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Statistical Characterization of Sea Surface Geometry for a Wave Slope Field Discontinuous in the Mean Square

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Statistics of two dimensional wave groups, of steep wave events, and of a cascade pattern manifested in the surface geometry in a developed sea state are derived. However, mathematical theories used to parameterize these as well as many other features of random surfaces have very limited ranges of validity. For example, high-order moments of wave spectra appearing in the calculations of wave slope statistics cannot be evaluated because of divergence of the corresponding integrals. In the present paper the restrictions are reviewed and the difficulties are shown to be due to a pseudo-fractal geometry of the sea surface whose spectrum is known only within a limited range of frequency (characterized by either the resolution of a measuring technique or the constraints of a theoretical model). An approach is presented that solves the problem: treating the surface elevation field as specified on a spatial (temporal) running grid, an averaging procedure is developed employing the Taylor microscale as the mesh size. The technique is illustrated by first exposing errors in direct calculations of the effective surface impedance for a coherently reflected L band radio wave. The errors arise from the use of wave spectra whose high-frequency tail is identified with the Phillips saturation range. The technique is then employed in the study of wave groups and steep waves for a Gaussian, two-dimensional, time-varying surface. In particular, it is found that wave groups are not observable in a developed sea. Finally, the theory is applied to estimating breaking wave statistics. A comparison with field observations is presented.

1. Introduction

A number of statistical characteristics of the rough sea surface have been studied by *Longuet-Higgins* [1957, 1962]. Many new results on the geometry of random fields have been obtained in recent years [*Nosko*, 1969; *Belyaev*, 1970; *Belyaev and Nosko*, 1974; *Adler*, 1981; *Vanmarcke*, 1983]. Some of these results appear to be useful in various geophysical applications and particularly in wave studies, remote sensing, and acoustics. The present work uses recent achievements of random field theory to study geometrical features of a two-dimensional, moving sea surface. Among our goals is the development of statistical theories for wave group and wave slope (surface's vertical acceleration) fields. This development requires generalization of certain results available in the theory of random fields. We also provide instructive interpretation of many mathematical relationships, emphasizing their physical meaning.

So far, most results of random field theory remained largely unemployed despite their high practical value. This is so partly because many important characteristics are expressed via spectral moments M_i

$$M_i = \int_0^\infty S(\omega)\omega^i \, d\omega \tag{1}$$

Starting from some i, this integral will not converge unless the spectrum $S(\omega)$ decays sufficiently fast. For example, all spectral moments of order $i \geq 4$ diverge in the case of wave spectra with Phillips' equilibrium range. Wave spectra with a less rapid decay in the high-frequency range, e.g., the spectrum $\sim\omega^{-4}$ proposed by *Zakharov and Filonenko* [1966] (also *Toba* [1973], *Kawai et al.* [1977], and *Phillips* [1985]) yield infinite spectral moments starting from $i = 3$. Moments L_i defined with respect to two-dimensional, spatial spectra diverge at even smaller i. Some of the difficulties have recently been resolved by redefining several statistical characteristics of the sea surface so as to avoid diverging integrals [*Longuet-Higgins*, 1983, 1984]. However, this approach does not help when the high-order moments constitute an essential feature of the problem. Examples are given by problems of acoustic wave and radio wave reflection from a random sea surface treated in the so-called stochastic Fourier transform approach [*Brown*, 1982, 1985], where the mean square continuity of the surface slope $\nabla\zeta$ and of higher-order derivative fields $\nabla^n\zeta$ is an essential requirement. Another example is the parameterization of breaking wave statistics, when values of M_i have to be found for i from 4 [*Snyder and Kennedy*, 1983; *Ochi and Tsai*, 1983] up to 8 [*Glazman*, 1985]. Usual means of handling problems involving high-order spectral moments include the use of a frequency cutoff in which the upper limit in (1) is replaced with some finite value. However, such a treatment has serious theoretical deficiencies and leads to inconsistent quantitative results.

One of the goals of the present work is to develop a general approach to problems involving spectral moments so as to make results of random field theory accessible for use in practical applications and particularly in wave studies. In what follows we present a systematic view of the spectral moment problem, highlight its physical meaning, and show its relationship with the theory of random fields. This is done in sections 2 and 3. A solution of the problem that is appropriate to the case of wind-generated surface waves is suggested in sections 4 and 5. The standpoint assumed in the present work, and originally proposed by *Vanmarcke* [1983] in the context of random field theory, relates the divergence of integrals like equation (1) to the fact that actual, either theoretical or experimental, spectra are fundamentally inadequate in the high-frequency domain and require low-pass filtering. The filtering technique developed in sections 4 and 5 generalizes the Vanmarcke approach to be applicable in the case of arbitrary order derivatives of a random field. Estimates of "filtered"

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spectral moments and spectral width measures are given in section 6 for the case of wave spectra characterized by the Phillips equilibrium range.

As an illustration of the inadequacy of the "raw" spectra, we investigate errors in the coherent reflection coefficient for vertically polarized L band radio waves, which arise when an unfiltered wave spectrum is employed, in section 7. Further applications of the filtering technique are presented (as a vital part of the theoretical development) when deriving statistical characteristics of the spatial and temporal variations of the wave envelope (section 8 and Appendix A) and wave slope (section 9 and Appendix B) fields. The present theory of the envelope statistics represents an extension to a spatial case of the corresponding one-dimensional theory developed by *Longuet-Higgins* [1957, 1962, 1984] (who also stressed [*Longuet-Higgins*, 1984] the necessity of low-pass filtering in studies of wave groups, although on different grounds).

The wave envelope and wave slope statistics are important in connection with mathematical modeling of breaking wave occurrence. A sample prediction of steep wave rates is presented and compared with field observations in section 9.

The consideration of the wave slope field allows one to expose a fractal pattern appearing in the sea surface when similarity laws like the Phillips saturation spectrum are employed without due regard for their applicability range. Thus in section 9 we also consider a cascade nature of sea surface geometry and illustrate the notions of the mean square differentiability and continuity of random fields by direct calculations.

Finally, in appendices A and B a simple interpretation of certain mathematical relationships appearing in modern theory of random fields is presented to facilitate their use and highlight their physical meaning.

2. Covariance Functions and Mean Square Continuity

In order to facilitate the subsequent analysis, some basic relationships are summarized in this section. For details the reader is referred to *Monin and Yaglom* [1975], *Kinsman* [1965], *Phillips* [1977], *Adler* [1981], *Rytov* [1976], and *Rytov et al.* [1978].

The surface is considered to be statistically stationary and homogeneous, which permits presenting it as (the real part of) the Fourier-Stieltjes integral:

$$\zeta(\mathbf{r}, t) = \iiint_{R^3} \exp\,[i(\omega t + \mathbf{kr})]\, dZ\,(\omega, \mathbf{k}) \tag{2}$$

where the vertical displacement ζ is reckoned from the zero-valued mean level $z = 0$; ω and $\mathbf{k}$ designate the temporal frequency and the spatial wave number vector, respectively, for wave components with orthogonal (complex) amplitude increments dZ. Vector $\mathbf{r} = (x_1, x_2)$ fixes a position on the plane $z = 0$. The integration in the three-dimensional space R^3 implies that ω and (k_1, k_2) vary from $-\infty$ to $+\infty$. Properties of dZ can be described by the following symbolic formula:

$$\langle dZ\,(\mathbf{A}_a)\, dZ^*\,(\mathbf{A}_b)\rangle = \delta(\mathbf{A}_a - \mathbf{A}_b)\Psi(\mathbf{A}_a)\, d\mathbf{A}_a\, d\mathbf{A}_b \tag{3}$$

in which the asterisk denotes a complex conjugate quantity. (The usefulness of extending the surface displacement field onto the complex plane is realized mainly when deriving statistics of the wave envelope (see section 8 and Appendix A). Until then the reader may ignore the fact that some of the quantities introduced possess an imaginary part.) The angular brackets denote an operator of ensemble averaging, $\delta(\ \ )$ is a three-dimensional Dirac delta function, $\mathbf{A}$ is a vector with components ω, k_1, k_2; and $\Psi(\mathbf{A})$ is a (real) three-dimensional spectral density function related to one- and two-dimensional spectra by

$$\Phi(\omega) = \iint \Psi(\omega, \mathbf{k})\, d\mathbf{k} \tag{4a}$$

$$X(\mathbf{k}) = \int \Psi(\omega, \mathbf{k})\, d\omega \tag{4b}$$

Where not indicated otherwise, the limits of integration are infinite.

Employing (2) through (4), one can relate the (complex) covariance functions U and W to the one- and two-dimensional wave spectra:

$$U_\zeta(\tau) = \langle\zeta(\mathbf{r}, t)\zeta^*(\mathbf{r}, t + \tau)\rangle = \int \exp\,(i\omega\tau)\Phi(\omega)\, d\omega \tag{5a}$$

$$W_\zeta(\boldsymbol{\rho}) = \langle\zeta(\mathbf{r}, t)\zeta^*(\mathbf{r} + \boldsymbol{\rho}, t)\rangle = \iint \exp\,(i\mathbf{k}\boldsymbol{\rho})X(\mathbf{k})\, d\mathbf{k} \tag{5b}$$

Furthermore, differentiating (2) and its complex conjugate with respect to either time or space coordinates, one finds covariance functions for the field's derivatives. In particular, the covariance function of the wave slope field becomes

$$W_{\nabla\zeta}(\boldsymbol{\rho}) = \langle\nabla\zeta(\mathbf{r}, t)\nabla\zeta^*(\mathbf{r} + \boldsymbol{\rho}, t)\rangle = \iint k^2 X(\mathbf{k}) \exp\,(i\mathbf{k}\boldsymbol{\rho})\, d\mathbf{k} = -\Delta W_\zeta(\boldsymbol{\rho}) \tag{6}$$

where $\Delta = \nabla \cdot \nabla$. Similar calculation leads to a generalization

$$W_{\nabla_n\zeta}(\boldsymbol{\rho}) = \iint k^{2n} X(\mathbf{k}) \exp\,(i\mathbf{k}\boldsymbol{\rho})\, d\mathbf{k} = (-1)^n \Delta^n W_\zeta(\boldsymbol{\rho}) \tag{7a}$$

For time derivatives we have

$$U_{\zeta^{(n)}}(\tau) = \int \omega^{2n}\Phi(\omega) \exp\,(i\omega\tau)\, d\omega = (-1)^n \partial^{2n} U_\zeta(\tau)/\partial\tau^{2n} \tag{7b}$$

Here $\zeta^{(n)}$ denotes an nth-order time derivative of ζ, and the subscripts in (5)–(7) point at a field with respect to which a given covariance function is defined.

