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# Are rogue waves really unexpected?

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ABSTRACT

We present a third-order nonlinear model for the statistics of unexpected waves drawing on the work of Gemmrich and Garrett (2008). The model is verified by way of Monte Carlo simulations of Gaussian seas and comparisons to oceanic measurements. In particular, the analysis of oceanic data suggests that both skewness and kurtosis effects must be accounted for to obtain accurate predictions. As a specific application, the unexpectedness of the Andrea and WACSIS rogue wave events is examined in detail. Observations indicate that the crests of these waves have nearly the same amplitude ratio $h/H_s \sim 1.6$, where $H_s$ is the significant wave height. Both waves appeared without warning and they were nearly two-times larger than the surrounding $O(10)$ waves, and thus unexpected according to Gemmrich and Garrett (2008). The model developed here predicts that the two rogue waves are stochastically similar as they occur on average once every $10^4$ waves. Further, the maximum crest height actually observed is nearly the same as the threshold $h_{10^6} \sim 1.62H_s$ exceeded by the $1/10^6$ fraction of largest crests. These results imply that rogue waves are likely occurrences of unexpected events in weakly nonlinear random seas.

## 1. Introduction

A rogue wave is a wave whose crest-to-trough height is at least 2.2 times the significant wave height $H_s$ or whose crest height exceeds $1.34H_s$, where $H_s = 4\sigma$ and $\sigma$ is the standard deviation of the surface elevation (Dysthe et al. 2008). Evidences given for the occurrence of such waves in nature include the Draupner and Andrea events. In particular, the Andrea wave was measured on November 9 2007 by a LASAR system mounted on the Ekofisk platform in the North Sea in a water depth of $d = 74$ m (Magnusson and Donelan 2013). The Draupner freak wave was measured by Statoil at a nearby platform in January 1995 (Haver 2001).

The Andrea wave occurred during a sea state with significant wave height $H_s = 4\sigma = 9.2$ m, mean period $T_0 = 13.2$ s and wavelength $L_0 = 220$ m. Its crest height is $h = 1.63H_s = 15$ m and the crest-to-trough height $H = 2.3H_s = 21.1$ m. In the last decade, the properties of the Draupner and Andrea waves have been extensively studied (Dysthe et al. 2008; Osborne 1995; Magnusson and Donelan 2013; Bitner-Gregersen et al. 2014; Dias et al. 2015). Further, observations of such large extreme waves show that they tend to extend above the surrounding smaller waves without warning and thus unexpectedly. Further, both waves were twice as high as the immediately preceding as well as following groups of waves.

In describing the unexpectedness of wave extremes, Gemmrich and Garrett (2008) define as unexpected a wave $\alpha$-times larger than a set of one-sided (preceding) waves or two-sided (preceding and following) waves (see Fig. 1). By means of Monte Carlo simulations of typical oceanic sea states characterized by the JONSWAP spectrum, they estimated that a wave with height at least twice that of any of the preceding 30 waves (corresponding to 21 peak periods in their simulations) occurs once every $7 \times 10^4$ waves on average, giving a return period of 8 days if the peak period of the waves is 10 s. Also unexpected crest heights are more probable than unexpected wave heights as the return period is 4 days ($\sim 3.5 \times 10^4$ waves) for linear waves and 2 days ($\sim 10^4$ waves) if second order bound nonlinearities are accounted for.

The preceding studies provide the principal motivation here to consider a theoretical model for describing unexpected waves and their rogueness. In particular, we introduce a new statistics for the prediction of unexpected waves that accounts for both second- and third-order nonlinearities. First, we present analytical solutions for the return period of unexpected waves and associated unconditional and conditional averages for crest and wave heights. Then, the conceptual framework is validated by way of Monte Carlo simulations of Gaussian seas and the theoretical predictions are compared to oceanic measurements.

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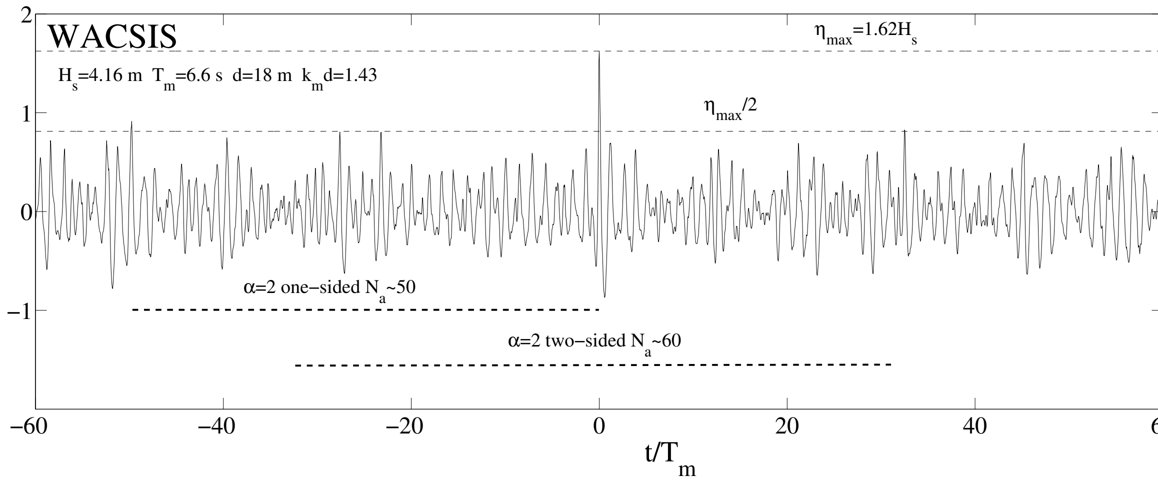


FIG. 1. WACSIS measurements: the observed largest crest height is $\alpha = 2$ -times larger than the crests of the one-sided (two-sided) $N_a \sim 50$ (180) waves. Wave parameters $H_s = 4.16$ m, $T_m = 6.6$ s, depth $d = 18$ m (Forristall et al. 2004).

As a specific application here, we capitalize on the numerical simulations of the Andrea sea state (Bitner-Gregersen et al. 2014; Dias et al. 2015) and examine the unexpectedness of the Andrea wave in detail. Summary and conclusions follow subsequently.

