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PROPAGATOR MATRICES IN ELASTIC WAVE AND VIBRATION PROBLEMS†

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The boundary value problems most frequently encountered in studies of elastic wave propagation in stratified media can be formulated in terms of a finite number of linear, first order, ordinary differential equations with variable coefficients. Volterra (1887) has shown that solutions to such a system of equations are conveniently represented by the product integral, or propagator, of the matrix of coefficients.

In this paper we summarize some of the better known properties of propagators plus numerical methods for their computation. When the dispersion relation is some mth order minor of the integral matrix it is possible to deal with mth minor propagators so that the dispersion relation is a single element of the mth minor integral matrix. In this way one of the major sources of loss of numerical accuracy in computing the dispersion relation is avoided.

Propagator equations for SH and for P-SV waves are given for both isotropic and transversely isotropic media. In addition, the second minor propagator equations for P-SV waves are given. Matrix polynomial approximations to the propagators, obtained from the method of mean coefficients by the Cayley-Hamilton theorem and the Lagrange-Sylvester interpolation formula, are derived.

1. FIRST ORDER EQUATIONS

Most computations for stratified elastic wave-guides are made with the use of some approximation to the stratification. The most common approximation is to represent the stratification as a sequence of homgeneous layers. Such an approximation leads to the application of matrix methods attributed to Thomson (1950) and Haskell (1953), and modified by Rosenbaum (1964), Dunkin (1965), and others, or to an equivalent method (Knopoff, 1964).

In this paper we describe a general method, first used by Volterra (1887). It includes the Thomson-Haskell method and Knopoff's method as special cases.

To fix ideas and to minimize algebraic details we examine the problem of SH wave propagation in a stratified, isotropic half-space. In a rectangular cartesian coordinate system (x, y, z) the surface of the half-space is normal to the z axis. The density, ρ, and the Lamé parameters, λ and μ, are piecewise continuous functions of z. The displacement has only a y component, $u(x, z, t)$. The doubly transformed displacement is

$$U(k, z, \omega) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt\, u(x, z, t) \cdot \exp(i\omega t - ikx). \tag{1.1}$$

The constitutive relation is

$$T = \mu dU/dz, \tag{1.2}$$

and the linearized equation for the conservation of linear momentum is

$$dT/dz = (\mu k^2 - \rho\omega^2)U. \tag{1.3}$$

Equations (1.2) and (1.3) are written in matrix form

$$\frac{d}{dz}\begin{bmatrix} U \\ T \end{bmatrix} = \begin{bmatrix} 0 & \mu^{-1} \\ \mu k^2 - \rho\omega^2 & 0 \end{bmatrix} \cdot \begin{bmatrix} U \\ T \end{bmatrix}, \tag{1.4}$$

which is a special case of

$$\frac{d}{dz}\begin{bmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{bmatrix}. \tag{1.5}$$

2. PROPAGATORS FOR ORDINARY LINEAR DIFFERENTIAL EQUATIONS

Let $\mathbf{A}(z)$ be an $n\times n$ matrix of complex valued functions $A_{ij}(z)$ of the real variable z. Let $\mathbf{f}(z)$ be an $n\times 1$ column matrix of complex valued func-

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tions $f_i(z)$ of the real variable z. The matrix equation

$$d\mathbf{f}/dz = \mathbf{A}(z)\mathbf{f}(z) \qquad (2.1)$$

is a system of n linear homogeneous ordinary differential equations for the functions $f_i(z)$, $i=1, \cdots, n$. An $n\times n$ matrix $\mathbf{F}(z)$ of complex functions $F_{ij}(z)$ is called an integral matrix of (2.1) if

$$d\mathbf{F}/dz = \mathbf{A}(z)\mathbf{F}(z). \qquad (2.2)$$

The integral matrices of (2.1) are those $n\times n$ matrices each of whose columns is a solution of (2.1).

An integral matrix of (2.1) is called a fundamental matrix of (2.1) (Coddington and Levinson, 1955, 68) if it is nonsingular for every z in its domain of definition. An integral matrix, $\mathbf{F}(z)$, of (2.1) is called a propagator from z_0 if $\mathbf{F}(z_0)$ is the $n\times n$ identity matrix. We prefer the word "propagator" to Gantmacher's word "matricant" (1959, *2*, 113) because the former is more descriptive and is sanctioned by analogy with other initial-value problems.