Owing to the statistical stationarity and homogeneity of ζ,

$$W_{\nabla_n\zeta}(\boldsymbol{\rho}) = \int_0^\infty \int_0^\infty k^{2n} G_\zeta(\mathbf{k}) \exp\,(i\mathbf{k}\boldsymbol{\rho})\, d\mathbf{k} \tag{8a}$$

$$U_{\zeta^{(n)}}(\tau) = \int_0^\infty \omega^{2n} S_\zeta(\omega) \exp\,(i\omega\tau)\, d\omega \tag{8b}$$

and for standard deviations we have

$$\sigma_{\nabla_n\zeta}{}^2 = \int_0^\infty \int_0^\infty k^{2n} G_\zeta(\mathbf{k})\, d\mathbf{k} = (-1)^n \lim_{\rho\to 0} \Delta^n W_\zeta(\boldsymbol{\rho}) = L_{\zeta,2n} \tag{9a}$$

$$\sigma_{\zeta^{(n)}}{}^2 = \int_0^\infty \omega^{2n} S_\zeta(\omega)\, d\omega = (-1)^n \lim_{\tau\to 0} \partial^{2n} U_\zeta(\tau)/\partial\tau^{2n} = M_{\zeta,2n} \tag{9b}$$

Here $M_{\zeta,i}$ is a spectral moment defined with respect to a "one-sided" spectral density function $S_\zeta(\omega) = 2\Phi(\omega)$, where subscript ζ denotes the field whose spectrum is implied. $L_{\zeta,i}$ is a similar property for a one-sided two-dimensional spatial spectrum

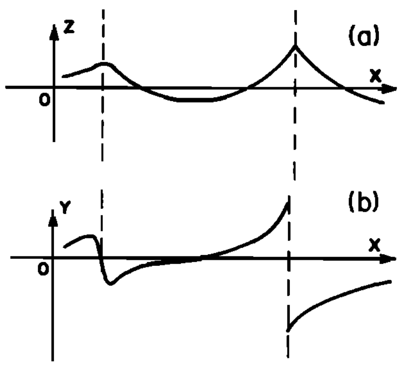

Fig. 1. (*a*) The surface elevation profile $\zeta(x)$. (*b*) The surface's derivative profile $Y(x) = \partial\zeta/\partial x$.

$G_\zeta(k_1, k_2) = 4X(k_1, k_2)$ of $\zeta(x_1, x_2; t)$. Both $S_\zeta(\omega)$ and $G_\zeta(k_1, k_2)$ are assumed to be zero for negative values of their arguments.

If improper integrals in (9) are finite, then so are the integrals in (8), and as is shown by (9), the finite spectral moments guarantee the existence of mean square derivatives (i.e., σ with appropriate subscripts) of a random field. The mean square differentiability of a random field ensures continuity, in the mean square sense, of a field obtained by differentiating. The mean square continuity, in turn, is necessary for the existence of the Fourier-Stieltjes representation for the derivative field. If, for instance, $L_{\zeta,2}$ is infinite, the wave slope field $\nabla\zeta$ has neither representation in the form of (2) nor continuous spectrum density functions of the type given by (4).

Let m be the dimension of a spectrum defined for field ζ, and let n be the order of ζ's derivative. Then the rate of decay of the spectrum as $A \to \infty$ (A is the m-dimensional generalized frequency) must be greater than A^{-2n-m} to ensure the mean square continuity of the nth-order derivative field. Particularly, for $\langle|\nabla\zeta|^2\rangle$ to exist, $G_\zeta(\mathbf{k})$ must decay faster than k^{-4} as $k \to \infty$, and for $\langle|\partial^2\zeta/\partial t^2|^2\rangle$ to exist, $S_\zeta(\omega)$ must decay faster than ω^{-5}.

Concluding this summary, we relate a one-dimensional frequency spectrum with a two-dimensional wave vector spectrum. From (4) it follows that

$$\int \Phi(\omega)\, d\omega = \iint X(\mathbf{k})\, d\mathbf{k} = \int_0^\infty k\, dk \int_0^{\pi/2} G(k, \Theta)\, d\Theta$$
$$= \int_0^\infty (dk/d\omega)k(\omega)\, d\omega \int_0^{\pi/2} G(k(\omega), \Theta)\, d\Theta \qquad (10)$$

where $\mathbf{k}$ is presented in polar coordinates, and a dispersion relation $k = k(\omega)$ is assumed to exist. Equation (10) means that

$$S_\zeta(\omega) = k(\omega)\, dk/d\omega \int_0^{\pi/2} G(k(\omega), \Theta)\, d\Theta \qquad (11)$$

In many applications, $G(k, \Theta)$ is approximated by a product of a directionality factor $\Upsilon(\Theta)$ and a wave number modulus factor $\chi(k)$. The commonly used, although not very accurate, form of $\Upsilon(\Theta)$ is given by

$$\begin{aligned} \Upsilon(\Theta) &= (4/\pi)\cos^2\Theta \qquad &&\text{for } 0 \le \Theta \le \pi/2 \\ \Upsilon(\Theta) &= 0 &&\text{otherwise} \end{aligned} \qquad (12)$$

thus allowing one to obtain rough estimates of many statistical properties of surface geometry when only $S_\zeta(\omega)$ is known. The scaling coefficient in (12) assures $\int_0^\infty \chi(k)k\, dk = \sigma_\zeta^2$.

3. Ill-Posed Problems of Sea Surface Statistical Geometry

Starting from some frequency value, any wave spectrum, either theoretical or experimental, becomes invalid. An illustration is provided by common spectral models whose high-frequency tail is identified with Phillips' equilibrium range.

In the course of reaching an equilibrium, the spectrum width increases, and the surface ceases to be a well-behaved function with finite high-order derivatives. A limiting shape of the sea surface corresponding to the energy supply-dissipation equilibrium is characterized by sharp crests whose local curvature radii tend to zero (Figure 1). Such crests are ubiquitous (see section 9), and the resulting discontinuity (in the mean square sense) of the wave slope field represents a basic property of the developed sea. *Kitaigorodskii et al.* [1975] even take this property as a starting point for deriving one-dimensional frequency spectra corresponding to particular types of dispersion relationships. Hence (in order to violate the mean square differentiability conditions of section 2) the spatial spectrum of a developed sea must fall off at large wave numbers at least as slowly as k^{-4} (a law originally obtained on dimensional grounds [*Phillips*, 1958]).

Though appropriate for gravity waves, a k^{-4} spectrum is not realistic at very high wave numbers. For instance, infinitely small radii of curvature at wave crests are unrealistic because they require an infinitely large capillary pressure. Being interested in large-scale features, one is reluctant to complicate a spectrum model by adding capillary-gravity, viscocapillary and other higher-frequency subranges. The fact that (5)–(9) call for such information, even for rather small n, means that the problems of statistical geometry have to be treated as ill-posed problems whose solution is unstable with respect to small perturbations in input data. This statement can be readily formalized by substituting into equation (9*b*) $S_a(\omega) = S_0(\omega) + \delta S(\omega)$, where S_a is the spectrum actually available, S_0 is the "true" spectrum, and $\delta S(\omega)$ is the error which, unlike $S_0(\omega)$, does not necessarily decrease as the frequency increases. Therefore the inadequacy of actual spectra at high frequencies in conjunction with a high sensitivity of the integrals in (5)–(9) to the high-frequency content of wave spectra necessitate special treatment.

One approach, that of *Vanmarcke* [1983], reduces to employing a low-pass filter, (22) or (14), which suppresses small-scale oscillations, thereby making the result of integration independent of the indeterminate component. A different approach was undertaken by *Belyaev and Piterbarg* [1972], who studied overshoot statistics for a random field that does not satisfy the mean square differentiability requirement. The resulting infinitely large local density of overshoot events was forced by these authors to become finite by introducing a procedure that in effect combines a cluster of multiple overshoots into a single isolated event. Another approach was formulated by *Adler* [1981], who treated "erraticism" of random fields by employing the Hausdorff ("fractal" in *Mandelbrot*'s [1982] terminology) dimension concept.

The approach proposed in the present work extends the filtering technique to arbitrary order derivatives by treating the surface elevation field as specified on a spatial (temporal) running grid. A mesh size, entering the calculations, represents a "yardstick" (Mandelbrot's term) that makes the statistical description consistent in terms of limitations of the (physical) theory that has yielded a given (spectral) model of sea waves. This yardstick is determined as an intrinsic property of a given theoretical model of sea waves. It characterizes a high-frequency bound of the model's validity.

