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Wentzel–Kramers–Brillouin approximation for atmospheric waves

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Ray and Wentzel–Kramers–Brillouin (WKB) approximations have long been important tools in understanding and modelling propagation of atmospheric waves. However, contradictory claims regarding the applicability and uniqueness of the WKB approximation persist in the literature. Here, we consider linear acoustic–gravity waves (AGWs) in a layered atmosphere with horizontal winds. A self-consistent version of the WKB approximation is systematically derived from first principles and compared to *ad hoc* approximations proposed earlier. The parameters of the problem are identified that need to be small to ensure the validity of the WKB approximation. Properties of low-order WKB approximations are discussed in some detail. Contrary to the better-studied cases of acoustic waves and internal gravity waves in the Boussinesq approximation, the WKB solution contains the geometric, or Berry, phase. The Berry phase is generally non-negligible for AGWs in a moving atmosphere. In other words, knowledge of the AGW dispersion relation is not sufficient for calculation of the wave phase.

Key words: compressible flows, elastic waves, internal waves

1. Introduction

The Wentzel–Kramers–Brillouin (WKB) approximation, also known as the WKBJ, JWKB, or Liouville–Green approximation (Heading 1962; Frömann & Frömann 1965; Olver 1974; Maslov & Fedoriuk 1981; Fedoryuk 1987) is an important tool in the theoretical analysis of waves in continuously stratified media (Brekhovskikh 1960; Bretherton 1968; Gossard & Hooke 1975; Grimshaw 1975; Ursin 1983; Ostashev 1997; Brekhovskikh & Godin 1998, 1999; Nazarenko, Kevlahan & Dubrulle 1999). Generally speaking, it is applicable provided that the spatial scale of variations of the propagation medium parameters are large compared to the spatial scales of variation of the wavefield. In this respect, the WKB approximation is very similar to the application of the ray theory to layered media. In this paper, the WKB method is understood to be an asymptotic technique for solving ordinary differential equations, specifically, one-dimensional wave equations.

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We are interested in the application of the WKB method to linear acoustic–gravity waves (AGWs) in fluids, especially to waves in the Earth's atmosphere. AGWs are mechanical waves in compressible fluids in a gravity field. They encompass gravity waves and infrasound. These waves play an important role in the dynamics of stars and planetary atmospheres (Duvall *et al.* 1993; Podesta 2005; Fouchet *et al.* 2008), and have been extensively studied in meteorology, climate research, and space physics contexts (Gossard & Hooke 1975; Hargreaves & Gadsden 1992; Fritts & Alexander 2003). Several developments have motivated a renewed interest in AGW theory. Ground-breaking observational techniques (Nishida, Kobayashi & Fukao 2013; Garcia *et al.* 2014) have provided new insights into atmospheric waves. AGWs couple wave processes in the solid earth and the ocean with those in the ionosphere and thermosphere (Watada 2009; Godin & Fuks 2012; Ardhuin & Herbers 2013; Godin, Zabotin & Bullett 2015) and, thus, underlie radar observations of earthquakes (Maruyama *et al.* 2012; Astafyeva *et al.* 2013) and satellite detection of tsunamis (Makela *et al.* 2011; Occhipinti *et al.* 2013; Garcia *et al.* 2014; Coïsson *et al.* 2015). Vertical transport of horizontal momentum by atmospheric waves plays a crucial role in large-scale circulation of the middle and upper atmosphere, and the development of climate models requires vastly improved parameterizations of the momentum flux (Vadas & Liu 2009; Geller *et al.* 2013; Jia *et al.* 2014 and Schirber *et al.* 2014). On the other hand, accurate, data-assimilating atmospheric models are becoming available (Liu *et al.* 2010; Fuller-Rowell *et al.* 2010; Akmaev 2011) that provide a sufficiently detailed description of the physical parameters of the background atmosphere to enable quantitative comparison of theoretical predictions of wavefields to AGW observations in the middle and upper atmosphere.

The wave equation for linear AGWs is more involved than the acoustic wave equation (Brekhovskikh & Godin 1998, 1999) or the wave equation for gravity waves in incompressible fluids in the Boussinesq approximation (Lighthill 1978; Gill 1982; Miropol'sky 2001). The difference in structure of the wave equation for AGWs from wave equations for sound and gravity waves has important implications for construction of the WKB and ray approximations. Care needs to be exercised in introducing the large parameter, which underlies an asymptotic analysis of atmospheric waves, in a way that is consistent with the physics of the atmosphere.

Various prescriptions for wavefield calculations, all claiming to represent the WKB approximation, can be found in the literature on atmospheric waves. Moreover, it is often claimed that even positions of turning points are not uniquely defined and are somehow dependent on the chosen 'flavour' of the WKB approximation (Pitteway & Hines 1965; Einaudi & Hines 1970, 1974; Gossard & Hooke 1975; Fritts & Alexander 2003). This unnecessary confusion makes applicability of the WKB approximation to atmospheric waves questionable and its predictions uncertain. Here, we derive the WKB approximation as an asymptotic solution of the wave equation and demonstrate that various mathematically legitimate approaches lead to the same WKB approximation, which is markedly different from the *ad hoc* solutions presented in Pitteway & Hines (1965), Einaudi & Hines (1970, 1974), Gossard & Hooke (1975), Jones & Georges (1976) and Fritts & Alexander (2003).

The paper is organized as follows. Governing equations for AGWs are formulated in § 2. The WKB approximation is derived for atmospheric waves in § 3.1 and in appendix A using two alternative forms of the wave equation. Asymptotic results are compared to *ad hoc* solutions, and properties of the geometric phase of AGWs, which is missing from the *ad hoc* solutions, are discussed in § 3.2. The conditions of validity of the WKB approximation are considered in § 3.3. The transition of the AGW asymptotics into those for gravity waves in incompressible fluids and for sound is addressed in § 3.4. The gravity waves are further considered in appendix B, where we also discuss an alternative way of introducing the large parameter in the AGW problem and demonstrate the significance of the physics-based scaling. In § 4, we show that AGW energy and wave action are conserved exactly in the first WKB approximation. In § 5, the asymptotic solutions are compared to known exact solutions of the wave equation, and it is shown how new exact analytic solutions can the obtained as special cases of the WKB series. Several examples of exact analytic solutions for waves in both a moving and a quiescent atmosphere are derived in appendix C. Section 6 summarizes our findings.

2. Governing equations

We consider linear waves in a compressible fluid in a uniform gravity field with acceleration $\boldsymbol{g}$. In a Cartesian coordinate system with horizontal coordinates x and y and a vertical coordinate z increasing upward, background, i.e. unperturbed by waves, pressure p_0, density ρ, sound speed c and flow velocity $\boldsymbol{u}$ in the fluid are smooth functions of z; the background flow is horizontal: $\boldsymbol{u} = (u_x, u_y, 0)$; and $\boldsymbol{g} = (0, 0, -g)$. The fluid is stationary (i.e. its parameters are independent of time t) in the absence of waves. The background pressure and density are related by the hydrostatic equilibrium equation

$$\mathrm{d}p_0/\mathrm{d}z = -\rho g, \tag{2.1}$$

while $c(z)$ and $\boldsymbol{u}(z)$ can be arbitrary. We neglect the Earth's rotation and curvature, which makes the theory inapplicable to atmospheric waves with periods longer than a few hours. Wave propagation is assumed to be an adiabatic thermodynamic process; we disregard irreversible processes associated with viscosity, thermal conductivity, and diffusion. For a discussion of AGW absorption that results from the irreversible processes, see Vadas & Nicolls (2012) and Godin (2014*b*) and references therein.

In layered media, wavefields can be represented as a superposition of continuous waves with harmonic dependence $\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r} - \mathrm{i}\omega t)$ on horizontal coordinates and time, which are sometimes referred to as quasi-plane waves (Brekhovskikh & Godin 1998). Here, $\boldsymbol{r} = (x, y, z)$ is the position vector, and $\boldsymbol{k} = (k_x, k_y, 0)$ and ω are the horizontal wavevector and wave frequency. Below, the dependence $\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r} - \mathrm{i}\omega t)$ of the wave-induced pressure p and fluid velocity $\boldsymbol{v}$ perturbations, vertical displacement of fluid parcels w, etc. on horizontal coordinates and time is assumed and suppressed.

In quasi-plane waves, the vertical dependences of p and w satisfy a set of two first-order ordinary differential equations,

$$\frac{\mathrm{d}p}{\mathrm{d}z} + \frac{g}{c^2}p = \rho(\omega_d^2 - N^2)w, \tag{2.2}$$

$$\frac{\mathrm{d}w}{\mathrm{d}z} - \frac{g}{c^2}w = \left(\frac{k^2}{\omega_d^2} - \frac{1}{c^2}\right)\frac{p}{\rho}, \tag{2.3}$$

where the Doppler-shifted, or intrinsic, wave frequency ω_d and the buoyancy frequency N are given by

$$\omega_d = \omega - \boldsymbol{k}\cdot\boldsymbol{u}, \quad N^2 = -g^2c^{-2} - g\rho^{-1}\,\mathrm{d}\rho/\mathrm{d}z. \tag{2.4a,b}$$

The intrinsic frequency has the meaning of the wave frequency in the reference frame uniformly moving with the local background flow; like the flow velocity, ω_d is a function of z. In terms of p and w, fluid velocity perturbation is given by the equation

$$\boldsymbol{v} = \frac{\boldsymbol{k}p}{\omega_d \rho} - w\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}z} - \mathrm{i}\omega_d w\,\hat{\boldsymbol{z}}. \tag{2.5}$$

Equations (2.2), (2.3), and (2.5) are obtained from the Euler, continuity, and state equations linearized with respect to wave amplitude (see e.g. Godin 2012*b*). Similar governing equations can be found in Pierce (1965), Pitteway & Hines (1965), Bretherton (1969) and Tatarskiy (1979).

The quantity

$$\tilde{p} = p - \rho g w \tag{2.6}$$

has the meaning of the Lagrangian pressure perturbation, i.e. the wave-induced pressure perturbation in a moving fluid parcel (Lamb 1932), as opposed to the (Eulerian) pressure perturbation p at a fixed point in space. Using $\tilde{p}$ as an unknown function instead of p, (2.2) and (2.3) become

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}z} + \frac{gk^2}{\omega_d^2}\tilde{p} = \rho\left(\omega_d^2 - \frac{g^2k^2}{\omega_d^2}\right)w, \tag{2.7}$$

$$\frac{\mathrm{d}w}{\mathrm{d}z} - \frac{gk^2}{\omega_d^2}w = \left(\frac{k^2}{\omega_d^2} - \frac{1}{c^2}\right)\frac{\tilde{p}}{\rho}. \tag{2.8}$$

Note that all the effects of the background flow are described though the intrinsic frequency ω_d. In both representations, (2.2), (2.3) and (2.7), (2.8), the governing equations in moving fluids differ from the respective equations in quiescent fluids only by substitution of ω_d for ω.

By eliminating one of the dependent variables from either (2.2) and (2.3) or (2.7) and (2.8), one readily obtains closed-form wave equations for various characteristics of AGW fields:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\omega_d^2\,\mathrm{d}\tilde{p}/\mathrm{d}z}{\rho(\omega_d^4 - g^2k^2)}\right] + \left[\frac{1}{c^2} - \frac{k^2\omega_d^2}{\omega_d^4 - g^2k^2} + g\rho\frac{\mathrm{d}}{\mathrm{d}z}\frac{k^2}{\rho(\omega_d^4 - g^2k^2)}\right]\frac{\tilde{p}}{\rho} = 0, \tag{2.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\mathrm{d}p/\mathrm{d}z}{\rho(\omega_d^2 - N^2)}\right] + \left[\frac{\omega_d^2}{c^2(\omega_d^2 - N^2)} - \frac{k^2}{\omega_d^2} + \frac{\mathrm{d}}{\mathrm{d}z}\frac{g}{c^2(\omega_d^2 - N^2)}\right]\frac{p}{\rho} = 0, \tag{2.10}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\rho\omega_d^2\,\mathrm{d}w/\mathrm{d}z}{\omega_d^2c^{-2} - k^2}\right] + \left[\omega_d^2 - \frac{g^2k^2}{\omega_d^2 - k^2c^2} - \frac{g}{\rho}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\rho k^2}{\omega_d^2c^{-2} - k^2}\right)\right]\rho w = 0. \tag{2.11}$$

A simple relation $\rho c^2 \nabla\cdot\boldsymbol{v} = \mathrm{i}\omega_d\tilde{p}$ between the Lagrangian pressure perturbation and velocity divergence follows from (2.5), (2.6), and (2.8). Using this relation, it is easy to see that, at $\boldsymbol{u} \equiv 0$, (2.9) reduces to Lamb's (1932) wave equation, where $\nabla\cdot\boldsymbol{v}$ is the dependent variable. An inspection shows that one-dimensional wave equations (2.9)–(2.11) are consistent with more general, three-dimensional wave equations for AGWs in moving fluids (Godin 1987; Ostashev 1987, 1997).