When $A_{ij}(z)$ are continuous functions of z it is well known (Coddington and Levinson, 1955, 20) that for any complex $n\times 1$ column matrix $\mathbf{b}$ and any z_0 there is exactly one solution, $\mathbf{f}(z)$, of (2.1) such that $\mathbf{f}(z_0)=\mathbf{b}$. It follows that for any $n\times n$ matrix $\mathbf{B}$ and any z_0 there is exactly one solution, $\mathbf{F}(z)$, of (2.2) such that $\mathbf{F}(z_0)=\mathbf{B}$. In particular. for any z_0, (2.1) has a unique propagator from z_0 which we will denote by $\mathbf{P}(z, z_0)$. The matrices $\mathbf{F}=\mathbf{P}(z, z_0)$ and $\mathbf{F}=\mathbf{P}(z, z_1)\mathbf{P}(z_1, z_0)$ are both solutions of (2.2) and are equal when $z=z_1$. Therefore the uniqueness theorem assures their equality for all z. For any z_1, z_2, z_3,

$$\begin{aligned}\mathbf{P}(z_3, z_1) &= \mathbf{P}(z_3, z_2)\mathbf{P}(z_2, z_1)\\ &= \prod_{l=2}^{3} \mathbf{P}(z_l, z_{l-1}). \qquad (2.3)\end{aligned}$$

Since $\mathbf{P}(z_1, z_1)$ is the identity matrix, it follows from (2.3) that the inverse of $\mathbf{P}(z_2, z_1)$ is $\mathbf{P}(z_1, z_2)$. In particular, for any z, $\mathbf{P}(z, z_0)$ has an inverse. Hence any propagator is a fundamental matrix.

If $\mathbf{M}(z)$ is any fundamental matrix for (2.1), then not only $\mathbf{M}(z)$ but also $\mathbf{M}(z)\mathbf{M}^{-1}(z_0)$ satisfies (2.2). Since $\mathbf{M}(z_0)\mathbf{M}^{-1}(z_0)$ is the identity matrix, $\mathbf{M}(z)\mathbf{M}^{-1}(z_0)$ must be the propagator from z_0. That is, if $\mathbf{M}(z)$ is any fundamental matrix for (2.1),

$$\mathbf{P}(z, z_0) = \mathbf{M}(z)\mathbf{M}^{-1}(z_0), \qquad (2.4)$$

Therefore, the propagator from any point z_0 can be calculated immediately by matrix inversion, once n linearly independent $n\times 1$ column matrix solutions of (2.1) are known.

The name *propagator* derives from the fact that if $\mathbf{f}(z)$ is an $n\times 1$ column matrix solution of (2.1), then

$$\mathbf{f}(z) = \mathbf{P}(z, z_0)\mathbf{f}(z_0). \qquad (2.5)$$

This is a consequence of the uniqueness theorem.

The propagator can be used to solve the inhomogeneous system

$$d\mathbf{f}/dz = \mathbf{A}(z)\mathbf{f}(z) + \mathbf{g}(z), \qquad (2.6)$$

where $\mathbf{g}(z)$ is given $n\times 1$ column matrix function of z. The solution, verified by direct substitution, is

$$\mathbf{f}(z) = \mathbf{F}(z)\left[\int_{z_0}^{z} \mathbf{F}^{-1}(\zeta)\mathbf{g}(\zeta)d\zeta + \mathbf{F}^{-1}(z_0)\mathbf{f}(z_0)\right], \qquad (2.7)$$

where $\mathbf{F}(z)$ is any fundamental matrix of (2.1). We may take $\mathbf{F}(z)=\mathbf{P}(z, z_0)$ in (2.7), so that

$$\mathbf{f}(z) = \int_{z_0}^{z} \mathbf{P}(z, \zeta)\mathbf{g}(\zeta)d\zeta + \mathbf{P}(z, z_0)\mathbf{f}(z_0). \qquad (2.8)$$

