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Towards a Space-Time Centred TVD Scheme for the Linear Advection Equation

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Abstract

A new approach to solving the hyperbolic, linear advection equation is presented. It is based on using a finite volume scheme centred both in space and time. Spatial discretisation is performed via a conventional flux limited scheme. Temporal discretisation relies on a TVD extrapolation in time to achieve second order accuracy. The derivation of this extrapolation scheme and a TVD analysis of the resulting numerical scheme is presented in detail. The method is compared with conventional explicit and implicit schemes. The scheme is demonstrated to be second order accurate for CFL numbers less than or equal to one provided a suitable limiter function is used. The reason for the loss of accuracy at higher CFL numbers is explained. An approach to circumvent this problem is outlined.

Contents

1 Introduction

Over the past 30 years much work has been performed in developing numerical techniques and the associated theory to accurately calculate time-dependent compressible flow fields. Excellent reviews of this material are presented in the books of LeVeque [1], Hirsch [2] and Toro [3]. The most widely used techniques are those that involve finite-volumes in space and time but with the associated solution values at the centre of the control volume in space and at the edge in time. To ensure the non-physical oscillations do not occur in the solution then spatial gradients are nonlinearly *limited*. Recently Sidilikover [4] has presented a scheme which involves limiting gradients in space and time, though again solution values coincide with the time integration steps.

The above techniques may also be applied to calculate steady flow fields. Usually time-stepping is used to obtain a steady state solution with the assistance of acceleration techniques such as multigrid. Occasionally Newton solvers are used, though usually in the context of a backward Euler time stepping technique and by letting the time-step grow as the solution error decreases.

For steady supersonic flows the space-marching technique is often used. In supersonic inviscid flow fields the Euler equations are hyperbolic in the dominant flow direction and may be integrated much as a time-dependent problem would be. For viscous problems the Parabolized Navier-Stokes or PNS equations are parabolic in the dominant flow direction and may also be integrated using the techniques from time-dependent problems. A review of such techniques is presented in Lawrence [5].

A different approach first proposed by Newsome and Walters [6] and also reviewed in Lawrence [5] involves discretising the Euler or PNS equations using a conventional cell centred in space technique and using approximately factored time-marching to obtain a steady solution at each cross flow plane of the calculation. In this approach advantage is taken of the fact that there is no upstream influence in the solution in order to decouple the downstream part of the domain from the current solution plane. The close relationship between such solution techniques and space-marching was investigated by Chang and Merkle [7]. There is however a problem, accentuated when flux vector splitting schemes are used, in the streamwise direction. That is how to accurately determine the streamwise flux based solely on upstream information. Normally limiting would be used to obtain second order accuracy without introducing oscillations but there is no downstream gradient to limit against. Thompson and Matus [8] examined various unlimited 'extrapolations' and concluded that first order upwind extrapolation give the most consistent, stable method.

If the conventional Vigneron [9] splitting of the streamwise flux is used and upstream influences ignored then the PNS equations are truly parabolic and the problem reduces to extrapolating to the adjacent downstream cell face from the current cell centre.

Recently Forth [10] showed that for the Euler equations a linear extrapolation in the streamwise direction results in a scheme that is non-TVD and oscillations in shocked flow fields result. He implemented a TVD extrapolation procedure that was shown to suppress these oscillations. However the resulting scheme was demonstrated not to be as accurate as hoped.

In this report we analyse a scheme that is cell-centred in space **and** time for the hyperbolic linear advection equation. The discretisation is given in section 2. The approach of Forth [10] is made more rigorous for this simple equation and a new condition of the TVD limiters for the time-like extrapolation (analogous to streamwise in space marched schemes) results. The resulting implicit scheme is compared to various standard schemes in section 4. Results are encouraging for CFL numbers less than or equal to one but deteriorate thereafter. The reason for this deterioration is discussed in section 5. Conclusions and prospects for improvements are given in section 6.

2 A TVD Scheme

Consider the linear advection equation for a variable $u(x,t)$,

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{1}$$

for flux function,

$$f(u) = au,$$

with a constant.

2.1 Space-Time Finite Volume Scheme

We define a mesh in space-time with spacing Δx in space and Δt in time and consider a space-time finite-volumes containing a solution value u_j^n labelled with j in space and n in time. This is depicted in figure 1.

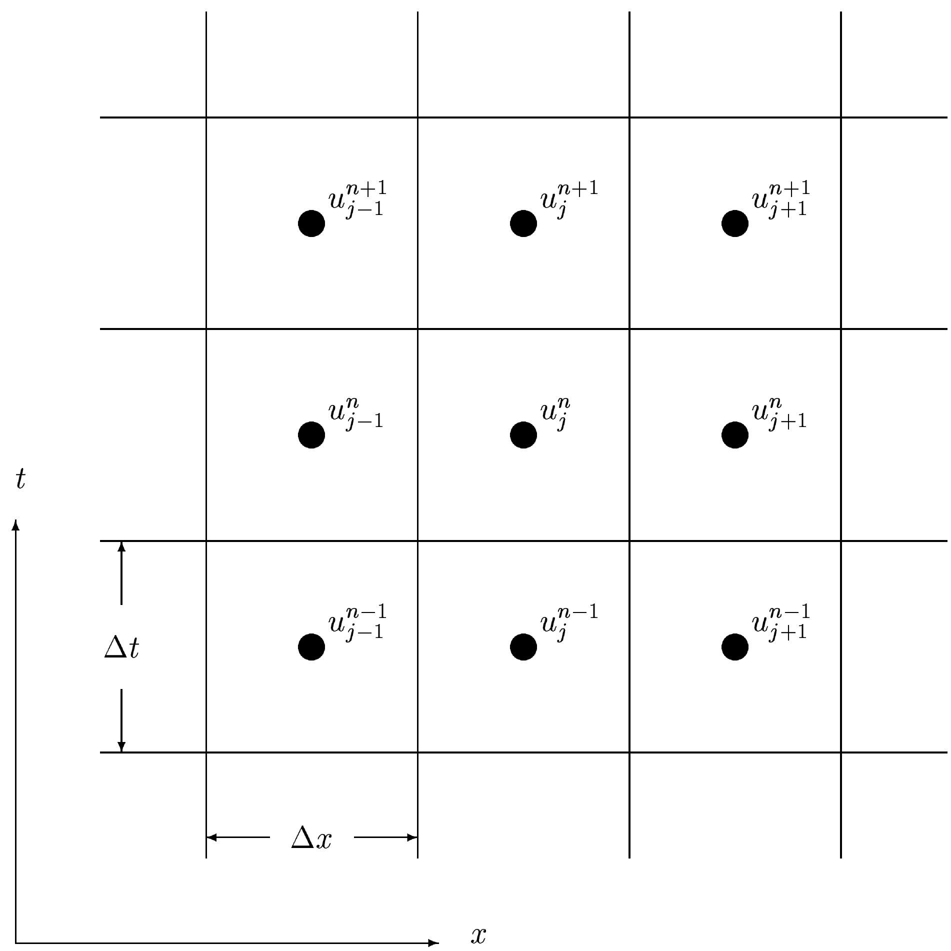

Figure 1: Space-Time Computational Mesh

By writing equation (1) in conservation form,

$$\nabla.(u, f(u)) = 0,$$

with,

$$\nabla = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right),$$

and applying Gauss's theorem over the volume containing the value u_j^n then we obtain,

$$\Delta x \left(u_j^{n+\frac{1}{2}} - u_j^{n-\frac{1}{2}} \right) + \Delta t \left(f_{j+\frac{1}{2}}^n - f_{j-\frac{1}{2}}^n \right) = 0, \tag{2}$$

where,

$$\left. \begin{aligned} u_j^{n\pm\frac{1}{2}} &= \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t = t_{n\pm\frac{1}{2}})\, dx \\ f_{j\pm\frac{1}{2}}^n &= \frac{1}{\Delta t} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} f\left(u(x = x_{j\pm\frac{1}{2}}, t) \right) dt \end{aligned} \right\}.$$

2.1.1 Spatial Fluxes

We approximate the spatial fluxes via a second order accurate flux limited scheme,

$$\left. \begin{aligned} f_{j+\frac{1}{2}}^n &= \frac{1}{2} \left(a(u_j^n + u_{j+1}^n) - |a|(1 - \phi_{j+\frac{1}{2}}^n)(u_{j+1}^n - u_j^n) \right) \\ f_{j-\frac{1}{2}}^n &= \frac{1}{2} \left(a(u_{j-1}^n + u_j^n) - |a|(1 - \phi_{j-\frac{1}{2}}^n)(u_j^n - u_{j-1}^n) \right) \end{aligned} \right\} \tag{3}$$

where $\phi_{j\pm\frac{1}{2}}^n$ are spatial flux limiters yet to be defined. We note that if $\phi_{j\pm\frac{1}{2}}^n = 0$ then we obtain an upwind scheme for the fluxes and if $\phi_{j\pm\frac{1}{2}}^n = 1$ we obtain a central difference scheme.