All the wave equations (2.9)–(2.11) are linear second-order ordinary differential equations. Their coefficients are rather different, however, leading to distinct, inconsistent 'WKB approximations', when *ad hoc* techniques (Einaudi & Hines 1970, 1974; Gossard & Hooke 1975) are applied to different wave equations, see § 3.2. In the next section, we show that this problem is eliminated when a systematic asymptotic approach is used to solve the wave equations for AGWs.

3. Derivation of the WKB approximation

From the equation of state of an ideal gas and (2.1) it follows (Lamb 1932; Pierce 1965) that

$$p_0(z) = p_0(0)\exp\left(-\int_0^z \frac{\mathrm{d}z_1}{h(z_1)}\right), \quad \rho(z) = \rho(0)\frac{h(0)}{h(z)}\exp\left(-\int_0^z \frac{\mathrm{d}z_1}{h(z_1)}\right), \tag{3.1a,b}$$

$$c^2 = \gamma p_0/\rho = \gamma RT/\mu, \quad h = c^2/\gamma g. \tag{3.2a,b}$$

Here T, R, μ, and γ are the absolute temperature of the gas, the universal gas constant, the molecular weight, and the ratio of specific heats at constant pressure and constant volume; h is the scale height of the atmosphere. For diatomic ideal gases, $\gamma = 1.4$.

We now make the key assumption that certain parameters of the propagation medium vary gradually and slowly with height z. It is important to note that the spatial scales of variation of the unperturbed density and sound speed are generally distinct. Below, we assume that the spatial scale L of variation of the flow velocity as well as of the gas temperature and composition (and, therefore, of h and c) is large. No restrictions are placed on h. The scale L can be large or small compared to the density scale height h. For instance, L is infinitely large in the isothermal atmosphere of constant composition, while h is a constant ∼8 km.

A convenient way to formalize the assumption that h, $\boldsymbol{u}$, and c have a spatial scale L is to consider these as arbitrary smooth functions of $\zeta = z/L$. We want to find an asymptotic solution for the AGW wavefield that is valid at large L (formally, at $L \to \infty$). To do so, we follow the approach described by Brekhovskikh & Godin (1998) for acoustic waves.

3.1. *Derivation based on the wave equation for Lagrangian pressure perturbations*

We search for a solution to (2.9) in the form

$$\tilde{p}(z) = \tilde{p}(0)\exp\left(\mathrm{i}L\int_0^\zeta \varphi(\zeta_1, L)\,\mathrm{d}\zeta_1\right), \quad \zeta = z/L, \tag{3.3}$$

where φ is an unknown function of the dimensionless variable ζ and the parameter L. In a fluid with the background density profile (3.1), substitution of (3.3) into the wave equation (2.9) gives a nonlinear first-order ordinary differential equation (more specifically, a Riccati equation):

$$\frac{\mathrm{i}}{L}\frac{\mathrm{d}\varphi}{\mathrm{d}\zeta} + \mathrm{i}\varphi\left[\frac{1}{h} + \frac{1}{L}\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln\left(\frac{\omega_d^2 h}{\omega_d^4 - g^2k^2}\right)\right] = \varphi^2 - m^2 - \frac{1}{4h^2} - \frac{gk^2}{L\omega_d^2}\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln\left(\frac{h}{\omega_d^4 - g^2k^2}\right), \tag{3.4}$$

where

$$m^2 = \frac{\omega_d^2}{c^2} - k^2 - \frac{1}{4h^2} + \frac{gk^2}{\omega_d^2}\left(\frac{1}{h} - \frac{g}{c^2}\right). \tag{3.5}$$

Note that $m(\zeta) = O(1)$ and does not contain terms that are small or large with respect to the parameter L. For definiteness, it will be implied that $m = |m|$, when $m^2 \geqslant 0$, and $m = \mathrm{i}|m|$, when $m^2 < 0$.

We represent the unknown function φ in terms of a power series:

$$\varphi(\zeta, L) = f + \mathrm{i}/2h, \quad f = \sum_{n=0}^{\infty} f_n(\zeta)L^{-n} \tag{3.6}$$

in the small parameter of the problem, L^{-1}, and note that

$$f^2=\sum_{n=0}^{\infty}L^{-n}\sum_{l=0}^{n}f_{n-l}f_l=f_0^2+\sum_{n=1}^{\infty}L^{-n}\left(2f_0f_n+\sum_{l=1}^{n-1}f_{n-l}f_l\right). \tag{3.7}$$

By substituting (3.6) and (3.7) into the Riccati equation (3.4), and equating terms of the same order in L^{-1}, we find

$$f_0=\pm m, \tag{3.8}$$

$$f_1=\frac{\mathrm{i}}{2}\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln\left(\frac{\omega_d^2hf_0}{\omega_d^4-g^2k^2}\right)+\frac{gk^2}{2f_0\omega_d^2}\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln\left(\frac{h}{\omega_d^4-g^2k^2}\right)+\frac{1}{4f_0h}\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln\left(\frac{\omega_d^4-g^2k^2}{\omega_d^2}\right), \tag{3.9}$$

$$f_n=\frac{1}{2f_0}\left[\mathrm{i}f_{n-1}\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln\left(\frac{\omega_d^2hf_{n-1}}{\omega_d^4-g^2k^2}\right)-\sum_{l=1}^{n-1}f_lf_{n-l}\right],\quad n\geqslant 2. \tag{3.10}$$

This is an exact solution of the wave equation as long as $m\neq 0$, $\omega_d\neq 0$, and $\omega_d^2\neq gk$. From (3.5) and the identity

$$m^2+\left(\frac{1}{2h}-\frac{gk^2}{\omega_d^2}\right)^2=\left(1-\frac{g^2k^2}{\omega_d^4}\right)\left(\frac{\omega_d^2}{c^2}-k^2\right) \tag{3.11}$$

it follows that all f_n are finite when $0<m^2<\infty$.

When $m\neq 0$, the solutions (3.3) corresponding to the upper and lower signs in (3.8) are linearly independent, and (3.6), (3.8)–(3.10) define the full asymptotic solution of the problem. This is the WKB asymptotic solution for AGWs. The solution breaks down in the vicinity of a point or points where $m=0$. Such points are referred to as turning points (Brekhovskikh & Godin 1998).

WKB approximations of various order are obtained by retaining a finite number of terms in the series (3.6). In the first WKB approximation, one retains only terms f_0 and f_1. Then, (3.6), (3.8) and (3.9) give two linearly independent approximate solutions of the wave equation:

$$\tilde{p}(z)=\tilde{p}(0)\sqrt{\frac{\rho m(0)(\omega_d^2-g^2k^2/\omega_d^2)}{\rho(0)m[\omega_d^2(0)-g^2k^2/\omega_d^2(0)]}}\exp\left(\pm\mathrm{i}\left(\int_0^z m\,\mathrm{d}z_1+\chi\right)\right), \tag{3.12}$$

$$\chi=\int_0^z\frac{\mathrm{d}z_1}{2m}\left[\frac{gk^2}{\omega_d^2}\frac{\mathrm{d}}{\mathrm{d}z_1}\ln\left(\frac{h}{\omega_d^4-g^2k^2}\right)+\frac{1}{2h}\frac{\mathrm{d}}{\mathrm{d}z_1}\ln\left(\frac{\omega_d^4-g^2k^2}{\omega_d^2}\right)\right]. \tag{3.13}$$

Here, we have returned to integration over dimensional height z, and the scale L no longer enters the solution explicitly. The first WKB approximation gives the solution to the wave equation with accuracy up to the factor $1+O(L^{-1})$. To the same accuracy, from (2.6), (2.7), and (3.12) we get

$$w(z)=\frac{gk^2\omega_d^{-2}\pm\mathrm{i}m-1/2h}{(\omega_d^2-g^2k^2\omega_d^{-2})\rho}\tilde{p}(z), \tag{3.14}$$

$$p(z)=\frac{\omega_d^2\pm\mathrm{i}mg-g/2h}{\omega_d^2-g^2k^2\omega_d^{-2}}\tilde{p}(z) \tag{3.15}$$

for the vertical displacement and the Eulerian pressure perturbation, respectively. Equations (3.14) and (3.15) as well as their corollary

$$w(z) = \frac{gk^2\omega_d^{-2} \pm \mathrm{i}m - 1/2h}{(\omega_d^2 \pm \mathrm{i}mg - g/2h)\rho} p(z) \tag{3.16}$$

can be viewed as the WKB polarization relations. These do not contain derivatives of any environmental parameters and coincide with the polarization relations for plane AGWs in a uniformly moving fluid with height-independent c and h (e.g. Godin & Fuks 2012). The WKB polarization relations involving the oscillatory velocity $\boldsymbol{v}$ follow readily from (2.5) and (3.14)–(3.16).

When $m^2 > 0$, the asymptotic solutions that are obtained by choosing either the upper or lower sign in (3.12)–(3.16) have the meaning of obliquely propagating locally plane waves; the waves propagate vertically in opposite directions in the particular case $k = 0$. When $m^2 < 0$, the two asymptotic solutions describe exponentially growing with height and exponentially decreasing with height (evanescent) locally plane waves that propagate horizontally. Note that in each of the locally plane waves, different physical quantities p/ρ, w, $\tilde{p}/\rho$, $\nabla \cdot \boldsymbol{v}$, etc. share the same exponential factor and differ by the factor before the exponential. These pre-exponential factors describe polarization relations for AGWs. Also, p, w, $\tilde{p}$, and $\boldsymbol{v}$ diverge in the vicinity of the turning points where $m \to 0$. It should be emphasized that, in contrast to what is often claimed in the literature (Einaudi & Hines 1970, 1974; Gossard & Hooke 1975; Fritts & Alexander 2003) but in agreement with physical expectations, the wave turning point is the same whether it is defined using p, w, $\tilde{p}$, or $\boldsymbol{v}$.

In addition to the turning points, where $m^2 = 0$, the WKB approximation may diverge and become inapplicable in the vicinity of points where either

$$\omega_d = 0 \tag{3.17}$$

or

$$\omega_d^2 = kg. \tag{3.18}$$

Neither of these conditions can be met when $k = 0$ (vertical propagation). When (3.17) holds, we have $\omega = \boldsymbol{k} \cdot \boldsymbol{u}$, and the intrinsic frequency vanishes. Such points are referred to as points of wave-flow synchronism (Brekhovskikh & Godin 1998) or critical levels (Gossard & Hooke 1975), and are singular points of the WKB approximation for sound waves (i.e. at $g = 0$) as well. However, the physics of resonance interaction with flow in the vicinity of points of wave–flow synchronism is distinct for sound and AGWs. For sound, $m^2 < 0$ when the condition (3.17) is met. Unlike sound, for AGWs in a stably stratified fluid, wave–flow synchronism occurs for propagating waves ($m^2 > 0$, $m \to \infty$) (Gossard & Hooke 1975) rather than waves that are evanescent in a vertical direction.

The WKB solutions are inapplicable in the vicinities of the turning points and points of wave-flow synchronism; the vertical extent of the vicinities depends on the singularity type but is small compared to the spatial scale L (see e.g. chapter 9 in Brekhovskikh & Godin 1998 where acoustic waves are considered, and references therein). The applicability of the WKB approximation for AGWs in the vicinity of a turning point is discussed in more detail in § 3.3.

Apparent singularities of the type (3.18) are specific to AGWs and can occur only in fluids with inhomogeneous background flow. It follows from (3.5) that $m^2 < 0$ and waves are evanescent when (3.18) holds. Condition (3.18) coincides with the dispersion equation of incompressible wave motion (Godin 2012*a*, 2014*a*, 2015) in uniformly moving fluids and is related to the existence of non-trivial wave solutions, in which Lagrangian pressure perturbations are identically zero. As discussed in appendix A, the WKB solutions are not necessarily singular when the condition (3.18) is met.

