omega_2^2\beta_2^2}\right)$$
$$\times(\tilde{P}_2W_1+\tilde{P}_1W_2)-\left[\rho_1\left(\omega_1^2\beta_1^2-\frac{g_1^2q_1^2}{\omega_1^2\beta_1^2}\right)-\rho_2\left(\omega_2^2\beta_2^2-\frac{g_2^2q_2^2}{\omega_2^2\beta_2^2}\right)\right]W_1W_2\Bigg\}. \tag{15}$$

With integration extended over the whole vertical extent of the fluid, identities (14) and (15) will be referred to as generalized orthogonality relations for normal modes of AGWs. When $g_1=g_2=0$ and the boundary $z=0$ is a free surface, Eq. (15)



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reduces to the known generalized orthogonality relation for acoustic waves.[22,24] Identities (14) and (15) are corollaries of the equations of motion and boundary conditions. The identities have no direct physical meaning but prove to be rather convenient mathematical tools for establishing a number of properties of AGW normal modes. In particular, mode orthogonality conditions are obtained below as special cases of Eqs. (14) and (15).

Assume temporarily that $c_j(z)$, $\mathbf{u}_j(z)$, $\rho_j(z)$ are continuously differentiable functions. Then buoyancy frequencies $N_j$ are defined for all $z > 0$ and are continuous. Quite similarly to derivation of Eqs. (14) and (15), from Eqs. (5) and (6) we find another generalized orthogonality relation:

$$(P_1W_2 - P_2W_1)\big|_{z=0} = \int_0^{+\infty} dz\left\{\left(\frac{q_1^2c_1^2-\omega_1^2\beta_1^2}{\omega_1^2\beta_1^2\rho_1c_1^2} - \frac{q_2^2c_2^2-\omega_2^2\beta_2^2}{\omega_2^2\beta_2^2\rho_2c_2^2}\right)P_1P_2 + \left(\frac{g_1}{c_1^2}-\frac{g_2}{c_2^2}\right)\right.$$
$$\left.\times (P_2W_1 + P_1W_2) - [\rho_1(\omega_1^2\beta_1^2 - N_1^2) - \rho_2(\omega_2^2\beta_2^2 - N_2^2)]W_1W_2\right\}. \quad (16)$$

Equation (16) is similar to Eq. (15) but generally does not reduce to it. We see that the generalized orthogonality relation for AGWs is not unique. Below, we will utilize mainly Eq. (15) rather than Eq. (16) because the former applies to general layered fluids and allows for fluid-fluid interfaces.

Let us apply the generalized orthogonality relation (15) to the particular case where the two layered media are identical ($c_1 = c_2 = c$, $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}$, $\rho_1 = \rho_2 = \rho$, $g_1 = g_2 = g$), and the two normal modes have the same frequency $\omega$ and differ only by their horizontal wave vectors. After simple algebra, from Eqs. (7), (10), and (15) we obtain

$$\int_0^{+\infty}\left[\left(\frac{q_n^2}{\beta_n^2}-\frac{q_m^2}{\beta_m^2}\right)\frac{P_nP_m}{\rho\omega^2} - \rho\omega^2(\beta_n^2-\beta_m^2)W_nW_m\right]dz = 0. \quad (17)$$

This is a mode orthogonality (but not mode orthonormality) relation. Here we distinguish two normal modes, which were assigned subscripts $j = 1, 2$ in Eq. (15), by their mode indexes $n$ and $m$, which can take various integer values. If desired, by using Eqs. (5) and (6), the integrand in Eq. (17) can be expressed in terms of either $P_n, P_m, dP_n/dz$, and $dP_m/dz$ or $W_n, W_m, dW_n/dz$, and $dW_m/dz$. In quiescent fluids, $\beta_n = \beta_m = 1$, and Eq. (17) simplifies to

$$\int_0^{+\infty}\frac{dz}{\rho}P_nP_m = 0, \quad n \neq m. \quad (18)$$

Thus, vertical dependencies of pressure in AGW normal modes of different order are orthogonal with weight $\rho^{-1}$, just as in acoustic normal modes in motionless fluids.[23,24] When $g = 0$, $W = (\rho\omega^2\beta^2)^{-1}dP/dz$ according to Eq. (5), and the orthogonality relation (17) of AGW normal modes reduces to the known orthogonality relation[22,24] of acoustic normal modes in layered moving media.

When wave vectors $\mathbf{q}_n$ and $\mathbf{q}_m$ of normal modes are parallel, we have $\omega(\beta_n - \beta_m) = (q_m - q_n)\tilde{u}$, where $\tilde{u} = u_x\cos\psi + u_y\sin\psi$ is the projection of the background flow velocity on the direction of the wave vectors. Then, dividing the integrand in (17) by $q_m - q_n$, the mode orthogonality relation can be written as

$$\int_0^{+\infty}\left[\frac{(q_m\beta_n + q_n\beta_m)P_nP_m}{\rho\beta_n^2\beta_m^2} + \omega^3\rho\tilde{u}(\beta_n+\beta_m)W_nW_m\right]dz = 2q_m\delta_{mn}, \quad (19)$$

where $\delta_{mn}$ is the Kronecker symbol. The ability to normalize the normal modes, as in Eq. (19) with $n = m$, is due to the fact that, according to Eq. (12), integrals of $\rho^{-1}|P_nP_m|$ and $\rho^{-1}|W_nW_m|$ over the vertical extent of the fluid are finite for proper normal modes with arbitrary orders $n$ and $m$.[29] Of course, the choice of the non-zero factor in front of the Kronecker symbol in Eq. (19) is arbitrary. It reflects the freedom in choosing normal mode normalization. An alternative derivation of Eq. (19), which is based on wave energy conservation law[24,26] rather than the generalized orthogonality relation (15), is given in the Appendix.

Independence of the weighting function $\rho^{-1}$ in Eq. (18) of the mode indices is heavily relied upon in various coupled-mode theories[23,24,30–33] of sound propagation in irregular waveguides in motionless media. For applications of the mode orthogonality to derivation of mode coupling equations in range-dependent waveguides and, more generally, for modal decomposition of generic AGW fields in moving media, it is important to represent the mode orthogonality relation (17) as an integral of a weighted product of mode shape functions with the weight being independent of the mode indices $n$, $m$. To achieve such a representation, we follow an earlier analysis of the corresponding acoustic problem[34] and will characterize AGW wave fields by a state vector $\mathbf{S}$.

Consider a 2-D problem, where $u_y = 0$ and CW AGW fields are functions of the horizontal coordinate $x$ and the vertical coordinate $z$. The state vector is defined by the equation

$$\mathbf{S} = (p, w_1, dw_1/dt, w_3, dw_3/dt)^T, \quad (20)$$

with superscript $T$ denoting matrix transposition. When the wave field is due to a single normal mode of order $m$, from Eqs. (3) and (4) (where now $q_y = 0$, $q_x = q_m$) we have $\mathbf{S} = \mathbf{S}_m(z)\exp(iq_mx)$ and

$$\mathbf{S}_m = \left(P_m, \frac{iq_mP_m}{\rho\omega^2\beta_m^2}, \frac{q_mP_m}{\rho\omega\beta_m}, W_m, -i\omega\beta_mW_m\right)^T. \quad (21)$$



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Here, it is convenient to characterize normal modes with the $x$ component of their wave vector $\mathbf{q}_m$, rather than with $|\mathbf{q}_m|$ and the azimuthal angle $\psi$, which takes only values 0 and $\pi$ in the 2-D problem. So, $q_m$ can be positive or negative in Eq. (21).

