The evolution of large ocean waves: the role of local and rapid spectral changes.

By R. S. Gibson and C. Swan†

Department of Civil and Environmental Engineering, Imperial College London, London SW7 2AZ

Appendix A

The purpose of this appendix is to provide comparisons between the Krasitskii (1994) derivation of the Zakharov (1968) equation (ZE) for three well known wave solutions; the first having been detailed by Badulin (2003). Using these results as examples, further information is given concerning comparisons between ZE and the fully-nonlinear wave-model referred to as BST.

(*a*) *Nonlinear regular waves*

Fenton (1985) defines the water surface elevation, $\eta(x)$, of a third-order Stokes wave in a steady frame of reference (moving with the wave) as

$$\begin{aligned}\eta(x) &= (a - \frac{3}{8}a^3k^2)\cos(kx) + (\frac{1}{2}a^2k)\cos(2kx) \\ &+ (\frac{3}{8}a^3k^2)\cos(3kx)\,.\end{aligned} \tag{A.1}$$

Krasitskii (1994) gives the corresponding solutions as

$$\begin{aligned}\eta(x) &= \frac{M}{\pi}[B + B^3(B^{(2)} + B^{(3)})]\cos(kx) \\ &+ \frac{2kM}{\pi}[B^2(A^{(1)} + A^{(3)})]\cos(2kx) \\ &+ \frac{3kM}{\pi}[B^3(B^{(1)}]\cos(3kx)\,.\end{aligned} \tag{A.2}$$

where B is the canonical conjugate variable described in §3 and the kernels $A^{(i)}$ and $B^{(i)}$ define the *bound* wave interactions, at second- and third-order respectively. If $B = \frac{\pi}{M}A$, where A is the amplitude of the wave component, then $\eta(x)$ is given by

$$\begin{aligned}\eta(x) &= (A - \frac{1}{8}A^3k^2)\cos(kx) + (\frac{1}{2}A^2k)\cos(2kx) \\ &+ (\frac{3}{8}A^3k^2)\cos(3kx)\,.\end{aligned} \tag{A.3}$$

† Corresponding author.

After appropriate normalisation, to ensure the wave profiles are defined in terms of consistent parameters ($A = a - \frac{1}{4}a^3k^2$), equations A.1 and A.3 are identical to third-order.

However, at third-order the Stokes solution also includes a well-known change in the phase velocity. In terms of ZE this change arises as a resonant interaction, recovered by the four wave (third-order) degenerate interaction between the wave component and itself. The change in the phase of the wave component, of wave-number k, can be modelled using ZE by time-marching equation 3.7, which in this case reduces to

$$i\frac{\delta B_k}{\delta t} = \tilde{V}^{(2)}_{k,k,k,k} B^*_k B^2_k \,. \tag{A.4}$$

However, since the change in the phase velocity is constant, the wave frequency may be altered according to

$$\omega = \omega^{(1)}(k) + \tilde{V}^{(2)}_{k,k,k,k}|B|^2 \,, \tag{A.5}$$

where $\omega^{(1)}(k)$ is the frequency determined from the linear dispersion relationship and $\tilde{V}^{(2)}_{k,k,k,k}$ the four-wave interaction kernel when one component, of wave-number k, interacts with itself. Comparison between this result and the third-order Stokes solution are given in figure A.1.

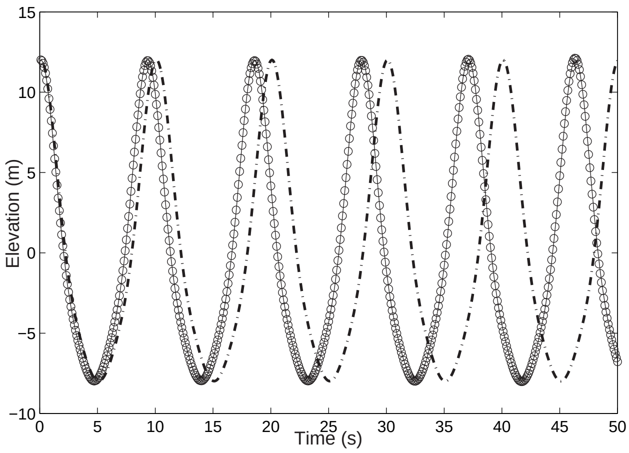


Figure A.1: Comparison between the third-order Stokes and ZE solutions for a single wave component of steepness $ak = 0.4$ using the third-order dispersion relationship: $(\cdot - \cdot)$ third-order Fenton (1985) solution with linear dispersion relationship, (—) third-order Fenton (1985) solution with nonlinear dispersion relationship, ($\circ$) ZE to third-order.