3.2. *Comparison of the asymptotic and ad hoc solutions. Geometric phase*

In the literature on atmospheric waves, it is often argued (e.g. Einaudi & Hines 1970, 1974; Jones & Georges 1976) that the WKB approximation can be derived by applying the following prescription. Any linear second-order ordinary differential equation, which governs AGWs with harmonic dependence $\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}-\mathrm{i}\omega t)$ on the horizontal coordinates and time, is first reduced to its normal form,

$$\mathrm{d}^2\psi/\mathrm{d}z^2+q^2\psi=0. \tag{3.19}$$

Then, the function $q^2(z,\boldsymbol{k},\omega)$ is interpreted as the vertical wavenumber squared, $m^2=q^2(z,\boldsymbol{k},\omega)$ is interpreted as the dispersion equation of AGWs with the wavevector $(\boldsymbol{k},m)$, and the functions

$$\psi_{1,2}(z)=\text{const.}\ q^{-1/2}\exp\left(\pm\mathrm{i}\int_0^z q\,\mathrm{d}z_1\right) \tag{3.20}$$

are claimed to be the solutions of the wave equation in the WKB approximation (Einaudi & Hines 1970, 1974; Jones & Georges 1976). Such an interpretation of the WKB approximation is rather widely accepted in the atmospheric wave community (Gossard & Hooke 1975; Fritts & Alexander 2003). To distinguish the functions (3.20) from the asymptotic solutions considered in § 3.1 and appendix A, we refer to the functions $\psi_{1,2}$ as *ad hoc* solutions.

The *ad hoc* solutions coincide with the first WKB approximation for solutions of the time-independent non-relativistic Schrödinger equation for a particle in a potential field (e.g. Fedoryuk 1987), for horizontally polarized electromagnetic waves (e.g. Brekhovskikh 1960), and for acoustic waves in fluids with a constant density (e.g. Brekhovskikh & Godin 1998) but, as has been mentioned already, are problematic when applied to AGWs. For atmospheric waves, the function q^2 in (3.19) depends on the choice of the dependent variable ψ. For instance, for the choices $\psi(z)=[(k^2c^2-\omega^2)\rho h]^{1/2}w(z)$ and $\psi(z)=kc[(N^2-\omega^2)\rho h]^{-1/2}p(z)$ one obtains (Einaudi & Hines 1970, 1974)

$$q_w^2=m^2+\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{2h}+\frac{gk^2}{2\omega^2}\ln\frac{c^{2\gamma}}{(c^2k^2-\omega^2)}\right]-\frac{3}{4}\left[\frac{\mathrm{d}}{\mathrm{d}z}\ln(c^2k^2-\omega^2)\right]^2+\frac{\mathrm{d}^2(c^2k^2-\omega^2)/\mathrm{d}z^2}{2(c^2k^2-\omega^2)} \tag{3.21}$$

and

$$\begin{aligned}q_p^2 &= m^2+g\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{c^2}+\frac{k^2}{\omega^2}\ln c^2-\frac{2+\gamma}{2c^2}\ln(N^2-\omega^2)-\frac{\gamma(\gamma-1)g^2}{2\omega^2c^4}\right]\\ &\quad+\left[\frac{k}{g}\frac{\mathrm{d}c^2}{\mathrm{d}z}+\frac{\gamma(\gamma-1)g^3}{\omega^2c^4}+\frac{\mathrm{d}}{\mathrm{d}z}\right]\frac{\mathrm{d}}{\mathrm{d}z}\ln(N^2-\omega^2)+\frac{1}{4}\left[\frac{\mathrm{d}}{\mathrm{d}z}\ln c^2(N^2-\omega^2)\right]^2\\ &\quad-\frac{\gamma g^2}{\omega^2c^2}\left[2\frac{\mathrm{d}^2\ln c^2}{\mathrm{d}z^2}+\left(\frac{\mathrm{d}\ln c^2}{\mathrm{d}z}\right)^2\right]-\frac{1}{2c^2}\frac{\mathrm{d}^2c^2}{\mathrm{d}z^2},\end{aligned} \tag{3.22}$$

respectively. The atmosphere was assumed to be quiescent by Einaudi & Hines (1970, 1974). Generally, functions q^2 for various choices of ψ in (3.19) differ from m^2, which is defined by (3.5), and from each other by terms $O(L^{-1})$. Equations (3.21) and (3.22) illustrate the fact that the *ad hoc* approach (Einaudi & Hines 1970, 1974; Gossard & Hooke 1975; Jones & Georges 1976; Fritts & Alexander 2003) gives different dispersion equations of AGWs and different positions of turning points, where the *ad hoc* solutions (3.20) are singular, for different choices of the dependent variable ψ.

The actual WKB solutions for AGWs generally do not have the form (3.20) regardless of the choice of the function $q(z)$, see (3.12) and (A 7). In addition to the phase integral (eikonal)

$$S(z) = \int_0^z m(z_1)\, \mathrm{d}z_1, \tag{3.23}$$

the exponent in the solution (3.12) for $\tilde{p}$ in the first WKB approximation contains the term $\pm \mathrm{i}\chi$. When AGWs are propagating waves, i.e. $m^2 > 0$, between heights 0 and z, $\chi(z)$ is real-valued, and $\pm(S+\chi)$ has the meaning of the phase of two linearly independent solutions. Both $S(z)$ and $\chi(z)$ are purely imaginary, when $m^2 < 0$ between heights 0 and z, see (3.13). Turning points, where $m = 0$, generally do not lead to singularities in $\chi(z)$. Unlike S given by (3.23), the integrand in (3.13) for χ contains first derivatives of the environmental parameters, namely, h and $\boldsymbol{u}$. For wave propagation over distances $O(L)$ or larger, the increments of χ are $O(1)$, and retaining $\chi(z)$ in the asymptotic solution (3.12) is necessary to approximate exact solutions to the wave equation.

An additional phase term, which is quite analogous to χ, is present also in the first WKB approximation for w; it is given by the second and third terms in the square brackets in the integrand in (A 7). This additional phase term has properties quite similar to those of χ but looks somewhat different because of the phase shift between $\tilde{p}$ and w, which is obvious in the polarization relation (3.14).

The additional phase terms in (3.12) and (A 7) are but a manifestation of a much more general phenomenon. Analogous phase terms arise in wavefunctions describing adiabatic transitions in quantum systems, have important implications in numerous physical problems, and are usually referred to as the geometric phase or Berry phase (Berry 1984; Shapere & Wilczek 1989; Bohm *et al.* 2003; Berry 2010). The geometric phase arises also in the WKB-type approximations for various types of waves, including waves in fluids and solids (Babich 1961; Karal & Keller 1964; Bretherton 1968; Berry 1990; Tromp & Dahlen 1992; Babich & Kiselev 2004). In a study that was limited to the first WKB approximation, an expression for the geometric phase of atmospheric waves, which is equivalent to (3.13), was previously derived by Budden & Smith (1976). They referred to the geometric phase as the 'additional memory'.

For AGWs, the geometric phase χ, (3.13), vanishes for vertically propagating waves, i.e. at $k = 0$. When $k \neq 0$, but there is no wind and $\gamma = \mathrm{const.}$, the quantity under the integral in (3.13) is a full differential, and

$$\chi(z) = \frac{gk}{2\omega^2}\left[\arcsin\left(\frac{2kh(z)-b}{\sqrt{b^2-1}}\right) - \arcsin\left(\frac{2kh(0)-b}{\sqrt{b^2-1}}\right)\right], \quad b = \frac{(\gamma-1)gk}{\gamma\omega^2} + \frac{\omega^2}{\gamma gk}. \tag{3.24}$$

Then, the geometric phase increment $\chi(z_3) - \chi(z_2)$ can be expressed in terms of the environmental parameters at just the two heights, z_2 and z_3. However, when there is a variable wind $\boldsymbol{u}(z)$ and/or γ is not constant due to a continuous change in the composition of the atmosphere, the integrand in (3.13) is not a full differential, and

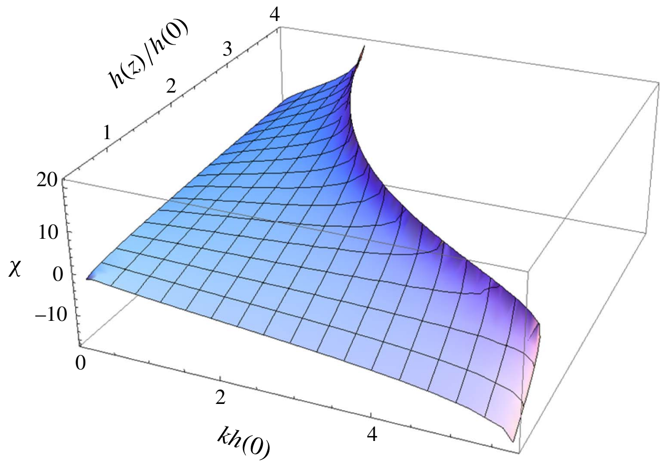

FIGURE 1. (Colour online) Geometric phase of AGWs in a quiescent atmosphere. The geometric phase $\chi(z)$ is shown in radians as a function of dimensionless parameters $h(z)/h(0)=c^2(z)/c^2(0)$ and $kh(0)$ in an atmosphere with the ratio $\gamma=1.4$ of specific heats at constant pressure and constant volume; $kg/\omega^2=20$.

the geometric phase increment depends on the values of the environmental parameters c, h, and $\boldsymbol{u}$ at all heights between z_2 and z_3. In this respect, the geometric phase differs from the wave amplitude, which is given by the factor in front of the exponential in (3.12) and depends only on the local values of the environmental parameters. When a wave is launched from a height z_2, is reflected at a turning point, and returns to the height z_2, the wave amplitude is unchanged, but both the eikonal and the geometric phase gain finite increments. While the wave amplitude can be found from the energy conservation law (see § 4) and knowledge of the dispersion relation is sufficient to calculate the eikonal, an asymptotic analysis has been necessary to derive (3.13) for the geometric phase and, therefore, to calculate the AGW phase at oblique propagation.

As a rule, the AGW field cannot be approximated without taking the geometric phase into account. Figure 1 illustrates the significance of the geometric phase χ, (3.24), in the simple case of a quiescent atmosphere with a constant γ. In figure 1, the dimensionless parameter $h(z)/h(0)$ can be interpreted as the ratio of absolute temperatures at heights z and $z=0$. The range of sound speeds in the figure, from $c(0)/1.5$ to $2c(0)$, is close to their range in the real atmosphere; χ is shown for propagating waves, for which $m^2(z)>0$ and $m^2(0)>0$. For propagation between heights with sufficiently different sound speeds (or, equivalently, air temperatures), the increment of the geometric phase can be as large as 10π (figure 1).

It follows from the dispersion equation (3.5) that propagating AGWs (i.e. waves with real-valued $\boldsymbol{k}$ and m) exist when either ${\omega_d}^2>\Omega^2$ or ${\omega_d}^2<N_0^2$, where

$$\Omega=\frac{c}{2h}=\frac{\gamma g}{2c},\quad N_0^2=\frac{g}{h}-\frac{g^2}{c^2}=\frac{(\gamma-1)g^2}{c^2}. \tag{3.25a,b}$$

Since $1<\gamma<2$ in ideal gases, we have $0<N_0<\Omega$. Waves with $\omega_d^2>\Omega^2$ and $\omega_d^2<N_0^2$ form the acoustic and buoyancy branches of AGWs, respectively. (For a discussion of the acoustic and buoyancy branches, see Hines 1960 and Gossard & Hooke 1975. In

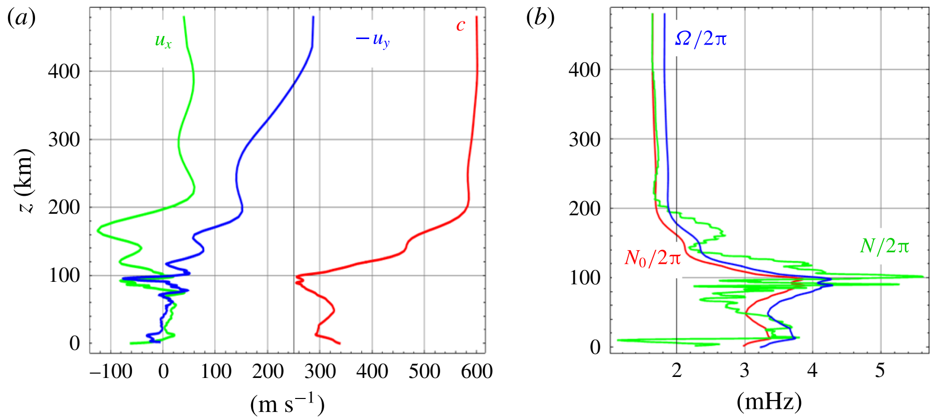

FIGURE 2. (Colour online) Atmospheric stratification. (*a*) An example of the sound speed c and wind velocity profiles predicted by the Whole Atmosphere Model (Fuller-Rowell *et al.* 2008, 2010; Zabotin *et al.* 2014); u_x and u_y are the zonal (from west to east) and meridional (from south to north) components of the wind velocity. (*b*) Atmospheric parameters that constrain the frequencies of propagating AGWs. Buoyancy frequency N, the effective buoyancy frequency N_0, and the acoustic cutoff frequency Ω are defined in the text and are calculated for the sound-speed profile shown in (*a*) assuming that $\gamma = 1.4$ and $g = 9.8$ m s^{-2}.

the literature on atmospheric waves, AGWs on the buoyancy and acoustic branches are sometimes loosely referred to as gravity waves and infrasound, respectively.) Under the conditions of figure 1, $\omega < N_0$, and the results pertain to AGWs on the buoyancy branch.