2.1.2 Time Fluxes

We propose to use upwind fluxes in time,

$$\left. \begin{aligned} u_j^{n+\frac{1}{2}} &= u_j^n + \frac{1}{2} \psi_j^{n+\frac{1}{2}} (u_j^n - u_j^{n-1}) \\ u_j^{n-\frac{1}{2}} &= u_j^{n-1} + \frac{1}{2} \psi_j^{n-\frac{1}{2}} (u_j^{n-1} - u_j^{n-2}) \end{aligned} \right\} \tag{4}$$

where the $\psi_j^{n\pm\frac{1}{2}}$ are temporal flux limiters yet to be defined.

2.2 Second Order Accuracy in Space

If we use first order upwinding in time so that $\psi_j^{n\pm\frac{1}{2}} = 0$ in equations (4) then the finite-volume scheme (2) is implicit and given by,

$$\begin{aligned} u_j^n \; &+ \frac{a^+ \Delta t}{2\Delta x} \left(2 - \phi_{j-\frac{1}{2}}^n + \frac{\phi_{j+\frac{1}{2}}^n}{R_{j+\frac{1}{2}}^n} \right) (u_j^n - u_{j-1}^n) \\ &+ \frac{a^- \Delta t}{2\Delta x} \left(2 - \phi_{j+\frac{1}{2}}^n + \frac{\phi_{j-\frac{1}{2}}^n}{R_{j-\frac{1}{2}}^n} \right) (u_{j+1}^n - u_j^n) \quad = u_j^{n-1} \end{aligned} \tag{5}$$

where,

$$a^{\pm} = \frac{a \pm |a|}{2},$$

are the positive and negative parts of a with $a^+ \geq 0$ and $a^- \leq 0$. Also,

$$\left.\begin{array}{lll} R^n_{j+\frac{1}{2}} & = \dfrac{u^n_j - u^n_{j-1}}{u^n_{j+1} - u^n_j} & \text{for } a \geq 0 \\ R^n_{j+\frac{1}{2}} & = \dfrac{u^n_{j+2} - u^n_{j+1}}{u^n_{j+1} - u^n_j} & \text{for } a < 0 \end{array}\right\} \qquad (6)$$

We may now use a theorem due to Harten [11].

Theorem 2.1 (Harten) *Consider the scheme,*

$$\begin{aligned} & V_j + \theta \left(c_{j-\frac{1}{2}}(V_j - V_{j-1}) - d_{j+\frac{1}{2}}(V_{j+1} - V_j) \right) \\ & \quad = \quad U_j - (1-\theta) \left(C_{j-\frac{1}{2}}(U_j - U_{j-1}) - D_{j+\frac{1}{2}}(U_{j+1} - U_j) \right), \end{aligned}$$

with $\theta \in [0,1]$. Then the following conditions are sufficient for the scheme to be TVD,

$$\begin{array}{ll} C_{j-\frac{1}{2}}, D_{j+\frac{1}{2}} & \geq 0 \quad \forall j \\ c_{j-\frac{1}{2}}, d_{j+\frac{1}{2}} & \geq 0 \quad \forall j \\ (1-\theta)\left(C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}}\right) & \leq 1 \quad \forall j \end{array}$$

Putting $\theta = 1$ we see that

$$\begin{aligned} c_{j-\frac{1}{2}} &= \frac{a^+ \Delta t}{2\Delta x} \left(2 - \phi^n_{j-\frac{1}{2}} + \frac{\phi^n_{j+\frac{1}{2}}}{R^n_{j+\frac{1}{2}}} \right) \\ d_{j+\frac{1}{2}} &= \frac{-a^- \Delta t}{2\Delta x} \left(2 - \phi^n_{j+\frac{1}{2}} + \frac{\phi^n_{j-\frac{1}{2}}}{R^n_{j-\frac{1}{2}}} \right) \end{aligned}$$

and so the TVD conditions are,

$$0 \leq 2 - \phi(R) + \frac{\phi(S)}{S},$$

for arbitrary R, S. Defining Φ as,

$$\Phi = ||\phi|| = \max_S \frac{\phi(S)}{S},$$

then the TVD condition becomes,

$$\phi(R) \leq 2 + \Phi. \qquad (7)$$

Note that this does not rely on $\phi(R) = 0$ for $R \leq 0$.

We see then that the scheme (5) is unconditionally TVD provided the limiter function ϕ satisfies equation (7). This unconditional TVD stability is of course well known for backward Euler integration with TVD limiters. It is also well known (See Hirsch [2] p558-559) that a conventional Crank-Nicolson type integration equally combining explicit and implicit integration is TVD only for time-steps less than twice the maximum explicit time-step.

2.3 TVD Extrapolation

As was first noted by Forth [10], in the context of space-marching, equation (4) may be regarded as an extrapolation procedure. We may impose conditions on the limiters ψ to ensure that the extrapolation is TVD, i.e. that,

$$TV(u^{n+\frac{1}{2}}) \leq TV(u^n). \tag{8}$$

To do this we write,

$$\psi_j^{n+\frac{1}{2}} = \Psi(r_j^n, s_j^n) + \Psi(s_j^n, r_j^n), \tag{9}$$

with,

$$\left.\begin{aligned} r_j^n &= \frac{u_{j+1}^n - u_j^n}{u_j^n - u_j^{n-1}} \\ s_j^n &= \frac{u_{j-1}^n - u_j^n}{u_j^n - u_j^{n-1}} \end{aligned}\right\}, \tag{10}$$

to give,

$$u_j^{n+\frac{1}{2}} = u_j^n + \frac{1}{2}\left(\Psi(r_j^n, s_j^n) + \Psi(s_j^n, r_j^n)\right)(u_j^n - u_j^{n-1}) \tag{11}$$

As noted by Forth [10] this form possesses the symmetry of invariance under the relabelling $j \pm 1 \to j \mp 1$ and so is independent of whether the mesh is labelled in space in the positive or negative directions.

We make use of Harten's theorem 2.1 to give conditions on Ψ. Note that for the moment we do not require conditions based on the second argument to Ψ, these will be introduced in section 2.4.

Theorem 2.2 (TVD Extrapolation)
The extrapolation procedure of equation (11) is TVD in the sense of equation (8) under the conditions,

$$\left.\begin{aligned} \Psi(r,s) &= 0 \quad \forall\ r \leq 0 \\ \Psi(r,s) &\geq 0 \quad \forall\ r > 0 \end{aligned}\right\}, \tag{12}$$

and,

$$\Psi(r,s) \leq r \ \ \forall\ \ r > 0. \tag{13}$$

Proof
We write equation (11) as,

$$u_j^{n+\frac{1}{2}} = u_j^n - \frac{\Psi(s_j^n, r_j^n)}{2s_j^n}(u_j^n - u_{j-1}^n) + \frac{\Psi(r_j^n, s_j^n)}{2r_j^n}(u_{j+1}^n - u_j^n),$$

where we have used equation (10). In Harten's theorem 2.1 we then identify $\theta = 0$,

$$C_{j-\frac{1}{2}} = \frac{\Psi(s_j^n, r_j^n)}{2s_j^n},$$

and,

$$D_{j+\frac{1}{2}} = \frac{\Psi(r_j^n, s_j^n)}{2r_j^n},$$

So firstly we need,

$$\left.\begin{array}{ll} \dfrac{\Psi(s_j^n, r_j^n)}{2s_j^n} & \geq 0 \quad \forall\, s_j^n \\ \dfrac{\Psi(r_j^n, s_j^n)}{2r_j^n} & \geq 0 \quad \forall\, r_j^n \end{array}\right\}$$

which are guaranteed by condition (12) of our theorem. Secondly we require,

$$\frac{\Psi(r_j^n, s_j^n)}{2r_j^n} + \frac{\Psi(s_{j+1}^n, r_{j+1}^n)}{2s_{j+1}^n} \leq 1.$$