Introduce auxiliary vectors[35]

$$\mathbf{Y}_m = \left( P_m, \frac{-iq_m P_m}{\rho\omega^2\beta_m^2}, \frac{-q_m P_m}{\rho\omega\beta_m}, W_m, -i\omega\beta_m W_m \right)^T. \tag{22}$$

In terms of the state vectors $\mathbf{S}_m$ and $\mathbf{Y}_m$, the orthogonality relation (19) (with $\tilde{u}$ replaced by $u_x$) of AGW normal modes becomes

$$\int_0^{+\infty} \mathbf{Y}_n^T \|\mathbf{B}\| \mathbf{S}_m dz = -\frac{2i}{\omega^2} q_m \delta_{mn}, \tag{23}$$

where $\|\mathbf{B}\|$ is a sparse $5 \times 5$ matrix:

$$\|\mathbf{B}\| = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & \rho u_x & 0 & 0 \\ 0 & \rho u_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho u_x \\ 0 & 0 & 0 & \rho u_x & 0 \end{pmatrix}. \tag{24}$$

The AGW mode orthogonality relation (23) differs from the orthogonality relation[34] for acoustic modes in moving media by the appearance of corresponding state vectors, which explicitly contain functions $W_{n,m}$ in the AGW case. Equation (23) is largely similar to its counterpart (18) for AGWs in quiescent fluids but differs in two respects. Modes are characterized by state vectors in the moving fluids rather than scalar shape functions $P_n$ in Eq. (18); the mode state vectors are orthogonal with a matrix weight $\|\mathbf{B}\|$ rather than scalar weight $\rho^{-1}$ in quiescent fluids. In both cases, the weight depends on $z$ but not on mode indices.

As an example of application of the mode orthogonality condition (23), consider the problem of modal decomposition of AGW fields in a 2-D waveguide. Let the wave field consist of normal modes with unknown amplitudes $\eta_n$. Then

$$\mathbf{S}(x,z) = \sum_m \eta_m \mathbf{S}_m(z) \exp(iq_m x). \tag{25}$$

Properties of the waveguide and, therefore, mode state vectors $\mathbf{S}_m$ and $\mathbf{Y}_m$ are assumed known. To find amplitudes $\eta_n$, we multiply both sides of Eq. (25) by $\mathbf{Y}_n^T\|\mathbf{B}\|$ from the left, integrate over the vertical extent of the waveguide, apply Eq. (23), and obtain

$$\eta_n = \frac{i\omega^2}{2q_n} e^{-iq_n x_0} \int_0^{+\infty} \mathbf{Y}_n^T(z) \|\mathbf{B}\| \mathbf{S}(x_0, z) dz. \tag{26}$$

Mode decomposition (26) can be effected at an arbitrary vertical cross-section $x = x_0$ of the waveguide.

So far, we considered sets of proper normal modes having the same frequency $\omega$ and different horizontal wave vectors $\mathbf{q}_m$, as is usually done in acoustics.[23,24] In studies of internal gravity waves, one typically considers sets of normal modes with the same horizontal wave vectors $\mathbf{q}$ and different frequencies $\omega_m$.[36] Orthogonality relations for modes having the same frequency and for modes having the same wave vector can be different. To derive the orthogonality relation for proper normal modes with a given horizontal wave vector, we apply the generalized orthogonality relation (15) to the particular case, where the two layered media are identical ($c_1 = c_2 = c$, $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}$, $\rho_1 = \rho_2 = \rho$, $g_1 = g_2 = g$), and the two normal modes differ only by their frequency and corresponding shape functions. Using the same notation as in Eq. (17), from Eqs. (7)–(10), and (15) we obtain

$$\begin{aligned}(\omega_n - \omega_m) &\int_0^{+\infty} (\omega_n + \omega_m - 2\mathbf{q}\cdot\mathbf{u}) \\ &\times \left[ \frac{q^2 P_n P_m}{\rho(\omega_n - \mathbf{q}\cdot\mathbf{u})^2(\omega_m - \mathbf{q}\cdot\mathbf{u})^2} + \rho W_n W_m \right] dz \\ &= P_m(0)W_n(0) - P_n(0)W_m(0). \end{aligned} \tag{27}$$

The right-hand side of Eq. (27) vanishes when the boundary $z = 0$ is free, rigid, or has an impedance inversely proportional to frequency. In the case of a quiescent fluid with such a boundary, the AGW mode orthogonality relation (27) simplifies to

$$\int_0^{+\infty} dz \left( \frac{q^2 P_n P_m}{\rho\omega_n^2\omega_m^2} + \rho W_n W_m \right) = 0, \quad n \neq m \tag{28}$$

and is clearly distinct from Eq. (18). An inspection shows that, in an appropriate limit, Eq. (28) reduces to the mode orthogonality condition derived in Sec. 3.1 of Ref. 36 for internal gravity waves.

## IV. PERTURBATION THEORY

Consider variations of the wave vector $\mathbf{q}_m = q_m$ ($\cos\psi$, $\sin\psi$, 0) of a normal mode due to small variations of waveguide parameters when wave frequency $\omega$ and azimuthal angle $\psi$, which determines the direction of the wave vector, are kept constant. We apply the generalized orthogonality relation (15) to the modes of the same order in waveguides with slightly different parameters. Let $q_1 = q_m$, $q_2 = q_m + \delta q_m$, $\psi_1 = \psi_2 = \psi$, $c_1 = c$, $c_2 = c + \delta c$, $\rho_1 = \rho$, $\rho_2 = \rho + \delta\rho$, $\mathbf{u}_1 = \mathbf{u}$, $\mathbf{u}_2 = \mathbf{u} + \delta\mathbf{u}$, $Z_1 = Z$, $Z_2 = Z + \delta Z$, $g_1 = g$, $g_2 = g + \delta g$. We want to calculate both sides of Eq. (15) with accuracy up to the terms of second order in the small perturbations. Note that coefficients in front of $\tilde{P}_1\tilde{P}_2$, $\tilde{P}_2 W_1 + \tilde{P}_1 W_2$, and $W_1 W_2$ in the integrand in the right-hand side of Eq. (15) are of the first order in the perturbations and vanish when the two waveguides become identical. Thus, one can disregard perturbations of the mode shape functions and replace $\tilde{P}_1\tilde{P}_2$, $\tilde{P}_2 W_1 + \tilde{P}_1 W_2$, and $W_1 W_2$ with $P_m^2$, $2P_m W_m$, and $W_m^2$, respectively. This also applies to the left-hand side of Eq. (15) after $\tilde{P}_{1,2}$ are expressed in terms of $W_{1,2}$ and impedances $Z_{1,2}$ by using the boundary condition (10). By equating first-order terms in the right- and left-hand sides of Eq. (15), after simple algebra we obtain



