The Stochastic Parametric Mechanism for Generation of Surface Water Waves by Wind

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Theoretical understanding of the generation and growth of wind driven surface water waves has been based on two distinct mechanisms; the first being stochastic excitation by wave incoherent random atmospheric pressure fluctuations unrelated to wave amplitude (Eckart 1953; Phillips 1957) and the second instability arising from wave induced coherent atmospheric pressure fluctuations proportional to wave amplitude (Helmholtz 1868; Kelvin 1871; Jeffreys 1925; Miles 1957). Incoherent wave independent forcing produces growth of surface height variance linear in time while coherent forcing proportional to wave amplitude produces exponential growth. While observed wave developments can be fit to these forms, and despite broad agreement on the underlying physical process of momentum transfer from the atmospheric boundary layer shear flow to the water waves by atmospheric pressure fluctuations, quantitative agreement between theory and field observations of wave growth has proved elusive. Accepting that the dominant contribution to wave growth results from an at least statistical proportionality and coherent phase lag between atmospheric pressure fluctuations and surface elevation, at issue is the mechanism by which this relationship is produced and maintained. Mechanisms previously proposed by Jeffreys and Miles have been extensively examined and found to be in substantial variance with observations (Snyder and Cox 1966). In this work an alternative mechanism is proposed which unites the wave incoherent atmospheric forcing process, with its essentially turbulent character and linear in time variance growth, with the wave induced coherent atmospheric forcing with its exponential growth. The mechanism produces exponential growth that exceeds that produced by laminar critical layer instability and growth rate dependencies on wave number more in accord with observations. This stochastic parametric instability is an example of the universal instability arising from the nonnormality of nearly all time-dependent flows (Farrell & Ioannou 1999).

1. Introduction

Theories proposed to explain the phenomenon of surface water wave excitation by wind involve either wave incoherent stochastic forcing by random atmospheric pressure fluctuations (Eckart 1953; Phillips 1957) or wave coherent forcing by wave induced atmospheric pressure fluctuations (Helmholtz 1868; Kelvin 1910; Jeffreys 1925,1926; Miles 1957,1959a,b). These mechanisms together with a parametrization of nonlinear interactions were incorporated into a general prediction equation form by Hasselmann (1960). Various empirical versions of this prediction equation are presently used for operational wave forecasting purposes (WAMDI group 1988). These prediction equations incorporate a variance growth linear in time to account for wave incoherent forcing and a growth exponential in time to account for wave induced coherent forcing. Coherent forcing by wave induced atmospheric pressure fluctuations proportional to wave amplitude is generally accepted as necessary to account for observed rates of wave development.

The surface water wave generation problem can be thought of as a shear stability problem in the presence of a flexible lower boundary. Perhaps the most familiar example of this class of stability problems is Kelvin-Helmholtz instability resulting from Bernoulli suction coherently 180 degrees out of phase with surface elevation. However, Kelvin-Helmholtz instability requires that this suction exceed the gravitational restoring force in turn requiring wind speeds in excess of those observed to be associated with wave generating and it follows that Kelvin-Helmholtz instability is not generally regarded as an important mechanism for water wave generation. The instability mechanisms of Jeffreys and Miles result from the positive momentum flux from the atmosphere to the water wave that occurs when a component of wave induced atmospheric perturbation pressure lags surface water elevation by 90^o. While Jeffreys' theory postulates that turbulent flow separation produces the required atmospheric pressure to surface elevation phase lag, Miles' is a linear laminar theory which has no direct role for turbulence in producing the phase lag required for instability. In fact the phase lag and the growth rate are determined by the ratio of the curvature of the atmospheric velocity profile to its shear at the height where the wind velocity component in the direction of the wave velocity and the wave velocity are equal. In general this linear critical layer theory substantially under predicts observed growth rates and also requires an unobserved vanishing of growth rate as wave speed approaches the maximum wave parallel wind speed (Snyder & Cox 1966).

It must be noted that any laminar instability theory must take some account of the turbulent nature of atmospheric boundary layer pressure fluctuations so that the phase lag between atmospheric pressure fluctuations and wave height must at least be interpreted as a statistical average quantity. The attraction of the laminar instability mechanism is in large part that it predicts a properly phased component of atmospheric pressure fluctuations systematically proportional to wave amplitude and as a result growth exponential in time. In contrast wave incoherent turbulence reduces pressure fluctuations unrelated to wave amplitude and therefore gives rise to variance growth linear in time and even coherent resonant pressure fluctuations produce variance growth at most quadratic in time (Phillips 1957). There is no doubt as to the existence in laminar flow over a flexible boundary of the instability described by Miles (cf Farrell & Ioannou, 2005). However, theory and observation could be brought into better agreement if there existed in a turbulent shear flow over a flexible boundary another instability mechanism exponential in time and substantially dominant in growth rate when compared with the laminar mechanism. We demonstrate such a mechanism which uses either wave incoherent atmospheic turbulence or atmosphere incoherent wave turbulence to produce a statistically coherent component of atmospheric pressure fluctuations proportional to and properly phased with surface elevation to produce exponential growth . The mechanism can be viewed as an incoherent parametric instability and is an example of the universal instability arising from the nonnormality of nearly all time-dependent flows (Zel'dovich et al 1984; Farrell & Ioannou 1996, 1999). We note that this instability mechanism that involves the full spectrum of the underlying dynamical operator is entirely distinct from that based on the convexity of the modal instability growth rate as a function of wind speed (Nikolayeva & Tsimring 1986; Jannsen 1994; ; Miles 1997; Miles & Ierley 1997).