4. Partial Averaging and Spectral Moments

The use of filters, including the "perfect" filter given by (14) and (22), in obtaining spectrum estimates has a long history [e.g., *Tukey*, 1959]. However, until Vanmarcke's [1983] work the filtering was not used as a means for dealing with mean square nondifferentiability of random fields. We shall call the procedure of applying a low-pass filter "partial averaging", thereby emphasizing that the wave field for which a model spectrum is available is viewed as a macroscale component $\bar{\zeta}$ of the actual field:

$$\zeta = \bar{\zeta} + \zeta' \tag{13}$$

4.1. *Spatial Averaging*

Suppose we have a two-dimensional spectrum $X(\mathbf{k})$ as the model of a random field. The macroscale component is defined as

$$\bar{\zeta}(x_1, x_2; t) = (1/A_1A_2) \iint_{\Sigma x_1, x_2} \zeta(X_1', X_2'; t)\, dx_1'\, dx_2' \tag{14}$$

Field ζ' will be referred to as microscale. Surface area Σ_{x_1,x_2} represents a running rectangular window with coordinates $(x_1 \pm A_1/2,\ x_2 \pm A_2/2)$, where A_i, with $i = 1, 2$, are such that all spectral components with wave numbers $k_i \le 2\pi/A_i$ pertain to the large-scale field whose spectral model $X(\mathbf{k})$ is specified. Substituting (2) into (14), one derives a Fourier-Stieltjes representation for the averaged field:

$$\bar{\zeta}(x_1, x_2, t) = \iiint V(k_1A_1)V(k_2A_2) \exp\,[i(\omega t + \mathbf{kr})]\, dZ\,(\omega, \mathbf{k}) \tag{15}$$

where

$$V(a) = \sin\,(a/2)/(a/2) \tag{16}$$

The calculations that previously led to (5) now yield

$$\bar{U}_\zeta(\tau) = \langle \bar{\zeta}(\mathbf{r}, t)\bar{\zeta}^*(\mathbf{r}, t + \tau)\rangle = \int \exp\,(i\omega\tau)\, d\omega \iint \Psi(\omega, \mathbf{k})V^2(k_1A_1)V^2(k_2A_2)\, d\mathbf{k} \tag{17}$$

$$\bar{W}_\zeta(\rho) = \iiint \Psi(\omega, \mathbf{k})V^2(k_1A_1)V^2(k_2A_2) \exp\,(i\mathbf{k}\rho)\, d\mathbf{k}\, d\omega \tag{18}$$

In (18) and below, an overbar signifies that the quantity pertains to the macroscopic field. Defining the macroscopic spectra by

$$\bar{X}(\mathbf{k}) = X(\mathbf{k})V^2(k_1A_1)V^2(k_2A_2) \tag{19a}$$

$$\bar{\Phi}(\omega) = \iint \Psi(\omega, \mathbf{k})V^2(k_1A_1)V^2(k_2A_2)\, d\mathbf{k} \tag{19b}$$

it is easy to show that they are related to the corresponding covariance functions by the Wiener-Khinchine relationships, and an equation analogous to (4) holds

$$\bar{\Phi}(\omega) = \iint \bar{\Psi}(\omega, \mathbf{k})\, d\mathbf{k} \tag{20a}$$

where

$$\bar{\Psi}(\omega, \mathbf{k}) = \Psi(\omega, \mathbf{k})V^2(k_1A_1)V^2(k_2A_2) \tag{20b}$$

Macroscopic analogs of (10) and (11) also follow in a straightforward way: all spectral functions simply acquire a bar. In particular, one finds

$$\bar{S}_\zeta(\omega) = k(\omega)\frac{dk}{d\omega}\left\{\int_0^{\pi/2} G(k, \Theta)V^2(k_1A_1)V^2(k_2A_2)\, d\Theta\right\}_{k=k(\omega)} = \left\{k\frac{dk}{d\omega}\int_0^{\pi/2} \bar{G}(k, \Theta)\, d\Theta\right\}_{k=k(\omega)} \tag{21}$$

However, derivation of macroscopic analogs for (7)–(9) requires additional effort. This will be described later in this section, when we introduce the finite difference representation of a field's derivatives.

Obviously, the function V given by (16) has only a small effect on the macroscale, energy-containing portion of the spectra. However, when $k_iA_i \gg 2$, spectrum $\bar{G}$ falls off as $G(\mathbf{k})k^{-4}$. Hence the contribution of the high-frequency portion becomes unimportant, and moments of wave number spectrum $\bar{G}(\mathbf{k})$ exist up to order $(n + 3)$, where $n = 4$ in the case of the Phillips spectrum. This makes possible estimation of spatial covariance functions for the wave slope $\nabla\bar{\zeta}$ and the surface "curvature" $\Delta\bar{\zeta}$ fields. The question arises as to the higher-order moments' existence. Intuitively one may anticipate that the spectral moments of a smoothed field should remain finite to an arbitrary order. This, however, is not confirmed by straightforward computations using (19*a*). Later in this section we shall clarify this important matter by constructing a general method of partial averaging.

4.2. *Time Averaging and High-Order Spectral Moments*

In many applications the wave statistics are expressed via one-dimensional frequency spectra because such spectra are best studied. Let us present the procedure for the evaluation of arbitrary order spectral moments for this particular case. However, this development can be readily extended to higher-dimension fields. For the sake of convenience we continue denoting the macroscopic quantities by an overbar, though a new notation ought to be introduced to distinguish time-averaged quantities from the space-averaged ones.

The one-dimensional version of (15) and (17) is derived by using time averaging

$$\bar{\zeta}(t) = (1/T) \int_{t-T/2}^{t+T/2} \zeta(t')\, dt' \tag{22}$$

along with a one-dimensional version of (2) and (3). The final results are

$$\bar{\zeta}(t) = \int V(\omega T) \exp\,(i\omega t)\, dZ\,(\omega) \tag{23}$$

$$\bar{U}_\zeta(\tau) = \int V^2(\omega T) \exp\,(i\omega\tau)\Phi(\omega)\, d\omega \tag{24}$$

Obviously, the one-dimensional macroscopic spectrum must be defined as

$$\bar{\Phi}(\omega) = \Phi(\omega)V^2(\omega T) = \tfrac{1}{2}S_\zeta(\omega)V^2(\omega T) \tag{25}$$

in order to satisfy the Wiener-Khinchine relationships. One can readily prove the existence of macroscopic analogs for (4) and (10)–(12), wherein the time-averaged version of the three-dimensional spectrum is now

$$\bar{\Psi}(\omega, \mathbf{k}) = \Psi(\omega, \mathbf{k})V^2(\omega T) \tag{26}$$

The first five moments of a wave spectrum characterized by the Phillips equilibrium range (ω^{-5}), averaged in accord with equation (25), can be easily computed:

$$\bar{M}_{\zeta,i} = \int_0^\infty V^2(\omega T)\omega^i S_\zeta(\omega)\, d\omega \tag{27}$$

The sixth-order moment computed in such a straightforward way tends to infinity. However, as was noted by *Glazman* [1985], direct application of (27) for $i \geq 4$ is inconsistent with the essence of the partial averaging. Let us, recalling (9), obtain even order spectral moments as variances of the $\bar{\zeta}$ derivatives. Apparently (Leibnitz's rule), the definition of the averaging process (equation (22)) is equivalent to

$$\partial\bar{\zeta}/\partial t = \frac{1}{T}[\zeta(t + T/2) - \zeta(t - T/2)] \tag{28}$$

In other words, the derivative of the averaged process is related to variation over a finite time period T. For this reason, for example, an equation

$$(\partial/\partial t)(\partial\bar{\zeta}/\partial t) = \frac{1}{T}\left[\partial\zeta/\partial t\bigg|_{t+T/2} - \partial\zeta/\partial t\bigg|_{t-T/2}\right] \tag{29}$$

does not represent the second-order derivative of the averaged process, for it involves derivatives $\partial\zeta/\partial t$ which carry information about the variations of the (unfiltered) process $\zeta(t)$ over an infinitesimal time interval. A consistent definition of the second-order derivative of the averaged process is

$$\begin{aligned}\partial^2\bar{\zeta}/\partial t^2 &= \frac{1}{T}\left[\partial\bar{\zeta}/\partial t\bigg|_{t+T/2} - \partial\bar{\zeta}/\partial t\bigg|_{t-T/2}\right]\\ &= T^{-2}[\zeta(t+T) - 2\zeta(t) + \zeta(t-T)]\end{aligned} \tag{30}$$

Thus we in effect explicitly introduce a (temporal) "yardstick" and discard all oscillations with time scales under T.

The Fourier-Stieltjes representation of the derivatives can now be obtained using the finite-difference formulae, equations (28), (30), etc. After a little algebra, one arrives at

$$\partial\bar{\zeta}/\partial t = i\int \omega \exp(i\omega t)V(\omega T)\, dZ(\omega) \tag{31a}$$

$$\partial^2\bar{\zeta}/\partial t^2 = -\int \omega^2 \exp(i\omega t)V^2(\omega T)\, dZ(\omega) \tag{31b}$$

And in general,

$$\partial^n\bar{\zeta}/\partial t^n = i^n\int \omega^n \exp(i\omega t)V^n(\omega T)\, dZ(\omega) \tag{31c}$$

Using (3) and (31), the macroscopic analogs of (7*b*), (8*b*), and (9*b*) can now be obtained for all $n \geq 1$. Specifically,

$$\begin{aligned}\bar{U}_{\zeta^{(n)}}(\tau) &= \langle \bar{\zeta}^{(n)}(t)\bar{\zeta}^{*(n)}(t+\tau)\rangle\\ &= \int \omega^{2n}V^{2n}(\omega T)\Phi(\omega)\exp(i\omega\tau)\, d\omega\end{aligned} \tag{32}$$

A macroscopic analog of (9*b*) yields

$$\bar{M}_{\zeta,2n} = \int_0^\infty \omega^{2n}V^{2n}(\omega T)S_\zeta(\omega)\, d\omega \tag{33}$$

Hence the variances $\sigma_{\bar{\zeta}^{(n)}}{}^2 = \bar{M}_{\zeta,2n}$ of the macroscopic field derivatives differ in the power of the averaging function V. It is easy to see that the even order moments given by (33) exist for all n provided that the spectrum S_ζ decays faster than ω^{-1}. Therefore the ultimate limitation imposed on the ζ field is the requirement of its continuity in the mean square sense.

We define odd order moments, to be consistent with (32), as

$$\bar{M}_{\zeta,2n+1} = \int_0^\infty \bar{S}_{\zeta^{(n)}}(\omega)\omega\, d\omega \qquad (= \bar{M}_{\zeta^{(n)},1}) \tag{34}$$

where

$$\bar{S}_{\zeta^{(n)}}(\omega) = V^{2n}(\omega T)S_{\zeta^{(n)}}(\omega) \tag{35}$$

is the macroscopic spectrum of the n-order derivative process. Unlike M_{2n}, the odd order moments cannot be presented as statistical properties of linear transforms of a random process. They emerge only when nonlinear transformations (e.g., a wave envelope) are introduced.

In a similar manner, the averaging procedure can be extended to two-dimensional quantities to obtain spectral moments of $\bar{G}_\zeta(\mathbf{k})$. Apparently, the finite difference representation of a random field (equation (30) and the like) provides a convenient formalism generalizing the "running" average (equations (14) and (22)).

Having shown that the fundamental relationships of section 2 hold for macroscopic spectra as well, we can employ (4) through (12) in the subsequent development.

5. Intrinsic Microscales

In some applications the averaging scales are clearly indicated by the problem under consideration. For instance, studying acoustic wave or radio wave reflection in the framework of a two-scale model [*Kur'yanov*, 1962], one separates large-scale surface features $\bar{\zeta}$ from small-scale roughness ζ' on the basis of the acoustic wavelength or radio wavelength and incidence angle. The values of A_i may then be related to the axes of the first-order Fresnel zone on the mean surface or in some other fashion [e.g., *Bahar et al.* [1983]).