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$$\delta q_m = -q_m\langle q_m^2 P_m^2 + \rho^2\omega^4(1-\beta_m)\beta_m^4 W_m^2\rangle^{-1}\left\{\frac{i}{2}\omega^3 W_m^2(0)\delta Z + q_m^2\langle\rho\beta_m P_m W_m\rangle\delta g + \left\langle\frac{\omega^2\beta_m^3}{c^3}(P_m-\rho g W_m)^2\delta c\right.\right.$$
$$\left.\left. +(q_m^2 P_m^2 + \rho^2\omega^4\beta_m^4 W_m^2)\frac{\mathbf{q}_m\cdot\delta\mathbf{u}}{\omega} + \left[\left(\frac{\omega^2\beta_m^2}{c^2}-q_m^2\right)(P_m-\rho g W_m)^2 - \rho^2(\omega^4\beta_m^4 - g^2 q_m^2)W_m^2\right]\frac{\beta_m\delta\rho}{2\rho}\right\rangle\right\}. \quad (29)$$

To write the somewhat cumbersome result in a relatively compact form, here we denote $\langle F\rangle = \int \rho^{-1}\beta_m^{-3}F dz$, where the integral is taken over the full vertical extent of the fluid.

It should be emphasized that no perturbations in the mode shape functions enter Eq. (29). It is this property that makes the expression (29) for mode wave number perturbation useful in applications. Equation (29) has been derived for a waveguide occupying half-space $z>0$. In a different geometry, where fluid occupies half-space $z<0$ or a finite layer with variable properties of the upper boundary, Eq. (29) remains unchanged, except $\delta Z$ should be replaced with $-\delta Z$.

Generally, $\delta c$, $\delta\rho$, and $\delta\mathbf{u}$ are piecewise continuous functions of $z$ in the right-hand side of Eq. (29). Discontinuities may occur at fluid-fluid interfaces within the waveguide and necessarily occur when the positions and/or the number of the interfaces in the original and perturbed waveguides do not coincide.

Equation (29) allows one to quantify the sensitivity of AGW normal modes to small variations in the sound speed, density, and flow velocity profiles as well as in the impedance of the boundary $z=0$ and in the acceleration of gravity. Obviously, $\delta q_m$ (29) is not affected by the way the mode shape functions are normalized. Note that sensitivity to sound speed, flow velocity, and density variations around any given altitude $z_0$ is proportional to the magnitude squared of the modal shape functions at $z=z_0$. Particular combinations of $P_m(z_0)$ and $W_m(z_0)$ are different for $\delta c$, $\delta\rho$, and $\delta\mathbf{u}$. Since the acceleration of gravity is spatially uniform, sensitivity to $\delta g$ is determined by an integral of $\beta_m^{-2}P_m W_m$. Sensitivity to $\delta Z$ variations is proportional to $W_m^2(0)$. When the properties of the waveguide's boundary are close to that of a rigid surface, where $Z$ is infinite, it is more appropriate to consider small variations in the reciprocal quantity $Z^{-1}$. Then the first term in braces in the right-hand side of Eq. (29) is replaced by $0.5i\omega\tilde{P}_m^2(0)\delta(Z^{-1})$,and the sensitivity to the perturbation in the boundary properties is proportional to $\tilde{P}_m^2(0)$. In the derivation of Eq. (29) it has been assumed that $Z$ and $Z^{-1}$ in the unperturbed waveguide [but not necessarily the perturbations $\delta Z$ and $\delta(Z^{-1})$] are independent of $q$.

According to Eq. (29), sound-speed perturbations do not affect $q_m$, to first order, if $P_m=\rho g W_m$, i.e., the Lagrangian pressure perturbation $\tilde{P}_m\equiv 0$, in the normal mode. Such a mode can indeed exist in motionless and uniformly moving fluids.[37,38] It corresponds to an incompressible wave motion of compressible fluids and, in fact, its properties are independent of the sound speed to all orders.[37]

If the ratio $\delta\rho/\rho$ is independent of $z$, the density perturbation does not affect the mode wave number in unbounded fluids or fluids with free and rigid boundaries. Indeed, from Eqs. (8) and (9) it follows that

$$\frac{d}{dz}(\tilde{P}_m W_m) = \rho W_m^2\left(\omega^2\beta_m^2 - \frac{g^2 q_m^2}{\omega^2\beta_m^2}\right) + \frac{\tilde{P}_m^2}{\rho}\left(\frac{q_m^2}{\omega^2\beta_m^2} - \frac{1}{c^2}\right), \quad (30)$$

and $\delta\rho/\rho$ is multiplied by $\langle\beta_m(\omega^2\beta_m^2 c^{-2} - q_m^2)\tilde{P}_m^2 - \rho^2\beta_m(\omega^4\beta_m^4 - g^2 q_m^2)W_m^2\rangle = 0$in Eq. (29). In fact, multiplication of the density profile $\rho(z)$ by a constant $1+\xi$ does not change $q_m$ to all orders in $\xi$. It is easy to see that, if $P_m(z)$ and $W_m(z)$ satisfy Eqs. (8) and (9) and boundary conditions at fluid-fluid interfaces in the original waveguide, then $(1+\xi)P_m(z)$ and $W_m(z)$ satisfy these equations and boundary conditions with unchanged $q_m$ in a waveguide with the density $(1+\xi)\,\rho(z)$. When the original waveguide has an impedance boundary, it follows from Eqs. (10) and (29) that $q_m$ will be unchanged if the boundary impedance $Z$ is multiplied by the same factor $1+\xi$ as the density profile.

When $g=\delta g=0$ and $\delta Z=0$, Eq. (29) reduces to results obtained earlier[22,24] from a generalized orthogonality relation for acoustic waveguides in moving media. Using a different approach, Pierce[9] considered perturbations in the wave number $q_m$ for AGW normal modes in an atmospheric waveguide with a rigid boundary, assuming that profiles of sound speed and density are smooth. The fluid was modeled as an ideal gas with a height-independent ratio of specific heats at constant pressure and constant volume. Under these assumptions, sound speed and density perturbations are no longer independent since the density profile can be expressed in terms of the sound-speed profile. An inspection shows that, for the waveguides treated in Ref. 9, Eq. (29) is equivalent to the results obtained by Pierce.