If $\mathbf{F}(z)$ is any solution of (2.2), let $\mathbf{F}^a(z)$ denote the classical adjoint of $\mathbf{F}$, that is, $F_{ij}{}^a$ is the cofactor of F_{ji}. Then $\mathbf{F}^a\mathbf{F}=\mathbf{I}$ det $(\mathbf{F})$ where $\mathbf{I}$ is the $n\times n$ identity matrix. Also,

$$\begin{aligned}d(\det(\mathbf{F}))/dz &= (dF_{ij}/dz)F_{ji}{}^a\\ &= A_{ik}F_{kj}F_{ji}{}^a\\ &= A_{ik}\delta_{ki}\det(\mathbf{F}). \qquad (2.9)\end{aligned}$$

Therefore,

$$d(\det(\mathbf{F}))/dz = \mathrm{tr}\,(\mathbf{A})\det(\mathbf{F})$$

$$\det(\mathbf{F}(z)) = \det(\mathbf{F}(z_0))\int_{z_0}^{z} \mathrm{tr}\,(\mathbf{A}(\zeta))d\zeta. \qquad (2.10)$$

In particular,

$$\det(\mathbf{P}(z, z_0)) = \int_{z_0}^{z} \operatorname{tr}(\mathbf{A}(\zeta))d\zeta. \quad (2.11)$$

In case $\mathbf{A}(z)$ and $\int_{z_0}^{z} \mathbf{A}(\zeta)d\zeta$

commute for every z,

$$\mathbf{P}(z, z_0) = \exp \int_{z_0}^{z} \mathbf{A}(\zeta)d\zeta. \quad (2.12)$$

Equation (2.12) is applicable in particular if $\mathbf{A}(z)$ is independent of z.

For many elastic wave and vibration problems the secular function, whose roots are the eigenvalues, is some minor of the integral matrix $\mathbf{F}$ of (2.1) that has met prescribed initial conditions. A difficulty that frequently arises in such problems is that even when the minor is of moderate size, the elements of $\mathbf{F}$ which enter into its composition may be so large that the minor is computed with severe loss of numerical accuracy. One way to circumvent this difficulty is to compute the minor directly rather than from the elements of $\mathbf{F}$.

A square matrix $\mathbf{F}$ of order n has $\binom{n}{m}^2$ minors of order $m \leq n$. When the minors of order m are arranged in a square array in some definite order, the array is called the mth minor matrix of $\mathbf{F}$, which we denote by $\mathfrak{F}^{(m)}$. Gantmacher (1959, *1*, 19–20) calls $\mathfrak{F}^{(m)}$ the mth compound matrix.

An mth order minor of $\mathbf{F}$ is

$$F\begin{pmatrix} i_1 & \cdots & i_m \\ k_1 & \cdots & k_m \end{pmatrix} = F_{i_1 j_1} \cdots F_{i_m j_m} \epsilon_{k_1 \cdots k_m}^{j_1 \cdots j_m}, \quad (2.13)$$

where

$$\epsilon_{k_1 \cdots k_m}^{j_1 \cdots j_m}$$

is defined to be one if all the k's are different and the j's are some even permutation of the k's; is defined to be -1 if all the k's are different and the j's are some odd permutation of the k's; and is zero otherwise.

If the matrix $\mathbf{F}$ satisfies (2.2), then

$$dF\begin{pmatrix} i_1 & \cdots & i_m \\ k_1 & \cdots & k_m \end{pmatrix} \Big/ dz = A_{i_1 l} F\begin{pmatrix} l & \cdots & i_m \\ k_1 & \cdots & k_m \end{pmatrix} + \cdots + A_{i_m l} F\begin{pmatrix} i_1 & \cdots & l \\ k_1 & \cdots & k_m \end{pmatrix}. \quad (2.14)$$

Therefore $\mathfrak{F}^{(m)}$, the mth minor matrix of $\mathbf{F}$, is a solution of

$$d\mathfrak{F}^{(m)}/dz = \mathbf{A}(z)\mathfrak{F}^{(m)}(z), \quad (2.15)$$

where $\mathfrak{A}$ is a square matrix of order $\binom{n}{m}$ each of whose elements is a linear combination of the elements of $\mathbf{A}$. The propagator of $\mathfrak{A}$ is the mth minor matrix of the propagator of $\mathbf{A}$ and is called the mth minor propagator of $\mathbf{A}$. The fact that the secular function, some element of $\mathbf{F}^{(m)}$ in (2.15), is computed directly rather than from (2.13) means that its accuracy is independent of the magnitude of the elements of $\mathbf{F}$.