As was noted in section 3, actual wave spectra become inadequate in the high-frequency domain. A question arises as to the limiting frequency value ($2\pi/T$), characterizing an individual spectrum, beyond which the inadequate spectral information is to be discarded. Specifically, one is tempted to select T (or A_i) as corresponding to a low-frequency boundary of an additional range spanning frequencies higher than those under primary consideration. For instance, being interested in wind-generated gravity waves, one tends to relate T to a characteristic period of the longest ripples affected by surface tension. Then the T and thereby all subsequent (averaged) properties will contain a parameter (surface tension coefficient) irrelevant to the given problem. Alternatively [*Glazman*, 1985], the averaging scale can be introduced as an intrinsic property of a given spectral model. Indeed, the characteristic period of the most rapid oscillations parametrized by a spectral model is determined by the rate of change of the corresponding autocorrelation coefficient $b(\tau) = U(\tau)/U(0)$ in the vicinity of the origin. Owing to the assumption of statistical stationarity, $[db/d\tau]_{\tau=0} = 0$, this rate is fully characterized by $[d^2b/d\tau^2]_{\tau=0}$, which yields T as

$$\begin{aligned}T &= [-(d^2b/d\tau^2)^{-1}]_{\tau=0}{}^{1/2} = [-U(\tau)/(d^2U/d\tau^2)]_{\tau=0}{}^{1/2}\\ &= (M_0/M_2)^{1/2}\end{aligned} \tag{36}$$

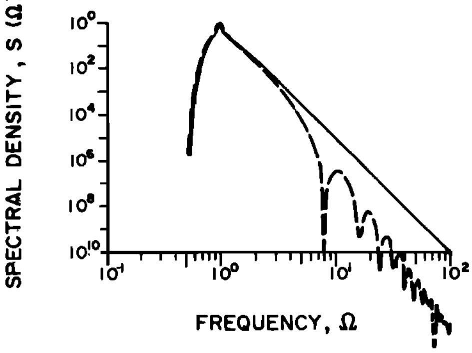

Fig. 2. Dimensionless JONSWAP spectrum. The solid line represents the original spectrum, and the dashed line represents the spectrum averaged in accord with (14).

Now, the "yardstick" T is extracted from the autocorrelation function. We shall call it a Taylor microscale. This equation (and, generally, a Taylor microscale of any random field) is meaningful only if function ζ is mean square differentiable.

Thus we have formally expanded the definition of the Taylor microscale to encompass not only spatial but also temporal variations, and we have applied it to the ζ rather than fluid velocity field. In place of usual hydrodynamical arguments underlying the Taylor microscale concept [e.g., *Batchelor*, 1953], we have employed formal properties of random functions. Hence the present approach can be viewed as rather heuristic.

The direct spatial analog of (36) is the surface area of the averaging window $\Sigma = -W_\zeta(0)/\Delta W_\zeta(0) = L_0/L_2$. If $X(\mathbf{k})$ falls off as k^{-4} or slower, this quantity equals zero, which signifies that for a developed sea the $\zeta(\mathbf{r})$ field is not mean square differentiable. Technically, the absence of a Taylor microscale means that the inertial subrange extends to infinitely large frequencies. Since this is impossible, we have to redefine the spatial microscales A_i and Σ so as to avoid calculation of the spatial derivatives of ζ. To this end, note that although $\Sigma = 0$ for the Phillips spectra, T remains finite. This happens because (as we find again later, in sections 8 and 9) for gravity water waves, spatial variations of surface vertical displacement are much more erratic than temporal variations. For capillary waves the situation is reversed because the phase speed increases with the wave number.

We will consider T as a well-defined microscale and obtain values of A_i as its products. This task reduces to employing dispersion relationship to express A_i via T, and is straightforward if the spectrum can be represented as a wave number modulus factor $\chi(k)$ times a directionality factor $\Upsilon(\Theta)$. If, however, the spectrum is more complex, we can only determine Σ (equal to $A_1 \times A_2$), setting it equal to $(gT^2/2\pi)^2$.

A remarkable feature of the averaging scales identified with the Taylor microscale is that the macroscopic spectra preserve self-similarity. An example is given in Figure 2, where the "raw" and macroscopic Joint North Sea Wave Project (JONSWAP) spectra are plotted in a normalized form (equation (38*b*)). For a Gaussian process, the T is $1/2\pi$ times the mean period of ζ zero upcrossings.

6. Examples for Gravity Wind Wave Spectra

In practice, one deals with spectra $S(\omega)$ of the real part of $\zeta(t)$, defined for $\omega \geq 0$. The correct relationship between $S(\omega)$ and $S_\zeta(\omega)$, which becomes important in section 8, is given by $2S(\omega) = S_\zeta(\omega)$. Respectively, the spectral moments M_i defined for $S(\omega)$ are found from

$$2M_i = M_{\zeta,i} \tag{37a}$$

In particular,

$$M_{\zeta,0} = \sigma_\zeta^2 = 2\sigma^2 \tag{37b}$$

Let us consider the Pierson-Moskowitz (P-M) and the "mean" JONSWAP spectra. Introducing a dimensionless frequency $\Omega = \omega/\omega_0$, one has

$$S(\omega) = \alpha g^2 \omega_0^{-5} s(\Omega) \tag{38a}$$

where

$$s(\Omega) = E(\Omega)\Omega^{-5} \tag{38b}$$

is the dimensionless spectral density function, α is the Phillips constant, ω_0 is the spectral peak frequency, and $E(\Omega) \to 1$ for $\Omega \gg 1$. For the P-M spectrum

$$E(\Omega) = \exp(-1.25\Omega^{-4})$$

and for the mean JONSWAP spectrum

$$E(\Omega) = \exp(-1.25\Omega^{-4})p^{\exp[-(\Omega-1)^2/2q^2]}$$

with the peak-enhancing factor $p = 3.3$, $q = 0.07$ for $\Omega \leq 1$, and $q = 0.09$ for $\Omega > 1$. Finally, we define dimensionless spectral moments m_i as follows:

$$M_i = \alpha g^2 \omega_0^{i-4} m_i \tag{39a}$$

$$m_i = \int_0^\infty s(\Omega)\Omega^i \, d\Omega \tag{39b}$$

and a dimensionless averaging scale $\tau = (m_0/m_2)^{1/2}$; τ is related to T (equation (36)) by $T = \tau/\omega_0$. The dimensionless spectral moments of the averaged record (note that (33)–(35) do not change their form when the real part of ζ is implied) will be denoted by $\bar{m}_i$. Calculations yield $\tau = 0.710$ and $\tau = 0.777$ for the P-M and the JONSWAP spectra, respectively. In Table 1, numerical values for moments of order 0 to 8 are presented. One finds that the zero-order moments of the averaged spectra differ from their raw counterparts by only 7.62% and 7.65% for the P-M and JONSWAP spectra, respectively. As the order of the moment increases, the discrepancy grows.

Longuet-Higgins [1952] and *Cartwright and Longuet-Higgins* [1956] defined the spectrum width measures δ and ε:

$$\delta = (1 - m_1^2/m_0 m_2)^{1/2} \tag{40a}$$

$$\varepsilon = (1 - m_2^2/m_0 m_4)^{1/2} \tag{40b}$$

Values of these measures, based on the data of Table 1, are presented in Table 2 for the raw spectrum $S(\omega)$ and for the spectrum of the averaged record $\bar{S}(\omega)$. In Table 2 these mea-

TABLE 1. Values for Nondimensional Spectral Moments m_i

	P-M Spectrum		JONSWAP Spectrum	
Order i	m_i	$\bar{m}_i$	m_i	$\bar{m}_i$
0	0.2000	0.1848	0.3050	0.2816
1	0.2591	0.2316	0.3656	0.3279
2	0.3963	0.3263	0.5046	0.4196
3	0.8572	0.5389	0.9679	0.6192
4		0.7998		0.8255
5		1.7420		1.6079
6		2.9284		2.4465
7		7.9703		6.0465
8		14.2859		9.8784

TABLE 2. Spectral Width Measures (Equation (40))

	P-M Spectrum		JONSWAP Spectrum	
	Raw	Averaged	Raw	Averaged
δ	0.391	0.332	0.363	0.300
ε	1	0.529	1	0.493

sures are given based on the data of Table 1. The averaging produces only a small effect on the δ measure, whereas the ε measure changes from an uninformative 1 (corresponding to a white noise spectrum) to physically meaningful values. Owing to the peak enhancement factor, the JONSWAP spectrum is narrower than the P-M spectrum. The averaging does not change the difference, since it does not affect the energy-containing portion of the spectra. Moreover, it is due to the averaging that the bandwidth difference between the P-M and the JONSWAP spectra becomes apparent not only in terms of the δ measure but also in terms of the ε measure.

In the following sections we shall illustrate the effect of averaging for several cases where the order of spectral moments spans the entire range of our Table 1.

7. COHERENT REFLECTION COEFFICIENT EVALUATION

In this section we demonstrate that although the corresponding integrals of the raw spectrum may be finite, the averaging is still necessary in order to ensure the consistency of the result with respect to a theoretical model selected.

In a small perturbation approximation the complex reflection coefficient for the coherent part of a specularly reflected, vertically polarized radio wave at shallow grazing incidence is given [*Bass and Fuks*, 1979] by $R = (\sin\psi - \eta_e)/(\sin\psi + \eta_e)$, where ψ is the grazing angle and η_e is the effective impedance of the rough surface. For a statistically isotropic surface

$$\eta_e = \eta + (k/16\pi)^{1/2}(i-1)\int_0^\infty r^{-3/2}(dW(r)/dr)\,dr \tag{41}$$

where η is the component due solely to dielectrical properties of seawater, $k = 2\pi/\lambda$ with λ being the radio wavelength, and the second term in (41) describes the "geometric part" of the surface impedance parameterizing the loss of energy into an irregularly scattered field. $W(r)$ is the spatial covariance function for the statistically isotropic sea.

The problem of estimating a coherent reflection coefficient arises in remote interferometric measurements of dielectrical and geometrical properties of reflecting surfaces [e.g., *Tang et al.*, 1977; *Glazman*, 1982*a*, *b*] as well as in analysis of satellite communication links involving ocean surface reflection [e.g., *Fung et al.*, 1982; *Karasawa and Shiokawa*, 1984].