The perturbation result (29) also can be applied to quantify the influence of weak absorption on the mode wave numbers. Let wave energy absorption in the fluid be modeled by attributing small imaginary parts to the sound speed and density, so that in Eqs. (5)–(9) real-valued sound speed $c$ and density $\rho$ are replaced by $(1-i\alpha)c$ and $(1-i\zeta)\rho$, respectively. Here $|\alpha|\ll 1$, $|\zeta|\ll$; $\alpha$ and $\zeta$ are real-valued functions of height $z$ and wave frequency $\omega$. In addition, wave energy leakage through the waveguide boundary $z=0$ can contribute to mode attenuation. There is no time-averaged energy flux through the boundary when its impedance is purely reactive: $\mathrm{Re}Z=0$, see Eq. (A3) for AGW power flux density. It follows from Eq. (A3) and the boundary condition (10) that, when Im $\rho(0)=0$, wave energy is injected into the waveguide, if $\mathrm{Re}Z>0$, and wave energy leaves the waveguide, if $\mathrm{Re}Z<0$. We will assume that $0<-\mathrm{Re}Z\ll|\mathrm{Im}Z|$. Then, by taking the waveguide without energy losses as the unperturbed state and with the losses as the perturbed state, from Eq. (29) we find



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$$\delta q_m = iq_m \langle q_m^2 P_m^2 + \rho^2 \omega^4 (1-\beta_m)\beta_m^4 W_m^2 \rangle^{-1} \times \left\{ \left\langle \omega^2 \beta_m^3 c^{-2} (P_m - \rho g W_m)^2 \alpha \right\rangle + \left\langle \left[ \left( \frac{\omega^2 \beta_m^2}{c^2} - q_m^2 \right) (P_m - \rho g W_m)^2 - \rho^2 (\omega^4 \beta_m^4 - g^2 q_m^2) W_m^2 \right] \frac{\beta_m \zeta}{2} \right\rangle - \omega^3 W_m^2(0) \mathrm{Re} \frac{Z}{2} \right\}. \quad (31)$$

The mode wave number perturbation is purely imaginary and describes the mode attenuation. Note that weighting functions are rather different in Eq. (31) for contributions to the mode attenuation from the imaginary parts of the complex sound speed and density. An inspection shows that, in an acoustic waveguide ($g = 0$) in a moving fluid with free and/or rigid boundaries and $\zeta \equiv 0$, Eq. (31) reduces to previously established results,[22,24] which, in turn, contain as their special cases a number of earlier results for attenuation of acoustic normal modes in quiescent waveguides.

In problems involving fast gravity waves, such as in investigations of atmospheric manifestations of tsunamis,[12,14,16,18] it is important to know when the simpler description of the fluid as an incompressible one needs to be abandoned in favor of a complete yet more involved theory of AGWs. A qualitative answer to this question is usually obtained by either comparing terms with and without sound speed in the governing differential equations or by analyzing idealized problems which allow for plane-wave solutions.[1–3,39] Using Eq. (29), effects of weak compressibility can be quantified by taking a normal mode in the incompressible fluid ($c \to \infty$) as the unperturbed state and considering compressibility as the perturbation. Mode wave number in a compressible fluid differs from its value in the incompressible limit by

$$\delta q_m = 0.5\omega^2 q_m \langle q_m^2 P_m^2 + \rho^2 \omega^4 (1-\beta_m)\beta_m^4 W_m^2 \rangle^{-1} \times \langle \beta_m^3 (P_m - \rho g W_m)^2 c^{-2} \rangle . \quad (32)$$

Similarly, in investigations of guided propagation of very low-frequency infrasound in the ocean and atmosphere,[39–43] the question arises whether treatment of the wave as an acoustic wave rather than AGW is justified. While, at best, a general answer can give an order of magnitude of the "transition" frequency, effects of gravity on a specific normal mode are readily quantified by Eq. (29) by choosing the solution in the absence of gravity ($g = 0$) as the unperturbed mode. Then, the change of the mode wave number due to gravity is given by

$$\delta q_m = -0.5 g \omega^{-2} q_m^3 \langle q_m^2 P_m^2 + (1-\beta_m)(dP_m/dz)^2 \rangle^{-1} \times \langle \beta_m^{-1} dP_m^2/dz \rangle. \quad (33)$$

In Eq. (33) we took into account that $W_m = (\omega^2 \rho \beta_m^2)^{-1} \partial P_m / \partial z$ when $g = 0$.

When the flow velocity $\mathbf{u}$ is much smaller than the phase speed $c_m = \omega / q_m$ of a normal mode, the influence of the flow on the mode wave number is given by Eq. (29), where the normal mode in the motionless waveguide is chosen as the unperturbed state:

$$\delta q_m = -q_m \frac{\mathbf{q}_m \cdot \mathbf{u}_{av}}{\omega}, \quad \mathbf{u}_{av} = \int_0^{+\infty} \frac{dz}{\rho} (P_m^2 + q_m^{-2} \omega^4 \rho^2 W_m^2) \mathbf{u} \Big/ \int_0^{+\infty} \frac{dz}{\rho} P_m^2 . \quad (34)$$

Here, $\mathbf{u}_{av}$ is independent of the direction of $\mathbf{q}_m$ and has the meaning of height-averaged flow velocity. Note that the azimuthal dependence of the mode wave number, $q_m + \delta q_m$, in slowly moving fluid is completely determined by the vector $\mathbf{u}_{av}$ (34). Active[44] and passive[45] measurements of the nonreciprocity of modal phase $\Phi_m$, i.e., the difference in phases of $m$th normal mode at propagation in opposite directions between two points, can be used for tomographic reconstruction of the flow velocity profile $\mathbf{u}$.[24,44] For the phase nonreciprocity, from Eq. (34) we find $\Phi_m = -2\omega^{-1} (\mathbf{q}_m \cdot \mathbf{u}_{av}) q_m R [1 + O(u/c_m)]$. Here $R$ is the horizontal separation between the points where measurements are made.

Consider a quiescent fluid half-space $z > 0$ with a constant sound speed $c_H$, arbitrary density profile $\rho(z)$, and rigid boundary $z = 0$. The solution with

$$P_m(z) = P_m(0) \exp(-gz/c_H^2), \quad W_m(z) \equiv 0, \quad q_m = \omega / c_H \quad (35)$$

meets equations of motion (8)–(9), conditions at infinity and boundary conditions at $z = 0$ and, hence, specifies a proper normal mode. This is a Lamb wave,[1,2] which is usually considered assuming an isothermal atmosphere with height-independent ratio $\gamma$ of the specific heats. In the latter case, the density profile is exponential and is given by Eq. (11) with $H = 0$ and $\mu = \gamma g / 2 c_H^2$.[1,2] When the sound speed $c$ and/or flow velocity $\mathbf{u}$ vary with height, no exact solution is available for the Lamb wave, but for its phase speed $c_m$ from Eqs. (29) and (35) we find

$$c_m(\psi) = \int_0^{+\infty} \frac{dz}{\rho} e^{-2gz/c^2} (c + u_x \cos\psi + u_y \sin\psi) \Big/ \int_0^{+\infty} \frac{dz}{\rho} e^{-2gz/c^2}. \quad (36)$$

This is generally an approximation valid to first order in the small parameters $|1 - c/c_H| \ll 1$ and $|\mathbf{u}/c_H| \ll 1$. However, Eq. (36) becomes exact when $c$ and $\mathbf{u}$ are independent of $z$.[46] According to Eq. (36), the Lamb wave remains nondispersive in waveguides with weakly stratified sound speed and flow velocity, and its phase speed is given by a weighted height average of the effective sound speed[47] $c + \mathbf{q} \cdot \mathbf{u}/q$. For a near-isothermal atmosphere, similar results were obtained by Garrett[10] and Bretherton.[11]