To illustrate the dependence of the geometric phase on the AGW frequency and horizontal wavevector in the real atmosphere, we use the wind velocity and temperature profiles generated by the Whole Atmosphere Model, or WAM (Fuller-Rowell *et al.* 2008, 2010). The WAM description of the atmosphere above southern Iceland, which is shown in figure 2(*a*), was previously used in modelling long-range propagation of atmospheric waves generated by the 2010 eruption of Eyjafjallajökull volcano (Matoza *et al.* 2011; Zabotin *et al.* 2014). Note that below about 250 km the effective buoyancy frequency N_0, which enters the AGW dispersion relation (3.5), varies much more smoothly with height than the buoyancy frequency N (figure 2*b*). Unlike N, N_0 is always positive and smaller than the acoustic cutoff frequency Ω; N and N_0 are indistinguishable at very high altitudes, where the atmosphere becomes nearly isothermal (figure 2*b*).

In the real atmosphere, the geometric phase χ, (3.13), of AGWs with frequencies $\omega_d < N_0$ can be both positive and negative (figure 3). Its sign is mostly opposite that of the eikonal S, (3.23), at heights $0 < z < 100$ km. Variations of the geometric phase with height can exceed 2π and, because of the wind, strongly depend on the direction of the horizontal wavevector $\boldsymbol{k}$ unless the trace velocity $C = \omega/k$ is large compared to the wind speed. (The trace velocity is the reciprocal of the magnitude of the horizontal slowness $\boldsymbol{k}/\omega$ of the wave and has the meaning of the phase speed of the trace of the wave on the horizontal plane.) More rapid variations of χ are associated with stronger winds and larger temperature and wind velocity gradients (figures 2 and 3).

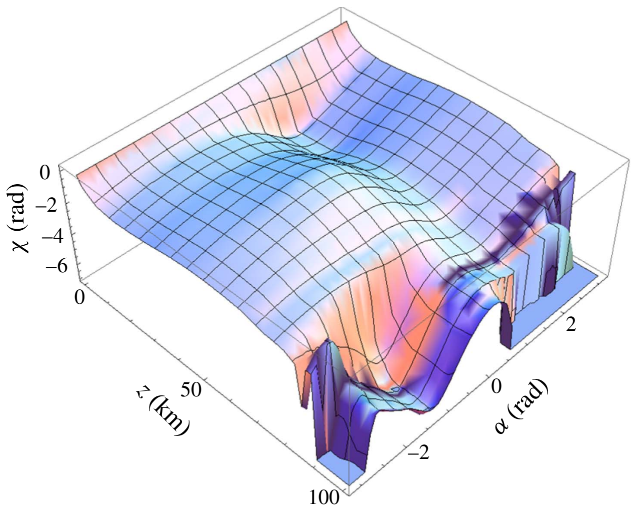

FIGURE 3. (Colour online) Geometric phase at various heights and azimuthal directions of propagation for atmospheric waves on the buoyancy branch of the AGWs. Wave frequency $f = 2$ mHz and horizontal wavevector $\boldsymbol{k} = 2\pi f C^{-1}(\cos\alpha, \sin\alpha, 0)$ with the trace speed $C = 60$ m s^{-1}. Parameters of the atmosphere are shown in figure 2.

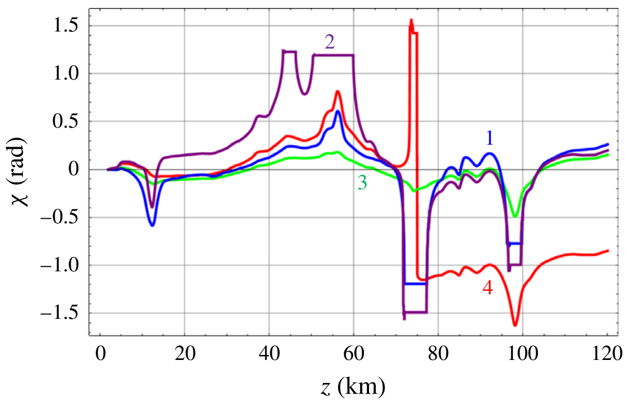

FIGURE 4. (Colour online) Geometric phase as a function of height for waves on the acoustic branch of the AGWs in the atmosphere with parameters shown in figure 2. Wave frequencies $f = \omega/2\pi$ and trace speeds $C = \omega/k$ are 4.5 mHz and 400 m s^{-1} (1); 4.65 mHz and 380 m s^{-1} (2); 5 mHz and 400 m s^{-1} (3); and 5 mHz and 380 m s^{-1} (4). Horizontal wavevector $\boldsymbol{k} = (2\pi f C^{-1}, 0, 0)$.

On the acoustic branch of AGWs, the geometric phase χ tends to zero in the limit of high frequencies, but remains significant when the wave frequency is of the order of the acoustic cutoff frequency (figure 4). The intervals of heights where waves are evanescent, i.e. $m^2 < 0$, contribute to neither χ, (3.13), nor the eikonal S, (3.23), and account for the horizontal segments of lines 1 and 2 in figure 4. The geometric phase is sensitive to small variations in wave frequencies and trace speed, especially when a turning point approaches the heights where gradients of air temperature or wind velocity are large (figure 4).

3.3. *Validity of the WKB solutions when turning points are present*

While the WKB solutions diverge at turning points $z = z_t$, where $m(z_t) = 0$, the solutions remain valid outside a layer $|z - z_t| < d_t$, which proves to be narrow compared to the spatial scale L of c, $\boldsymbol{u}$, and h variations. Consider a turning point $z = z_t$, such that there are no other singular points of the WKB solutions in the vicinity $|z - z_t| \ll L$ of z_t. To evaluate d_t and find the domain of applicability of the first WKB approximation in the vicinity of the isolated turning point, we will follow Brekhovskikh & Godin (1998) and require that the difference between the first and second WKB approximations is small.

In the vicinity $|z - z_t| \ll L$ of the isolated turning point, from (3.5) we have

$$m^2 = \pm\mu^2 \cdot (\zeta - \zeta_t)[1 + O(\zeta - \zeta_t)], \tag{3.26}$$

where $\zeta_t = z_t/L$ and μ^2 is a representative value of $|m^2|$ away from the turning point. Equations (3.9), (3.10), and (3.26) give $f_1 = \mu^2 O(m^{-1})$ and $f_2 = \mu^4 O(m^{-5})$. The second WKB approximation differs from the first approximation (3.12) by the factor

$$E_2(\zeta) = L^{-1} \int_0^{\zeta} f_2(\zeta_1)\, \mathrm{d}\zeta_1, \tag{3.27}$$

see (3.3) and (3.6). Here, f_2 and E_2 diverge, when $\zeta \to \zeta_t$; $E_2 = (\mu L)^{-1} O((\zeta - \zeta_t)^{-3/2})$. For the first WKB approximation to be applicable, it is necessary that $|E_2| \ll 1$ and, therefore, $|\zeta - \zeta_t| \gg (\mu L)^{-2/3}$. In terms of dimensional height z, the applicability condition becomes

$$|z - z_t| \gg (\mu^{-2} L)^{1/3}, \tag{3.28}$$

or $d_t \sim (\mu^{-2} L)^{1/3}$. Thus, the vicinity of the turning point, where the first WKB approximation is invalid, is large compared to the vertical scale μ^{-1} of the wavefield variation away from the turning point but is small compared to the scale L of the c, $\boldsymbol{u}$, and h variations.

Within its domain of applicability, the WKB approximation predicts a large increase of the wave amplitude in the vicinity of the turning point. According to (3.12), (3.15), (3.26), and (3.28), $|\rho^{-1/2}\tilde{p}|$ and $|\rho^{-1/2}p|$ in the vicinity of the turning point are amplified by the large factor $O((\mu L)^{1/3})$ compared to their values away from the turning point on its 'illuminated' side, where $m^2 > 0$ and the AGWs are propagating (as opposed to evanescent) waves.

Let us show that the condition (3.28) applies also to higher-order WKB approximations. Assume that

$$f_n = \mu^{2n} O(m^{1-3n}) \tag{3.29}$$

for $n = 1, 2, \ldots, N - 1$. We have already seen that (3.29) holds at $n = 0$, 1, and 2. Then, it follows from (3.10) that (3.29) is valid for $n = N$. Hence, (3.29) is valid for all natural n. According to (3.3) and (3.6), the nth WKB approximation for $\tilde{p}$ differs from the $(n - 1)$th approximation by the factor $\exp(E_n(\zeta))$, where

$$E_n(\zeta) = L^{1-n} \int_0^{\zeta} f_n(\zeta_1)\, \mathrm{d}\zeta_1. \tag{3.30}$$

When $\zeta \rightarrow \zeta_t$, it follows from (3.26), (3.29), and (3.30) that $E_n = (\mu L)^{1-n} O((\zeta - \zeta_t)^{3(1-n)/2})$. The necessary condition $|E_{n+1}| \ll 1$ of applicability of the nth approximation with $n > 1$ is again given by (3.28). Since (3.28) ensures the smallness of contributions of all higher-order approximations, it is not only a necessary but also a sufficient condition of validity of the first WKB approximation (as an asymptotic solution in the limit $L \rightarrow \infty$). A different approach is necessary to obtain rigorous estimates of the deviation of the first WKB approximation from exact solutions at finite L, see Olver (1974), Fedoryuk (1987), Brekhovskikh & Godin (1998) and references therein.

The condition (3.28) of the WKB approximation validity can be written in a rather intuitive form in terms of the phase integral S, see (3.23). In follows from (3.26) and (3.28) that the WKB approximation is applicable at heights z, where the increment of the phase integral between the current height and the turning point is large compared to unity:

$$|S(z) - S(z_t)| \gg 1. \tag{3.31}$$

The WKB applicability conditions (3.28) and (3.31) for AGWs have the same form as in the case of acoustic waves (Brekhovskikh & Godin 1998, § 8.1). The condition (3.28) has been derived from the asymptotic series (3.3), (3.6), (3.8)–(3.10) for the Lagrangian pressure perturbation $\tilde{p}$. As expected, the same result follows from the series (A 1), (A 3)–(A 6) for the vertical displacement w. The necessary reasoning is essentially unchanged and is based on the observation that, as follows from (A 4)–(A 6) and (3.26), the same estimate (3.29) applies to functions F_n in (A 3) as to functions f_n.

In various *ad hoc* approximations, which were referred to as 'WKB approximations' in Pitteway & Hines (1965), Einaudi & Hines (1970, 1974), Gossard & Hooke (1975), Jones & Georges (1976) and Fritts & Alexander (2003), the value of m^2 differs from our (3.5) by the terms $\mu O(L^{-1})$, which originate from the terms containing derivatives $\mathrm{d}h/\mathrm{d}z$, $\mathrm{d}c/\mathrm{d}z$, and/or $\mathrm{d}\omega_d/\mathrm{d}z$. In particular, an additional term $\mu O(L^{-1})$ results from replacement of $g(h^{-1} - gc^{-2})$ with N^2, the buoyancy frequency squared, in the last term on the right-hand side of (3.5). The difference between the m^2 values arising in various *ad hoc* approximations (Einaudi & Hines 1970, 1974) are also $\mu O(L^{-1})$, see (3.21) and (3.22). The *ad hoc* approximations diverge at their respective turning points. An addition $\mu O(L^{-1})$ to m^2 shifts the turning point height, i.e. the location of a zero of the function $m^2(z)$, by $\mu^{-1}O(L^0)$, see (3.26). Such a shift is much larger than the vicinity $|z - z_t| < d_t$ of the actual turning point, where the WKB approximation is not applicable. To rationalize the difference between various *ad hoc* approximations, Einaudi & Hines (1970, 1974) argued that these approximations differ significantly only when the WKB approximation is not valid. Our analysis of the conditions of the WKB approximation applicability shows that Einaudi & Hines' argument is fallacious.