In addition to a small-perturbation assumption, (41) presumes that $\lambda \ll \Lambda$, where Λ characterizes the length scale of the surface spatial variation. That is, small-length-scale surface roughness is ignored in this theory. For L band systems this means that the subject of interest is the impact of dominant wind waves, while the effect of ripples on the coherent reflection is insignificant.

Reducing the geometric term in (41) to an integral over the sea wave frequency [see *Glazman*, 1982*a*]) one finds that

$$\eta_e = \eta + (i-1)\frac{k^{1/2}\Gamma(1/4)}{8(2\pi)^{3/2}\Gamma(7/4)}\int_0^\infty (\omega^2/g)^{3/2}S(\omega)\,d\omega \tag{42}$$

The third-order spectral moment can be evaluated by a straightforward integration if the wave spectrum decays faster than ω^{-4}. However, Table 1 shows that m_3 is greater than $\bar{m}_3$ by about 60%. Therefore if the raw spectrum is employed, the impact of gravity wind waves on the coherent reflection will be greatly exaggerated as a result of inadequacy of the raw spectral models.

8. WAVE GROUP STATISTICS ON THE PLANE

The knowledge of wave group size and time duration is important in many problems [*Longuet-Higgins*, 1984]. In this section we extend to a two-dimensional spatial configuration the treatment of wave group statistics presented by *Longuet-Higgins* [1957, 1984] for a one-dimensional case. This extension demonstrates that even with a "lower-order" definition of the length of a wave group, employed by Longuet-Higgins to avoid the infinite fourth-order moment (appearing in the case of *Rice*'s [1945] definition), consideration of the M_4 moment is unavoidable when studying a spatial wave field.

A wave group is determined using the mean rate and length, or area, of excursions by the envelope R of a random process (or field) beyond some fixed level H (Figure 3). For a one-dimensional case the basic result of Rice is given by (see, for example, *Longuet-Higgins* [1984])

$$N_1 = \frac{1}{\pi^{1/2}}\left(\frac{M_{\dot{R},0}}{M_{R,0}}\right)^{1/2}\frac{H}{M_0^{1/2}}\exp\left(-\frac{H^2}{2M_0}\right) \tag{43}$$

where N_1 is the mean number of excursions by $R(t)$ beyond level H in unit time. $M_{\dot{R},0}$ is defined as $\langle(\partial R/\partial t)^2\rangle$, and $M_{R,0} = \langle R^2\rangle$. For a Gaussian process, $M_{\dot{R},0} = \delta^2 M_{\zeta,0}$, where $M_{\dot{\zeta},0} = \langle(\partial\zeta/\partial t)^2\rangle = M_{\zeta,2} = 2M_2$; δ is the spectral width measure defined by (40*a*). Within the framework of section 2, the envelope R of a surface elevation field can be defined as

$$R^2(\mathbf{r}, t) = \zeta\zeta^* = \xi^2 + \eta^2 \tag{44}$$

where real functions ξ and η appearing in the decomposition $\zeta = \xi + i\eta$ are related by a pair of Hilbert transforms. A definition of the Hilbert transform for two-dimensional random fields is given by *Adler* [1978]. Equations (2)–(5) and (37), employed with the first equality in (44), readily yield basic statistics of R: $\langle R^2\rangle = 2\sigma^2$, $\langle R\dot{R}\rangle = 0$, and $\langle R\partial R/\partial x_i\rangle = 0$.

However, much more complicated analysis is necessary in order to obtain a two-dimensional generalization of Rice's envelope equation (43) [*Adler*, 1978]. For high levels of H ($H^2 \gg M_{R,0}$), the Adler result takes the form

$$N_2 = \frac{1}{\pi}\frac{|\mu|^{1/2}}{M_{R,0}}\frac{H^2}{M_0}\exp\left(-\frac{H^2}{2M_0}\right) \tag{45}$$

The 2×2 matrix $\{\mu\}$ is given (see Appendix A) by (A10) so that

$$2\mu_{ij} = \int_0^\infty \int_0^{\pi/2} (k_1 - k_{01})^i(k_2 - k_{02})^j G(k, \theta)k\,dk\,d\theta \tag{46a}$$

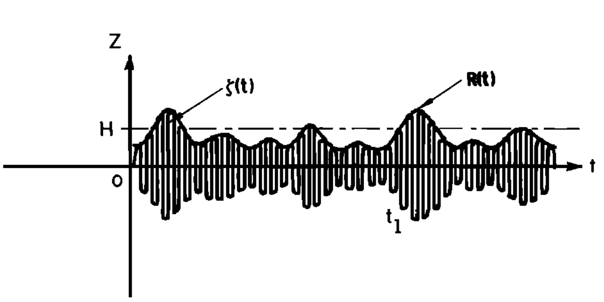

Fig. 3. The envelope of a random process crossing a constant level H; only three upcrossings occur during the time interval $(0, t_1)$.

with $i + j = 2$, where the elements are ordered as

$$\mu_{ij} = \begin{bmatrix} \mu_{20} & \mu_{11} \\ \mu_{11} & \mu_{02} \end{bmatrix} \tag{46b}$$

and $k_1 = k \cos \theta$, $k_2 = k \sin \theta$.

According to Adler, vector $\mathbf{k}_0 = (k_{01}, k_{02})$ represents a two-dimensional (spatial) frequency corresponding to the spectrum peak, whereas the two-dimensional spectrum $G_\zeta(k_1, k_2)$ is assumed to be symmetric about the peak. The latter limitation, however, can be relaxed (see Appendix A), provided that the spectrum is sufficiently narrow. As a result, $\mathbf{k}_0$ becomes the "mean" frequency, defined as

$$\mathbf{k}_0 = \iint \mathbf{k} G_\zeta(\mathbf{k})\, d\mathbf{k} \Big/ \iint G_\zeta(\mathbf{k})\, d\mathbf{k} \tag{47}$$

Now the theory is fully compatible with traditional treatments, and the speed of envelope translation can be shown to coincide with wave group velocity [*Longuet-Higgins*, 1957]. Matrix $\{\mu\}$ appears to be nearly diagonal (see Appendix A), and its determinant is approximately

$$|\mu| = \mu_{20}\mu_{02} = L_2^{(1)}L_2^{(2)}\Delta^2 \tag{48}$$

where $\Delta = \delta_1\delta_2$ denotes a product of "directional" spectral width measures.

$$\delta_i = (1 - [L_1^{(i)}]^2/[L_2^{(i)}L_0])^{1/2} \tag{49}$$

and the marginal spectral moments are given by

$$2L_n^{(i)} = \int_0^\infty \int_0^\infty G_\zeta(k_1, k_2) k_i^n \, dk_1 \, dk_2 \tag{50}$$

Obviously, $L_0^{(1)} = L_0^{(2)} = L_0 = M_0 = \sigma^2$.

If $G_\zeta(\mathbf{k})$ decays at large k as k^{-4} or slower, the second-order spectral moments $L_2^{(i)}$ tend to infinity, and the familiar problem of the mean square discontinuity of the field $\nabla\zeta$ arises. Formally, the envelope level-crossing rate, (equation (45)) for an unfiltered field becomes infinitely large, while (43) still yields a finite temporal rate. Therefore spatial variations of the field R are more erratic than its temporal oscillations recorded at a fixed location.

Now we can estimate the ratio $Q_2 = \nu_{\zeta,2}/N_2$ of the mean excursion rate $\nu_{\zeta,2}$ of the surface elevation field to the mean excursion rate N_2 of the wave envelope field. This ratio is interpreted as the mean number of waves in a wave group (i.e., the clump size in *Vanmarcke*'s [1983] terminology or the run of high waves in *Longuet-Higgins*' [1984]). In Appendix B an asymptotic expression for $\nu_{\zeta,2}$ is presented. For high levels of H the clump size becomes (see also *Lyon* [1961] and *Vanmarcke* [1983]):

$$Q_2(H) = \nu_{\zeta,2}/N_2 = \frac{1}{(2\pi)^{1/2}\Delta} \frac{\sqrt{M_0}}{H} \tag{51}$$

For a $\cos^2 \Theta$ dependence on wind direction (equation (12)), and the deep water dispersion relationship, equations (11), (49), and (50) yield (we omit the calculations, as they are simple although rather lengthy)

$$\delta_1^{\,2} = 1 - \frac{256}{27\pi^2} \frac{M_2^{\,2}}{M_0 M_4} \tag{52a}$$

$$\delta_2^{\,2} = 1 - \frac{64}{9\pi^2} \frac{M_2^{\,2}}{M_0 M_4} \tag{52b}$$

The result is now presented via the well-known wave spectrum moments. Assuming that the moments entering (52) pertain to the macroscopic field, one can employ the data of Table 1. Specifically, for the averaged P-M spectrum we get $Q_2(H) = 1.036\sqrt{M_0}/H$, and for the JONSWAP spectrum we get $Q_2(H) = 1.134\sqrt{M_0}/H$. Therefore the observer whose eye distinguishes only waves of significant height ($H \approx 2\sqrt{M_0}$) will not see wave clusters on the surface: the fully developed sea is too erratic to exhibit a group structure. For wave groups to appear, the spectral width product Δ must be noticeably smaller than $1/(2\sqrt{2\pi})$. This may be the case in the initial stages of wave field development.

For a one-dimensional wave record one can readily establish, using the Rice theory, that (51) reduces to

$$Q_1(H) = \nu_{\zeta,1}/N_1 = \frac{1}{(2\pi)^{1/2}\delta} \frac{\sqrt{M_0}}{H} \tag{53}$$

where δ is given by (40*a*) and subscript 1 designates temporal (one-dimensional) rates. Employing values in Table 2, one finds that the mean clump size for a point record is nearly the same as that for a two-dimensional wave field. This agreement can be viewed also as a credit to the averaging period T selection made in section 5. Without the averaging, $\Delta = 1$ although $\delta < 1$.

9. Steep Wave Occurrence and Fractal Geometry of the Surface

Occurrence of waves whose steepness exceeds a certain specified value γ is of great interest. For example, such waves are commonly associated with events of large-scale wave breaking [*Longuet-Higgins and Smith*, 1983]. Here a theory of steep wave occurrence proposed by *Glazman* [1985] is revised, and further insight into the statistical geometry of the sea surface is offered.