As one example, we take $n=4$, $m=2$. Arranging the second order minors of $\mathbf{F}$ in the array

$$\mathfrak{F}^2 = \begin{bmatrix} F\binom{12}{12} & F\binom{12}{13} & F\binom{12}{14} & F\binom{12}{23} & F\binom{12}{24} & F\binom{12}{34} \\ F\binom{13}{12} & & & & & \cdot \\ F\binom{14}{12} & & & & & \cdot \\ F\binom{23}{12} & & & \text{etc.} & & \cdot \\ F\binom{24}{12} & & & & & \cdot \\ F\binom{43}{12} & \cdot & \cdot & \cdot & \cdot & F\binom{34}{34} \end{bmatrix}, \quad (2.16)$$

we find the matrix $\mathfrak{A}$ to be

$$\mathfrak{A} = \begin{bmatrix} A_{11}+A_{22} & A_{23} & A_{24} & -A_{13} & -A_{14} & 0 \\ A_{32} & A_{11}+A_{33} & A_{34} & A_{12} & 0 & -A_{14} \\ A_{42} & A_{43} & A_{11}+A_{44} & 0 & A_{12} & A_{13} \\ -A_{31} & A_{21} & 0 & A_{22}+A_{33} & A_{34} & -A_{24} \\ -A_{41} & 0 & A_{21} & A_{43} & A_{22}+A_{44} & A_{23} \\ 0 & -A_{41} & A_{31} & -A_{42} & A_{32} & A_{33}+A_{44} \end{bmatrix}. \qquad (2.17)$$

3. PRODUCT INTEGRALS

Consider $\mathbf{P}(z, z_0: \mathbf{A})$, the propagator for some coefficient matrix $\mathbf{A}$

$$d\mathbf{P}/dz = \mathbf{AP} \qquad (3.1)$$

in the interval $z_0 \leq z \leq z_k$. Divide the interval into k parts by introducing the intermediate points $z_1 \leq z_2 \leq \cdots \leq z_{k-1}$. Let $\gamma_l = z_l - z_{l-1}$, $l = 1, \cdots, k$. Then, from (2.3),

$$\mathbf{P}(z_k, z_0) = \prod_{l=1}^{k} \mathbf{P}(z_l, z_{l-1}). \qquad (3.2)$$

In the interval $z_{l-1} \leq z \leq z_l$ we choose an intermediate point ζ_l. Regarding γ_l as a small quantity we make the approximation

$$\mathbf{A}(z) \simeq \mathbf{A}(\zeta_l); \quad z_{l-1} \leq z \leq z_l. \qquad (3.3)$$

Then

$$\mathbf{P}(z_l, z_{l-1}) \simeq \exp\,[\mathbf{A}(\zeta_l)\gamma_l]$$

$$\mathbf{P}(z_k, z_0) \simeq \prod_{l=1}^{k} \exp\,[\mathbf{A}(\zeta_l)\gamma_l]. \qquad (3.4)$$

Thus,

$$\mathbf{P}(z_k, z_0) = \mathcal{P}\!\!\int_{z_0}^{z_k} [\mathbf{I} + \mathbf{A}(\zeta)d\zeta]$$

$$= \lim_{k\to\infty} \prod_{l=1}^{k} [\mathbf{I} + \mathbf{A}(\zeta_l)\gamma_l]. \qquad (3.5)$$

In (3.5) the limit is called the product integral of $\mathbf{A}$ (Volterra, 1887; Birkhoff, 1937). In (3.4) the product is called the Π approximant to the product integral of $\mathbf{A}$. In (3.1) the propagator of $\mathbf{A}$ is the product integral of $\mathbf{A}$.

When $\mathbf{A}(\zeta_l)\gamma_l$ in (3.4) is replaced by

$$\int_{z_{l-1}}^{z_l} \mathbf{A}(\zeta)d\zeta$$

the Π approximant to the product integral is said to be computed by the method of mean coefficients (Frazer, Duncan, and Collar, 1960, 232–245).