## 2. Statistics of unexpected waves

Drawing on Tayfun and Fedele (2007), the exceedance probability distribution of wave crests characterized by third-order nonlinearities is described by

$$P(x) = \exp\left(-\frac{x_0^2}{2}\right)\left[1+\frac{\Lambda}{64}x_0^2\left(x_0^2-4\right)\right], \qquad (1)$$

where $x = h/\sigma$ is the crest amplitude $h$ scaled by the standard deviation $\sigma$ and $x_0$ follows from the quadratic equation (Tayfun 1980)

$$x = x_0 + \frac{\mu}{2}x_0^2.$$

Here, the Tayfun wave steepness $\mu = \lambda_3/3$ relates to the skewness of surface elevations (Fedele and Tayfun 2009) and the parameter

$$\Lambda = \lambda_{40} + 2\lambda_{22} + \lambda_{04} \qquad (2)$$

is a measure of third-order nonlinearities as a function of the fourth order cumulants $\lambda_{nm}$ of the wave surface $\eta$ and its Hilbert transform $\hat{\eta}$ (Tayfun and Fedele 2007). Drawing on Mori and Janssen (2006), we assume the following relations between cumulants

$$\lambda_{22} = \lambda_{40}/3, \qquad \lambda_{04} = \lambda_{40}, \qquad (3)$$

which, to date, have been proven to hold for second-order NB waves only (Tayfun and Lo 1990). Then, $\Lambda$ can be approximated in terms of the excess kurtosis $\lambda_{40}$ by

$$\Lambda_{\mathrm{appr}} = \frac{8\lambda_{40}}{3}, \qquad (4)$$

which will be used in this work. Then, Eq. (1) reduces to a modified Edgeworth-Rayleigh (MER) distribution (Mori and Janssen 2006). For realistic oceanic seas the kurtosis $\lambda_{40}$ is mainly affected by bound nonlinearities (Fedele 2015b,a).

Consider now a time interval $\tau$ during which $N_w = \tau/T_m$ waves occur on average, where $T_m$ is the mean zero-upcrossing period. Define the event $E$ of an unexpected wave whose crest is $\alpha$-times larger than the surrounding $N_a$ waves. We assume that neighboring waves are stochastically independent. Then, in a sample of $N_a + 1$ successive waves it is irrelevant what wave is the unexpected wave larger than the surrounding waves. Indeed, any wave in the sample could be "$p$-sided" unexpected, i.e. $\alpha$-times larger than the previous $m$ waves and following $N_a - m$ waves, with $m = 1, ...N_a/2$ and $p = N_a/m$. For instance, the last wave in the sample could be larger than the preceding (one-sided) $N_a$ waves ($m = N_a$ and $p = 1$), or the central wave could extend above the preceding and following (two-sided) $m = N_a/2$ waves ($p = 2$ and $N_a$ even) (see Fig. 1). Clearly, the statistics of one- and two-sided unexpected waves, or more generally the $p$-sided statistics are the same if stochastic independence of successive waves holds. On this basis, the fraction of waves $n(x;\alpha,N_a)$ that have a dimensionless crest height $\xi = h/\sigma$ within the interval $(x, x+dx)$ and that is $\alpha$-times larger than any of the surrounding $N_a$ waves is given by

$$n(x;\alpha,N_a)dx = \left[1-P\left(\frac{x}{\alpha}\right)\right]^{N_a} p(x)dx, \qquad (5)$$

where $P(x)$ is the exceedance probability given in Eq. (1) and

$$p(x) = -\frac{dP}{dx} \tag{6}$$

is the pdf of $x$. Then the probability that the crest height $\xi$ is in $(x, x+dx)$ follows as

$$p_h(x;\alpha,N_a)dx = \frac{n(x;\alpha,N_a)dx}{n(\alpha,N_a)}, \tag{7}$$

where $n(\alpha,N_a)$ is the mean number of waves whose crest height is $\alpha$-times larger than the surrounding $N_a$ waves, namely

$$n(\alpha,N_a) = \int_0^\infty n(x;\alpha,N_a)dx = \int_0^\infty \left[1 - P\left(\frac{x}{\alpha}\right)\right]^{N_a} p(x)dx. \tag{8}$$

By definition, the return period $R$ or the average time interval between two consecutive occurrences of the unexpected event $E$ is

$$R(\alpha,N_a) = \frac{\tau}{N_w n(\alpha,N_a)} = \frac{N_w T_m}{N_w n(\alpha,N_a)} = \frac{T_m}{n(\alpha,N_a)}. \tag{9}$$

Since $T_m$ is the mean wave period, $E$ occurs on average once every $N_R$ waves where

$$N_R(\alpha,N_a) = \frac{1}{n(\alpha,N_a)}. \tag{10}$$

The associated mean crest height of a wave $\alpha$-times larger than the surrounding $N_a$ waves follows from Eq. (7) as

$$\overline{h}_{\alpha,N_a} = \sigma \int_0^\infty x p_h(x;\alpha,N_a)dx. \tag{11}$$

We rely on a quantile-type approach to describe rare occurrences of unexpected waves. In particular, we consider the threshold $h_{q,\alpha,N_a}$ exceeded with probability $q$ by an unexpected crest height $h$ $\alpha$-times larger than the surrounding $N_a$ waves. This satisfies

$$P_h(h_{q,\alpha,N_a}/\sigma;\alpha,N_a) = q \tag{12}$$

where

$$P_h(x;\alpha,N_a) = \int_x^\infty p_h(s;\alpha,N_a)ds \tag{13}$$

is the exceedance probability from Eq. (7). The conditional mean $\overline{h|h > h_{q,\alpha,N_a}}$ follows by integration by parts as

$$\overline{h}_{q,\alpha,N_a} = h_{q,\alpha,N_a} + \frac{\sigma}{q}\int_{h_{q,\alpha,N_a}}^\infty P_h(x;\alpha,N_a)dx. \tag{14}$$

In applications, we will use numerical integration to solve for the above statistical quantities. For comparison purposes, we also consider the standard statistics $\overline{h}_{\max,n}$, $h_n$ and $h_{1/n}$ for crest heights (Tayfun and Fedele 2007). In particular, $\overline{h}_{\max,n}$ is the mean maximum crest height of a sample of $n$ waves, namely

$$\overline{h}_{\max,n} = \sigma \int_0^\infty [1 - P(x)]^n dx, \tag{15}$$

which can be approximated via Gumbel-type asymptotics (see, for example Tayfun and Fedele (2007); Fedele (2015a)). The threshold $h_n$ is exceeded by the $1/n$ fraction of largest crest heights and it satisfies $P(h_n) = 1/n$. The conditional mean $h_{1/n}$, given $h > h_n$ then follows by definition from

$$h_{1/n} = h_n + \sigma n \int_{h_n}^\infty P(x)dx. \tag{16}$$

One can show that $\overline{h}_{\max,n}$ is always smaller than $h_{1/n}$ and they tend to be the same as $n$ increases (Tayfun and Fedele 2007). It is straightforward to check that $h_n$ ($h_{1/n}$) coincides with $h_{q,\alpha,N_a}$ ($\overline{h}_{q,\alpha,N_a}$) for $q = 1/n$ and $\alpha = 1$.