This is true if we ensure separately that,

$$\frac{\Psi(r_j^n, s_j^n)}{2r_j^n} \leq \frac{1}{2} \text{ and } \frac{\Psi(s_{j+1}^n)}{2s_{j+1}^n} \leq \frac{1}{2},$$

which both reduce to,

$$\frac{\Psi(r, s)}{r} \leq 1,\ \forall\, r,$$

and hence true for $r < 0$ since then $\Psi(r) = 0$ and assured for $r \geq 0$ under condition (13) of our theorem. □

We note that condition (13) differs from that obtained for flux limiting which is,

$$\phi(r) \leq 2r\ \forall\, r > 0. \tag{14}$$

2.4 Second Order Accuracy in Space and Time

Use of the limited spatial fluxes of equation (3) and limited time fluxes of equation (4) results in the implicit scheme,

$$\begin{aligned} u_j^n \quad & + \frac{1}{2}\left[\frac{u_{j+1}^n - u_j^n}{r_j^n}\Psi(r_j^n, s_j^n) - \frac{u_j^n - u_{j-1}^n}{s_j^n}\Psi(s_j^n, r_j^n)\right] \\ & + \frac{a^+\Delta t}{2\Delta x}\left(2 - \phi_{j-\frac{1}{2}}^n + \frac{\phi_{j+\frac{1}{2}}^n}{R_{j+\frac{1}{2}}^n}\right)(u_j^n - u_{j-1}^n) \\ & + \frac{a^-\Delta t}{2\Delta x}\left(2 - \phi_{j+\frac{1}{2}}^n + \frac{\phi_{j-\frac{1}{2}}^n}{R_{j-\frac{1}{2}}^n}\right)(u_{j+1}^n - u_j^n) \\ & = u_j^{n-1} + \frac{1}{2}\left[\frac{u_{j+1}^{n-1} - u_j^{n-1}}{r_j^{n-1}}\Psi(r_j^{n-1}, s_j^{n-1}) - \frac{u_j^{n-1} - u_{j-1}^{n-1}}{s_j^{n-1}}\Psi(s_j^{n-1}, r_j^{n-1})\right] \end{aligned}$$

This may be trivially arranged to give,

$$\begin{aligned} u_j^n \quad & + \left[\frac{a^+\Delta t}{2\Delta x}\left(2 - \phi_{j-\frac{1}{2}}^n + \frac{\phi_{j+\frac{1}{2}}^n}{R_{j+\frac{1}{2}}^n}\right) - \frac{\Psi(s_j^n, r_j^n)}{2s_j^n}\right](u_j^n - u_{j-1}^n) \\ & + \left[\frac{a^-\Delta t}{2\Delta x}\left(2 - \phi_{j+\frac{1}{2}}^n + \frac{\phi_{j-\frac{1}{2}}^n}{R_{j-\frac{1}{2}}^n}\right) + \frac{\Psi(r_j^n, s_j^n)}{2r_j^n}\right](u_{j+1}^n - u_j^n) \\ & = u_j^{n-1} - \frac{\Psi(s_j^{n-1}, r_j^{n-1})}{2s_j^{n-1}}(u_j^{n-1} - u_{j-1}^{n-1}) + \frac{\Psi(r_j^{n-1}, s_j^{n-1})}{2r_j^{n-1}}(u_{j+1}^{n-1} - u_j^{n-1}) \end{aligned} \tag{15}$$

In order to obtain TVD conditions we generalise Harten's theorem 2.1 to give,

Theorem 2.3 *Consider the scheme,*

$$\begin{aligned} & V_j + c_{j-\frac{1}{2}}(V_j - V_{j-1}) - d_{j+\frac{1}{2}}(V_{j+1} - V_j) \\ = \; & U_j - C_{j-\frac{1}{2}}(U_j - U_{j-1}) + D_{j+\frac{1}{2}}(U_{j+1} - U_j). \end{aligned}$$

Then the following conditions are sufficient for the scheme to be TVD,

$$\begin{array}{lll} C_{j-\frac{1}{2}}, D_{j+\frac{1}{2}} & \geq 0 & \forall\, j \\ c_{j-\frac{1}{2}}, c_{j+\frac{1}{2}} & \geq 0 & \forall\, j \\ C_{j-\frac{1}{2}} + D_{j-\frac{1}{2}} & \leq 1 & \forall\, j \end{array}$$

Proof

Subtracting the scheme for V_j and V_{j-1} we obtain,

$$\begin{aligned} V_j - V_{j-1} \quad & + c_{j-\frac{1}{2}}(V_j - V_{j-1}) - c_{j-\frac{3}{2}}(V_{j-1} - V_{j-2}) \\ & - d_{j+\frac{1}{2}}(V_{j+1} - V_j) + d_{j-\frac{1}{2}}(V_j - V_{j-1}) \\ = \; & U_j - U_{j-1} - C_{j-\frac{1}{2}}(U_j - U_{j-1}) + C_{j-\frac{3}{2}}(U_{j-1} - U_{j-2}) \\ & + D_{j+\frac{1}{2}}(U_{j+1} - U_j) - D_{j-\frac{1}{2}}(U_j - U_{j-1}), \end{aligned}$$

or,

$$\begin{aligned} & \left(1 + c_{j-\frac{1}{2}} + d_{j-\frac{1}{2}}\right)(V_j - V_{j-1}) \\ = \; & c_{j-\frac{3}{2}}(V_{j-1} - V_{j-2}) + d_{j+\frac{1}{2}}(V_{j+1} - V_j) \\ & + \left(1 - C_{j-\frac{1}{2}} - D_{j-\frac{1}{2}}\right)(U_j - U_{j-1}) + C_{j-\frac{3}{2}}(U_{j-1} - U_{j-2}) + D_{j+\frac{1}{2}}(U_{j+1} - U_j). \end{aligned}$$

The conditions of the theorem ensure that all the coefficients in the above are positive and hence,

$$\begin{aligned} & \left(1 + c_{j-\frac{1}{2}} + d_{j-\frac{1}{2}}\right)|V_j - V_{j-1}| \\ \leq \; & c_{j-\frac{3}{2}}|V_{j-1} - V_{j-2}| + d_{j+\frac{1}{2}}|V_{j+1} - V_j| \\ & + \left(1 - C_{j-\frac{1}{2}} - D_{j-\frac{1}{2}}\right)|U_j - U_{j-1}| + C_{j-\frac{3}{2}}|U_{j-1} - U_{j-2}| + D_{j+\frac{1}{2}}|U_{j+1} - U_j|, \end{aligned}$$

summing,

$$\begin{aligned} & \sum_j \left(1 + c_{j-\frac{1}{2}} + d_{j-\frac{1}{2}}\right)|V_j - V_{j-1}| \\ \leq \; & \sum_j c_{j-\frac{3}{2}}|V_{j-1} - V_{j-2}| + \sum_j d_{j+\frac{1}{2}}|V_{j+1} - V_j| \\ & + \sum_j \left(1 - C_{j-\frac{1}{2}} - D_{j-\frac{1}{2}}\right)|U_j - U_{j-1}| + \sum_j C_{j-\frac{3}{2}}|U_{j-1} - U_{j-2}| + \sum_j D_{j+\frac{1}{2}}|U_{j+1} - U_j|, \end{aligned}$$

and then reordering four of the sums,

$$\begin{aligned} & \sum_j \left(1 + c_{j-\frac{1}{2}} + d_{j-\frac{1}{2}}\right)|V_j - V_{j-1}| \\ \leq \; & \sum_j c_{j-\frac{1}{2}}|V_j - V_{j-1}| + \sum_j d_{j-\frac{1}{2}}|V_j - V_{j-1}| \\ & + \sum_j \left(1 - C_{j-\frac{1}{2}} - D_{j-\frac{1}{2}}\right)|U_j - U_{j-1}| + \sum_j C_{j-\frac{1}{2}}|U_j - U_{j-1}| + \sum_j D_{j-\frac{1}{2}}|U_j - U_{j-1}|, \end{aligned}$$