When n is not large in (2.1) and (3.1) the Cayley-Hamilton theorem and the Lagrange-Sylvester interpolation formula can be used to compute

$$\mathbf{P}(z_l, z_{l-1}) \simeq \exp\left[\int_{z_{l-1}}^{z_l} \mathbf{A}(\zeta)d\zeta\right]. \qquad (3.6)$$

Alternatively, some purely numerical method, such as the fourth order Runge-Kutta-Gill method, can be used to solve (3.1).

In (3.4) the approximation of taking $\mathbf{A}$ constant in subintervals of z is equivalent to the approximation of taking (λ, μ, ρ) constant in the subintervals. The subintervals are sometimes called homogeneous layers, and in this case the methods of Thomson (1950) and Haskell (1953) are usually used to compute the propagator. It is clear from (3.5) that the Thomson-Haskell procedure yields the product integral as the limit in (3.5) is taken.

4. THE METHOD OF MEAN COEFFICIENTS

Let

$$\mathbf{B} = \int_{z_{l-1}}^{z_l} \mathbf{A}(\zeta)d\zeta.$$

Then

$$\mathbf{P}(z_l, z_{l-1}) \simeq \exp \mathbf{B}. \qquad (4.1)$$

For SH waves $\mathbf{A}$ is given by (1.4). The characteristic equation for $\mathbf{B}$ is

$$\gamma^2 - B_{21}B_{12} = 0. \qquad (4.2)$$

Using the Cayley-Hamilton theorem and the Lagrange-Sylvester interpolation formula, we have

$$\exp \mathbf{B} = a_1\mathbf{B} + a_0\mathbf{I}, \qquad (4.3)$$

where $a_1 = \text{sh}q/q$, $a_0 = \text{ch}q$, $q = (B_{21}B_{12})^{1/2}$. In the special case where μ and ρ are constant (4.1) is exact and (4.3) is the propagator matrix derived by Haskell (1953) for SH waves.

For P-SV waves $n=4$ in (2.1). Taking f_1 to be the z displacement, f_2 to be the x or y displacement, f_3 to be the z stress, and f_4 to be the z_x or z_y stress, it is simple to show that the coefficient matrix is

$$\mathbf{A} = \begin{bmatrix} 0 & \lambda k(\lambda+2\mu)^{-1} & (\lambda+2\mu)^{-1} & 0 \\ -k & 0 & 0 & \mu^{-1} \\ -\rho\omega^2 & 0 & 0 & k \\ 0 & -\rho\omega^2 + 4k^2\mu(\lambda+\mu)(\lambda+2\mu)^{-1} & -\lambda k(\lambda+2\mu)^{-1} & 0 \end{bmatrix}. \tag{4.4}$$

Therefore the characteristic equation for $\mathbf{B}$ is

$$\gamma^4 - p\gamma^2 + q = 0, \tag{4.5}$$

where

$$\begin{aligned} p &= B_{42}B_{24} + B_{43}B_{34} + B_{12}B_{21} \\ &\quad + B_{13}B_{31}, \\ q &= B_{12}B_{43}(B_{21}B_{34} - B_{31}B_{24}) \\ &\quad + B_{13}B_{42}(B_{31}B_{24} - B_{21}B_{34}), \end{aligned} \tag{4.6}$$

Thus

$$\exp \mathbf{B} = \sum_{n=0}^{3} a_n \mathbf{B}^n, \tag{4.7}$$

where

$$\begin{aligned} a_3 &= (\text{sh}\,\gamma_1/\gamma_1 - \text{sh}\,\gamma_2/\gamma_2)/\epsilon, \\ a_2 &= (\text{ch}\,\gamma_1 - \text{ch}\,\gamma_2)/\epsilon, \\ a_1 &= (\gamma_1^2\,\text{sh}\,\gamma_2/\gamma_2 - \gamma_2^2\,\text{sh}\,\gamma_1/\gamma_1)/\epsilon, \\ a_0 &= (\gamma_1^2\,\text{ch}\,\gamma_2 - \gamma_2^2\,\text{ch}\,\gamma_1)/\epsilon, \\ \epsilon &= \gamma_1^2 - \gamma_2^2, \\ \gamma_{1,2} &= [\tfrac{1}{2}(p \pm (p^2 - 4q)^{1/2})]^{1/2}. \end{aligned} \tag{4.8}$$