For deep water waves in the absence of currents, the wave slope field $\nabla\zeta$ is statistically similar to the field of the local vertical acceleration $Y = -\partial^2\zeta/\partial t^2$. Indeed, employing (2)–(4) along with the dispersion relationship $\omega^2 = kg$, one finds

$$\begin{aligned} W_Y(\boldsymbol{\rho}) &= \left\langle \frac{\partial^2\zeta(\mathbf{r}, t)}{\partial t^2} \frac{\partial^2\zeta^*(\mathbf{r}+\boldsymbol{\rho}, t)}{\partial t^2} \right\rangle \\ &= g^2 \iint k^2 \exp(i\mathbf{k}\boldsymbol{\rho}) X(\mathbf{k})\, d\mathbf{k} \\ &= g^2 W_{\nabla\zeta}(\boldsymbol{\rho}) = -g^2 \Delta W_\zeta(\boldsymbol{\rho}) \end{aligned} \tag{54}$$

Also, $U_Y(\tau) = g^2 U_{\nabla\zeta}(\tau)$. Consequently, if the wave slope (modulus) is assigned a critical value γ (equal to ak), the corresponding threshold for the downward acceleration becomes γg. Then one can identify the temporal frequency of steep wave events with the mean rate $\nu_{Y,1}$ of excursions by the process $Y(t)$ beyond level $\Gamma = \gamma g$. (Actually, a small correction ought to be introduced into the value of γ to account for the impact of hydrodynamic nonlinearity on the dispersion relationship.) The final result, based on the *Rice* [1944] solution of the level crossing problem, is given by

$$\nu_{Y,1}(\Gamma) = (1/2\pi)(M_6/M_4)^{1/2} \exp(-\Gamma^2/2M_4) \tag{55}$$

If the spectrum is narrow, the factor $P_{Y,1}(\Gamma) = \exp(-\Gamma^2/2M_4)$ can be interpreted [*Glazman*, 1985] as the probability of encountering in the Y record an oscillation whose amplitude exceeds the threshold value Γ, whereas factor $\tau_Y = (2\pi)(M_4/M_6)^{1/2}$ can be viewed as the amount of space allo-

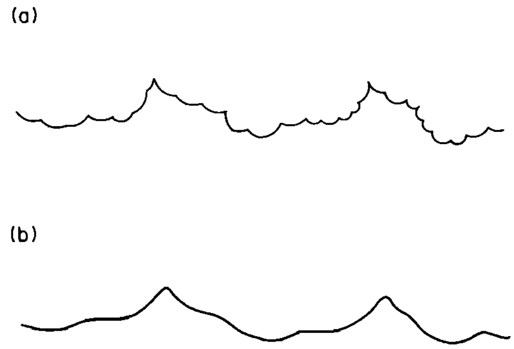

Fig. 4. A qualitative illustration of a surface elevation profile when the surface can be envisioned as a fractal: (*a*) the profile presented as a cascade of similar "elementary" wave shapes characterized by sharp crests and shallow troughs, and (*b*) the same surface after partial averaging.

cated to the mean Y wave in the time domain. In other words, $\nu_{Y,1}(\Gamma) = P_{Y,1}(\Gamma)/\tau_Y$, which affords an additional insight into the geometrical meaning of the slope field mean square discontinuity. Namely, a wave spectrum decaying as ω^{-5} yields $P_{Y,1}(\Gamma) \to 1$ for arbitrarily high values of Γ, while $\tau_Y \to 0$. In the language of fractal geometry, the plane (t, Y) is completely filled by the function $Y(t)$, and therefore Y's fractal (Hausdorff) dimension is 2. *Pfeifer* [1984] gives a more formal way of evaluating the fractal dimension for cases similar to ours, and his formula leads to the same value 2.

Equation (55) with M_i of the averaged wave record allows one to estimate the mean number of wave crests per "basic" wave (Figure 4). Apparently, this number can be defined as the ratio of the mean period of a full wave cycle τ_ζ to the mean period of a full $Y(t)$ cycle. The former is found as the mean period of zero upcrossings by $\zeta(t)$. Ultimately,

$$n_1 = \tau_\zeta/\tau_Y = \nu_{Y,1}(0)/\nu_{\zeta,1}(0) = (M_0 M_6/M_2 M_4)^{1/2} \tag{56}$$

For the "mean" JONSWAP spectrum, Table 1 yields $n_1 = 1.41$, and it is easy to show that the "raw" spectra would yield n_1 going to infinity.

A two-dimensional generalization of the Rice formula is sketched in Appendix B, and an interpretation is given to the factors determining ν_2. Raising the order of all the spatial spectrum moments entering (B3) by 2 [as is dictated by (54)], one arrives at the surface density of steep wave events:

$$\nu_{Y,2}(\Gamma) = \frac{1}{(2\pi)^{3/2}} \frac{|\Lambda|^{1/2}}{L_{Y,0}} \frac{\Gamma}{L_{Y,0}{}^{1/2}} \exp\left(-\frac{\Gamma^2}{2L_{Y,0}}\right) \tag{57}$$

where by steep waves we imply again waves whose local downward acceleration at the crest exceeds Γ. Matrix $\{\Lambda\}$ consists of elements Λ_{ij}

$$2\Lambda_{ij} = -\frac{\partial^2 W_Y(0, 0)}{\partial x_1{}^i \, \partial x_2{}^j} = g^2 \iint k^2 k_1{}^i k_2{}^j X(k_1, k_2)\, dk_1\, dk_2 \tag{58a}$$

ordered similarly to the elements in (46*b*), and

$$2L_{Y,0} = g^2 \iint k^2 X(\mathbf{k})\, d\mathbf{k} = g^2 L_{\zeta,2} \tag{58b}$$

The diagonal elements of $\{\Lambda\}$ can be presented in the form $2\Lambda_{ij} = L_{Y,2}{}^{(i)}$, where

$$2L_{Y,2}{}^{(i)} = g^2 \int_0^\infty \int_0^\infty G_\zeta(k_1, k_2) k^2 k_i{}^2\, dk_1\, dk_2 \qquad i = 1, 2 \tag{59}$$

The mean surface area occupied by a full, two-dimensional cycle of the $Y(x_1, x_2)$ field (a Y wave) is obtained as the product of orthogonal wave lengths (see Appendix B for detail):

$$\Sigma_Y = \frac{(2\pi)^2 L_{Y,0}}{[L_{Y,2}{}^{(1)} L_{Y,2}{}^{(2)}]^{1/2}} \left(= \frac{4\pi^2 L_{Y,0}}{|\Lambda|^{1/2}}\right) \tag{60}$$

The number of wave crests per basic wave on the surface, obtained in the fashion of (56), is

$$n_2 = \Sigma_\zeta/\Sigma_Y = \frac{L_{\zeta,0}}{L_{Y,0}} \frac{|\Lambda_{ij}|^{1/2}}{|\lambda_{ij}|^{1/2}} \tag{61a}$$

where Σ_ζ is given by (B1).

In a developed, "theoretical" sea the $\zeta(x_1, x_2)$ field is not differentiable at any point on the surface, which is the first sign of a surface's fractal geometry [*Mandelbrot*, 1982]. One then envisions the $\zeta(x_1, x_2)$ as a random field characterized by cusps whose spatial frequency of occurrence is infinitely large. This can appear only as a cascade pattern wherein each surface wave carries smaller-scale waves and each of those is again a carrier of yet smaller waves and so on. At each scale a wave is characterized by a sharp crest, and higher-order waves are similar to their carrier (Figure 4*a*). As a result, the mean square discontinuity of the $Y(\mathbf{r})$ field is due to irregular Y jumps taking place with probability of 1 within any arbitrarily small domain. Such behavior is peculiar to a Brownian sheet [see *Adler*, 1981]. The power form (i.e., k^{-p}) of $X(\mathbf{k})$, together with the fact that the exponent p of the spectrum decay remains unchanged as k increases, results in a self-similarity of wave shapes at different scales. In the Phillips case, $p = 4$, which determines the shape of the "generator" (i.e., sharp crests).

In order to estimate the quantities expressed by (57)–(61*a*), one may employ (11) and (12) to reduce the spatial spectral moments to the frequency spectrum moments. After rather lengthy calculations one arrives at

$$n_2 = M_0 M_8/M_4{}^2 \tag{61b}$$

$$\nu_{Y,2} = \frac{1}{(2\pi)^{3/2}} \frac{\sqrt{3}}{8g^2} \frac{M_8}{M_4} \frac{\Gamma}{M_4{}^{1/2}} \exp\left(-\frac{\Gamma^2}{2M_4}\right) \tag{62}$$

Using dimensionless moments of the averaged spectra (section 6), one finds that for the P-M spectrum case, $n_2 = 4.13$ and for

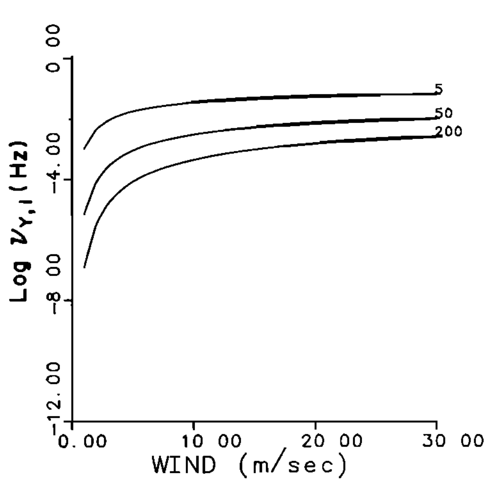

Fig. 5. Temporal rate of steep wave occurrence as a function of wind speed and fetch. The values of the wind fetch (in kilometers) are plotted on the right of each curve. The limiting slope $\gamma = 0.3$.