Let a Lamb wave with wave vector $q_L(\psi)(\sin\psi, \cos\psi, 0)$ propagate in a stratified, moving atmosphere over an ocean. We model the ocean as a quiescent half-space $z < 0$ with a constant sound speed $c_W$ and an exponential density profile $\rho(z) = \rho_W \exp(-2\mu_W z)$. Then the AGW field in water is



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given by Eq. (12) (with the lower sign), where now $c_H = c_W$, $\rho_H = \rho_W$, $\mu = \mu_W > 0$, $s = s_W(q_L) > 0$, $H = 0$, and $\beta_H = 1$. According to Eq. (13),

$$s_W(q) = \sqrt{q^2 + \mu_W^2 - g\omega^{-2}(2\mu_W - gc_W^{-2})q^2 - \omega^2/c_W^2}. \tag{37}$$

For the impedance $Z$ defined by Eq. (10), from Eqs. (7) and (12) we obtain

$$Z(q) = i\omega^{-1}\rho_W[\omega^2 - g\mu_W - gs_W(q)]/[gc_W^{-2} - \mu_W + s_W(q)] \tag{38}$$

on the water side of the air-water interface. Since $Z$ is continuous at fluid-fluid interfaces, Eqs. (10) and (38) define the boundary condition at $z=0$ for the AGW in air. The impedance $Z$ (38) tends to infinity when $\rho_W \to \infty$, and, due to the large density contrast between air and water, the air-water interface is a weak perturbation relative to the rigid surface $z=0$ for AGWs in the atmosphere.[48] In terms of the shape functions $P_m$, $W_m$ and the wave number $q_m$ of a Lamb wave in the atmosphere with a rigid boundary, for the Lamb wave over the ocean we find

$$q_L = q_m - \frac{q_m\omega^2[gc_W^{-2} - \mu_W + s_W(q_m)]}{2\rho_W[\omega^2 - g\mu_W - gs_W(q_m)]} \times \langle q_m^2 P_m^2 + \rho^2\omega^4(1-\beta_m)\beta_m^4 W_m^2\rangle^{-1} P_m^2(0) \tag{39}$$

from Eqs. (29) and (38). Equation (39) is valid up to terms of the second order in the small ratio $r = \rho(0)/\rho_W$ of the densities of air and water at $z=0$.

In the particular case, where $\mathbf{u} \equiv 0$ and the atmosphere is described by Eq. (11) with $H=0$, we have $q_m = \omega/c_H$, and the shape functions $P_m$, $W_m$ are given by Eq. (12). Then, Eq. (39) simplifies to

$$q_L = \frac{\omega}{c_H}\left[1 - r(\mu c_H^2 - g)\frac{gc_W^{-2} - \mu_W + s_W(\omega/c_H)}{\omega^2 - g\mu_W - gs_W(\omega/c_H)} + O(r^2)\right]. \tag{40}$$

An inspection shows that Eq. (40) agrees with results obtained from direct analyses[38,49] of surface AGWs at an interface of the ocean and atmosphere modeled as isospeed half-spaces.

In addition to weak perturbations in environmental parameters considered above, the perturbation theory can be applied to quantify effects of strong environmental perturbations if the latter occur in a thin layer. An important example of such perturbations is a small shift of a fluid-fluid interface. Consider two waveguides, piece-wise continuous parameters $c_j(z)$, $\rho_j(z)$, and $\mathbf{u}_j(z)$ ($j=1, 2$) of which differ only in a layer $D - \delta D/2 < D < D + \delta D/2$. While $\delta D$ is assumed to be small, $c_1 - c_2$, $\rho_1 - \rho_2$, and $\mathbf{u}_1 - \mathbf{u}_2$ are arbitrary. (In particular, one waveguide may have an interface at $D_1 = D - \delta D/2$ and the other at $D_2 = D + \delta D/2$.) Retaining only first-order perturbations in the generalized orthogonality relation (15), similarly to derivation of Eq. (29) we obtain $\delta q_m = q_2 - q_1 = (\partial q_m/\partial D)\delta D + O((\delta D)^2)$, where

$$\frac{\partial q_m}{\partial D} = -\frac{q_m}{2}\langle q_m^2 P_m^2 + \rho^2\omega^4(1-\beta_m)\beta_m^4 W_m^2\rangle^{-1} \times \left\{\left(\frac{\omega^2}{\rho_2 c_2^2} - \frac{\omega^2}{\rho_1 c_1^2} - \frac{q_m^2}{\rho_2\beta_2^2} + \frac{q_m^2}{\rho_1\beta_1^2}\right)\tilde{P}_m^2 - 2gq_m^2(\beta_2^{-2} - \beta_1^{-2})\tilde{P}_m W_m - \left[g^2 q_m^2(\rho_2\beta_2^{-2} - \rho_1\beta_1^{-2}) - \omega^4(\rho_2\beta_2^2 - \rho_1\beta_1^2)\right]W_m^2\right\}. \tag{41}$$

In particular, if an interface is located at $z=D$ in a waveguide and subscripts 1 and 2 stand for the values that corresponding parameters take just below and just above the interface, Eq. (41) gives the partial derivative of AGW mode wavenumber with respect to position of the interface and, at $g=0$, reduces to a known result[50] for acoustic waves. For example, Eq. (41) can be applied to quantify the sensitivity of dispersion relations of AGW normal modes to ocean depth, provided the ocean bottom is modeled as a fluid. Alternatively, if the seafloor is modeled as a rigid surface, we have $W_m(D) = 0$, $\beta_1 = 1$, $\beta_2 = \beta_m$, $\rho_1 \to \infty$, and Eq. (41) simplifies to

$$\frac{\partial q_m}{\partial D} = \langle q_m^2 P_m^2 + \rho^2\omega^4(1-\beta_m)\beta_m^4 W_m^2\rangle^{-1} \times \left(\frac{q_m^2}{\beta_m^2} - \frac{\omega^2}{c^2}\right)\frac{q_m P_m^2}{2\rho}\bigg|_{z=D}. \tag{42}$$