giving,

$$\sum_j |V_j - V_{j-1}| \leq \sum_j |U_j - U_{j-1}|,$$

or,

$$TV(\mathbf{V}) \leq TV(\mathbf{U}),$$

and the scheme is TVD. □

Now returning to equation (15) and invoking theorem 2.3 we obtain,

$$\left.\begin{aligned}
C_{j-\frac{1}{2}} &= \frac{\Psi(s_j^{n-1}, r_j^{n-1})}{2s_j^{n-1}} \\
D_{j+\frac{1}{2}} &= \frac{\Psi(r_j^{n-1}, s_j^{n-1})}{2r_j^{n-1}} \\
c_{j-\frac{1}{2}} &= \frac{a^+\Delta t}{2\Delta x}\left(2 - \phi_{j-\frac{1}{2}}^n + \frac{\phi_{j+\frac{1}{2}}^n}{R_{j+\frac{1}{2}}^n}\right) - \frac{\Psi(s_j^n, r_j^n)}{2s_j^n} \\
d_{j+\frac{1}{2}} &= \frac{-a^-\Delta t}{2\Delta x}\left(2 - \phi_{j+\frac{1}{2}}^n + \frac{\phi_{j-\frac{1}{2}}^n}{R_{j-\frac{1}{2}}^n}\right) - \frac{\Psi(r_j^n, s_j^n)}{2r_j^n}
\end{aligned}\right\}$$

We see that $C_{j-\frac{1}{2}} > 0$ and $D_{j+\frac{1}{2}} > 0$ and $C_{j-\frac{1}{2}} + D_{j-\frac{1}{2}} \leq 1$ provided Φ satisfies the conditions of theorem 2.2 for TVD extrapolation. The conditions $c_{j-\frac{1}{2}}, d_{j+\frac{1}{2}} \geq 0$ are harder to enforce due to the term corresponding to TVD extrapolation being negative.

To circumvent this we use the identities,

$$\left.\begin{aligned}
\frac{u_{j+1}^n - u_j^n}{r_j^n} &= -\frac{u_j^n - u_{j-1}^n}{s_j^n} \\
\frac{u_j^n - u_{j-1}^n}{s_j^n} &= -\frac{u_{j+1}^n - u_j^n}{r_j^n}
\end{aligned}\right\},$$

to rewrite equation (15) as,

$$\begin{aligned}
u_j^n &+ \left[\frac{a^+\Delta t}{2\Delta x}\left(2 - \phi_{j-\frac{1}{2}}^n + \frac{\phi_{j+\frac{1}{2}}^n}{R_{j+\frac{1}{2}}^n}\right) - \frac{\Psi(r_j^n, s_j^n)}{2s_j^n}\right](u_j^n - u_{j-1}^n) \\
&- \left[\frac{-a^-\Delta t}{2\Delta x}\left(2 - \phi_{j+\frac{1}{2}}^n + \frac{\phi_{j-\frac{1}{2}}^n}{R_{j-\frac{1}{2}}^n}\right) - \frac{\Psi(s_j^n, r_j^n)}{2r_j^n}\right](u_{j+1}^n - u_j^n) \\
&= u_j^{n-1} - \frac{\Psi(s_j^{n-1}, r_j^{n-1})}{2s_j^{n-1}}(u_j^{n-1} - u_{j-1}^{n-1}) + \frac{\Psi(r_j^{n-1}, s_j^{n-1})}{2r_j^{n-1}}(u_{j+1}^{n-1} - u_j^{n-1}) \quad (16)
\end{aligned}$$

Now,

$$\left.\begin{aligned} C_{j-\frac{1}{2}} &= \frac{\Psi(s_j^{n-1}, r_j^{n-1})}{2s_j^{n-1}} \\ D_{j+\frac{1}{2}} &= \frac{\Psi(r_j^{n-1}, s_j^{n-1})}{2r_j^{n-1}} \\ c_{j-\frac{1}{2}} &= \frac{a^+\Delta t}{2\Delta x}\left(2 - \phi^n_{j-\frac{1}{2}} + \frac{\phi^n_{j+\frac{1}{2}}}{R^n_{j+\frac{1}{2}}}\right) - \frac{\Psi(r_j^n, s_j^n)}{2s_j^n} \\ d_{j+\frac{1}{2}} &= \frac{-a^-\Delta t}{2\Delta x}\left(2 - \phi^n_{j+\frac{1}{2}} + \frac{\phi^n_{j-\frac{1}{2}}}{R^n_{j-\frac{1}{2}}}\right) - \frac{\Psi(s_j^n, r_j^n)}{2r_j^n} \end{aligned}\right\} \tag{17}$$

so we see that provided condition (7) holds for the spatial limiters we have the additional condition,

$$-\frac{\Psi(r,s)}{2s} \geq 0 \text{ for all } r, s.$$

We see that since $\Psi(r, s) \geq 0$ for all r, s that we must have

$$\Psi(r, s) = 0 \text{ for } s > 0 \tag{18}$$

2.4.1 Second Order Accuracy

So far we have only considered properties of the extrapolation limiters $\Psi(r, s)$ needed to ensure that the numerical scheme produced is TVD. To summarise these are,

$$\left.\begin{aligned} \Psi(r,s) &= 0 & \forall r \leq 0 \\ \Psi(r,s) &\geq 0 & \forall r > 0 \\ \Psi(r,s) &\leq r & \forall r > 0 \\ \Psi(r,s) &= 0 & \forall s > 0 \end{aligned}\right\}. \tag{19}$$

We now need to consider what properties such a limiter must have in order to attain second order accuracy. To do this we follow Forth [10] and consider conditions on $\Psi(r, s)$ such that a linear field may be recovered by the extrapolation scheme.

Consider the linear field,

$$u(t, x) = At + Bx,$$

on a uniform mesh spacing $\Delta t, \Delta x$. Then equation (10) gives,

$$\left.\begin{aligned} r_j^n &= \frac{B\Delta x}{A\Delta t} \\ s_j^n &= -\frac{B\Delta x}{A\Delta t} \end{aligned}\right\},$$

and equation (11) then yields,

$$\frac{A\Delta t}{2} = \frac{1}{2}\left(\Psi\left(\frac{B\Delta x}{A\Delta t}, -\frac{B\Delta x}{A\Delta t}\right) + \Psi\left(-\frac{B\Delta x}{A\Delta t}, \frac{B\Delta x}{A\Delta t}\right)\right)\frac{A\Delta t}{2}.$$

Now since the two arguments to the limiters are of different sign we see that for $\frac{B\Delta x}{A\Delta t} > 0$ we recover,

$$\frac{A\Delta t}{2} = \frac{1}{2}\Psi\left(\frac{B\Delta x}{A\Delta t}, -\frac{B\Delta x}{A\Delta t}\right)\frac{A\Delta t}{2},$$

and for $\frac{B\Delta x}{A\Delta t} < 0$ we obtain,

$$\frac{A\Delta t}{2} = \frac{1}{2}\Psi\left(-\frac{B\Delta x}{A\Delta t}, \frac{B\Delta x}{A\Delta t}\right)\frac{A\Delta t}{2},$$

both of which reduce to,

$$\frac{A\Delta t}{2} = \frac{1}{2}\Psi\left(\left|\frac{B\Delta x}{A\Delta t}\right|, -\left|\frac{B\Delta x}{A\Delta t}\right|\right)\frac{A\Delta t}{2},$$

or,

$$\Psi(r, -r) = 1 \;\; \forall\, r > 0. \tag{20}$$

2.5 Limiter Functions

The conditions that our limiter function $\Psi(r, s)$ needs to satisfy are the TVD conditions of equation (19) and the second order accuracy condition (20). We therefore set,

$$\left.\begin{array}{lll} \Psi(r,s) & = \psi(r) & \forall s \leq 0 \\ \Psi(r,s) & = 0 & \forall s > 0 \end{array}\right\} \tag{21}$$

where $\psi(r)$ satisfies,

$$\left.\begin{array}{ll} \psi(r) = 0 & \forall r \leq 0 \\ 0 \leq \psi(r) \leq r & \forall r > 0 \end{array}\right\} \tag{22}$$

to ensure the scheme is TVD and further that,

$$\psi(r) = 1 \tag{23}$$

where possible to give second order accuracy. In figure 2 the shaded area shows in $(r, \psi(r))$ space the requirements on the limiter function to be TVD. The solid line shows the requirement for second order accuracy. We see that for $r \geq 1$ it is possible for a TVD scheme to be second order accurate. For $r < 1$ the line defining second order accuracy is not possible.