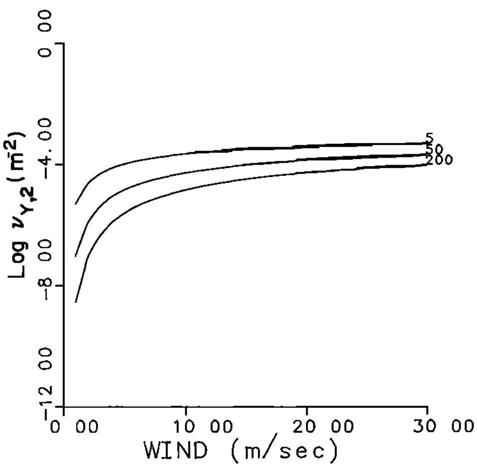

Fig. 6. Spatial rate of steep wave occurrence as a function of wind speed and fetch. The values of the wind fetch (in kilometers) are plotted on the right of each curve. The limiting slope $\gamma = 0.3$.

the mean JONSWAP spectrum, $n_2 = 4.08$. The role of the partial averaging (equations (14) and (22)) is to truncate the infinite cascade of sharp-crested waves and to round off the crests of the retained waves, as in Figure 4*b*. Reducing the value of T would increase both n_1 and n_2, making them infinite in the limit of $T \to 0$.

Furthermore, (62) can be written in terms of the Phillips constant α and the spectral peak frequency ω_0 if the wave spectrum satisfies (38). Equations (38) and (39) yield

$$\nu_{Y,2} = \frac{\beta \omega_0{}^4}{g^2} \frac{m_8}{m_4} \frac{\gamma}{(\alpha m_4)^{1/2}} \exp\left(-\frac{\gamma^2}{2\alpha m_4}\right) \tag{63}$$

where $\beta = \sqrt{3}/[8(2\pi)^{3/2}]$, and an overbar is omitted although (temporal) averaging is implied. The factors that determine the probability of steep waves include, in this particular case, only the Phillips constant α. A departure from the k^{-4} behavior (as, for example, that proposed recently by *Phillips* [1985]) would bring about additional parameters (including ω_0) in the probability of steep wave occurrence. Therefore the *Phillips* [1958] equilibrium (ω^{-5} or k^{-4}) represents an approximation that in a certain sense is consistent with the principles of the breaking wave statistics models (briefly reviewed by *Glazman* [1985]) based on a wave steepness (or vertical acceleration) threshold.

In Figures 5 and 6 the temporal and spatial frequencies of steep wave occurrence are plotted for different wind velocities and fetch lengths. The mean JONSWAP spectrum was employed along with empirical laws for α and ω_0 as functions of wind speed and fetch, suggested by *Hasselmann et al.* [1976]. The 0.3 value of slope was selected as a guess for a "criterion" of wave breaking. The guessing was based largely on *Longuet-Higgins*' [1985] analysis of the distribution of the vertical acceleration along the wave profile and also on the fact that the averaging procedure employed leads to a reduction in the slope (and vertical acceleration) of the macroscopic field: such a reduction is greater at the crests than at the troughs. In Figure 7 the temporal frequency is presented for a fixed value of fetch roughly corresponding to the conditions of *Thorpe and Humphries*' [1980] observations in Loch Ness. Three different values of the limiting slope are used. The comparison with the observations suggests that by appropriately adjusting γ one may be able to forecast breaking wave statistics by identifying them with statistics of the waves whose steepness exceeds γ. The adequacy of such an approach depends on whether the "critical" γ is a "universal constant" (for the sea at equilibrium).

The discrepancy with the experimental data, shown in Figure 7, is mainly due to the fact that the measurements account not only for the breaking events commencing at the moment of the measurement but also for the breaking events that started within the preceding t s, where t is the (mean) duration of the wave breaking process.

10. CONCLUSIONS

Sea surface geometry is amenable to statistical characterization, although the spectral models supplied by physical theories or direct measurements are valid only within a limited range of frequencies. The approach developed in sections 4 and 5 is in a sense opposite to that of fractal geometry: we explicitly introduced into the statistical description a temporal/spatial scale ("yardstick") characterizing the "resolution" of a given physical theory or observational technique. Upon transforming to the macroscopic quantities, all basic relationships among various spectra remain in force. Therefore one can apply recent achievements of random field theory to problems of sea surface statistical geometry, remote sensing, acoustics, etc.

In a developed sea the "raw" spectra yield a greatly overestimated (although converging) spectral moment M_3, which may lead to considerable errors in applications. Hence the averaging may be necessary not only as a means of dealing with diverging integrals but also as a way of making quantitative estimates consistent with respect to the assumptions of an underlying physical theory.

The *Adler* [1978, 1981] theory for the envelope of a Gaussian random field, extended to the case of an asymmetric spectral density function (Appendix A of the present paper), allowed us to develop a statistical description of wave groups on the two-dimensional sea surface. In particular, it was found that in a developed sea the wave groups are not observable: the surface displacement field is too erratic to exhibit the group structure.

On the basis of *Nosko*'s [1969] asymptotic rates of high-level excursions, we have derived statistics of a two-dimensional field of sea surface vertical acceleration (that provides an estimate of wave slope). This permitted in particular

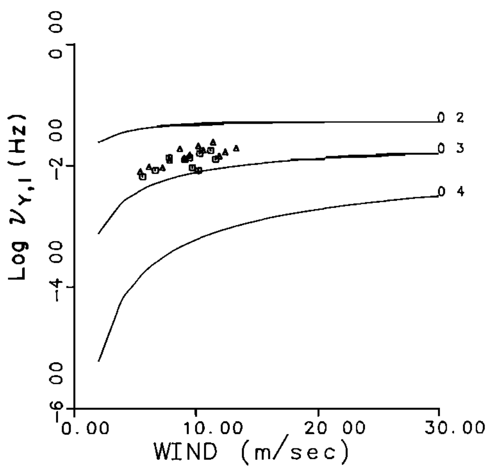

Fig. 7. Temporal rate of steep wave occurrence as a function of wind speed and limiting slope, at a fixed fetch of 10 km. The values of the limiting slope γ are plotted on the right of each curve. Marks represent *Thorpe and Humphries* [1980] measurements of breaking wave (triangles) and bubble cloud (squares) occurrence rates.

an evaluation of the mean number n_2 of "secondary waves" per "primary wave". Note that for a temporal case, an analogous quantity n_1, given by (56), can be easily (almost visually) inferred from an actual wave record. Furthermore, the temporal and spatial rates of steep wave occurrence, given by (55), (57), and (62), may be useful in predicting basic statistics of large-scale breaking waves. However, more work must be done, especially in the area of breaking wave observations, in order to see whether there exists a universal value of Γ effectively quantifying wave breaking conditions (for a developed sea). Such an effort appears to be worthwhile, in view of the fact that dynamically based theories involve considerable difficulties and may need much elaboration before yielding working formulas for engineering applications.

We have noted that high-level excursions of both the wave envelope and the wave slope fields, being Poisson processes, have a statistical prototype in the occurrence of gas molecules within a specified volume. An asymptotic theory for excursion rates also allowed us to illustrate the notion of mean square discontinuity of a random surface and to quantify a cascade pattern appearing in the surface topography. We hope that the techniques presented in this work may be useful in analyses of other random fields encountered in ocean and atmosphere studies.

APPENDIX A

Here the Adler result on the two-dimensional envelope excursion rates is extended to the case of an asymmetric wave number spectrum. Also, an instructive interpretation of the quantities entering (45) is given.

The derivation of the excursion set statistics for the envelope $R(\mathbf{r})$ field requires statistical independence between the real a and imaginary b parts of a slowly varying complex amplitude field $A(\mathbf{r}) = a(\mathbf{r}) + ib(\mathbf{r})$:

$$A(\mathbf{r}) = 2 \iint \exp\,[i(\mathbf{k} - \mathbf{k}_0)\mathbf{r}]\, dZ\,(\mathbf{k}) \tag{A1}$$

Adler [1981] has shown that the $a(\mathbf{r})$ field is statistically similar to the field $R(\mathbf{r})$ defined by (44). More precisely,

$$\left\langle \frac{\partial^i a(\mathbf{r})}{\partial x_1{}^i} \frac{\partial^j a(\mathbf{r}+\boldsymbol{\rho})}{\partial x_2{}^j} \right\rangle = \left\langle \frac{\partial^i R(\mathbf{r})}{\partial x_1{}^i} \frac{\partial^j R(\mathbf{r}+\boldsymbol{\rho})}{\partial x_2{}^j} \right\rangle \tag{A2}$$

where $i + j = 2$. The field A is related to the basic field ζ by

$$\zeta(\mathbf{r}) = \exp\,(i\mathbf{k}_0\mathbf{r})A(\mathbf{r}) \tag{A3}$$

where $\exp\,(i\mathbf{k}_0\mathbf{r})$ can be viewed as a carrier surface, and the integration in (A1) (and in (A4)) is done over an upper half plane of $\mathbf{k}$, which provides a properly defined Hilbert transform $\eta(\mathbf{r})$ for the $\zeta(\mathbf{r})$ field [*Adler*, 1978]. The cross correlation for the Hilbert transform pair is found, as usual, to be $B_{\eta\zeta}(\mathbf{r}) = \langle \zeta(\boldsymbol{\rho})\eta(\boldsymbol{\rho} + \mathbf{r})\rangle = \iint \sin\,(\mathbf{kr})\ \mathbf{X}\ (\mathbf{k})\ d\mathbf{k}$, with $B_{\eta\zeta}(-\mathbf{r}) = -B_{\eta\zeta}(\mathbf{r})$. These results are necessary in order to obtain

$$B_{ab}(\mathbf{r}) = \langle a(\boldsymbol{\rho})b(\boldsymbol{\rho} + \mathbf{r})\rangle = \iint \sin\,[(\mathbf{k} - \mathbf{k}_0)\mathbf{r}]X(\mathbf{k})\, d\mathbf{k} \tag{A4}$$

[*Adler*, 1981]. Obviously, $B_{ab}(\mathbf{r})$ becomes zero if the spectral density function $X(\mathbf{k})$ is symmetric about the peak (spatial) frequency $\mathbf{k}_0$. This determines the choice of $\mathbf{k}_0$ in (A1) and (A3), and the desired result is at hand (since the absence of correlation in this particular case leads to statistical independence) [*Adler*, 1978]. A much simpler, heuristic, derivation of the envelope excursion statistics has been proposed by *Vanmarcke* [1983]. It does not require the symmetry of $X(\mathbf{k})$ about the peak frequency. However, Vanmarcke's argument contains a flaw leading to an erroneous final result.