3.4. *Limiting cases*

In the limit $g \rightarrow 0$, the background fluid density ρ is no longer related to the background pressure p_0 by (2.1); the vertical dependencies of c, ρ, and $\boldsymbol{u}$ can be prescribed independently in a layered fluid with a generic equation of state. In this limit, $p_0 = \text{const.}$ and $h^{-1} \rightarrow 0$. For a fluid with a generic equation of state, equation (3.2) for h does not necessarily apply, and the ratio $h(0)/h(z)$ should be understood as $\rho(z)/\rho(0)$, see (3.1). The only type of mechanical wave supported by ideal compressible fluids in the absence of gravity is sound, i.e. acoustic waves. The difference vanishes between the Lagrangian, $\tilde{p}$, and Eulerian, p, pressure perturbations; wave equations (2.9) and (2.10) reduce to the well-known acoustic wave equation (e.g. Brekhovskikh & Godin 1998, § 1.2).

Asymptotic solution of the wave equation simplifies greatly in the acoustic case because parameters of the medium now have only one scale of spatial variations, L, instead of two distinct scales, L and h, in the case of AGWs. In this limit, (3.5), (3.8)–(3.10), which define the WKB series for AGWs, reduce to

$$m^2 = \frac{\omega_d^2}{c^2} - k^2; \quad f_0 = \pm m; \quad f_n = \frac{1}{2f_0}\left(\mathrm{i} f_{n-1} \frac{\mathrm{d}}{\mathrm{d}\zeta} \ln \frac{f_{n-1}}{\rho \omega_d^2} - \sum_{l=1}^{n-1} f_l f_{n-l} \right), \quad n \geqslant 1, \tag{3.32a–c}$$

and agree with the equations (Brekhovskikh & Godin 1998, § 8.1) which define the WKB series for acoustic waves and were derived by different means. In particular, in the first WKB approximation (3.12) and (3.13) give $\chi \equiv 0$ and

$$p(z) = p(0)\sqrt{\rho \omega_d^2 m(0)/\rho(0)\omega_d^2(0) m}\, \exp\left(\pm \mathrm{i} \int_0^z m \, \mathrm{d}z_1 \right) \tag{3.33}$$

in agreement with the acoustic result (Brekhovskikh & Godin 1998, § 8.2). Obviously, acoustic waves have no Berry phase. This is consistent with the general equation (3.13) for the Berry phase χ, where the integrand on the right-hand side vanishes in the acoustic limit. When $\boldsymbol{k}/\omega$ is independent of frequency, m/ω is also independent of frequency according to (3.32). Then, the amplitude and phase of propagating sound waves (3.33) are, respectively, independent of and proportional to ω. These properties reflect the well-known fact that sound, unlike AGWs, propagates without dispersion in fluids with gradually varying parameters (e.g. Brekhovskikh 1960; Brekhovskikh & Godin 1998, 1999).

In the *ad hoc* approach (Einaudi & Hines 1970, 1974; Gossard & Hooke 1975; Jones & Georges 1976; Fritts & Alexander 2003) it is implied that the WKB solutions have the form (3.20) with some function $q(z)$. Equation (3.33) shows that this assumption does not hold even for sound when the fluid is moving or its density varies with z. It follows from (3.21) and (3.22) that the functions $q_p(z)/\omega$ and $q_w(z)/\omega$ retain a dependence on frequency in the acoustic limit. Thus, the unjustified assumption, equation (3.20), about the form of the WKB solutions in the *ad hoc* approach leads to the unphysical prediction that sound waves are dispersive.

In the opposite limit, where gravity is the dominant restoring force and compressibility of the fluid is negligible, the WKB approach is often applied to study internal gravity waves in the Boussinesq approximation (Garrett 1968; Gill 1982, chapter 8; Miropol'sky 2001, chapter 3). For the Boussinesq approximation to be justified, relative changes of the background density $\rho(z)$ need to be small compared to unity. Therefore, this version of the WKB approximation is not relevant to the atmospheric waves unless the range of heights considered is limited to be within a fraction of the scale height h. A version of the WKB approach, which is suitable for internal gravity waves in incompressible fluids and does not rely on the Boussinesq approximation, is considered in appendix B.

4. Conservation of wave energy

Consider a general solution of the wave equation in the first WKB approximation:

$$p = \left[\left(\omega_d^2 - \frac{g^2k^2}{\omega_d^2}\right)\frac{m}{\rho}\right]^{-1/2}\left\{C_1\left(\omega_d^2 - \frac{g}{2h} + img\right)\mathrm{e}^{\mathrm{i}\psi} + C_2\left(\omega_d^2 - \frac{g}{2h} - img\right)\mathrm{e}^{-\mathrm{i}\psi}\right\}, \tag{4.1}$$

$$w = \left[\left(\omega_d^2 - \frac{g^2k^2}{\omega_d^2}\right)m\rho\right]^{-1/2} \times\left\{C_1\left(\frac{g^2k^2}{\omega_d^2} - \frac{1}{2h} + im\right)\mathrm{e}^{\mathrm{i}\psi} + C_2\left(\frac{g^2k^2}{\omega_d^2} - \frac{1}{2h} - im\right)\mathrm{e}^{-\mathrm{i}\psi}\right\}, \tag{4.2}$$

where

$$\psi = \int_{z_1}^{z}\frac{\mathrm{d}z_3}{2m}\left[2m^2 + \frac{gk^2}{\omega_d^2}\frac{\mathrm{d}}{\mathrm{d}z_3}\ln\left(\frac{h}{\omega_d^4 - g^2k^2}\right) + \frac{1}{2h}\frac{\mathrm{d}}{\mathrm{d}z_3}\ln\left(\frac{\omega_d^4 - g^2k^2}{\omega_d^2}\right)\right] \tag{4.3}$$

and constants C_1 and C_2 have the meaning of amplitudes of two linearly independent solutions. Here, we have used (3.12)–(3.15) and assume for definiteness that neither m^2 nor the quantity $\omega_d^2 - kg$ change their signs within the range $z_1 < z < z_2$ of heights considered. We also assume that $\omega_d \neq 0$. Note that ψ is real when $m^2 > 0$, and is purely imaginary when $m^2 < 0$.

For AGWs with periodic dependence on time and horizontal coordinates, the wave energy conservation law (Godin 1997; Brekhovskikh & Godin 1998) requires that the time-averaged vertical component of the power flux density is constant in the absence of absorption: $I_z = \text{const.}$, where

$$I_z = \omega\,\mathrm{Im}\,(p^* w). \tag{4.4}$$

Here and below, the asterisk * denotes complex conjugation. Equation (4.4) is exact for AGWs in media with horizontal flow (Brekhovskikh & Godin 1998). The ratio I_z/ω gives the vertical component of the wave action flux density (Brekhovskikh & Godin 1998).

When m is real, substitution of (4.1) and (4.2) into (4.4) gives, after some algebra,

$$I_z = \omega\,\mathrm{sgn}\,(\omega_d^2 - gk)(|C_1^2| - |C_2^2|). \tag{4.5}$$

No approximations are made in the derivation of (4.5) from (4.1) and (4.2). Equation (4.5) reveals several important properties of the power flux in the WKB approximation. First, it shows that wave energy is conserved exactly in the first WKB approximation. Second, as for plane waves (Godin & Fuks 2012), power fluxes are additive in the waves having the same horizontal wavevector $\boldsymbol{k}$ and propagating upward and downward. Waves with opposite signs of the vertical component of phase slowness have opposite signs of the vertical power flux density. Third, the directions of the phase increase with height and of the vertical component of the power flux coincide on the acoustic branch of AGWs, where $\omega_d^2 > N_0^2 \equiv gh^{-1} - g^2c^2$, and are opposite on the buoyancy branch, where $\omega_d^2 < N_0^2$.

To establish the third property of the power fluxes, we note that the definition of m^2, (3.5), can be rearranged to read

$$m^2+\left(\frac{1}{2h}-\frac{\omega_d^2}{g}\right)^2=\left(1-\frac{N_0^2}{\omega_d^2}\right)\left(\frac{\omega_d^4}{g^2}-k^2\right). \tag{4.6}$$

It follows from (4.6) that, when $m^2>0$, the quantities $\omega_d^2-N_0^2$ and ω_d^2-gk do not equal zero and have the same sign. Therefore, $\operatorname{sgn}(\omega_d^2-gk)$ can be replaced with $\operatorname{sgn}(\omega_d^2-N_0^2)$ in (4.5).

Now, consider the case where m is purely imaginary. Then ψ is purely imaginary and $\mathrm{e}^{\mathrm{i}\psi}$ and $\mathrm{e}^{-\mathrm{i}\psi}$ are real-valued in (4.1) and (4.2). After some algebra, from (4.1), (4.2), and (4.4) we find

$$I_z=2\omega\operatorname{sgn}(\omega_d^2-gk)\operatorname{Im}(C_1^*C_2). \tag{4.7}$$

Since we have assumed that the quantity ω_d^2-kg does not change its sign, I_z is independent of height. It follows from (4.5) that, in the regime of vertically inhomogeneous (evanescent) waves,

(i) wave energy is conserved exactly in the first WKB approximation;
(ii) waves which exponentially increase or decrease with height do not transport energy in the vertical direction;
(iii) non-zero vertical power flux results from interference of two evanescent waves that exponentially decrease and exponentially increase with z and have the same horizontal wavevector $\boldsymbol{k}$;
(iv) for a wavefield with given amplitude coefficients C_1 and C_2, the direction of the vertical power flux is determined by the sign of the quantity ω_d^2-gk.

Since $\operatorname{sgn}X$ is discontinuous at $X=0$, one might conclude, erroneously, that (4.7) indicates a discontinuity of the power flux at the height $z=z_c$, where $\omega_d^2(z_c)=gk$, which would clearly contradict the energy conservation law. In fact, (4.1) and (4.2) are inapplicable in the vicinity of z_c. To ascertain properties of the WKB approximation and the power flux in this vicinity, one should use another form of the WKB solution to the wave equation. From (3.16) and (A 7), we obtain

$$p=\sqrt{\left(\frac{1}{c^2}-\frac{k^2}{\omega_d^2}\right)\frac{\rho}{m}}\sum_{j=1}^{2}D_j\frac{\omega_d^2-(-1)^j\,\mathrm{i}mg-g/2h}{g^2k^2\omega_d^{-2}-(-1)^j\,\mathrm{i}m-1/2h}\exp[(-1)^{j+1}\mathrm{i}\Psi], \tag{4.8}$$

$$w=[(c^{-2}-k^2\omega_d^{-2})/m\rho]^{1/2}(D_1\mathrm{e}^{\mathrm{i}\Psi}+D_2\mathrm{e}^{-\mathrm{i}\Psi}), \tag{4.9}$$

where D_1 and D_2 are arbitrary constants and

$$\Psi=\int_{z_1}^{z}\frac{\mathrm{d}z_3}{2m}\left[2m^2+\frac{\mathrm{d}}{\mathrm{d}z_3}\left(\frac{1}{2h}-\frac{gk^2}{\omega_d^2}\right)-\left(\frac{1}{2h}-\frac{gk^2}{\omega_d^2}\right)\frac{\mathrm{d}}{\mathrm{d}z_3}\ln\left(\frac{h}{c^2}-\frac{k^2h}{\omega_d^2}\right)\right]. \tag{4.10}$$

Equations (4.8)–(4.10) give the general solution of the wave equation in the first WKB approximation. The solution is valid as long as neither m^2 nor the quantity $\omega_d^2-k^2c^2$ change their signs within the range $z_1<z<z_2$ of heights considered. We continue to assume that $\omega_d\neq 0$. Note that Ψ defined by (4.10) is real when $m^2>0$, and is purely imaginary when $m^2<0$.

When $m^2 > 0$, substitution of (4.8)–(4.10) in (4.4) gives

$$I_z = \omega \operatorname{sgn}(\omega_d^2 - k^2c^2)(|D_1^2| - |D_2^2|). \tag{4.11}$$

Since $\omega_d^2 - k^2c^2$ and $\omega_d^2 - gk$ have the same sign at $m^2 > 0$ (see (3.11)), (4.5) and (4.11) are equivalent, as expected. When $m^2 < 0$, substitution of (4.8)–(4.10) in (4.4) gives

$$I_z = 2\omega \operatorname{sgn}(\omega_d^2 - k^2c^2) \operatorname{Im}(D_1^* D_2). \tag{4.12}$$

Equation (4.12) proves that, in the first WKB approximation, the power flux remains continuous and, moreover, constant in the vicinity of points where $\omega_d^2 = gk$.

As a caveat, this reasoning does not apply if the conditions $\omega_d^2 = gk$ and $\omega_d^2 = k^2c^2$ are met simultaneously. In this degenerate case, $\omega = \boldsymbol{k}\cdot\boldsymbol{u}(z_c) \pm (gk)^{1/2}$ and $k = gc^{-2}(z_c)$. Neither of the representations (4.1)–(4.3) and (4.8)–(4.10) of the WKB solution holds then in the vicinity of z_c.