As expected we cannot attain global second order accuracy but seek a limiter function that guarantees the TVD property and is second order accurate where possible. We may for example use the usual minmod limiter,

$$\text{minmod}(r) = \left\{\begin{array}{ll} 0 & r \leq 0 \\ r & 0 < r < 1 \\ 1 & r \geq 1 \end{array}\right. . \tag{24}$$

As was noted by Forth [10] none of the other classical limiters such as superbee, Van Albada or Van Leer are suitable for extrapolation limiting since they violate the condition $\psi(r) < r$.

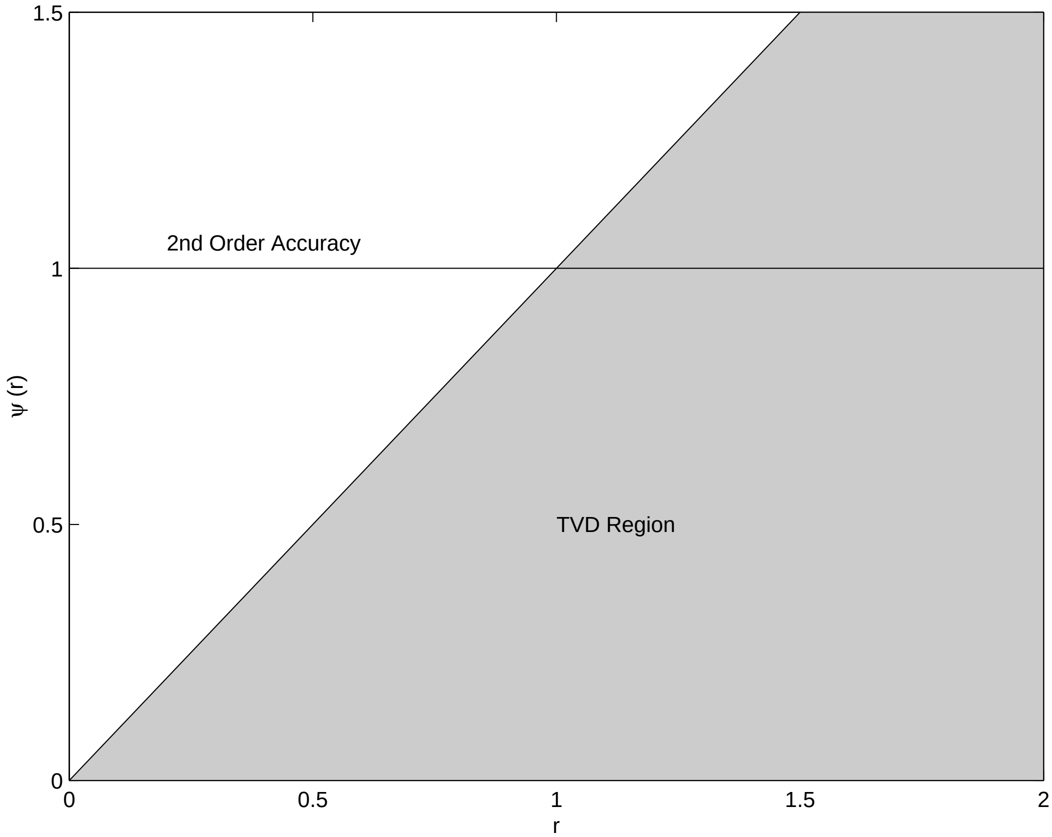

Figure 2: Region of $(r, \psi(r))$ space for which an extrapolation limiter is TVD (shaded) and second order accuracy requirement (solid line).

A second suitable limiter presented in Forth [10] is the so-called tanhquartic limiter defined by,

$$\text{tanhquartic}(r) = \begin{cases} 0 & r \leq 0 \\ \tanh\left(r + \frac{1}{2}r^4\right) & r > 0 \end{cases} \tag{25}$$

This has the advantage over minmod of being smooth for positive r.

Both the minmod and tanhquartic limiters are plotted in figure 3.

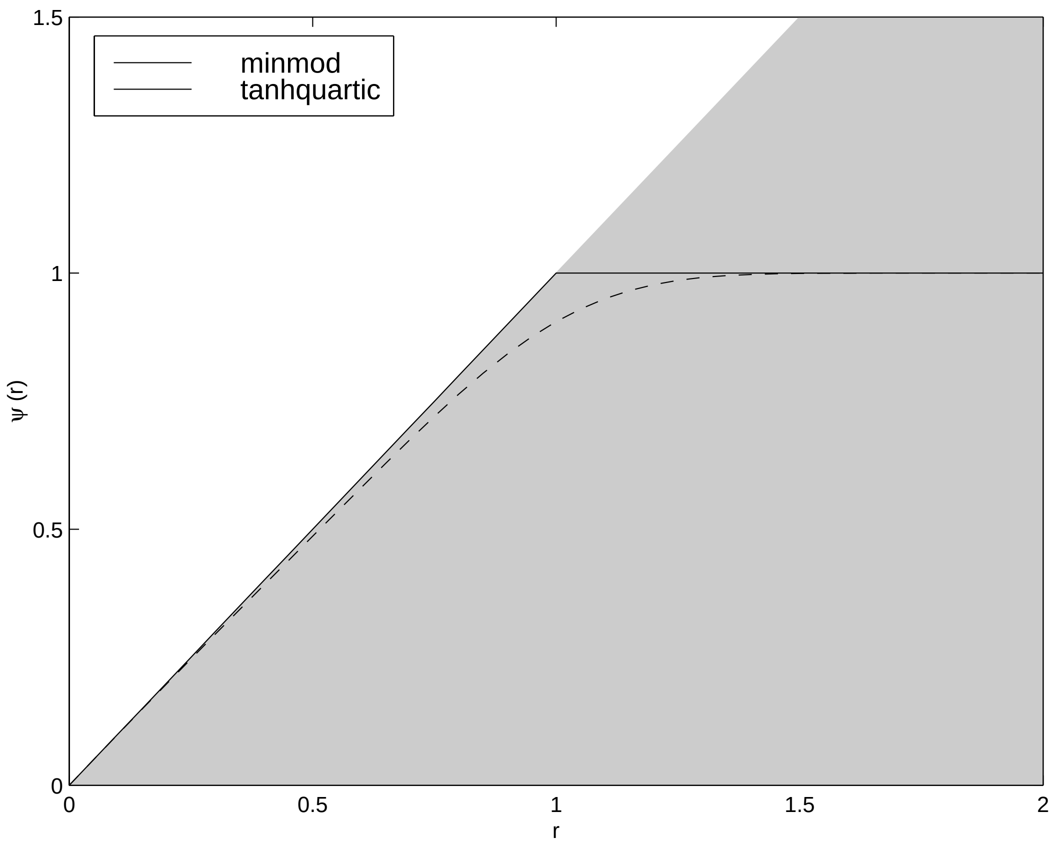

Figure 3: The minmod and tanhquartic limiters in $(r, \psi(r))$.

3 Numerical Implementation

A Matlab [12] program was written to solve the discretised equations of the previous section. In particular the residual or discrete version of the integral form of equation (2) was calculated. Spatial fluxes were calculated using the second order accurate flux limited scheme of equation (3) with Van Albada's continuous limiter function. Temporal fluxes were calculated using the techniques of section 2.4 with either an unlimited extrapolation or extrapolation using the minmod or tanhquartic limiters.