A statistical interpretation of the preceding is as follows. An unexpected wave whose crest height is $\alpha$-times larger than the surrounding $N_a$ wave crests occurs on average once every $N_R$ waves. At a probability level of $q$ and from a sample population of $N_R/q$ waves, only a smaller set of $1/q$ waves is unexpected. All the unexpected crests will be larger than the mean $\overline{h}_{\alpha,N_a}$ and only one crest in the set exceeds the threshold $h_{q,\alpha,N_a}$. For example, an unexpected Gaussian wave whose crest height is $\alpha = 2$-times larger than the surrounding $N_a = 40$ waves occurs on average once every $N_R = 10^5$ waves (see Fig. 1). The mean crest amplitude $\overline{h}_{2,40} \sim 1.1H_s$ and the threshold exceeded with probability $q = 1/10$ is $h_{1/10,2,40} \sim 1.3H_s$ (see Fig. 2). This means that in a sample of $N_R/q = 10^6$ waves a set of $1/q = 10$ waves are unexpected. All the waves in this set have crests that exceed $1.1H_s$, but only one wave crest exceeds $1.3H_s$. Further, $h_{q=1/10,2,40}$ is close to $h_{10^6}$, the threshold exceeded by the $1/10^6$ fraction of largest crests. This suggests that in general $\overline{h}_{\alpha,N_a}$ is nearly the same as the threshold $h_{N_R/q}$ exceeded by the $q/N_R$ fraction of largest crests, as confirmed later via comparisons to oceanic data.

The corresponding linear statistics follow by setting $\mu = 0$ and $\Lambda = 0$ in Eq. (1). These will hereafter be differentiated with the superscript $L$, namely as $\overline{h}^{(L)}_{\alpha,N_a}$, $h^{(L)}_{q,\alpha,N_a}$ and $\overline{h}^{(L)}_{\alpha,N_a}$.

Similar statistics for the crest-to-trough height $y = H/\sigma$ of unexpected waves follow by replacing the crest exceedance probability $P$ in Eq. (1) with the generalized Boccotti distribution (Alkhalidi and Tayfun 2013)

$$\begin{aligned} P_H(y) =& c_0 \exp\left(-\frac{1}{4}\frac{y^2}{1+\psi^*}\right) \\ & \times \left[1 + \frac{\Lambda}{64}\frac{y^2}{1+\psi^*}\left(\frac{y^2}{4(1+\psi^*)} - 2\right)\right], \end{aligned} \tag{17}$$

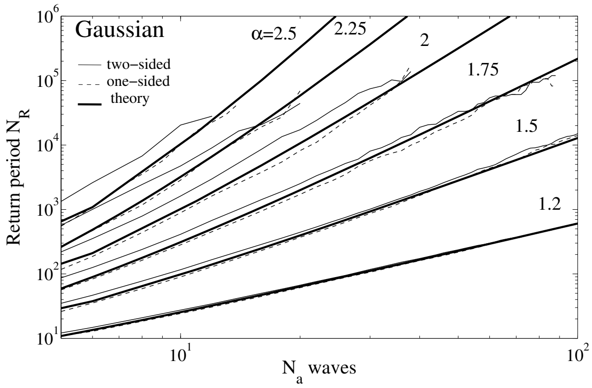


FIG. 2. Unexpected crest heights in Gaussian seas: empirical one-sided (thin dashed lines) and two-sided (thin solid lines, $N_a$ even) unexpected wave statistics versus predicted theoretical return period $N_R$ in number of waves (thick solid lines) of a wave whose crest height is $\alpha$-times larger than the surrounding $N_a$ waves for increasing values of $\alpha = 1.2, 1.3, 1.5, 2$ and 2.5. Sea state parameters: JONSWAP spectrum, mean period $T_m \sim 5$ s, spectral bandwidth $\nu = 0.35$, Boccotti parameter $\psi^* = 0.65$ and simulated $\sim 10^6$ waves.

where

$$c_0 = \frac{1+\ddot{\psi}^*}{\sqrt{2\ddot{\psi}^*(1+\psi^*)}},$$

and $\psi^* = \psi(\tau^*)$ is the absolute value of the first minimum of the normalized covariance function $\psi(\tau) = \overline{\eta(t)\eta(t+\tau)}/\sigma^2$ attained at $\tau = \tau^*$ and $\ddot{\psi}^*$ the corresponding second derivative (Boccotti 2000). The corresponding unexpected wave statistics will be referred to as $\overline{H}_{\alpha,N_a}$ and $H_{q,\alpha,N_a}$.

In the following, we will not dwell that much on unexpected wave heights, but our main focus will be the statistics of unexpected crests in typical oceanic sea states.

## 3. Verification and comparisons

### *a. Monte Carlo simulations*

We performed Monte Carlo simulations of a Gaussian sea described by the average JONSWAP spectrum for a total of $\sim 10^6$ waves. Fig. 2 shows the empirical return period $N_R = R/T_m$ in number of waves of both one-sided (thin dashed line) and two-sided (thin solid line) unexpected wave crests as a function of the surrounding $N_a$ waves for different values of $\alpha$ ($N_a$ is even for the two-sided statistics). The two statistics are roughly the same with two-sided unexpected waves slightly less frequent than the one-sided waves. A fair agreement with the theoretical predictions indicates that the stochastic independence of waves holds approximately.

Shown in Fig. 3 are the empirical statistics of mean crest heights and quantiles in comparison to the theoretical predictions. In particular, from the left panel of the figure we note that the mean crest height of two-sided unexpected waves is slightly smaller than the that of one-sided waves, especially as $\alpha$ increases. Nevertheless, both the statistics are in fair agreement with the theoretical predictions. Similar conclusions hold also for the threshold $h(1/10,\alpha,N_a)$ and conditional mean $\overline{h}(1/10,\alpha,N_a)$ shown in the other two panels of the same figure.

Finally, as regard to unexpected crest-to-trough heights, the present theoretical model fairly predicts the empirical wave height statistics from simulations as clearly seen in Fig. 4.

### *b. Oceanic observations*

We will analyze two data sets. The first comprises 9 h of measurements gathered during a severe storm in January, 1993 with a Marex radar from the Tern platform located in the northern North Sea in a water depth of $d = 167$ m. We refer to Forristall (2000) for further details on the data set, hereafter referred to as TERN. The second data set is from the Wave Crest Sensor Intercomparison Study (WACSIS) (Forristall et al. (2004)). It consists of 5 h of measurements gathered in January, 1998 with a Baylor wave staff from Meetpost Noordwijk in the southern North Sea (average water depth $d = 18$ m). Tayfun (2006) and Tayfun and Fedele (2007) elaborated both data sets and provided accurate estimates of statistical parameters, especially skewness and fourth-order cumulants which will be used in this work.