5. Comparison of the WKB and exact solutions

5.1. *Plane waves*

In the elementary case of a fluid where c, $\boldsymbol{u}$, and h are independent of height (in particular, in an isothermal ideal gas with constant γ), governing equations readily reduce to differential equations with constant coefficients (Pierce 1965); linearly independent solutions for $\rho^{-1/2}\tilde{p}$, $\rho^{-1/2}p$, and $\rho^{1/2}w$ are just $\exp(\pm imz)$ (Lamb 1932; Pierce 1965). Thus, continuous AGWs with a given horizontal wavevector $\boldsymbol{k}$ are a superposition of two plane waves.

In this case, $\chi \equiv 0$ in (3.12) and (3.13). For $n \geqslant 1$, $f_n \equiv 0$ in (3.9) and (3.10) and $F_n \equiv 0$ in (A 5) and (A 6). Then, as expected the WKB solutions (3.12), (3.14), (3.15), and (A 7) reduce to the well-known plane-wave solutions, as do various *ad hoc* approximations (Einaudi & Hines 1970, 1974). A more stringent test of the validity of the approximate solutions is provided by another exact solution (Godin 2012*a*, 2014*a*), which describes a particular type of AGWs in arbitrarily stratified fluids.

5.2. *Incompressible wave motion in compressible fluids*

Arbitrarily stratified compressible fluids with uniform background flow support AGWs with the dispersion equation (3.18), in which $\tilde{p} \equiv 0$,

$$p = g\rho w, \quad w(z) = w(0)e^{kz}. \tag{5.1a,b}$$

Fluid motion in these waves is incompressible in the sense that $\nabla\cdot\boldsymbol{v} \equiv 0$ in accordance with (2.5). With appropriate boundary conditions, solution (5.1) describes a surface wave which propagates horizontally (Godin 2012*a*, 2014*a*).

In the WKB approach, we have $m^2 = -(k - 1/2h)^2$ from (3.5) and (3.18). With $F_0 = i(k - 1/2h)$, (A 5)–(A 6) give $F_n \equiv 0$ for all $n \geqslant 1$. Then, the WKB equations (3.16) and (A 7) become exact and coincide with (5.1).

This should be contrasted with predictions of various *ad hoc* approximations considered by Einaudi & Hines (1970, 1974) and Jones & Georges (1976). None of these approximations reproduces the exact solution (5.1).

5.3. *The WKB series as an exact analytic solution*

When an asymptotic series defined by either (3.7)–(3.10) or (A 3)–(A 6) is convergent, the series gives an exact solution for AGWs in a continuously stratified atmosphere. In particular, this is the case when either series has only a finite number of non-zero terms. Therefore, by establishing conditions when the WKB series terminates, one can generate explicit, analytic solutions of the problem. Here, we consider a simple sufficient condition of the WKB series termination.

Let $f_2(\zeta) = \varepsilon s(\zeta)$, where ε is a small parameter and $s(\zeta)$ is an infinitely differentiable function. Then, it follows from (3.10) that $f_n(\zeta) = O(\varepsilon)$ for all $n \geqslant 3$ and all ζ such that neither of (3.17), (3.18) and $m = 0$ holds. Moreover, if $f_2(\zeta) \equiv 0$, then $f_n(\zeta) \equiv 0$ for all ζ and $n \geqslant 3$. Similarly, from (A 6) it follows that $F_n(\zeta) \equiv 0$ for all $n \geqslant 3$ in the series (A 3) when $F_2(\zeta) \equiv 0$. Thus, the first WKB approximation for the Lagrangian pressure perturbations, (3.12) and (3.13), gives exact solutions of the problem, when $f_2(\zeta) \equiv 0$; the first WKB approximation for the vertical displacement, (A 7), gives exact solutions of the problem, when $F_2(\zeta) \equiv 0$.

We have seen in §§ 5.1 and 5.2 that $f_2(\zeta) \equiv 0$ and $F_2(\zeta) \equiv 0$ for two known exact solutions. However, these equations can help in generating new exact solutions. Exploring the full set of such solutions is beyond the scope of this paper, and we limit ourselves to just a few examples. Three new exact solutions are derived in appendix C.

6. Conclusion

The assumptions that the sound speed, background flow velocity, and composition of the propagation medium vary gradually with height and that the spatial scale of these variations is large compared to the vertical spatial scale of wavefield variations, have allowed us to introduce a large parameter of the problem and systematically derive an asymptotic solution for the field of AGWs. The resulting WKB approximation for AGWs has been constructed to all orders in the large parameter of the problem; lower-order WKB approximations have been obtained in an explicit form. As expected, the WKB approximations, which were derived from various AGW wave equations, proved to be equivalent.

The asymptotic solution is consistent with the exact AGW solutions that are known in certain specific cases. The WKB solution reduces to the exact, plane-wave solution in the case of an isothermal atmosphere (Lamb 1932; Pierce 1965) and in the case of incompressible motion of arbitrarily stratified, compressible fluids (Godin 2012*a*, 2014*a*). New exact analytic solutions for AGWs in a continuously layered atmosphere have been derived from the requirement that the WKB series contains only a finite number of non-zero terms.

The WKB approximation ensures exact wave energy conservation in both propagating and evanescent waves. In upward- and downward-propagating waves, the vertical components of the power fluxes have opposite signs and are additive. Evanescent waves, the amplitudes of which increase or decrease exponentially with height, do not transport wave energy vertically. However, superposition of such evanescent waves leads to a non-zero vertical power flux.

In the limit of vanishing gravity, the asymptotic solution reduces to known WKB solutions for acoustic waves (Brekhovskikh & Godin 1998, § 8.1). In the opposite limit, where gravity is the dominant restoring force, the WKB approach is often applied in the literature to study internal gravity waves in the Boussinesq approximation. However, the latter approximation and its underlying assumption of small relative changes in the background density are not relevant to atmospheric waves unless the range of heights considered is within a fraction of the atmospheric scale height. A comparison of the WKB solutions that are derived with and without the Boussinesq approximation (see appendix B) shows that, when applied to atmospheric waves, the Boussinesq approximation leads to large amplitude errors and adequately describes the wave phase only for a part of the internal gravity wave spectrum.

When derived systematically from first principles, the WKB approximation for atmospheric waves does not suffer from any of the ambiguities or contradictions previously asserted in the literature (Pitteway & Hines 1965; Einaudi & Hines 1970, 1974; Gossard & Hooke 1975; Fritts & Alexander 2003). The contradictions and ambiguities had resulted from unnecessary assumptions about the form of the WKB solutions, which are found here to be incompatible with the physics of AGWs.

Care needs to be exercised when following the common practice of extending the dispersion equation of waves from the isothermal to generic atmosphere. The dispersion equation of AGWs in a smoothly varying atmosphere coincides with their dispersion equation in an isothermal, uniformly moving atmosphere provided the latter equation is expressed in terms of the sound speed or barometric height and not in terms of the buoyancy frequency.

Derivation of higher-order WKB approximations helped us to demonstrate self-consistency of the asymptotic approach and derive the validity conditions of the first WKB approximation. In particular, a rather intuitive condition has been established for the WKB approximation validity in the vicinity of a turning point. The WKB approximation becomes applicable when the absolute value of the phase integral between the observation point and the turning point is large compared to unity, see (3.31). The vertical extent of the vicinity of the turning point where the first WKB approximation is invalid is large compared to the vertical scale of the wavefield variation away from the turning point but is small compared to the spatial scale of the sound speed and wind velocity variations.

In addition to the eikonal and wave amplitude, calculations of the geometric, or Berry, phase are necessary for the WKB and ray solutions to approximate the acoustic–gravity wavefield in an inhomogeneous, moving atmosphere. Unlike the wave amplitude, which is determined by the initial amplitude and the fluid parameters at the beginning and the end of the propagation path, the geometric phase generally depends on the values of gradients of environmental parameters at every point along the path. While the geometric phase vanishes in the acoustic limit and for vertical propagation of AGWs, it has been found to readily reach significant, $O(1)$ values for oblique propagation of waves on both buoyancy and acoustic branches of atmospheric AGWs.

Further work is required to extend the above analysis to AGW propagation in a dissipative and horizontally inhomogeneous atmosphere.

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Appendix A. Derivation of the WKB approximation based on the wave equation for the vertical displacement

We search for a solution to (2.11) in the form (cf. (3.3))

$$w(z) = w(0) \exp\left(\mathrm{i}L \int_0^{\zeta} \Phi(\zeta_1, L)\, \mathrm{d}\zeta_1 \right), \tag{A 1}$$

where Φ is an unknown function. Substitution of (A 1) into the wave equation (2.11) gives the following Riccati equation for Φ:

$$\frac{\mathrm{i}}{L}\frac{\mathrm{d}\Phi}{\mathrm{d}\zeta} - \frac{\mathrm{i}\Phi}{h} - \frac{\mathrm{i}\Phi}{L}\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln\left[\frac{h}{\omega_d^2}\left(\frac{\omega_d^2}{c^2} - k^2\right)\right] = \Phi^2 - m^2 - \frac{1}{4h^2} - \frac{gk^2}{L\omega_d^2}\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln\left[h\left(\frac{\omega_d^2}{c^2} - k^2\right)\right], \tag{A 2}$$

where m is defined in (3.5). To solve (A 2), let us write Φ in terms of a power series in L^{-1} (cf. (3.6)):

$$\Phi(\zeta, L) = F - \mathrm{i}/2h, \quad F = \sum_{n=0}^{\infty} F_n(\zeta) L^{-n}. \tag{A 3}$$

Substitution of (A 3) into the Riccati equation (A 2) and equating terms of the same order in L^{-1}, we find

$$F_0 = \pm m, \tag{A 4}$$

$$F_1 = \frac{1}{2F_0}\left(\frac{gk^2}{\omega_d^2} - \frac{1}{2h}\right)\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln\left[\frac{h(\omega_d^2 c^{-2} - k^2)}{gk^2 - \omega_d^2/2h}\right] - \frac{\mathrm{i}}{2}\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln\left[\frac{h}{F_0}\left(\frac{1}{c^2} - \frac{k^2}{\omega_d^2}\right)\right], \tag{A 5}$$

$$F_n = -\frac{\mathrm{i}F_{n-1}}{2F_0}\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln\left[\frac{h}{F_{n-1}}\left(\frac{1}{c^2} - \frac{k^2}{\omega_d^2}\right)\right] - \frac{1}{2F_0}\sum_{l=1}^{n-1} F_l F_{n-l}, \quad n \geqslant 2. \tag{A 6}$$

This is an exact solution of the wave equation as long as $m \neq 0$, $\omega_d \neq 0$, and $\omega_d^2 \neq gk$. From (3.5) and (3.11) it follows that all F_n are finite when $0 < m^2 < \infty$.

In the first WKB approximation, retaining only terms F_0 and F_1 in (A 3), from (A 1), (A 5), and (A 6), we find

$$w(z) = w(0)\sqrt{\frac{[c^{-2}(z) - k^2\omega_d^{-2}(z)]\rho(0)m(0)}{[c^{-2}(0) - k^2\omega_d^{-2}(0)]\rho(z)m(z)}} \exp\left(\pm\mathrm{i}\int_0^z \frac{\mathrm{d}z_1}{2m}\left[2m^2 + \frac{\mathrm{d}}{\mathrm{d}z_1}\left(\frac{1}{2h} - \frac{gk^2}{\omega_d^2}\right) - \left(\frac{1}{2h} - \frac{gk^2}{\omega_d^2}\right)\frac{\mathrm{d}}{\mathrm{d}z_1}\ln\left(\frac{h}{c^2} - \frac{k^2 h}{\omega_d^2}\right)\right]\right). \tag{A 7}$$

AGW polarization relations follow from (2.3), (2.8), and (A 7) and prove to be identical to (3.14)–(3.16).

Using (3.11), it is straightforward to check that the two results for $w(z)$, (3.12)–(3.14) and (A 7), which have been obtained from wave equations for different AGW parameters, are equal exactly. (To perform such a check, it is sufficient to note that the two expressions agree at $z=0$ and that the derivative with respect to z of the ratio of the two expressions for $w(z)$ equals zero identically. The algebra is cumbersome but straightforward.) Similarly, it can be shown that the WKB solution of the wave equation (2.10), where the Eulerian pressure perturbation is the dependent variable, is equivalent to the WKB solutions (3.12)–(3.14) and (A 7), which are derived from other wave equations. These findings illustrate the fact that the WKB approximation for AGWs, when derived in a systematic manner, is the same regardless of the initial choice of dependent variable or variables. Furthermore, the equivalence of (3.12)–(3.14) and (A 7) shows that an apparent singularity of (3.12)–(3.14) at z such that $\omega_d^2 = gk$, which is not present in (A 7), is not a true singularity of the solution. Similarly, an apparent singularity of (A 7) at z such that $\omega_d^2 = k^2c^2$, which is not present in (3.12)–(3.14), is not a true singularity of the solution. In fact, the WKB approximation remains valid in the vicinity of the points where $\omega_d^2 = k^2c^2$ or $\omega_d^2 = gk$ as long as such points do not coincide with each other and are located sufficiently far from the turning points and the wave–flow synchronism points discussed in § 3.1.