## V. PHASE AND GROUP VELOCITIES OF NORMAL MODES

Generalized orthogonality relation (15) furnishes a simple way of calculating group velocities $\mathbf{c}^{(gr)} = \partial\omega/\partial\mathbf{q}$ of normal modes. Let the two states in the generalized orthogonality relation (15) refer to an AGW normal mode of the same order $m$ in the same waveguide but at different frequencies, so that $c_1 = c_2 = c$, $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}$, $\rho_1 = \rho_2 = \rho$, $g_1 = g_2 = g$, $Z_1 = Z_2 = Z$, $P_1 = P_m$, $W_1 = W_m$, $\omega_1 = \omega$, and $\mathbf{q}_1 = \mathbf{q}_m$. We differentiate both sides of Eq. (15) with respect to $\mathbf{q}_2$ and let $\omega_2 = \omega$, $\mathbf{q}_2 = \mathbf{q}_m$ in the result. It is convenient to use Eq. (10) and represent the left-hand side of Eq. (15) as $i[\omega_2 Z(\omega_2) - \omega_1 Z(\omega_1)]W_1(0)W_2(0)$. The coefficient in front of $W_1(0)W_2(0)$ equals zero at $\omega_1 = \omega_2$, and therefore derivatives of $W_2$ do not enter the result of differentiation. Quite similarly, only coefficients in front of $\tilde{P}_1\tilde{P}_2$, $\tilde{P}_2 W_1 + \tilde{P}_1 W_2$, and $W_1 W_2$ need to be differentiated in the integrand in the right-hand side of Eq. (15). Taking into account that $\partial(\omega\beta)/\partial\mathbf{q} = \mathbf{c}^{(gr)} - \mathbf{u}$ according to Eq. (4), we obtain

$$\mathbf{c}_m^{(gr)} = \frac{\langle(q_m^2 P_m^2 + \rho^2\omega^4\beta_m^4 W_m^2)\mathbf{u}\rangle + \omega\langle\beta_m P_m^2\rangle\mathbf{q}_m}{\langle q_m^2 P_m^2 + \rho^2\omega^4\beta_m^4 W_m^2\rangle - 0.5i(Z + \omega dZ/d\omega)W_m^2(0)}. \tag{43}$$



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It is assumed here that there is no physical dispersion in the medium, i.e., parameters $\rho$ and $c$ of the fluid are independent of frequency. In this section, we consider normal modes that propagate in a waveguide without attenuation, so that $q_m$ is real; the shape functions $P_m$ and $W_m$ are chosen to be real-valued. Since the impedance $Z$ is purely imaginary at reactive interfaces, the group velocity is real in normal modes propagating without attenuation, as expected. Mode group velocity is horizontal in layered media. In quiescent waveguides, the direction of the group velocity coincides with or is opposite to that of the wave vector. In moving fluids, directions of $\mathbf{c}_m^{(gr)}$ and $\mathbf{q}_m$ are generally different. In the case of unidirectional flow, $\mathbf{c}_m^{(gr)}$ and $\mathbf{q}_m$ are parallel only when the normal mode propagates up or down the flow.

When eigenvalues and mode shape functions of AGWs in a waveguide are found numerically, application of Eq. (43) obviates the need to calculate group velocities of normal modes at neighboring $\mathbf{q}_m$ and/or $\omega$ values and approximate derivatives in the group velocity definition with finite differences. Use of equations of the kind of Eq. (43) improves accuracy and decreases computation time in numerical simulations of guided propagation.[21] When the waveguide's boundary $z=0$ is either free or rigid, the second term in the denominator in Eq. (43) vanishes, and the latter reduces to results obtained earlier for acoustic waveguides in moving fluids[22,24] and for AGWs in atmospheric waveguides with smooth stratification.[9]

By taking into account an identity

$$\frac{\partial\omega}{\partial\mathbf{q}}=\left(\frac{\partial\omega}{\partial q}\right)_\psi\frac{\mathbf{q}}{q}+\left(\frac{\partial\omega}{\partial\psi}\right)_q\frac{(-\sin\psi,\cos\psi,0)}{q},$$

from Eq. (43) we readily find

$$\frac{1}{q_m}\left(\frac{\partial q_m}{\partial\omega}\right)_\psi=\frac{\langle q_m^2P_m^2+\rho^2\omega^4\beta_m^4W_m^2\rangle-0.5i(Z+\omega dZ/d\omega)W_m^2(0)}{\omega\langle q_m^2P_m^2+\rho^2\omega^4(1-\beta_m)\beta_m^4W_m^2\rangle},\tag{44}$$

$$\frac{1}{q_m}\left(\frac{\partial q_m}{\partial\psi}\right)_\omega=\frac{-1}{c_m}\left(\frac{\partial c_m}{\partial\psi}\right)_\omega=\frac{q_m\langle(q_m^2P_m^2+\rho^2\omega^4\beta_m^4W_m^2)(u_x\sin\psi-u_y\cos\psi)\rangle}{\omega\langle q_m^2P_m^2+\rho^2\omega^4(1-\beta_m)\beta_m^4W_m^2\rangle}.\tag{45}$$

As far as the effects of fluid flow on normal modes are concerned, while $q_m$ and the mode phase speed $c_m$ at fixed azimuthal angle $\psi$ depend only on the component of $\mathbf{u}$ that is parallel to $\mathbf{q}_m$, derivatives $\partial q_m/\partial\psi$ and $\partial c_m/\partial\psi$ are proportional to a height average of the flow velocity component that is orthogonal to $\mathbf{q}_m$. When the waveguide's boundary $z=0$ is either free or rigid, Eqs. (44) and (45) reduce to results obtained earlier for acoustic waveguides in moving fluids[22,24] and for AGWs in atmospheric waveguides with smooth stratification.[9]

In Sec. IV a perturbation theory was developed to quantify the effects of environmental perturbations on AGW mode wave numbers at a constant frequency. In an alternative representation, where normal modes are characterized by a set of eigenfrequencies $\omega_m$, which are functions of the wave vector $\mathbf{q}$,[36] perturbations of the eigenfrequencies can be found from Eq. (44), which relates infinitesimal changes in mode frequency and wave number, and Eq. (29):

$$\delta\omega_m=-\omega_m\left[\langle q^2P_m^2+\rho^2\omega_m^4\beta_m^4W_m^2\rangle-0.5i\left(Z+\omega_m\frac{dZ}{d\omega_m}\right)W_m^2(0)\right]^{-1}\left\{\frac{i}{2}\omega_m^3W_m^2(0)\delta Z+q^2\langle\rho\beta_mP_mW_m\rangle\delta g+\left\langle\frac{\omega_m^2\beta_m^3}{c^3}(P_m-\rho gW_m)^2\delta c+(q^2P_m^2+\rho^2\omega_m^4\beta_m^4W_m^2)\frac{\mathbf{q}\cdot\delta\mathbf{u}}{\omega_m}+\left[\left(\frac{\omega_m^2\beta_m^2}{c^2}-q^2\right)(P_m-\rho gW_m)^2-\rho^2(\omega_m^4\beta_m^4-g^2q^2)W_m^2\right]\frac{\beta_m\delta\rho}{2\rho}\right\rangle\right\}.\tag{46}$$

The same result, but after much lengthier algebra, follows directly from the generalized orthogonality relation (15) applied to normal modes with the same wave vector but different frequencies in the unperturbed and perturbed waveguides.