As regard to the WACSIS measurements, the left panel of Fig. 5 compares the theoretical nonlinear return period $N_R = R/T_m$ (solid line) of unexpected wave crests $\alpha$-times larger than the surrounding $N_a$ waves, linear predictions (dash lines) and the WACSIS empirical one-sided (+) and two-sided (□, $N_a$ even) statistics for $\alpha = 1.2, 1.5$ and 2. The right panel of the same figure shows similar comparisons for TERN. Two-sided unexpected waves are slightly less frequent than one-sided waves but both are close to the theoretical predictions, indicating that the assumption of stochastic independence of waves holds approximately. It is noticed that nonlinearities tend to increase the return period of unexpected waves and so their unconditional and conditional means. In particular, Fig. 6 compares the predicted nonlinear (solid line) and linear (dash line) mean crest heights $\overline{h}(\alpha,N_a)$ and $\overline{h}^{(L)}(\alpha,N_a)$ versus the WACSIS empirical one-sided statistics (squares) for $\alpha = 1.2$ and 1.5 (from the left, first and second panels respectively). The predicted nonlinear threshold $h(1/10,\alpha,N_a)$ (solid thin line) and conditional mean $\overline{h}(1/10,\alpha,N_a)$ (solid thick line) are compared against observations (squares) in the third and fourth panels of the same figure. The same comparisons for TERN are shown in Fig. 7. Clearly, linear predictions underestimate the observed crest amplitudes and nonlinearities must be accounted for to obtain reliable statistics of unexpected waves. Note that the empirical statistics tend to deviate from the theoretical predictions

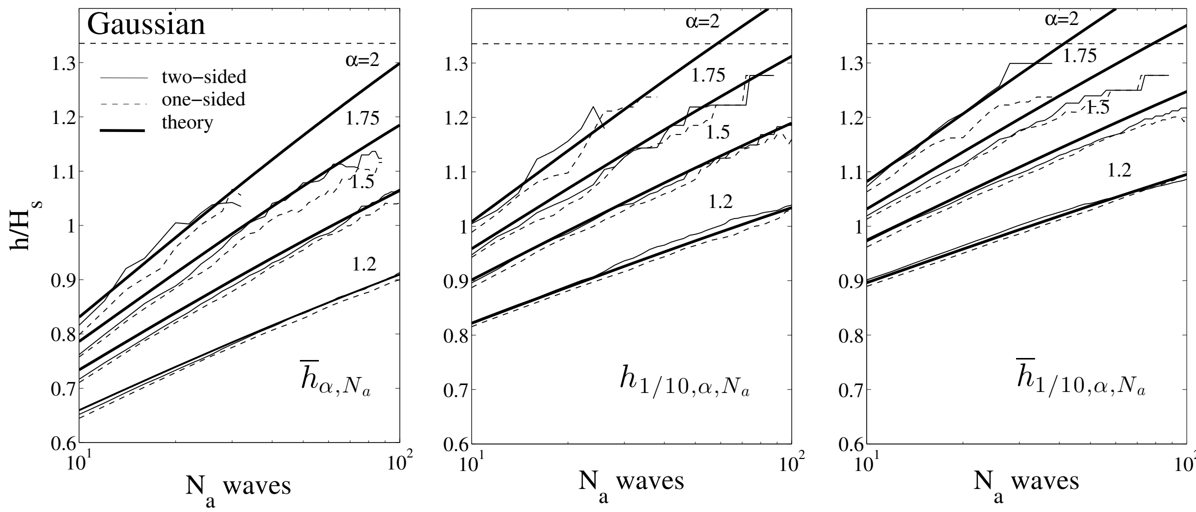


FIG. 3. Unexpected crest heights in Gaussian seas: empirical one-sided (thin dashed lines) and two-sided (thin dashed lines, $N_a$ even) unexpected wave statistics versus theoretical predictions (thick solid lines) of the (left) mean crest height $\overline{h}_{\alpha,N_a}$, (center) threshold $h_{1/10,\alpha,N_a}$ and (right) conditional mean $\overline{h}_{1/10,\alpha,N_a}$ of a wave whose crest height is $\alpha$-times larger than surrounding $N_a$ waves for $\alpha = 1.5, 2$, and $2.5$. Sea state parameters: JONSWAP spectrum, mean period $T_m \sim 5$ s, spectral bandwidth $\nu = 0.35$, Boccotti parameter $\psi^* = 0.65$ and simulated $\sim 10^6$ waves.

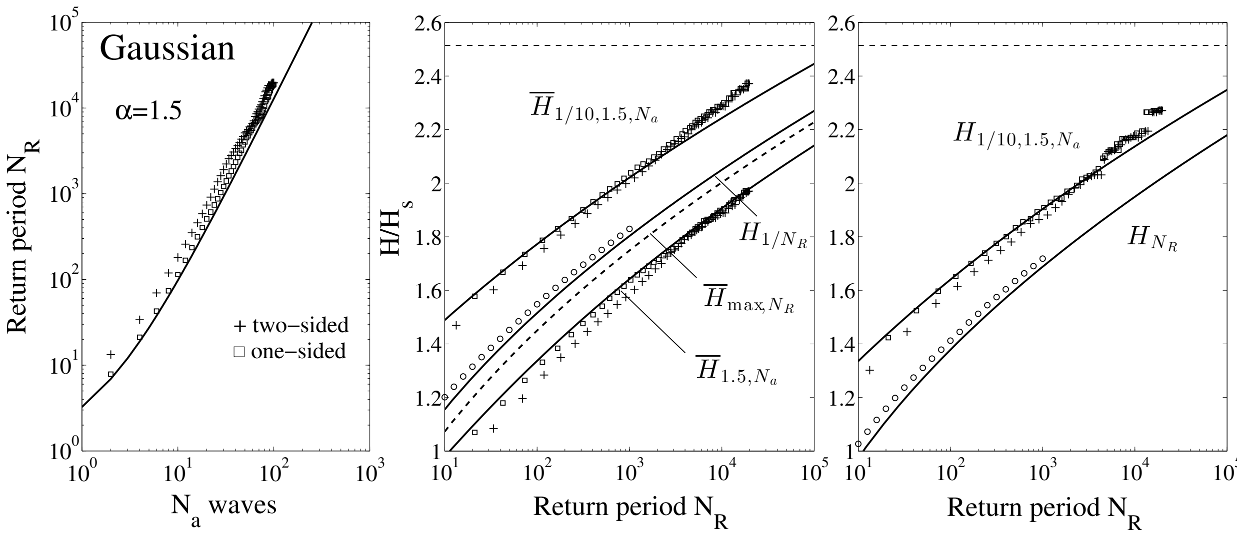


FIG. 4. Unexpected wave heights in Gaussian seas: (Left panel) predicted theoretical return period $N_R$ in number of waves (solid line) versus empirical one-sided (+) and two-sided (□, $N_a$ even) statistics as a function of the number $N_a$ of surrounding waves for $\alpha = 1.5$; (center panel) predicted mean unexpected wave height $\overline{H}_{1.5,N_a}$ and conditional mean $\overline{H}_{1/10,1.5,N_a}$ versus observations as a function of the return period $N_R$; (right panel) predicted threshold $H_{q=1/10,1.5,N_a}$ versus observations. For comparison purposes, the mean wave height $\overline{H}_{\max,N_R}$, conditional mean $H_{1/N_R}$ and threshold $H_{N_R}$ are also shown. The horizontal line denotes the observed maximum crest height $2.5H_s$. Sea state parameters: JONSWAP spectrum, mean period $T_m \sim 5$ s, spectral bandwidth $\nu = 0.35$, Boccotti parameter $\psi^* = 0.65$ and simulated $\sim 10^6$ waves.

for large values of $\alpha$ and $N_a$. In particular, for both TERN and WACSIS we could not produce statistically stable estimates of extreme values for $N_a > 10$ when $\alpha > 1.5$ or $q < 1/10$ due to the limited number of waves in the time series ($O(10^3)$ waves in comparison to the $10^6$ waves of the simulated Gaussian seas). Nevertheless, the agreement between the theory and observations is satisfactory.