At each time level the residual equation is linearised using the automatic differentiation library ADMIT of Coleman and Verma [13], and a Newton solver with backtracking (damped Newton) as recommended by Dennis and Schnabel [14] is used to drive the residual to zero. Occasional convergence problems were experienced but only on the coarsest meshes, otherwise the residuals were reduced to below 10^{-4} at each time-step.

For comparison purposes 3 standard finite-volume schemes for the linear advection equation were also coded. These were:

First Order Upwind Scheme : the explicit scheme,

$$u_j^{n+1} = u_j^n + \sigma\left(u_{j-1}^n - u_j^n\right). \tag{26}$$

Second Order Upwind Scheme : the explicit scheme,

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}\left(f_{j-\frac{1}{2}}^n - f_{j+\frac{1}{2}}^n\right), \tag{27}$$

with limited fluxes,

$$f_{j+\frac{1}{2}}^n = a\left[u_j^n + \frac{1}{2}(1-\sigma)(u_{j+1}^n - u_j^n)\Phi\left(\frac{u_j^n - u_{j-1}^n}{u_{j+1}^n - u_j^n}\right)\right], \tag{28}$$

where we have used Van Albada's limiter,

$$\Phi(R) = \frac{R(1+R)}{1+R^2}.$$

Implicit TVD Scheme : A so-called θ-scheme with $\theta = \frac{1}{2}$ giving a Crank-Nicolson type scheme,

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{2\Delta x}\left(f_{j-\frac{1}{2}}^{n+1} - f_{j+\frac{1}{2}}^{n+1} + f_{j-\frac{1}{2}}^n - f_{j+\frac{1}{2}}^n\right), \tag{29}$$

with fluxes given by the spatially second order accurate formula of equation (3) with Van Albada limiting.

Note that the cell centred schemes were initialised by performing 2, 4 or 8 steps of the implicit TVD scheme at the appropriate fraction of the required CFL number in order to obtain TVD solutions for the u_j^1 and $u_j^{1+\frac{1}{2}}$ with u_j^n for $n \geq 2$ determined using the cell centred scheme.

4 Results

Both of the following test cases correspond to the solution of the linear advection equation with unit advection velocity,

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \tag{30}$$

on the domain $[-1,1]$ subject to initial conditions $u(x,t=0) = u_0(x)$ and boundary conditions at the left hand boundary $u(x=-1,t) = u_l(t)$. The general solution to this problem is

$$u(x,t) = \begin{cases} u_0(x-t) & \text{for } x > -1+t \\ u_l(x-t) & \text{for } x < -1+t \end{cases} \tag{31}$$

4.1 Smooth Solution

The first test case we consider has initial condition,

$$u_0(x) = \frac{1}{2}\left(1 - \sin \pi x\right),$$

and boundary condition,

$$u_l(t) = \frac{1}{2}\left(1 + \sin \pi(1+t)\right),$$

carefully chosen so that the solution is continuous and differentiable,

$$u(x,t) = \frac{1}{2}\left(1 - \sin \pi(x-t)\right). \tag{32}$$

This was solved on equally spaced meshes of $N = 30, 60, 120$ and 240 mesh cells and for a range of CFL numbers. In all cases time integration took place from $t=0$ to $t=2$. Results for the solution error at the end of the integration are given in tables 1 to table 6 for a number of numerical schemes. We note that

	CFL number		
N	0.5	0.75	1
30	0.095242	0.050072	0.000000
60	0.049843	0.025591	0.000000
120	0.025527	0.012940	0.000000
240	0.012923	0.006507	0.000000

Table 1: Errors (Smooth Solution) for First Order Upwind Scheme

the two standard explicit schemes both have zero error for a CFL number σ of one. In this case both schemes exactly advect the solution and no numerical error results. Of course both of these schemes being explicit are subject to the CFL condition $\sigma \leq 1$ and so results for higher CFL numbers were not calculated.

N	CFL number 0.5	0.75	1
30	0.015396	0.008206	0.000000
60	0.003776	0.002057	0.000000
120	0.000892	0.000505	0.000000
240	0.000211	0.000123	0.000000

Table 2: Errors (Smooth Solution) for Second Order Upwind Scheme

N	CFL number 0.5	0.75	1	1.5	2	5
30	0.029431	0.027534	0.023639	0.019535	0.027897	0.193328
60	0.007707	0.007072	0.006222	0.005666	0.007152	0.045880
120	0.001844	0.001673	0.001482	0.001511	0.001883	0.011043
240	0.000440	0.000396	0.000343	0.000386	0.000478	0.002759

Table 3: Errors (Smooth Solution) for Second Order Implicit TVD Scheme

N	CFL number 0.5	0.75	1	1.5	2	5
30	0.024491	0.020870	0.029518	0.064725	0.113682	0.461750
60	0.006105	0.006040	0.007467	0.015976	0.029915	0.179261
120	0.001659	0.001555	0.002005	0.004030	0.007446	0.051181
240	0.000481	0.000601	0.000551	0.001031	0.001859	0.012831

Table 4: Errors (Smooth Solution) for Second Order Cell Centred (Unlimited) Scheme

N	CFL number 0.5	0.75	1	1.5	2	5
30	0.026259	0.034561	0.045964	0.104834	0.182314	0.453915
60	0.006618	0.010015	0.015395	0.053031	0.100583	0.302797
120	0.001964	0.002610	0.004327	0.027583	0.053630	0.182052
240	0.000544	0.000931	0.001102	0.014116	0.027729	0.101511

Table 5: Errors (Smooth Solution) for Second Order Cell Centred (Minmod limited) Scheme

	CFL number					
N	0.5	0.75	1	1.5	2	5
30	0.026962	0.035040	0.043852	0.111931	0.186587	0.454532
60	0.006738	0.009902	0.012903	0.056611	0.102994	0.303126
120	0.002041	0.002508	0.006488	0.029472	0.055072	0.182396
240	0.000564	0.004815	0.003525	0.015310	0.028650	0.101844

Table 6: Errors (Smooth Solution) for Second Order Cell Centred (tanhquartic limited) Scheme

If e_n is the numerical error on a mesh of n cells then

$$e_n = K(1/n)^p + \dots,$$

where p is the order of accuracy of the scheme and K is a constant. Similarly $e_{n/2}$ is given by,

$$e_{n/2} = K(2/n)^p + \dots.$$

Thus

$$\frac{e_{n/2}}{e_n} \approx 2^p,$$

and so,

$$p \approx \frac{\log\left(\frac{e_{n/2}}{e_n}\right)}{\log 2}.$$

In tables 7 to 12 we plot the numerical order of accuracy p calculated as above for the same test cases. Note that since we need the error on the current grid e_n and on a coarser grid $e_{n/2}$ we cannot calculate p on the coarsest mesh $N = 30$.

	CFL number	
N	0.5	0.75
60	0.934187	0.968368
120	0.965358	0.983800
240	0.982106	0.991769

Table 7: Order of Accuracy (Smooth Solution) for First Order Upwind Scheme

	CFL number	
N	0.5	0.75
60	2.027586	1.995869
120	2.081161	2.025346
240	2.076256	2.037171

Table 8: Order of Accuracy (Smooth Solution) for Second Order Upwind Scheme

We see that the standard schemes of tables 7 to 9 all attain the expected order

	CFL number					
N	0.5	0.75	1	1.5	2	5
60	1.933083	1.960918	1.925546	1.785709	1.963635	2.075119
120	2.062917	2.079151	2.069204	1.907271	1.924981	2.054745
240	2.065822	2.075965	2.109222	1.967379	1.977536	2.001116

Table 9: Order of Accuracy (Smooth Solution) for implicit TVD Scheme Scheme

of accuracy for this solution.