We now discuss some universal properties of the phase speed and group velocity of AGW normal modes in layered waveguides with ideal (free or rigid) boundaries or without boundaries. From the inner product of Eq. (43) and the wave vector $\mathbf{q}_m$, we obtain

$$\tilde{c}_m^{(gr)}/c_m=1-\langle\rho^2\omega^4\beta_m^5W_m^2\rangle/\langle q_m^2P_m^2+\rho^2\omega^4\beta_m^4W_m^2\rangle,\tag{47}$$

where $\tilde{c}_m^{(gr)}=\mathbf{c}_m^{(gr)}\cdot\mathbf{q}_m/q_m=(\partial\omega/\partial q_m)_\psi$ is the projection of the group velocity on the direction of the mode wave vector. Let us assume that flow velocity does not exceed the phase speed of the mode (but is not necessarily small compared to the sound speed). Then $\beta_m>0$,[28] the numerator and denominator in the right-hand side of Eq. (47) are non-negative and positive, respectively, and therefore

$$\tilde{c}_m^{(gr)}\le c_m.\tag{48}$$

While the projection of the group velocity on the direction of the mode wave vector cannot exceed $c_m$, the full group



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velocity, as in the acoustic case,[22,24] can be larger than $c_m$. For instance, for the Lamb wave with shape functions (35) in a half-space $z > 0$ with height-independent $c$ and $\mathbf{u}$, we find $c_m = c + \tilde{u}$ and $c_m^{(gr)} = (c_m^2 + u_\perp^2)^{1/2} \geq c_m$ from Eqs. (36) and (43). Here $\tilde{u} = u_x \cos\psi + u_y \sin\psi$ and $u_\perp = u_y \cos\psi - u_x \sin\psi$ are, respectively, the components of the flow velocity $\mathbf{u}$ along and across the wave vector $\mathbf{q}_m$.

In waveguides in quiescent fluids $\tilde{c}_m^{(gr)} = c_m^{(gr)}$ and $\beta_m = 1$. Then, according to Eq. (47), the strict inequality holds in Eq. (48) unless there is no vertical motion in the wave, $W_m \equiv 0$. It follows from Eqs. (8) and (9) and boundary conditions at $z = 0$ that such a wave will be a normal mode only if $c = \text{const.}$, $c_m = c$, and the boundary $z = 0$ is rigid. Thus, the only normal mode, for which $c_m^{(gr)} = c_m$, is the Lamb wave (35). For all AGW modes in all other stratified quiescent waveguides $c_m^{(gr)} < c_m$.

Since $\omega \partial c_m / \partial \omega = c_m (1 - c_m / \tilde{c}_m^{(gr)})$, it follows from Eq. (48) that mode phase velocity is a monotonous, non-increasing function of frequency in stratified waveguides. In the absence of a fluid flow, the Lamb wave (35) is the only non-dispersive AGW normal mode. For all other AGW normal modes in quiescent waveguides, $\partial c_m / \partial \omega < 0$ because $c_m^{(gr)} < c_m$. For a normal mode to be non-dispersive in a waveguide in a moving fluid, the motion in the mode should be purely horizontal, i.e., $W_m \equiv 0$, according to Eq. (47). For that to happen, any boundaries have to be rigid. In addition, it follows from Eqs. (8) and (9) that the effective sound speed $c + \tilde{u}$ should be height-independent. Then, $c_m = c + \tilde{u}$ in the non-dispersive mode. Note that sound speed and flow velocity may vary with height in a waveguide supporting the non-dispersive normal mode.

For waveguides with rigid and/or free boundaries in quiescent fluids, it follows from Eq. (30) that

$$\langle (c^{-2} - c_m^{-2}) \tilde{P}_m{}^2 \rangle = (\omega^2 - g^2 c_m^{-2}) \langle \rho^2 W_m^2 \rangle. \tag{49}$$

As discussed above, if $W_m \equiv 0$, $c$ is necessarily constant and $c_m = c$. If $W_m$ is not identically zero and $c_m < g/\omega$, the right-hand side in Eq. (49) is negative, and the factor $c^{-2} - c_m^{-2}$ in the left-hand side has to be negative within an interval of heights. Hence, $c_m < c_{max}$, where $c_{max}$ is the maximum value of the sound speed in the fluid. Similarly, if $c_m > g/\omega$ and $W_m$ is not identically zero, the right-hand side in Eq. (49) is positive, and the factor $c^{-2} - c_m^{-2}$ in the left-hand side has to be positive within an interval of heights. Hence, $c_m > c_{min}$, where $c_{min}$ is the minimum value of the sound speed in the fluid.

Additional properties of the phase speed of AGW normal modes in stratified waveguides in quiescent fluids can be established by assuming a smooth density stratification so that the buoyancy frequency is everywhere bounded.[51]

## VI. DISCUSSION

The main results of this work are the generalized orthogonality relation (15), orthogonality relations (18), (23), and (28) of normal modes in moving and quiescent media, Eq. (29) for variations in mode wave number due to small perturbations in environmental parameters, and integral expression (43) for the group speed as well as bounds for the phase speed and the group velocity.

A remarkably general integral relation (15) holds between fields of AGW normal modes in two waveguides in stratified fluids with piecewise continuous profiles of sound speed, density, and flow velocity. With the vertical displacement of fluid particles and Lagrangian perturbations in pressure chosen as primary characteristics of the AGW wave field, the derivation of the generalized orthogonality relation (15) is quite simple and rather similar to the derivation in the acoustic case.[22,24] We have assumed that the waveguide consists of an arbitrary stratified layer $0 < z < H$ located between an impedance boundary and a half-space $z > H$ with an exponentially stratified density and constant flow velocity and sound speed. The derivation applies equally to an unbounded fluid as well as to a fluid layer between ideal (rigid or free) or impedance boundaries. In all cases, integration in the right-hand side of Eq. (15) should be carried out over the full vertical extent of the fluid. When there are two impedance boundaries, the left-hand side of Eq. (15) should be replaced by the difference of the values that the quantity $\tilde{P}_1 W_2 - \tilde{P}_2 W_1$ takes on the lower and upper boundaries.

Similarly, the orthogonality relations (18), (23), (27), and (28) of normal modes remain valid in unbounded waveguides and waveguides of finite vertical extent, provided integration over $z$ is carried out over the full vertical extent of the fluid. Depending on the problem considered and, in particular, on the type of wave source, it may be more convenient to consider normal modes with either the same frequency and wave vectors being functions of the frequency $\omega$ and the mode order $m$ (as well as wave vector's direction), or the same wave vector $\mathbf{q}$ and frequencies being functions of $\mathbf{q}$ and the mode order $m$. While mode shape functions $P_m$ and $W_m$ are the same in both descriptions, when viewed as functions of $z$, $\mathbf{q}$, $\omega$, and $m$, the mode orthogonality is expressed by different equations when normal modes with the same frequency or the same wave number are compared (with the wave vector direction being kept constant in both representations). This is important to keep in mind when applying the orthogonality relations to mode decomposition of AGW fields or to calculation of amplitudes of normal modes generated by a given source.

Much like the generalized orthogonality relation (15), from which it has been obtained, the perturbation theory presented in Sec. IV remains valid in unbounded waveguides and waveguides of finite vertical extent, provided integrations over $z$ are carried out over the full vertical extent of the fluid. In the forward problem, Eq. (29) quantifies the sensitivity of AGW normal modes to variations in various environmental parameters. In the inverse problem, when dispersion curves of AGW modes are measured using either ambient noise interferometry or signals from a localized source, Eq. (29) can be used to determine unknown environmental parameters such as the wind profile, just like the counterparts of Eq. (29) are utilized in modal tomography in underwater acoustics[44] and surface wave tomography in seismology.[52]

We have demonstrated that the phase speed of each normal mode is a steady, non-increasing function of frequency.