In the special case where λ, μ, and ρ are constant (4.7) is the exact propagator and is equal to the matrix derived by Haskell (1953) for P-SV waves. When $\gamma_1^2 = \gamma_2^2$ in (4.8) we have

$$\begin{aligned} a_3 &= (\gamma\,\text{ch}\,\gamma - \text{sh}\,\gamma)/2\gamma^3, \\ a_2 &= \text{sh}\,\gamma/2\gamma, \\ a_1 &= (3\,\text{sh}\,\gamma - \gamma\,\text{ch}\,\gamma)/2\gamma, \\ a_0 &= \text{ch}\,\gamma - \gamma\,\text{sh}\,\gamma/2, \\ \gamma &= q^{1/4} = (p/2)^{1/2}, \end{aligned} \tag{4.9}$$

When $\gamma = 0$ in (4.9), $a_n = 1/n!$

Series expansions of (4.8) and (4.9) can be expressed in terms of the Lucas (1891) polynomials (Barakat, 1964)

$$\begin{aligned} a_3 &= \sum_{n=0}^{\infty} U_n(p,q)/(2n+1)! \\ a_2 &= \sum_{n=0}^{\infty} U_n(p,q)/(2n)! \\ a_1 &= -q \sum_{n=0}^{\infty} U_{n-1}(p,q)/(2n+1)! \\ a_0 &= -q \sum_{n=0}^{\infty} U_{n-1}(p,q)/(2n)! \end{aligned} \tag{4.10}$$

$$\begin{aligned} U_n(p,q) &= pU_{n-1}(p,q) - qU_{n-2}(p,q), \\ U_0(p,q) &= 0,\ U_1(p,q) = 1, \\ U_{-n}(p,q) &= -q^{-n}U_n(p,q). \end{aligned}$$

In many P-SV problems the secular function is a second order minor of an integral matrix of (2.1). The second minor propagator of $\mathbf{A}$, where $\mathbf{A}$ is given by (4.4), is then the propagator of $\mathcal{A}$ in (2.17). Let

$$\mathcal{B} = \int_{z_{l-1}}^{z_l} \mathcal{A}(\zeta)d\zeta.$$

Let $\mathcal{P}(z_l, z_{l-1})$ be the propagator of $\mathcal{A}$. Then

$$\mathcal{P}(z_l, z_{l-1}) \simeq \exp \mathcal{B}. \tag{4.11}$$

The characteristic equation for $\mathcal{B}$, when $\mathcal{A}$ is given by (2.17) and **A** is given by (4.4), is

$$\gamma^2(\gamma^2 - \sigma_1^2)(\gamma^2 - \sigma_2^2) = \gamma^6 - 2p\gamma^4 + (p^2 - 4q)\gamma^2 = 0, \quad (4.12)$$

where

$$\sigma_1 = \gamma_1 + \gamma_2, \ \sigma_2 = \gamma_1 - \gamma_2, \qquad \sigma_{1,2} = (p \pm 2q^{1/2})^{1/2}, \quad (4.13)$$

and γ_1 and γ_2 are given by (4.8).

Thus,

$$\exp \mathcal{B} = \sum_{n=0}^{5} a_n \mathcal{B}^n, \quad (4.14)$$

where

When $\sigma = 0$ in (4.16), $a_n = 1/n!$

Series expansions of (4.15) and (4.16) in terms of the Lucas polynomials are

$$\begin{aligned} a_5 &= \sum_{n=0}^{\infty} U_n(P, Q)/(2n + 3)! \\ a_4 &= \sum_{n=0}^{\infty} U_n(P, Q)/(2n + 2)! \\ a_3 &= -Q \sum_{n=0}^{\infty} U_{n-1}(P, Q)/(2n + 3)! \quad (4.17) \\ a_2 &= -Q \sum_{n=0}^{\infty} U_{n-1}(P, Q)/(2n + 2)! \\ P &= 2p, \ Q = p^2 - 4q. \end{aligned}$$