	CFL number					
N	0.5	0.75	1	1.5	2	5
60	2.004130	1.788644	1.982827	2.018379	1.926054	1.365050
120	1.879005	1.957656	1.897051	1.987162	2.006407	1.808373
240	1.784954	1.371313	1.862029	1.967149	2.001893	1.995994

Table 10: Order of Accuracy (Smooth Solution) for Cell Centred (Unlimited) Scheme

	CFL number					
N	0.5	0.75	1	1.5	2	5
60	1.988401	1.786896	1.577970	0.983195	0.858041	0.584071
120	1.752211	1.939955	1.830820	0.943063	0.907272	0.734000
240	1.851703	1.487312	1.972928	0.966487	0.951641	0.842710

Table 11: Order of Accuracy (Smooth Solution) for Cell Centred (Minmod Limited) Scheme

For the cell centred schemes we see that the streamwise unlimited scheme achieves high orders of accuracy close to 2. However the results for $\sigma = 0.75$ are poor. The minmod scheme appears to drop from first to second order accuracy for CFL numbers greater than 1. For the tanhquartic the effect is even worse with second order accuracy lost for a CFL number of 1. The reasons for this are discussed in section 5.1.

	CFL number					
N	0.5	0.75	1	1.5	2	5
60	2.000473	1.823123	1.764929	0.983466	0.857288	0.584461
120	1.722429	1.980781	0.991665	0.941723	0.903180	0.732842
240	1.854675	-0.940698	0.879975	0.944831	0.942762	0.840711

Table 12: Order of Accuracy (Smooth Solution) for Cell Centred (Tanhquartic Limited) Scheme

4.2 Discontinuous Solution

The second test case we consider has a step discontinuity in the initial conditions and hence for all time,

$$u_0(x) = \begin{cases} 1 & \text{for } x \leq -\frac{1}{2} \\ 0 & \text{for } x > -\frac{1}{2} \end{cases} .$$

The boundary condition at the left hand side $x = -1$ is,

$$u_l(t) = 1, \quad \forall t \geq 0.$$

The solution is therefore,

$$u(x,t) = \begin{cases} 1 & \text{for } x \leq -\frac{1}{2} + t \\ 0 & \text{for } x > -\frac{1}{2} + t \end{cases} ,$$

a step discontinuity propagating from left to right. We compute the solution numerically up to $t = 1$ so that the discontinuity remains within the domain.

In figures 13 to 18 we display the numerical error at time $t = 1$ for the computed solutions. Again the explicit schemes are only used for CFL

	CFL number		
N	0.5	0.75	1
30	0.144403	0.112520	0.000000
60	0.102577	0.072182	0.000000
120	0.072684	0.051271	0.000000
240	0.051449	0.036336	0.000000

Table 13: Errors (Discontinuous Solution) for First Order Upwind Scheme

	CFL number		
N	0.5	0.75	1
30	0.073934	0.082867	0.000000
60	0.045946	0.036955	0.000000
120	0.028204	0.022998	0.000000
240	0.017158	0.014142	0.000000

Table 14: Errors (Discontinuous Solution) for Second Order Upwind Scheme

numbers $\sigma \leq 1$. Note we have not run for $\sigma = 2$ in this case since a non-integer number of time-steps would be required.

	CFL number				
N	0.5	0.75	1	1.5	5
30	0.091966	0.108569	0.092877	0.103988	0.322379
60	0.056636	0.056485	0.056804	0.063859	0.235296
120	0.034607	0.034376	0.034468	0.038926	0.167252
240	0.021045	0.020808	0.020799	0.023741	0.116217

Table 15: Errors (Discontinuous Solution) for Second Order Implicit TVD Scheme

	CFL number				
N	0.5	0.75	1	1.5	5
30	0.090793	0.124102	0.121116	0.159979	0.300610
60	0.055717	0.063983	0.078964	0.108779	0.224964
120	0.033915	0.039552	0.050879	0.072721	0.159950
240	0.020545	0.024392	0.032724	0.048574	0.110015

Table 16: Errors (Discontinuous Solution) for cell Centred (streamwise unlimited) Scheme

	CFL number				
N	0.5	0.75	1	1.5	5
30	0.091518	0.119009	0.116993	0.159654	0.291055
60	0.056241	0.062764	0.074719	0.110635	0.245231
120	0.034186	0.038422	0.047232	0.076430	0.191780
240	0.020659	0.023360	0.029722	0.053092	0.141872

Table 17: Errors (Discontinuous Solution) for cell Centred (streamwise minmod) Scheme

	CFL number				
N	0.5	0.75	1	1.5	5
30	0.092136	0.119874	0.120114	0.162867	0.291296
60	0.056588	0.063997	0.077300	0.113364	0.245455
120	0.034528	0.039490	0.049498	0.078590	0.191970
240	0.021011	0.024214	0.031837	0.054664	0.142021

Table 18: Errors (Discontinuous Solution) for cell Centred (streamwise tanhquartic) Scheme

It is known (see LeVeque [1] p115) that for a discontinuous solution a method that is first order accurate for smooth solutions will converge with accuracy order $\frac{1}{2}$ for discontinuous solutions. Similarly a formally second order accurate scheme will give convergence of order $\frac{2}{3}$ on a discontinuous solution. In figures 19 to 24 we display the numerical order of accuracy calculated as described in section 4.1. For the standard schemes we see the expected orders $p \approx \frac{1}{2}$ for the first order

	CFL number	
N	0.5	0.75
60	0.493396	0.640467
120	0.496980	0.493491
240	0.498497	0.496744

Table 19: Order of Accuracy (Discontinuous Solution) for First Order Upwind Scheme

	CFL number	
N	0.5	0.75
60	0.686287	1.165034
120	0.704020	0.684252
240	0.716968	0.701519

Table 20: Order of Accuracy (Discontinuous Solution) for Second Order Upwind Scheme

	CFL number				
N	0.5	0.75	1	1.5	5
60	0.699378	0.942671	0.709328	0.703450	0.454278
120	0.710635	0.716445	0.720708	0.714158	0.492453
240	0.717565	0.724290	0.728771	0.713321	0.525196

Table 21: Order of Accuracy (Discontinuous Solution) for Implicit TVD Scheme

schemes. The second order schemes generally achieve order $p > \frac{2}{3}$ and are performing better than expected. The exception to this appears to be the implicit scheme with $\sigma = 5$. In fact this result is not surprising since this scheme is known to be TVD only for CFL numbers $\sigma \leq \frac{4}{2+\alpha}$ where α is a function of the spatial limiter function used,

$$\left|\Phi(r) - \frac{\Phi(s)}{s}\right| < \alpha,$$

as noted by Hirsch [2] p558-559. This is confirmed by the plot of figure 4

For the cell-centred schemes accuracy is of the order of $\frac{2}{3}$ only for CFL numbers less than or equal to 1. In particular for CFL numbers $\sigma = 5$ the numerical order of accuracy is slow to converge with the number of mesh cells and doesn't reach $\frac{1}{2}$ even on the finest mesh. The reason for this analysed in section 5.1.