## 4. How rogue are unexpected waves?

WACSIS observations indicate that the actual largest crest is nearly $\alpha = 2$-times larger than one-sided (two-sided) $N_a \sim 50$ (60) waves (see Fig. 1). From Eq. (11) theory predicts that an unexpected wave with a crest height 2-times larger than that of any of surrounding $N_a = 60$ waves

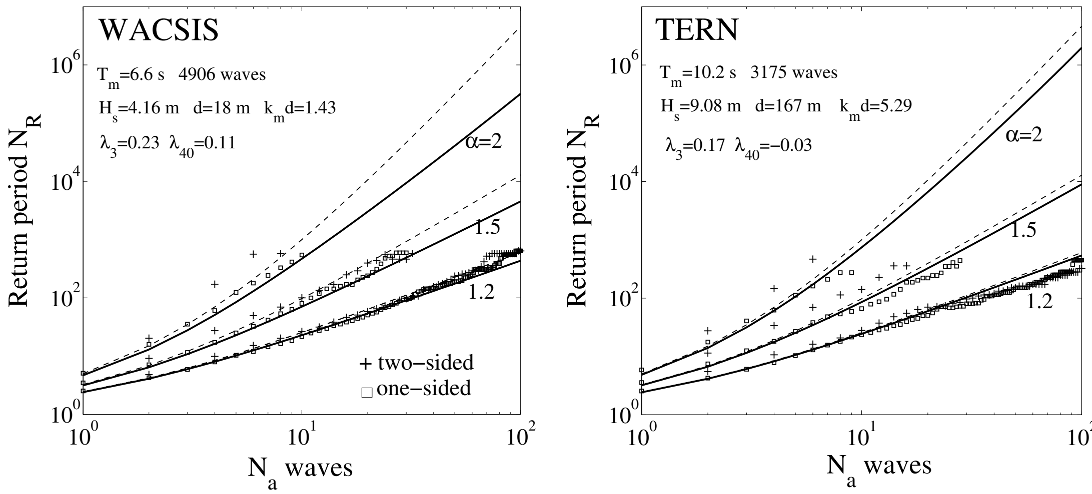


FIG. 5. WACSIS unexpected wave crest heights: (left panel) predicted theoretical nonlinear return period $N_R$ in number of waves (solid line) of a wave whose crest height is $\alpha$-times larger than the surrounding $N_a$ waves, linear predictions (dash lines) and empirical one-sided (+) and two-sided (□, $N_a$ even) observed statistics for $\alpha = 1.2, 1.5$ and 2. Right panel, same for TERN measurements. Statistical parameters, such as skewness and kurtosis among others, are taken from Tayfun (2006); Tayfun and Fedele (2007).

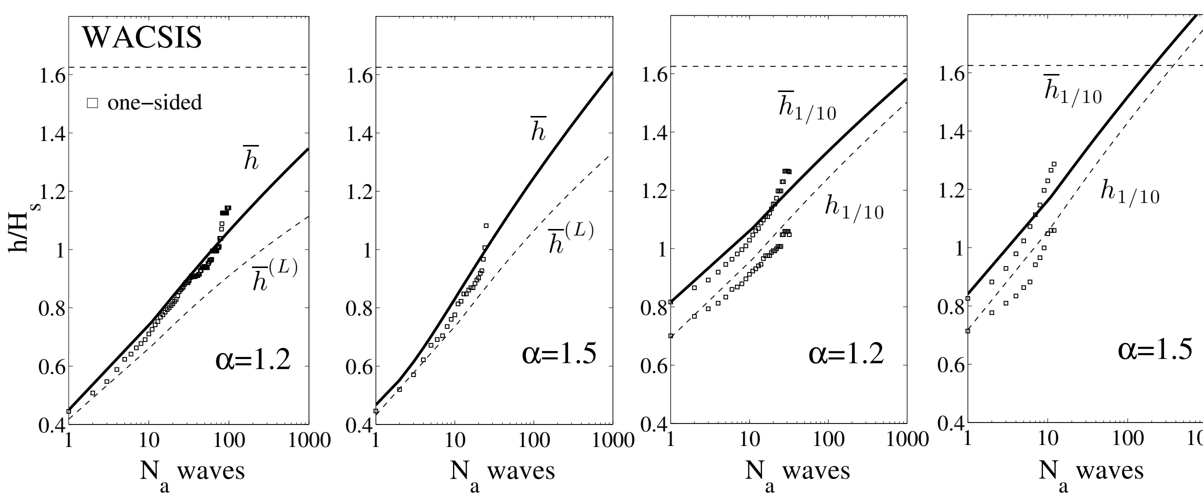


FIG. 6. WACSIS unexpected wave crest heights: predicted theoretical nonlinear (solid line) and linear (dash line) mean heights $\overline{h}_{\alpha,N_a}$ and $\overline{h}^{(L)}_{\alpha,N_a}$ as a function of the number $N_a$ of surrounding waves versus empirical one-sided statistics (squares) for (I panel from the left) $\alpha = 1.2$ and (II panel) 1.5 ; predicted nonlinear threshold $h_{1/10,\alpha,N_a}$ (solid thin line) and conditional mean $\overline{h}_{1/10,\alpha,N_a}$ (solid thick line) versus observed one-sided statistics (squares) for (III panel) $\alpha = 1.2$ and (IV panel) 1.5. The horizontal line denotes the observed maximum crest height $1.62H_s$. Wave parameters $H_s = 4.16$ m, $T_m = 6.6$ s, depth $d = 18$ m (Forristall et al. 2004). Statistical parameters, such as skewness and kurtosis, are taken from Tayfun (2006); Tayfun and Fedele (2007).