$$\begin{aligned} a_5 &= 1/\sigma_1^2\sigma_2^2 + \mathrm{sh}\sigma_1/\sigma_1^3(\sigma_1^2 - \sigma_2^2) - \mathrm{sh}\sigma_2/\sigma_2^3(\sigma_1^2 - \sigma_2^2), \\ a_4 &= 1/\sigma_1^2\sigma_2^2 + \mathrm{ch}\sigma_1/\sigma_1^2(\sigma_1^2 - \sigma_2^2) - \mathrm{ch}\sigma_2/\sigma_2^2(\sigma_1^2 - \sigma_2^2), \\ a_3 &= -(\sigma_1^2 + \sigma_2^2)/\sigma_1^2\sigma_2^2 - \sigma_2^2\mathrm{sh}\sigma_1/\sigma_1^3(\sigma_1^2 - \sigma_2^2) + \sigma_1^2\mathrm{sh}\sigma_2/\sigma_2^3(\sigma_1^2 - \sigma_2^2), \\ a_2 &= -(\sigma_1^2 + \sigma_2^2)/\sigma_1^2\sigma_2^2 - \sigma_2^2\mathrm{ch}\sigma_1/\sigma_1^2(\sigma_1^2 - \sigma_2^2) + \sigma_1^2\mathrm{ch}\sigma_2/\sigma_2^2(\sigma_1^2 - \sigma_2^2), \\ a_1 &= 1, \\ a_0 &= 1. \end{aligned} \quad (4.15)$$

In the special case where λ, μ and ρ are constant (4.11) is the exact second minor propagator. The elements of (4.11) are then the second order minors derived by Dunkin (1965) for *P-SV* waves. When $\sigma_1^2 = \sigma_2^2$ in (4.15) we have

$$\begin{aligned} a_5 &= (2\sigma - \sigma\ \mathrm{ch}\ \sigma - \mathrm{sh}\ \sigma)/2\sigma^5, \\ a_4 &= (2 - 2\ \mathrm{ch}\ \sigma - \sigma\ \mathrm{sh}\ \sigma)/2\sigma^4, \\ a_3 &= (5\ \mathrm{sh}\ \sigma - \sigma\ \mathrm{ch}\ \sigma - 4\sigma)/2\sigma^3, \\ a_2 &= (4\ \mathrm{ch}\ \sigma - 4 - \sigma\ \mathrm{sh}\ \sigma)/2\sigma^2, \quad (4.16) \\ a_1 &= 1, \\ a_0 &= 1, \\ \sigma &= p^{1/2}. \end{aligned}$$

For all examples given in this section each propagator has unit determinant as a consequence of (2.11).

Expressions (4.2), (4.5), and (4.12) are also valid for transversely isotropic media with the z axis being the axis of symmetry. For *SH* waves **A** in (1.4) is replaced by

$$\mathbf{A} = \begin{bmatrix} 0 & \mu^{-1} \\ -\rho\omega^2 + \mu' k^2 & 0 \end{bmatrix}. \quad (4.18)$$

For *P-SV* waves **A** in (4.4) is replaced by

$$\mathbf{A} = \begin{bmatrix} 0 & k\lambda\beta^{-1} & \beta^{-1} & 0 \\ -k & 0 & 0 & \mu^{-1} \\ -\rho\omega^2 & 0 & 0 & k \\ 0 & -\rho\omega^2 + k^2(\lambda' + 2\mu' - \lambda^2\beta^{-1}) & -k\lambda\beta^{-1} & 0 \end{bmatrix}, \quad (4.19)$$

where the five elastic parameters λ, λ', μ, μ', β enter the stress-strain relations as follows:

$$
\begin{aligned}
T_{33} &= \beta e_{33} + \lambda e_{kk}, \\
T_{3i} &= T_{i3} = 2\mu e_{3i}, \\
T_{ij} &= \lambda' e_{kk}\delta_{ij} + 2\mu' e_{ij} + \lambda e_{33}\delta_{ij}.
\end{aligned}
\tag{4.20}
$$

In (4.20) the letter indices take on the values 1, 2 and the summation convention is used. Also $x_1 = x$, $x_2 = y$, $x_3 = z$. In an isotropic medium $\lambda' = \lambda$, $\mu' = \mu$, $\beta = \lambda + 2\mu$. The T's and e's in (4.20) are the elements of the stress tensor and strain tensor, respectively.

In closing we remark that the characteristic roots of exp $\mathcal{B}$ in (4.14) are derivable from those of exp $\mathcal{B}$ in (4.7) by Kronecker's theorem (Gantmacher, 1959, *1*, 75).

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