Appendix B. An alternative scaling of the problem

Wave equations (2.9)–(2.11) are invariant with respect to multiplication of the background fluid density ρ by a constant. It is straightforward to cast the wave equations in the form where density enters their coefficients only through the logarithmic derivative $\mathrm{d}(\ln\,\rho)/\mathrm{d}z$. Therefore, in considering AGW propagation in fluids with gradually varying parameters, it appears natural to assume that $H \equiv -[\mathrm{d}(\ln\,\rho)/\mathrm{d}z]^{-1}$, just like the sound speed c and background flow velocity $\boldsymbol{u}$, is a smooth function of $\zeta = z/L$. Then,

$$\rho(z) = \rho(0)\exp\left(-\int_0^z \frac{\mathrm{d}z_1}{H(z_1)}\right), \tag{B 1}$$

and the buoyancy frequency squared $N^2 = gH^{-1} - g^2c^{-2}$ according to (2.4). Hence, the buoyancy frequency is also a smooth function of $\zeta = z/L$.

It will be assumed in §§ B.1 and B.2 that the large parameter L enters the density scale height H and, therefore, the buoyancy frequency N only through $\zeta = z/L$. As in the main text, we wish to find an asymptotic solution for the AGW wavefield that is valid at large L (formally, as $L \to \infty$). No restrictions are placed on H.

B.1. *Derivation based on the wave equation for Lagrangian pressure perturbations*

As in § 3.1, we search for a solution to (2.9) in the form (3.3) and let

$$\varphi(\zeta, L) = f + \mathrm{i}/2H, \quad f = \sum_{n=0}^{\infty} f_n(\zeta)L^{-n} \tag{B 2}$$

(cf. (3.6)). Substitution of (3.3) into the wave equation (2.9) gives the following Riccati equation:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\mathrm{i}f - \frac{1}{2H} + \frac{gk^2}{\omega_d^2}\right) - f^2 + M^2 = \left(\mathrm{i}f - \frac{1}{2H} + \frac{gk^2}{\omega_d^2}\right)\frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\omega_d^2 - \frac{g^2k^2}{\omega_d^2}\right), \tag{B 3}$$

where (cf. (3.5))

$$M^2 = \frac{\omega_d^2}{c^2} - k^2 - \frac{1}{4H^2} + \frac{gk^2}{\omega_d^2}\left(\frac{1}{H} - \frac{g}{c^2}\right) = \frac{\omega_d^2}{c^2} - \frac{1}{4H^2} + k^2\left(\frac{N^2}{\omega_d^2} - 1\right). \tag{B 4}$$

By substituting (3.3) and (B 2) into the Riccati equation (B 3) and equating terms of the same order in L^{-1}, we find that

$$f_0 = \pm M, \tag{B 5}$$

$$f_1 = \frac{1}{2f_0}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\mathrm{i}f_0 - \frac{1}{2H} + \frac{gk^2}{\omega_d^2}\right) - \frac{1}{2f_0}\left(\mathrm{i}f_0 - \frac{1}{2H} + \frac{gk^2}{\omega_d^2}\right)\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln\left(\omega_d^2 - \frac{g^2k^2}{\omega_d^2}\right), \tag{B 6}$$

$$f_n = \frac{1}{2f_0}\left[\mathrm{i}f_{n-1}\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln\left(\frac{\omega_d^2 f_{n-1}}{\omega_d^4 - g^2k^2}\right) - \sum_{l=1}^{n-1} f_l f_{n-l}\right], \quad n \geqslant 2. \tag{B 7}$$

This is an exact solution of the wave equation as long as $m \neq 0$, $\omega_d \neq 0$, and $\omega_d^2 \neq gk$. From (B 5) and the identity

$$M^2 + \left(\frac{1}{2H} - \frac{gk^2}{\omega_d^2}\right)^2 = \left(1 - \frac{g^2k^2}{\omega_d^4}\right)\left(\frac{\omega_d^2}{c^2} - k^2\right), \tag{B 8}$$

it follows that all f_n are finite when $0 < M^2 < \infty$. Aside from the AGW propagation being an adiabatic thermodynamic process, no assumptions about the equation of state of the fluid have been made in deriving the asymptotic solutions.

In the first WKB approximation, retaining only terms f_0 and f_1 in (B 2), from (3.3), (B 5), and (B 6) we find

$$\begin{aligned}\tilde{p}(z) &= \tilde{p}(0)\sqrt{\frac{[\omega_d^2(z) - g^2k^2\omega_d^{-2}(z)]\rho(z)M(0)}{[\omega_d^2(0) - g^2k^2\omega_d^{-2}(0)]\rho(0)M(z)}}\exp\left(\pm\mathrm{i}\int_0^z \frac{\mathrm{d}z_1}{2M}\right. \\ &\quad \times \left.\left[2M^2 - \frac{\mathrm{d}}{\mathrm{d}z_1}\left(\frac{1}{2H} - \frac{gk^2}{\omega_d^2}\right) + \left(\frac{1}{2H} - \frac{gk^2}{\omega_d^2}\right)\frac{\mathrm{d}}{\mathrm{d}z_1}\ln\left(\omega_d^2 - \frac{g^2k^2}{\omega_d^2}\right)\right]\right).\end{aligned} \tag{B 9}$$

The first WKB approximation gives the solution to the wave equation with accuracy up to the factor $1 + O(L^{-1})$. To the same accuracy, from (2.7) and (B 9) we obtain AGW polarization relations, which differ from (3.14) to (3.16) only by replacement of h by H.

B.2. *Derivation based on the wave equation for the vertical displacement*

Let $w \neq 0$ at some height, say, $z = 0$. (In the opposite case, where $w \equiv 0$, we have the Lamb wave, the possibility of existence of which in stratified fluids is considered in Godin 2012*b*.) We search for a solution to the wave equation in the form (A 1), where now

$$\Phi(\zeta, L) = F - \mathrm{i}/2H, \quad F = \sum_{n=0}^{\infty} F_n(\zeta)L^{-n} \tag{B 10}$$

(cf. (A 3)). Substitution of (A 1) and (B 10) into the wave equation (2.11) gives the following Riccati equation for F:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\mathrm{i}F + \frac{1}{2H} - \frac{gk^2}{\omega_d^2}\right) - F^2 + M^2 = \left(\mathrm{i}F + \frac{1}{2H} - \frac{gk^2}{\omega_d^2}\right)\frac{\mathrm{d}}{\mathrm{d}z}\ln\left(c^{-2} - \frac{k^2}{\omega_d^2}\right), \tag{B 11}$$

where M is defined in (B 4). With F given by the power series (B 10), equating terms of the same order in L^{-1} in (B 11), we find

$$F_0 = \pm M, \tag{B 12}$$

$$F_1 = \frac{1}{2F_0}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\mathrm{i}F_0 + \frac{1}{2H} - \frac{gk^2}{\omega_d^2}\right) - \frac{1}{2F_0}\left(\mathrm{i}F_0 + \frac{1}{2H} - \frac{gk^2}{\omega_d^2}\right)\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln\left(c^{-2} - \frac{k^2}{\omega_d^2}\right), \tag{B 13}$$

$$F_n = -\frac{\mathrm{i}F_{n-1}}{2F_0}\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln\left(\frac{\omega_d^2 - k^2c^2}{c^2\omega_d^2 F_{n-1}}\right) - \frac{1}{2F_0}\sum_{l=1}^{n-1} F_l F_{n-l}, \quad n \geqslant 2, \tag{B 14}$$

$$F_n = \frac{1}{2F_0}\left(\mathrm{i}\dot{F}_{n-1} - \sum_{l=1}^{n-1} F_l F_{n-l}\right) - \frac{\mathrm{i}F_{n-1}}{2F_0}\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln\left(c^{-2} - \frac{k^2}{\omega_d^2}\right), \quad n \geqslant 2. \tag{B 15}$$

From (B 4) and (B 8) it follows that all F_n are finite when $0 < M^2 < \infty$.

In the first WKB approximation, retaining only terms F_0 and F_1 in (B 10), from (A 1) and (B 12) and (B 13) we find

$$\begin{aligned} w(z) &= w(0)\sqrt{\frac{[c^{-2}(z) - k^2\omega_d^{-2}(z)]\rho(0)M(0)}{[c^{-2}(0) - k^2\omega_d^{-2}(0)]\rho(z)M(z)}}\exp\left(\pm\mathrm{i}\int_0^z \frac{\mathrm{d}z_1}{2M}\right. \\ &\quad \times \left.\left[2M^2 + \frac{\mathrm{d}}{\mathrm{d}z_1}\left(\frac{1}{2H} - \frac{gk^2}{\omega_d^2}\right) - \left(\frac{1}{2H} - \frac{gk^2}{\omega_d^2}\right)\frac{\mathrm{d}}{\mathrm{d}z_1}\ln\left(c^{-2} - \frac{k^2}{\omega_d^2}\right)\right]\right). \end{aligned} \tag{B 16}$$

As with (3.12) in the main text and (A 7), using the AGW polarization relations, it is easy to check that the WKB solution (B 16), which is obtained from the wave equation for the vertical displacement, is identical to the WKB solution (B 9), which is obtained from the wave equation for the Lagrangian pressure perturbation.

It is instructive to compare (B 16) to solutions obtained in the Boussinesq approximation. For internal gravity waves in quiescent, incompressible fluids, Lighthill (1978, § 4.1) derived an equation in the Boussinesq approximation, which can be written as

$$\frac{\mathrm{d}^2Q}{\mathrm{d}z^2} + s^2Q = 0, \quad s^2 = k^2\left(\frac{N^2}{\omega^2} - 1\right) \tag{B 17}$$

for waves with harmonic dependence on horizontal coordinates and time. The dependent variable $Q = -\mathrm{i}\omega\rho w$ in Lighthill's equation has the meaning of the vertical component of mass flux density (Lighthill 1978, § 4.1). Here, as in the main text, w is the wave-induced vertical displacement of fluid parcels. The same equation (B 17) is derived in (Miropol'sky 2001, § 3.3) but for the vertical component of fluid velocity, $Q = -\mathrm{i}\omega w$. Assuming a slow variation of the buoyancy frequency with height, solutions of (B 17) in the first WKB approximations are (Miropol'sky 2001, § 3.3)

$$Q(z) = Q(0)\sqrt{s(0)/s(z)}\exp\left(\pm\mathrm{i}\int_0^z s(z_1)\,\mathrm{d}z_1\right). \tag{B 18}$$

In the case of a quiescent, incompressible fluid, equations (B 4) and (B 16) simplify and become, respectively, $M^2 = s^2 - 0.25H^{-2}$ and

$$w(z) = w(0)\sqrt{\frac{\rho(0)M(0)}{\rho(z)M(z)}}\exp\left(\pm\mathrm{i}\int_0^z \mathrm{d}z_1\left(M + \frac{1}{4M}\frac{\mathrm{d}}{\mathrm{d}z_1}\frac{1}{H}\right)\right). \tag{B 19}$$

The term with $\mathrm{d}H^{-1}/\mathrm{d}z$ in the integrand on the right-hand side of (B 19) represents the geometric, or Berry, phase. No Berry phase arises in the Boussinesq approximation.

The WKB solutions (B 18) and (B 19) obtained within and without the Boussinesq approximation are obviously different. Aside from the factor $[\rho(0)/\rho(z)]^{1/2}$, which can be rather large for atmospheric waves but is assumed to be close to unity in the Boussinesq approximation, the wave amplitudes in (B 18) and (B 19) will be close provided $sH \gg 1$, or

$$(\omega^{-2}N^2 - 1)k^2g^2N^{-4} \gg 1. \tag{B 20}$$

This condition also ensures the proximity of the wave turning points in the two descriptions. For the phase discrepancy to be small compared to unity, one has to require additionally that

$$(\omega^{-2}N^2 - 1)k^{-2}g^{-2}\omega^4 \ll 1. \tag{B 21}$$

The latter inequality is obtained by replacing M with s in the last term in the integrand in (B 19) and then explicitly calculating the resulting integral for the geometric phase. For waves with frequencies $\omega \ll N$, both conditions (B 20) and (B 21) are satisfied, when $\omega \ll kg/N$.