occurs once every $N_R = 10^5$ waves (see the left panel of Fig. 8). Its mean amplitude is nearly $1.43H_s$ and smaller than the mean maximum crest height $\overline{h}_{\mathrm{max,N_R}} = 1.55H_s$ of $N_R$ waves and conditional mean $h_{1/N_R} = 1.59H_s$ of the $1/N_R$ fraction of largest crest heights (see the center panel of Fig. 8). These averages underestimate the actual crest amplitude $h_{obs} \sim 1.62H_s$ observed (dashed horizontal line in the plots). The nature of such extreme value can be described using quantiles. From Eq. (12), $h_{obs}$ nearly coincides with $h_{q=1/10,\alpha=2,N_a=60}$ (see left panel of Fig. 8), the threshold exceeded with probability $1/10$ by the crest height of an unexpected wave 2-times larger than surrounding 60 waves. Thus, in a sample of $N_R/q = 10^6$ waves on average we expect $1/q = 10$ unexpected waves and their crest height is 2-times larger than the surrounding 60 waves as their occurrence is once every $N_R = 10^5$ waves. Among the 10 unexpected waves, only one will have a crest that exceeds $h_{obs}$. This is also nearly the same

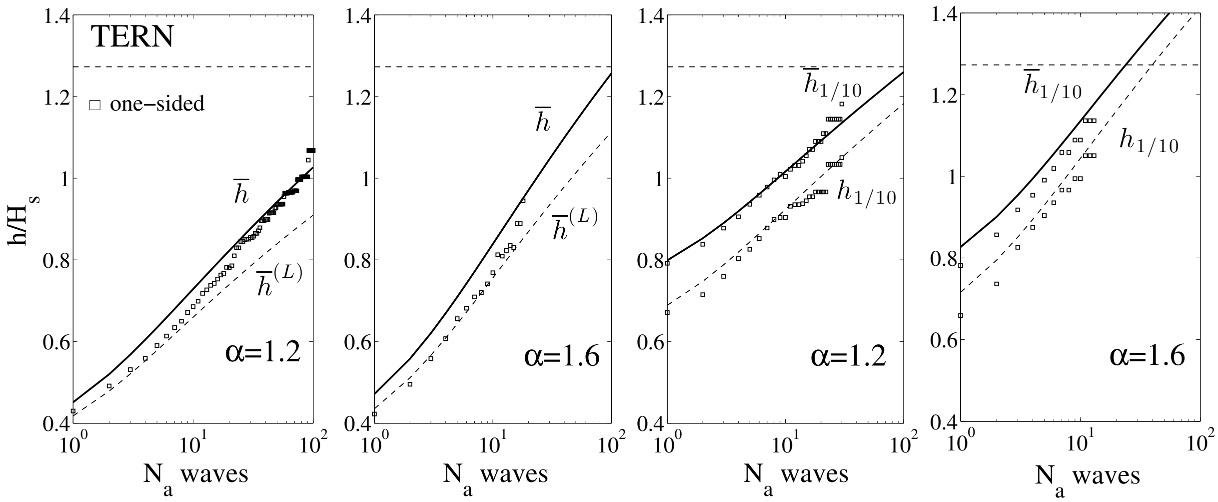


FIG. 7. TERN unexpected wave crest heights: predicted theoretical nonlinear (solid line) and linear (dash line) mean heights $\overline{h}_{\alpha,N_a}$ and $\overline{h}^{(L)}_{\alpha,N_a}$ as a function of the number $N_a$ of surrounding waves versus empirical one-sided statistics (squares) for (I panel from the left) $\alpha = 1.2$ and (II panel) 1.6 ; predicted nonlinear threshold $h_{1/10,\alpha,N_a}$ (solid thin line) and conditional mean $\overline{h}_{1/10,\alpha,N_a}$ (solid thick line) versus observed one-sided statistics (squares) for (III panel) $\alpha = 1.2$ and (IV panel) 1.6. The horizontal line denotes the observed maximum crest height $1.27H_s$. Wave parameters $H_s = 9.08$ m, $T_m = 10.2$ s, depth $d = 167$ m (Forristall 2000). Statistical parameters, such as skewness and kurtosis, are taken from Tayfun (2006); Tayfun and Fedele (2007).

as the threshold $h_{10^6}$ exceeded by the $1/10^6$ fraction of largest crests (see left panel of Fig. 8). These studies suggest that unexpected waves are rogue and rare occurrences of weakly nonlinear random seas (see also Fig. 9 for the WACSIS case of $\alpha = 1.5$). Similar conclusions can be drawn for TERN (see Fig. 10).

Finally, from Fig. 11 we conclude that there is fair agreement between the theoretical predictions and the observed statistics of unexpected wave heights.

## 5. The Andrea rogue wave and its unexpectedness

As a specific application of the present theoretical framework, the unexpected wave statistics of the 2007 Andrea rogue wave event is examined. Observations indicate that the large extreme appeared without warning and it was nearly two-times larger than the surrounding O(30) waves (see Fig. 12 in Magnusson and Donelan (2013)). For the hindcast Andrea sea state and from the the left panel of Fig. 12, theory predicts that a wave with a crest height at least twice that of any of surrounding $N_a = 30$ waves occurs on average once every $N_R = 10^4$ waves. Its mean amplitude is nearly $1.25H_s$ and lower than the mean maximum crest height $\overline{h}_{\max} = 1.36H_s$ of $N_R$ waves and conditional mean $h_{1/N_R} = 1.4H_s$ of the $1/N_R$ fraction of largest crest heights (see the center panel of the same Figure). As for WACSIS (see the center panel of Fig. 8), these averages underestimate the actual crest amplitude $h_{obs} \sim 1.63H_s$ observed at a point near the Ekofisk site. This nearly coincides with $h_{q=1/100,\alpha=2,N_a=30}$ and the threshold $h_{10^6}$ exceeded by the $1/10^6$ fraction of largest crests, as clearly seen in the right panel of Fig. 12. Thus, in a sample of $N_R/q = 10^6$ waves only one of the $1/q = 100$ waves $\alpha = 2$-times larger than the surrounding 30 waves exceeds $h_{obs}$.

## 6. Concluding remarks

A third-order nonlinear model for the statistics of unexpected waves is presented. Our data analysis and predictions indicate that the Andrea and WACSIS rogue wave events are stochastically similar. They appeared unexpected without warning and with crest heights two-times larger than the surrounding O(10) waves. According to the developed statistical model, such extremes occur on average every $10^4$ waves and the observed actual maximum crest height is nearly the same as the threshold $h_{10^6} \sim 1.62H_s$ exceeded by the $1/10^6$ fraction of largest crests. These results suggest that rogue events are rare occurrences of unexpected waves in weakly nonlinear random seas.