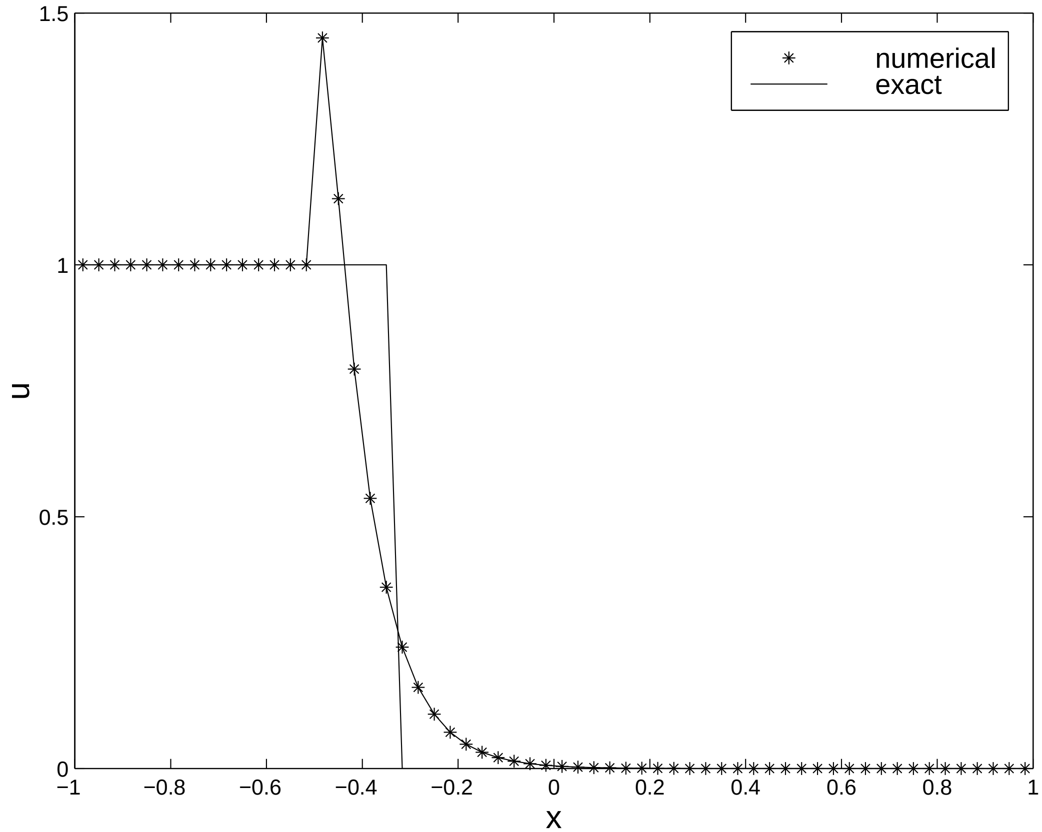

Figure 4: Solution for discontinuous test case after one time step for a CFL number $\sigma = 5$ on a mesh of 60 cells using the implicit TVD scheme.

	CFL number				
N	0.5	0.75	1	1.5	5
60	0.704464	0.955765	0.617126	0.556475	0.418200
120	0.716170	0.693910	0.634100	0.580960	0.492072
240	0.723116	0.697344	0.636723	0.582174	0.539927

Table 22: Order of Accuracy (Discontinuous Solution) for Cell Centred (streamwise unlimited) Scheme

	CFL number				
N	0.5	0.75	1	1.5	5
60	0.702420	0.923058	0.646874	0.529137	0.247148
120	0.718199	0.707986	0.661714	0.533596	0.354694
240	0.726623	0.717899	0.668197	0.525629	0.434858

Table 23: Order of Accuracy (Discontinuous Solution) for Cell Centred (streamwise Minmod) Scheme

	CFL number				
N	0.5	0.75	1	1.5	5
60	0.703271	0.905444	0.635865	0.522735	0.247027
120	0.712694	0.696507	0.643073	0.528531	0.354580
240	0.716634	0.705648	0.636649	0.523757	0.434777

Table 24: Order of Accuracy (Discontinuous Solution) for Cell Centred (streamwise tanhquartic) Scheme

5 Discussion

In this section we discuss the loss of accuracy displayed by the cell-centred schemes for higher CFL numbers and propose ways to circumvent this problem.

5.1 Loss of Accuracy for High CFL Numbers

From the results of the previous section we see that we are not attaining the desired second order accuracy of the cell centred schemes for CFL numbres greater than one, with the accuracy 'dropping off' with increasing CFL number. The reason for this can be explained by more fully analysing the accuracy of the extrapolation scheme of section 2.3.

If we expand in Taylor series about the value $u = u_j^n$ then the formulae (10) for the gradient r_j^n supplied to the streamwise limiter become,

$$
\begin{aligned}
r_j^n &= \frac{u_{j+1}^n - u_j^n}{u_j^n - u_j^{n-1}} \\
&= \frac{u + u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 - u + O\left((\Delta x)^3\right)}{u - \left(u - u_t \Delta t + O\left((\Delta t)^2\right)\right)} \\
&= \frac{u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + O\left((\Delta x)^3\right)}{u_t \Delta t} \left(1 + \frac{O\left((\Delta t)^2\right)}{u_t \Delta t}\right) \\
&= \frac{u_x \Delta x}{u_t \Delta t} + \frac{1}{2} \frac{u_{xx} (\Delta x)^2}{u_t \Delta t} + \frac{O\left((\Delta x)^3, (\Delta t)^2\right)}{u_t \Delta t}.
\end{aligned}
$$

But using $u_t = -a u_x$ this gives,

$$
r_j^n = -\frac{1}{\sigma} - \frac{1}{2} \frac{u_{xx} (\Delta x)}{u_x \sigma} + \frac{O\left((\Delta x)^3, (\Delta t)^2\right)}{u_t \Delta t}. \tag{33}
$$

Similarly we obtain,

$$
s_j^n = \frac{1}{\sigma} - \frac{1}{2} \frac{u_{xx} (\Delta x)}{u_x \sigma} + \frac{O\left((\Delta x)^3, (\Delta t)^2\right)}{u_t \Delta t}. \tag{34}
$$

Expanding the extrapolation formula (11) we obtain,

$$
u_j^{n+\frac{1}{2}} = u + \frac{1}{2} \left[\Psi\left(-\frac{1}{\sigma}, \frac{1}{\sigma}\right) + \Psi\left(\frac{1}{\sigma}, -\frac{1}{\sigma}\right) \right] u_t \Delta t + O\left(\frac{(\Delta t^2)}{u_t}, \frac{(\Delta x \Delta t)}{u_x \sigma}, \ldots\right).
$$

To obtain second order accuracy on a smooth solution (with $u_x, u_t \neq 0$) we therefore need,

$$
\Psi\left(\frac{1}{\sigma}, -\frac{1}{\sigma}\right) = 1, \tag{35}
$$

we see that this is only possible (see figure 3) for the streamwise minmod limiter for $\sigma \leq 1$ and the condition for the tanhquartic limiter is even more restrictive since this limiter only asymptotes to 1 (thought it does do so rather rapidly).

5.2 Improved Scheme

The analysis of section 5.1 indicates that we have a flaw in our TVD extrapolation analysis with the limiter working on values of $1/\sigma$ for smooth solutions. This may be rectified if we limit on ratios σr_j^n and σs_j^n. A more rigorous justification for this is that we may approximate the temporal gradient u_t either by $\frac{u_j^n - u_j^{n-1}}{\Delta t}$ as we have done here, or we may use,

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x},$$

from the differential equation, and so,

$$\frac{\partial u}{\partial t} \approx -\frac{a(u_j^n - u_{j-1}^n)}{\Delta x}.$$

Thus we might use,

$$\begin{aligned} u_j^{n+\frac{1}{2}} &= u_j^n - \frac{1}{2}\frac{a\Delta t(u_j^n - u_{j-1}^n)}{\Delta x} \\ &= u_j^n - \frac{1}{2}\sigma\left(u_j^n - u_{j-1}^n\right), \end{aligned}$$

So a more suitable limiter function might limit based on values of

$$s_j^n = \frac{\sigma\left(u_j^n - u_{j-1}^n\right)}{u_j^n - u_j^{n-1}}.$$

This approach has in fact been recently used by Sidilikover [4] but not in the context of schemes centred in time as is ours. However one of the advantages of the current framework over that of Sidilikover is that since the space-time limiter is applied only to temporal fluxes that second order accuracy in space should be maintained. However this author conjectures that applying such an approach may result in a TVD condition based on the CFL number.

6 Conclusions

In this report we have presented a new approach to solving the hyperbolic linear equation based on using a finite volume scheme centred both in space and time. The spatial discretisation was performed using a conventional flux limited scheme. Since this is performed at the same time level as the solution values it has zero contribution to the temporal component of the truncation error. Temporal discretisation relies on a TVD extrapolation in time to achieve second order accuracy. This extrapolation relies on limiter functions and properties of these limiter functions are discussed. A TVD analysis of the whole numerical scheme gives further restrictions on the limiter functions. The discretised equations were solved using a Newton scheme at each time-step with the exact linearisation performed using automatic differentiation.

The method was compared with first and second order accurate, explicit, flux limited schemes and an implicit Crank-Nicolson like TVD scheme. Numerical results demonstrate that the new scheme is second order accurate for CFL numbers one or less provided the minmod limiter is used for the temporal flux limiting. The scheme remains TVD for CFL numbers greater than one and is in this sense superior to the Crank-Nicolson type scheme but does lose an order of accuracy.

The loss of accuracy at high CFL numbers was analysed and shown to be due to a subtle inconsistency in the TVD extrapolation process. Comparison with the work of Sidilikover [4] show that this problem may be surmountable though it may be at the cost of a CFL type constraint.

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