(*b*) *Long-wave short-wave interactions at second-order*

Longuet-Higgins & Stewart (1960) provide a second-order solution for interactions between two wave components. These interactions can be modelled using the

$A^{(i)}$ kernels of ZE; $A^{(1)}$ and $A^{(3)}$ representing the wave-number sum terms and $A^{(2)}$ the wave-number difference terms. Comparisons between these results are given on figure A.2; part (a) describing the total wave surface elevation and part (b) the second-order *bound* waves.

(*c*) *Interaction of two wave components to third-order*

Longuet-Higgins & Phillips (1962) showed that when two wave components interact there is a third-order change in the phase speed of each component. In ZE this change is modelled as the degenerate four-wave (third-order) *resonant* interaction between two wave components, with wave-numbers k and l, by time-marching

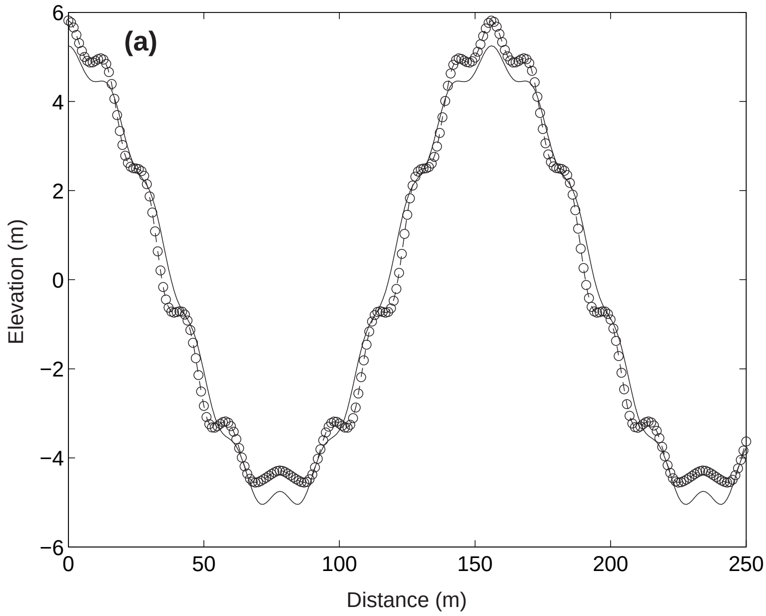
$$i\frac{\delta B_k}{\delta t} = \tilde{V}^{(2)}_{k,l,k,l} B^*_l B_k B_l \,. \tag{A.6}$$

However, since the rate of change of phase is constant this can again be represented by a change in frequency

$$\omega_k = \omega^{(1)}_k + 2\tilde{V}^{(2)}_{k,l,k,l}|B_l|^2 \,. \tag{A.7}$$

Figure A.3 concerns the change in frequency $\delta\omega = \omega_k - \omega^{(1)}_k$ of a wave component with wave-number $k = 0.040243$ interacting with a wave component of steepness $a_l l = 0.1$. There is perfect agreement between the results of Longuet-Higgins & Phillips (1962) and ZE.

In making these comparisons it is important to note that the transformation of the ZE results, from the canonical conjugate variables in which the solution is derived to physical variables, involves a re-normalisation to ensure that equivalent wave-fields are considered. This is explicitly noted in the first example in this appendix; with a similar process undertaken in the subsequent examples to ensure that the amplitude of the underlying linear wave components are consistent. In achieving the comparisons provided in §5 a similar renormalisation is required; the purpose again being to ensure that the underlying linear wave-fields are consistent, and hence, valid and meaningful comparisons achieved. Accordingly, the renormalisation is based upon the initial description of the wave-field, in its fully dispersed state, prior to time-marching either of the nonlinear wave models. As a result, the renormalisation does not detract from comparisons made at the focal locations and, more specifically, from evidence that the significant nonlinear wave evolutions arising in the vicinity of an extreme wave event may be accounted for in terms of *resonant* third-order wave interactions that are both rapid and local.