## 7. Acknowledgments

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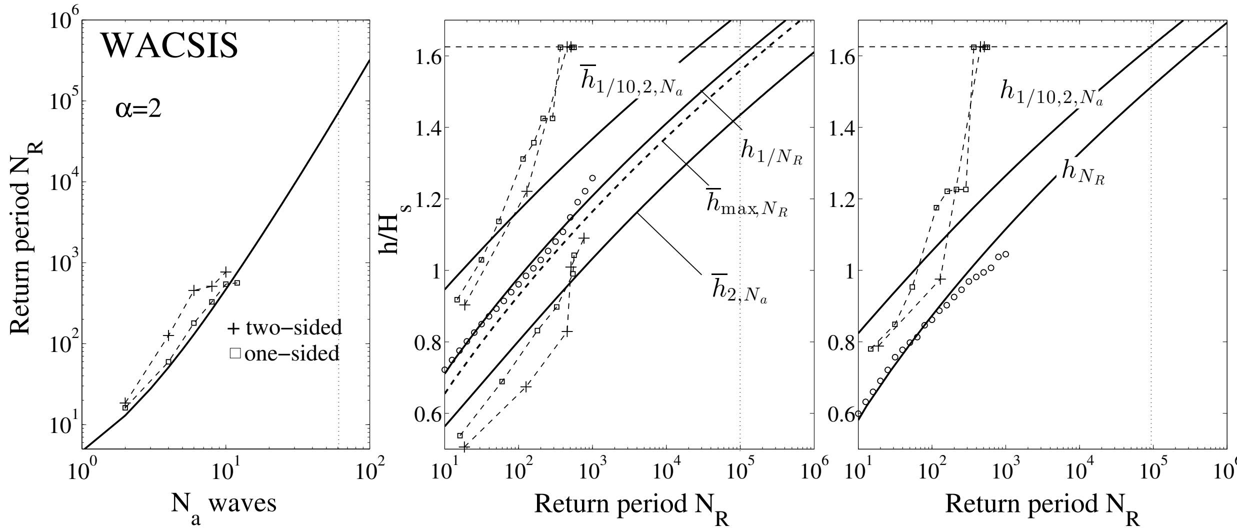


FIG. 8. WACSIS unexpected crest heights: (Left panel) predicted theoretical nonlinear return period $N_R$ in number of waves (solid line) versus empirical one-sided (+) and two-sided (□, $N_a$ even) statistics as a function of the number $N_a$ of surrounding waves for $\alpha = 2$; (center panel) predicted nonlinear mean unexpected crest height $\overline{h}_{1.5,N_a}$ and conditional mean $\overline{h}_{1/10,1.5,N_a}$ versus observations as a function of the return period $N_R$; (right panel) predicted threshold $h_{q=1/10,1.5,N_a}$ versus observations. For comparison purposes, the mean crest height $\overline{h}_{\max,N_R}$, conditional mean $h_{1/N_R}$ and threshold $h_{N_R}$ are also shown. The horizontal line denotes the observed maximum crest height $1.62H_s$.

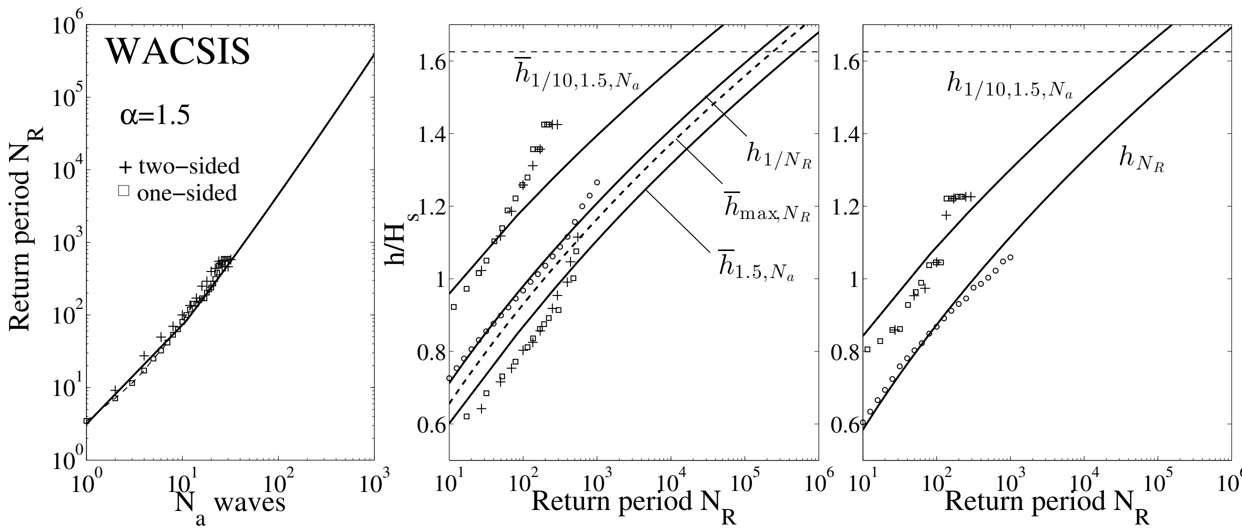


FIG. 9. WACSIS unexpected wave crest heights: same as caption of Fig. 8 but for $\alpha = 1.5$.

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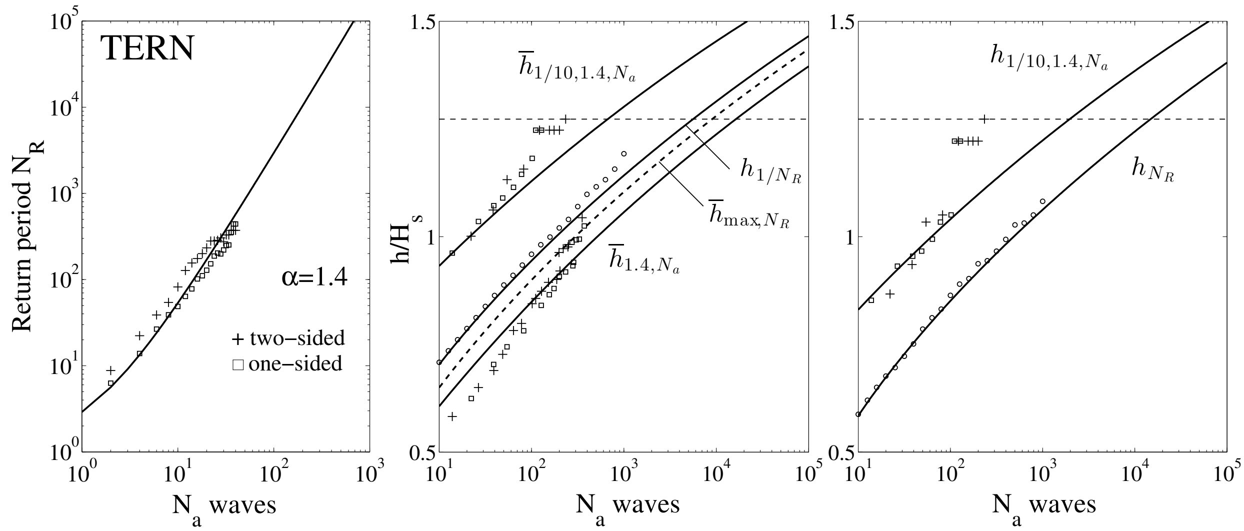


FIG. 10. TERN unexpected wave crest heights: same as caption of Fig. 8 but for $\alpha = 1.4$. The horizontal line denotes the observed maximum crest height $1.27H_s$.