The ocean wave spectra are essentially asymmetric, and the Adler theory is not applicable. However, for a sufficiently narrow spectrum the statistical independence between a and b can be obtained as an asymptotic property by an appropriate redefinition of $\mathbf{k}_0$. Assume that the spectrum falls off in all directions away from the spectrum peak frequency so rapidly that the sine factor in (A4) can be replaced by the first two terms in its Taylor expansion about this frequency:

$$2B_{ab}(\mathbf{r}) \approx \iint \sin\,[(\mathbf{k}_p - \mathbf{k}_0)\mathbf{r}]G_\zeta(\mathbf{k})\, d\mathbf{k} + \iint \cos\,[(\mathbf{k}_p - \mathbf{k}_0)\mathbf{r}]\mathbf{r}(\mathbf{k} - \mathbf{k}_p)G_\zeta(\mathbf{k})\, d\mathbf{k} \tag{A5}$$

Here we have employed the one-sided spectral density function $G_\zeta(\mathbf{k})$ and, consequently, the integration is carried out over the first quadrant of the $\mathbf{k}$ plane. The spectrum peak frequency is denoted by $\mathbf{k}_p$, whereas the "carrier" frequency $\mathbf{k}_0$ is yet to be determined. Naturally, one anticipates that $\mathbf{k}_0$ is very close to $\mathbf{k}_p$, which would permit replacing the sine and cosine factors in (A5) by the Taylor expansions about the infinitesimal phase. Neglecting $O([\mathbf{r}(\mathbf{k}_p - \mathbf{k}_0)]^2)$ compared with 1, which is permissible to do for all $\mathbf{r}$ within the radius of correlation, (A5) reduces to

$$2B_{ab}(\mathbf{r}) \approx (\mathbf{k}_p - \mathbf{k}_0)\mathbf{r} \iint G_\zeta(\mathbf{k})\, d\mathbf{k} + \mathbf{r} \iint (\mathbf{k} - \mathbf{k}_p)G_\zeta(\mathbf{k})\, d\mathbf{k} \tag{A6}$$

It is now evident that the desired result,

$$B_{ab}(\mathbf{r}) = 0 \tag{A7}$$

is obtained as an asymptotic formula (the more accurate, the narrower the wave spectrum) by selecting $\mathbf{k}_0$ as is indicated in (47) (which becomes equivalent to Adler's formulation in the special case of a spectrum symmetric about $\mathbf{k}_0$: $\mathbf{k}_0 = \mathbf{k}_p$).

The spatial covariance function of the real amplitude field $a(\mathbf{r})$ has been shown by Adler to be

$$2B_a(\mathbf{r}) = \iint \cos\,[(\mathbf{k} - \mathbf{k}_0)\mathbf{r}]G_\zeta(\mathbf{k})\, d\mathbf{k} \tag{A8}$$

Furthermore, it is easy to show that the covariance matrix of $a(\mathbf{r})$'s spatial derivatives is given by:

$$\langle(\partial^i a/\partial x_1{}^i)(\partial^j a/\partial x_2{}^j)\rangle = -\partial^2 B_a(\mathbf{r})/\partial x_1{}^i \partial x_2{}^j \tag{A9}$$

Obviously, $i + j = 2$. The matrix of spectral moments, μ_{ij}, is obtained as

$$\mu_{ij} = -\frac{\partial^2 B_a(0, 0)}{\partial x_1{}^i \partial x_2{}^j} \qquad \left(= \left\langle \frac{\partial^i R}{\partial x_1{}^i} \frac{\partial^j R}{\partial x_2{}^j} \right\rangle \right) \tag{A10}$$

The last equality has been written on the basis of (A2) as a reminder that the determinant $|\mu|$ plays in (45) a role similar to that of $M_{\dot{R},0}$ in (43). Upon substitution of (A8) into (A10), one arrives at (46).

The geometrical meaning of $|\mu|$ emerges when calculating a zero-level ($U = 0$) excursion rate for the two-dimensional field $a(\mathbf{r})$, or equivalently, for field $R(\mathbf{r})$. This rate has been shown by Adler to be

$$N_s(0) = \pi^{-1}|\mu|^{1/2}/M_{R,0} \tag{A11}$$

Therefore the covariance matrix determinant is inversely proportional to the mean area occupied on a horizontal plane by an individual two-dimensional oscillation cycle of field $a(\mathbf{r})$. Consequently, the factors multiplying $N_s(0)$ in equation (45) yield a relative number of the $a(\mathbf{r})$ waves crossing simultaneously the (high) level H. This interpretation is tightly connected with the assumption of Poisson behavior of the field's high excursions (see, for example, *Belyaev* [1970] and *Vanmarcke* [1983]): each such excursion can be likened to a gas molecule (in a two-dimensional space) that either occurs or does not occur within a specified area independently of all other particle positions at a given instant.

Equation (A7) makes the matrix $\{\mu\}$ diagonal. Its determinant, and thereby (the reciprocal of) the mean surface area occupied by an a field wave, is obtained as a product of two marginal spectral moments of order 2, each moment characterizing the mean spatial oscillation frequency of an envelope defined for a one-dimensional random function $\xi(x_i)$. Thus the relations between (45) and a spatial (one-dimensional) version of (43) become apparent, facilitating generalizations to higher dimensions, for example, in the manner of *Vanmarcke*'s [1983] heuristic treatment. However, the spatial frequency matrix $\{\mu\}$ should be determined in the fashion shown above and not as was erroneously proposed by Vanmarcke.

Now we have to check the applicability of the narrow-spectrum assumption, introduced when deriving (A7)–(A8) and (47)–(48), to the commonly accepted spectral models of sea waves. One finds that (46) and (47) yield $\mu_{ij} = L_2^{(1)}L_2^{(2)}\mu_{ij}^0$, where

$$\mu_{ij}^0 = \begin{vmatrix} \delta_1^2 & \beta\delta_{12} \\ \beta\delta_{12} & \delta_2^2 \end{vmatrix} \tag{A12}$$

with δ_i given by (49), and

$$\beta^2 = [L_{11}^{(12)}]^2/L_2^{(1)}L_2^{(2)} \tag{A13}$$

$$\delta_{12}^2 = 1 - L_1^{(1)}L_1^{(2)}/L_0L_{11}^{(12)} \tag{A14}$$

$$2L_{11}^{(12)} = \iint k_1k_2G_\zeta(\mathbf{k})\, d\mathbf{k} \tag{A15}$$

As usual, we express all spatial spectral moments L via moments M of the one-dimensional wave spectrum, employing again (11) and (12) along with the deep water dispersion relationship. After rather lengthy calculations one arrives at the following numerical values obtained for the averaged P-M spectrum case using Table 1:

$$\mu_{ij}^0 = \begin{vmatrix} 0.308 & 0.018 \\ 0.018 & 0.481 \end{vmatrix} \tag{A16}$$

This demonstrates that the contribution of nondiagonal elements is indeed negligibly small, even for such a relatively broad spectrum as the P-M, and that (46)–(48) provide a good approximation.

APPENDIX B

A two-dimensional version $\nu_{\zeta,2}$ of Rice's level-crossing rate $\nu_{\zeta,1}$ is presented here employing the results reported by *Nosko* [1969] (see also *Belyaev and Nosko* [1974], *Adler* [1981], and *Vanmarcke* [1983]). We set forth an interpretation of factors that determine $\nu_{\zeta,2}$ in the special case of high-level excursions when the excursion events occur in the manner of a Poisson process. As in all previous cases, the surface is treated as a Gaussian homogeneous random field.

The two-dimensional rate $\nu_{\zeta,2}$ can be viewed as a surface density, i.e., the mean number of events per unit area occurring simultaneously. Suppose that the wave spectrum is sufficiently narrow to make the $\zeta(r)$ field appear as a set of individual waves characterized by a (two-dimensional) spatial period and wave crest elevation h. The mean two-dimensional spatial period Σ_ζ for these waves is determined as a product of two mean one-dimensional spatial periods (wavelength components) of the ζ field variations in orthogonal directions:

$$\Sigma_\zeta = \frac{4\pi^2 L_0}{[L_2^{(1)}L_2^{(2)}]^{1/2}} \tag{B1}$$

where the one-dimensional spatial periods are defined as the zero-upcrossing mean periods. (A form invariant with respect to the rotation of coordinate axes involves the determinant of the spectral moments matrix $|\lambda|$ [e.g., *Vanmarcke*, 1983].) Let the probability density function for wave maxima, observed simultaneously on the sea surface, be $p(h)$. Then the relative number of the waves surpassing level H is given by $P_{\zeta,2}(H) = \int_H^\infty p(h)\, dh$. For high H (practically, $H \geq 2\sigma$) the excursion events behave as a Poisson process, and therefore their surface density can be presented as

$$\nu_{\zeta,2} = P_{\zeta,2}(H)/\Sigma_\zeta \tag{B2}$$

where $1/\Sigma_\zeta$ can be identified with the total number of waves per unit surface area.

Although $\nu_{\zeta,2}$ is usually derived in a direct way (while $P_{\zeta,2}$ is extremely difficult to get even for the large H), the heuristic argument leading to (B2) is useful. For instance, comparing the known formula for ν_2 [e.g., *Adler*, 1981, Theorem 5.4.1] with (B2), one can immediately arrive at an asymptotic expression for $p(h)$, which is more accurate the greater the value of h and the smaller the spectral width measure Δ.

The asymptotic formula for $\nu_{\zeta,2}$ at high H is

$$\nu_{\zeta,2} = \frac{1}{(2\pi)^{3/2}}\frac{|\lambda|^{1/2}}{L_0}\frac{H}{L_0^{1/2}}\exp\left(-\frac{H^2}{2L_0}\right) \tag{B3}$$

Here

$$2\lambda_{ij} = -\frac{\partial^2 W_\zeta(0,0)}{\partial x_1^i\, \partial x_2^j} = \iint k_1^i k_2^j X(\mathbf{k})\, d\mathbf{k} \tag{B4}$$

with the infinite integration limits, and $i + j = 2$. The matrix elements are ordered similarly to those in (46*b*). For a spatially homogeneous random field, nondiagonal elements can be made zero. Taking this into account and comparing (B1), (B2), and (B3), one infers

$$|\lambda|^{1/2}/(2\pi)^2 M_0 = 1/\Sigma_\zeta \tag{B5}$$

The probability of an excursion event is

$$P_{\zeta,2}(H) = \frac{(2\pi)^{1/2}H}{M_0^{1/2}}\exp\left(-\frac{H^2}{2M_0}\right) \tag{B6}$$

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