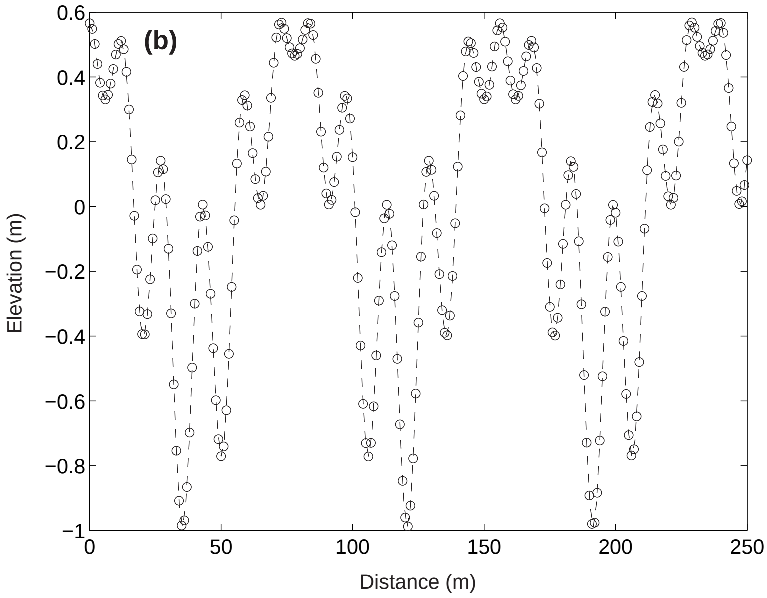


Figure A.2: Surface profile of two wave-components interacting to second-order (a) complete solution, (b) second-order terms: (—) linear, (- -) Longuet-Higgins & Stewart (1960) (○) ZE.

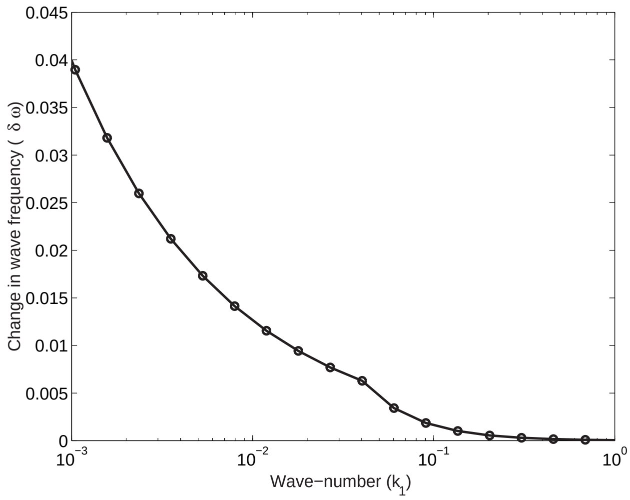


Figure A.3: Comparison between the third-order change in the frequency ($\delta\omega$) of a wave component with steepness $a_l l = 0.1$ interacting with a wave component with wave-number $k = 0.040243$: (—) Longuet-Higgins & Phillips (1962), (○) ZE. The change in slope is where $k = l$.

References

Badulin, S. 2003 Personal communication.

Fenton, J. D. 1985 A fifth order stokes' theory for steady waves. *J. Waterway, Port, Coastal and Ocean Eng.* **111**, 216–234.

Krasitskii, V. P. 1994 On reduced equations in the hamiltonian theory of weakly nonlinear surface waves. *Journal of Fluid Mechanics* **272**, 1–20.

Longuet-Higgins, M. & Phillips, O. 1962 Phase velocity effects in tertiary wave interactions. *Journal of Fluid Mechanics* **12**, 333–336.

Longuet-Higgins, M. S. & Stewart, R. W. 1960 Changes in the form of short gravity waves. *Journal of Fluid Mechanics* **8**, 565–583.

Zakharov, V. E. 1968 Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Journal of Applied Mechanics and Technical Physics* **9**, 190–194, english Translation.