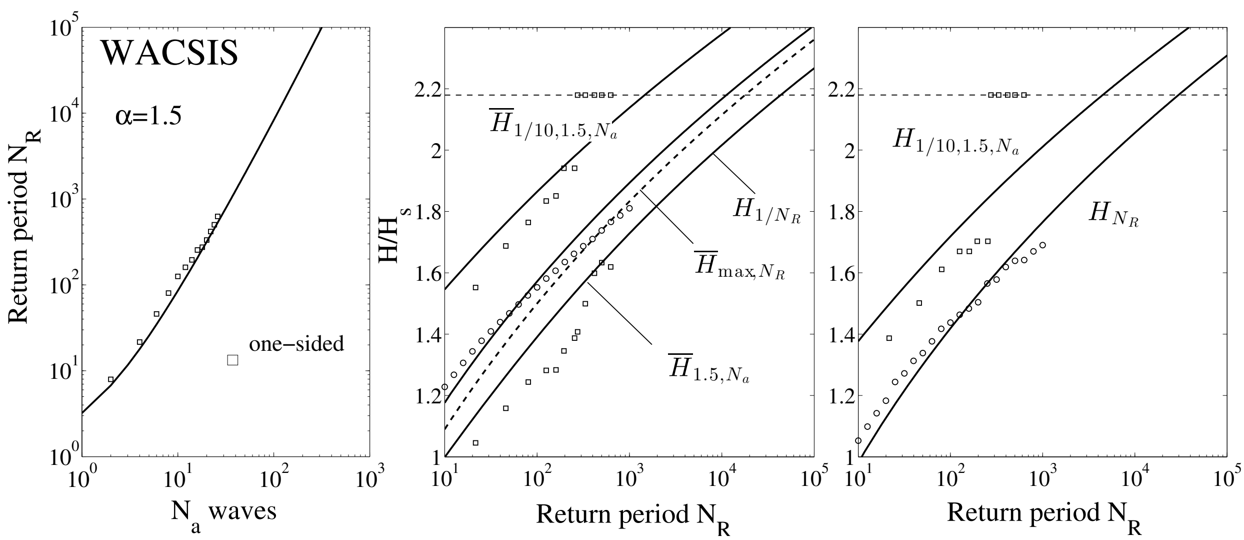


FIG. 11. WACSIS unexpected wave heights: (Left panel) predicted theoretical nonlinear return period $N_R$ in number of waves (solid line) versus empirical one-sided (+) and two-sided (□, $N_a$ even) statistics as a function of the number $N_a$ of surrounding waves for $\alpha = 1.5$; (center panel) predicted nonlinear mean unexpected wave height height $\overline{H}_{1.5,N_a}$ and conditional mean $\overline{H}_{1/10,1.5,N_a}$ versus observations as a function of the return period $N_R$; (right panel) predicted threshold $H_{q=1/10,1.5,N_a}$ versus observations. For comparison purposes, the mean crest height $\overline{H}_{\max,N_R}$, conditional mean $H_{1/N_R}$ and threshold $h_{N_R}$ are also shown. The horizontal line denotes the observed maximum crest height $\sim 2.2H_s$, Boccotti parameter $\psi^* = 0.67$.

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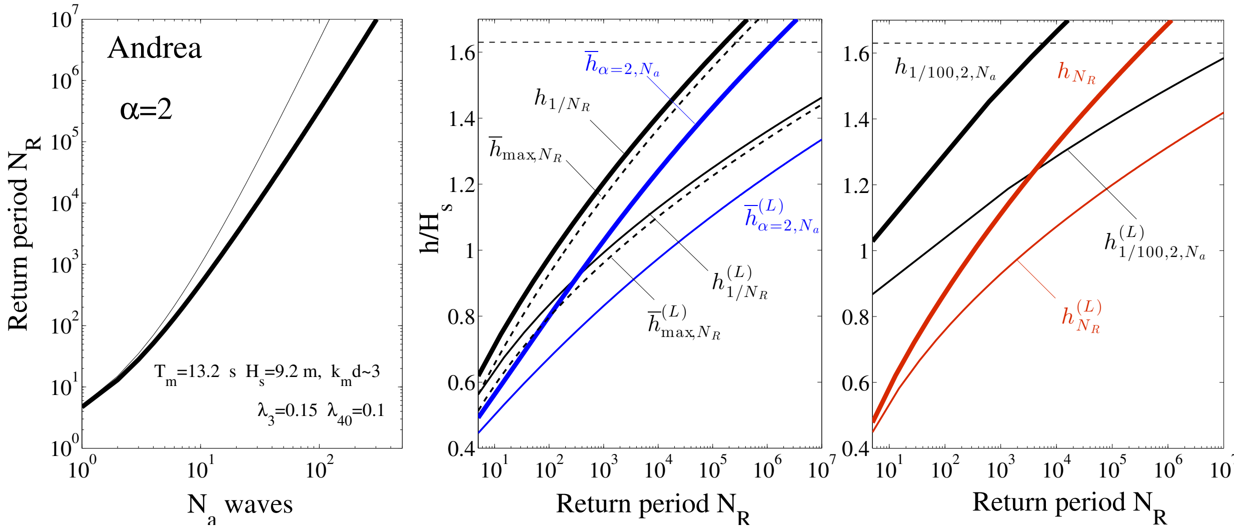


FIG. 12. Andrea wave: (left panel) predicted nonlinear return period $N_R$ in number of waves (solid line) of a wave whose crest height is $\alpha(=2)$-times larger than the surrounding $N_a$ waves and linear predictions (dashed line); (center panel) predicted nonlinear mean unexpected crest height $\overline{h}_{2,N_a}$ and conditional mean $h_{1/N_R}$ (solid thick lines), mean maximum crest height $\overline{h}_{\max,N_R}$ (dashed thick line), linear predictions $\overline{h}^{(L)}_{2,N_a}$, $\overline{h}^{(L)}_{\max,N_R}$ and $h^{(L)}_{1/N_R}$ (thin lines) as a function of the return period $N_R$; (right panel) predicted nonlinear crest height thresholds $h_{1/100,2,N_a}$ and $h_{N_R}$ (solid thick lines), linear predictions $h^{(L)}_{1/100,2,N_a}$ and $h^{(L)}_{N_R}$ (dashed thin lines) as a function of the return period $N_R$. Wave parameters $H_s = 9.2$ m, $T_m = 13.2$ s, depth $d = 74$ m (Magnusson and Donelan 2013), skewness $\lambda_3 = 0.15$ and excess kurtosis $\lambda_{40} = 0.1$ (Dias et al. 2015).

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