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It is well known that the Lamb wave in an isothermal atmosphere with a rigid boundary is non-dispersive.[1,2] We have shown that this is the only AGW normal mode which is non-dispersive in any finite interval of wave frequencies in a layered waveguide in motionless fluid with either free or rigid boundaries or without boundaries.

The bounds for mode phase and group speeds obtained in this paper are not as detailed or informative as in the acoustic case.[22,24] It remains an open question whether more restrictive bounds can be obtained for generic AGW waveguides.

Atmospheric and oceanic waveguides usually have to be treated as range-dependent or horizontally inhomogeneous when the long-range propagation of AGWs is considered. Much like its acoustic counterpart,[24,32,34,50] the theory of AGW normal modes in layered waveguides, which has been developed in this paper, provides important building blocks for understanding and quantifying horizontal refraction and mode coupling of AGWs in irregular waveguides.

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## APPENDIX: MODE ORTHOGONALITY AND ENERGY FLUX IN A WAVEGUIDE

Various symmetries of wave fields in non-dissipative systems can be derived from the energy conservation law.[19,32,53] Here we apply a version of a reasoning[19,32] previously used in investigations of irregular (range-dependent) acoustic waveguides to derive orthogonality relations of AGW normal modes.

Consider a 2-D waveguide in a layered moving fluid with $u_y = 0$. Normal modes propagate along horizontal coordinate $x$. The waveguide is either unbounded or has horizontal free and/or rigid boundaries. There is no wave energy dissipation. Let the monochromatic AGW field in the waveguide consist of $n$th and $m$th normal modes with amplitudes $\eta_n$ and $\eta_m$, respectively. Then the pressure perturbations and the vertical displacement of fluid particles in the waveguide are

$$p = \eta_m P_m(z)e^{iq_m x} + \eta_n P_n(z)e^{iq_n x},$$
$$w_z = \eta_m W_m(z)e^{iq_m x} + \eta_n W_n(z)e^{iq_n x}. \qquad \text{(A1)}$$

The time dependence $\exp(-i\omega t)$ of the wave field is assumed and suppressed. Below we will see that $q_m$ and $q_n$ are real. As in Eq. (21), $q_m$ and $q_n$ can be positive or negative.

Power flux density[24,26] in AGWs in inhomogeneous, moving fluids is

$$\mathbf{I} = \hat{p}\frac{\partial \hat{\mathbf{w}}}{\partial t} + \rho\mathbf{u}\left(\frac{\partial \hat{\mathbf{w}}}{\partial t}\cdot\frac{d\hat{\mathbf{w}}}{dt}\right), \qquad \text{(A2)}$$

where $d/dt = \partial/\partial t + \mathbf{u}\cdot\nabla$, and $\hat{p}$, $\hat{w}$ stand for the pressure perturbation and oscillatory particle displacement with arbitrary time dependence. For monochromatic waves with complex amplitudes $p$ and $\mathbf{w}$, $\hat{p}(\mathbf{r},t) = \mathrm{Re}[p(\mathbf{r})\exp(-i\omega t)]$ and $\hat{w}(\mathbf{r},t) = \mathrm{Re}[\mathbf{w}(\mathbf{r})\exp(-i\omega t)]$. For the power flux density averaged over the wave period, from Eq. (A2) one finds

$$\bar{\mathbf{I}} = \frac{\omega}{2}\mathrm{Im}\left[p^*\mathbf{w} - \rho\mathbf{u}\left(\mathbf{w}^*\cdot\frac{d\mathbf{w}}{dt}\right)\right], \qquad \text{(A3)}$$

where the asterisk denotes complex conjugation. According to Eq. (A3), there are no time-averaged power fluxes through free and rigid interfaces.[24,26] It follows from Eqs. (12) and (A3) that there are no power fluxes in proper normal modes to $z\to+\infty$ in the case of a waveguide occupying half-space $z > 0$.

Consider time-averaged power flux $D(x_0)$ through a vertical cross-section $x = x_0$ of a waveguide. $D(x_0)$ is given by an integral of $\bar{I}_x$ (A3) over the vertical extent of the waveguide. In the absence of power fluxes through the waveguide's boundaries, wave energy conservation law[24,26] requires that $\partial D/\partial x_0 = 0$. Let a single mode be propagating in the waveguide, e.g., $\eta_n = 0$ and $\eta_m \neq 0$ in Eq. (A1). Then, according to Eqs. (3), (4) and (A3) we have

$$D(x_0) = \frac{|\eta_m^2|}{2\omega}e^{-2x_0\mathrm{Im}q_m}\mathrm{Re}\int_0^{+\infty}\left[\frac{q_m|P_m^2|}{\rho\beta_m^2} + \omega^3\rho u_1\beta_m \times\left(\frac{|q_m P_m|^2}{\omega^4\rho^2|\beta_m^4|} + |W_m^2|\right)\right]dz. \qquad \text{(A4)}$$

For $D$ to be independent of $x_0$ and wave energy to be conserved, there should be $\mathrm{Im}q_m = 0$ for all energy-carrying normal modes. Then, $\beta_m$ and coefficients in Eqs. (5) and (6) for mode shape functions $P_m(z)$ and $W_m(z)$ are real, and the functions $P_m(z)$ and $W_m(z)$ can be chosen to be real-valued.[9,24] Below, we assume that this is the case.

When two normal modes with distinct wave vectors are propagating in the waveguide, from Eqs. (A1) and (A3) we find

$$D(x_0) = |\eta_m|^2 D_{mm} + |\eta_n|^2 D_{nn} + D_{nm}\mathrm{Re}[\eta_m\eta_n^* e^{ix_0(q_m-q_n)}], \qquad \text{(A5)}$$

where

$$D_{nm} = \frac{1}{4\omega}\int_0^{+\infty}\left[\frac{(q_m\beta_n + q_n\beta_m)P_nP_m}{\rho\beta_n^2\beta_m^2} + \omega^3\rho u_1(\beta_n + \beta_m)W_nW_m\right]dz \qquad \text{(A6)}$$

is independent of coordinates. The first two terms in the right-hand side of Eq. (A5) are power fluxes in the $m$th and $n$th normal modes, while the third term is due to inter-mode interference. For $D$ (A5) to be independent of $x_0$ and for wave energy to be conserved, it is necessary that $D_{nm} = 0$ at $n \neq m$. Comparison of Eqs. (19) and (A6) shows that this requirement is exactly the mode orthogonality condition



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derived in Sec. III for AGW normal modes having equal frequency. From the standpoint of wave energy analysis, mode orthogonality reflects the fact that time-averaged power fluxes carried by individual normal modes through each cross section of a waveguide are additive.

Quite similarly, the orthogonality relations for normal modes with the same wave vector and different frequencies can be derived from the requirement that power flux through a waveguide's cross section is independent of time when the AGW field is a superposition of two normal modes.


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