In terms of the vertical component of the wavevector of the internal gravity wave, the necessary conditions of the applicability of the Boussinesq approximation, (B 20) and (B 21), can be written as

$$g^{-1}N^2 \ll M \ll gk^2\omega^{-2}. \tag{B 22}$$

Inequalities (B 20)–(B 22) illustrate the restrictions that arise from adopting (e.g. Broutman, Rottman & Eckermann 2004) the unnecessary Boussinesq approximation in the WKB and ray theories of internal gravity waves.

B.3. *Comparison of asymptotic solutions*

The WKB solutions (B 5)–(B 7) and (B 12)–(B 15) for AGWs, which are derived in §§ B.1 and B.2, have a structure that is very similar to the structure of the WKB solutions in the main text. In particular, both sets of solutions feature the geometric phase, which is necessary for the asymptotic solution to approximate the exact solutions to the wave equation. The two sets of solutions coincide in the case of an isothermal atmosphere. Comparison of (3.12) and (3.13) with (B 9) or (A 7) with (B 16) shows, however, that the two sets of WKB solutions differ in important ways in fluids with generic inhomogeneities. For instance, the positions of the turning points, which are the heights where, respectively, $m = 0$ and $M = 0$, prove to be different as long as $h \neq H$, see (3.5) and (B 4). The differences extend to include the AGW dispersion relations, wave eikonals, and the geometrical phases. The differences between the two sets of asymptotic solutions illustrate that, while the WKB solutions are defined uniquely for a given scaling of the problem, different scalings (i.e. different sets of assumptions regarding environmental parameters) produce distinct WKB solutions.

The background density profiles in (3.1) and (B 1) coincide when

$$H = \frac{h}{1 + \mathrm{d}h/\mathrm{d}z}. \tag{B 23}$$

Thus, the difference between the two scalings can be formulated as follows. In the main text, it is assumed that, for fixed ζ, h^{-1} is independent of the large parameter L, while H^{-1}, and, therefore, N^2, is a sum of terms $O(1)$ and $O(L^{-1})$. Conversely, in §§ B.1 and B.2 we assume that, for fixed ζ, H^{-1} and N^2 are independent of the large parameter L, while h^{-1} is a sum of terms $O(1)$ and $O(L^{-1})$. The latter scaling is apparently inconsistent with (3.2) for h in an ideal gas, but may be appealing in studies of waves in fluids with other equations of state. In particular, it is the scaling that is typically implied in studies of internal gravity waves in the ocean (Garrett 1968; Gill 1982, chapter 8; Miropol'sky 2001, chapter 3; Broutman *et al.* 2004).

Appendix C. Exact analytic solution for AGWs in continuously layered fluids

When the WKB series defined by either (3.7)–(3.10) or (A 3)–(A 6) has only a finite number of non-zero terms, it provides an explicit analytic solution for linear AGWs in a continuously layered atmosphere. As shown in § 5.3 in the main text, the WKB series (3.7)–(3.10) and (A 3)–(A 6) have only two non-zero terms when $f_2(\zeta) \equiv 0$ or $F_2(\zeta) \equiv 0$, respectively. Here, we present three examples of exact solutions that are generated by equations $f_2(\zeta) \equiv 0$ and $F_2(\zeta) \equiv 0$.

First, consider AGWs with $k = 0$. In this case, the wavefield is independent of horizontal coordinates and is not affected by the background fluid flow (wind). When $k = 0$ and $f_2 \equiv 0$, equation (3.10) with $n = 2$ becomes a first-order differential equation $\mathrm{d}(hf_1)^{-1}/\mathrm{d}\zeta = \mathrm{i}h^{-1}$ for f_1. It is convenient to write its solution as

$$f_1 = \frac{\mathrm{i}L}{(a_1 + X)h}, \quad X = L\int_0^{\zeta} \frac{\mathrm{d}\zeta_1}{h(\zeta_1)} = \int_0^{z} \frac{\mathrm{d}z_1}{h} \tag{C 1}$$

and consider X as a new independent variable. It is related to background pressure as follows: $X = \ln[p_0(0)/p_0(z)]$, see (3.1). Note that X steadily increases from 0 to infinity as the height z increases from 0 to infinity; $h = \mathrm{d}z/\mathrm{d}X$. In (C 1), a_1 is an arbitrary dimensionless constant. We will assume that $a_1 > 0$ so that f_1 has no singularities at $z > 0$.

According to (3.2), (3.5), and (3.8), $f_0^2 = \omega^2/\gamma gh - (2h)^{-2}$ for atmospheric waves. Substitution of this expression and f_1 from (C 1) into (3.9) gives

$$\frac{4}{(a_1 + X)h} = \frac{\mathrm{d}}{\mathrm{d}z} \ln\left(\frac{4\omega^2 h}{\gamma g} - 1\right). \tag{C 2}$$

We solve (C 2) for the unknown function $z(X)$ and find

$$z = \frac{\gamma g}{4\omega^2}\left[X + \frac{a_2^2}{5}((X + a_1)^5 - a_1^5)\right], \quad c = \frac{\gamma g}{2\omega}\sqrt{1 + a_2^2(X + a_1)^4}. \tag{C 3a,b}$$

Here, a_2 is an arbitrary dimensionless constant, which will be assumed positive for definiteness. Equation (C 3) defines variation of the sound speed with height in a parametric form. Equation (3.12) give two linearly independent, exact solutions for AGWs in the atmosphere defined by (C 3):

$$\frac{\tilde{p}(z)}{\tilde{p}(0)} = \frac{a_1}{X + a_1} \exp\left[-\frac{X}{2} \pm \mathrm{i}\frac{a_2 X}{6}(X^2 + 3a_1 X + 3a_1^2)\right]. \tag{C 4}$$

The case of an isothermal atmosphere is recovered from (C 3) and (C 4) in the limit where ${a_1}^2 a_2 = \text{const.}$ and $a_1 \to \infty$. It is easy to verify the validity of the exact solution by a direct substitution of (C 4) into the wave equation (2.9).

Two AGWs, which are described by (C 4), do not have turning points and propagate vertically in opposite directions without reflections regardless of whether variations of the sound speed with height are fast or slow. The family (C 3) of non-reflective profiles has two free parameters, a_1 and a_2. Another two-parametric family of non-reflective profiles for vertically propagating AGWs has been found by Petrukhin, Pelinovsky & Batsyna (2011, 2012*a*,*b*) and Petrukhin, Pelinovsky & Talipova (2012*c*) from different considerations and discussed in the contexts of wave propagation in the Earth and solar atmospheres. The practical relevance of both families of non-reflecting profiles is limited, though, by the fact that exact analytic solutions are available and the profiles are reflectionless only when $k=0$.

As a second example, consider obliquely propagating acoustic waves in a moving fluid with a generic equation of state. Transition to the acoustic limit in the AGW solutions is discussed in § 3.4. As in § 3.4, the functions $c(z)$, $\rho(z)$, and $\boldsymbol{u}(z)$, which fully characterize the acoustic propagation environment, can be prescribed independently. (For an ideal gas, it implies strong variations of its composition.) When $g\to 0$ and $f_2\equiv 0$, (3.10) with $n=2$ becomes a first-order differential equation $\mathrm{d}(\rho\omega_d^2 f_1^{-1})/\mathrm{d}\zeta = \mathrm{i}\rho\omega_d^2$ for f_1. It gives

$$f_1 = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln(b_1+Y), \qquad Y=\frac{1}{\rho(0)\omega^2}\int_0^z \rho\omega_d^2\,\mathrm{d}z_1. \tag{C 5}$$

Defined by (C 5), Y has the meaning of a new vertical coordinate. It steadily increases from zero to infinity when z increases from zero to infinity; $Y=z$ in the case of a motionless fluid with a constant density.

According to (3.5) and (3.8), $f_0^2=\omega_d^2c^{-2}-k^2$ in the acoustic limit. With f_1 given by (C 5), (3.9) becomes

$$-4\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln(b_1+Y)=\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln\left(\frac{\omega_d^2c^{-2}-k^2}{\rho^2\omega_d^4}\right). \tag{C 6}$$

We solve (C 6) for the unknown sound speed and find

$$c=\left[\frac{k^2}{\omega_d^2}+\frac{\omega^2}{b_2^2\omega_d^2(Y+b_1)^4}\left(\frac{\mathrm{d}Y}{\mathrm{d}z}\right)^2\right]^{-1/2}. \tag{C 7}$$

Here $b_1\geqslant 0$ and $b_2>0$ are arbitrary constants. Equations (C 5) and (C 7) define the sound-speed profile $c(z)$ in a parametric form. For this sound-speed profile, (3.12) gives two linearly independent, exact solutions for obliquely propagating acoustic waves:

$$\frac{p(z)}{p(0)}=(Y+b_1)\exp\left[\pm\frac{\mathrm{i}\omega Y}{b_1b_2(Y+b_1)}\right]. \tag{C 8}$$

The exact solutions (C 8) are valid for arbitrary profiles of the density $\rho(z)$ and the background flow velocity $\boldsymbol{u}(z)$. The non-reflective sound-speed profile (C 7) depends on the horizontal wavevector $\boldsymbol{k}$ but is independent of the wave frequency ω as long as the horizontal slowness $\boldsymbol{k}/\omega$ is independent of ω. Then, the exact solutions (C 8) can easily be extended to the time domain. In the particular case of vertical propagation ($k=0$) of sound in a fluid of constant density, (C 7) gives the sound-speed profile $c=b_2(z+b_1)^2$, for which it was found earlier (Brekhovskikh & Godin 1998, § 8.2) that the WKB series terminates and gives an exact solution of the acoustic wave equation regardless of whether the wave phase variation is fast or slow compared to variation of the wave amplitude.

As the last example, consider obliquely propagating AGWs in an atmosphere with height-dependent winds. When $F_2 \equiv 0$, (A 6) with $n = 2$ becomes a first-order differential equation for F_1. Solving the equation, we find

$$F_1 = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln\left(\int_{\zeta_1}^{\zeta}\left(c^{-2} - \frac{k^2}{\omega_d^2}\right) h\,\mathrm{d}\zeta_2\right), \tag{C 9}$$

where ζ_1 is an arbitrary constant. Substitution of F_1 from (C 9) into (A 5) gives

$$\begin{aligned}\frac{1}{2F_0}\left(\frac{gk^2}{\omega_d^2} - \frac{1}{2h}\right)\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln\left[h\left(\frac{\omega_d^2}{c^2} - k^2\right)\right] - \frac{1}{2F_0}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{gk^2}{\omega_d^2} - \frac{1}{2h}\right) \\ = \frac{\mathrm{i}}{2}\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln\left[\frac{F_0}{h}\left(\frac{1}{c^2} - \frac{k^2}{\omega_d^2}\right)^{-1}\left(\int_{\zeta_1}^{\zeta}\left(c^{-2} - \frac{k^2}{\omega_d^2}\right) h\,\mathrm{d}\zeta_2\right)^2\right].\end{aligned} \tag{C 10}$$

We are interested in solutions with real-valued ω, $\boldsymbol{k}$, F_0, $\boldsymbol{u}$, c, and h. Then, the left- and right-hand sides of (C 10) are real and imaginary, respectively. From the requirement that the left-hand side equals zero, it follows that $\omega_d^2 = 2ghk^2$. Then, from (3.5) and (A 4) and the requirement that the right-hand side of (C 10) equals zero, we find $m = ka_3(1 + z/z_1)^{-2}$ and

$$c = \left(\frac{\gamma g}{2k}\right)^{1/2}\left[1 - \frac{\gamma a_3^2}{2-\gamma}\left(1 + \frac{z}{z_1}\right)^{-4}\right]^{-1/4}, \quad \frac{\boldsymbol{k}}{k}\boldsymbol{\cdot}\boldsymbol{u} = \frac{\omega}{k} \pm \left(\frac{2}{\gamma}\right)^{1/2} c. \tag{C 11a,b}$$

Here, $a_3 > 0$ and $z_1 > 0$ are arbitrary constants.

Equation (A 7) gives exact solutions

$$\frac{w(z)}{w(0)} = \left(1 + \frac{z}{z_1}\right)\left[1 - \frac{\gamma a_3^2}{2-\gamma}\left(1 + \frac{z}{z_1}\right)^{-4}\right]^{1/4}\sqrt{\frac{\rho(0)}{\rho(z)}}\exp\left(\pm\frac{\mathrm{i}ka_3z_1z}{z + z_1}\right), \tag{C 12}$$

which describe two AGWs that propagate without reflection in the atmosphere with parameters given by (C 11). It is straightforward to verify that the WKB solution (C 12) satisfies the wave equation (2.11) exactly. According to (C 11), the exact solutions (C 12) exist only when $k \neq 0$ and require that the component of the wind velocity along the wavevector is non-zero.

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