We begin by describing the stochastic time-dependent instability mechanism then apply it to the wave generation problem for representative boundary layer profiles. We then compare these results with predictions of laminar instability theory. Finally a comparison is made between the predictions of the stochastic parametric theory and observations of wave growth.

2. The time dependent stability problem

Consider an inviscid incompressible atmosphere of constant density in which the mean wind, $U(z,t)$, in the x direction varies both in the vertical direction, z, and with time, t, because of the gustiness of the mean wind. Assume, for the sake of generality, that the mean wind at $z=0$ is $U_0(t)$. Harmonic perturbations in the atmosphere with x wavenumber k and streamfunction $\psi_a(z,t)e^{ikx}$ are governed by the equation:

$$\left(\frac{\partial}{\partial t}+ikU(z,t)\right)D^2\psi_a-ikU''(z,t)\psi_a=0\ , \tag{2.1}$$

where D^2 is the Laplacian operator $D^2\equiv\partial^2/\partial z^2-k^2$, and $U''(z,t)\equiv\partial^2U(z,t)/\partial z^2$ denotes the curvature of the mean wind profile. Equation (2.1) is assumed to be valid in the region $z>0$ occupied by the atmosphere which has density ρ_a. The streamfunction ψ_a is assumed to decay to 0 as $z\to\infty$.

A semi-infinite incompressible fluid of density ρ_w occupies the region $z<0$. This fluid models the water; it is assumed to have no motion other than that associated with irrotational small amplitude surface waves with streamfunction form:

$$\psi_w(x,z,t)=\psi_w^0(t)e^{kz}e^{ikx}\ , \tag{2.2}$$

in which $\psi_w^0(t)$ is the streamfunction of the water at the mean air-water interface, $z=0$. The surface elevation, which is also assumed to take the harmonic form: $\eta=\hat\eta(t)e^{ikx}$, is a material boundary and satisfies in the small elevation limit the relations:

$$\frac{d\hat\eta}{dt}=-ikU_0(t)\hat\eta+ik\psi_a^0 \tag{2.3}$$

$$\frac{d\hat\eta}{dt}=ik\psi_w^0\ , \tag{2.4}$$

where $\psi_a^0(t)\equiv\psi_a(0,t)$. Subtracting the two conditions (2.3) and (2.4) we obtain that the streamfunctions in the air and water are given by:

$$\psi_w^0=\psi_a^0-U_0(t)\hat\eta\ . \tag{2.5}$$

Streamfunction (2.2) satisfies the momentum equations for $z<0$ with time dependence obtained by imposing (2.5) and continuity of normal stress, which on linearization at $z=0$, takes the form:

$$p_w-p_a=g(\rho_w-\rho_a)\hat\eta\ , \tag{2.6}$$

where $p_{w,a}$ are the values of the fourier coefficients of the pressure on the water and atmospheric side of the interface at $z=0$. In boundary condition (2.6) surface tension has been neglected.

Using the linearized momentum equation in the x direction both in the atmosphere and water layers the normal stress boundary condition at $z=0$ can be expressed as:

$$\frac{d}{dt}\left(\epsilon D\psi_a(0,t)-k\psi_a^0+kU_0(t)\hat\eta\right)=-ik\epsilon U_0(t)D\psi_a(0,t)+ik\epsilon\alpha(t)\psi_a^0-ik(1-\epsilon)g\hat\eta. \tag{2.7}$$

where $D\equiv\partial/\partial z$ and $\alpha(t)=U'(0,t)$ is the wind shear at $z=0$ and $\epsilon=\rho_a/\rho_w$; for the water-air interface $\epsilon\approx0.001$.

The linear temporal development of perturbations is determined by (2.1) together with the far field boundary condition $\psi_a(z,t) \to 0$ as $z \to \infty$ and the interface boundary conditions at $z = 0$ given by (2.3) and (2.7).

3. Instability due to the gustiness of the wind studied using the generalized Kelvin-Helmholtz stability problem

Consider in (2.1) the simple time dependent shear flow:

$$U(z,t) = \begin{cases} U_0(t) + \alpha(t) z, & z > 0; \\ 0, & z < 0. \end{cases} \tag{3.1}$$

This flow is a generalization of the time dependent version of the Kelvin-Helmholtz stability problem flow (Kelly (1965)) with the crucial inclusion of a time dependent shear in the atmosphere. Although shear has been included the streamfunction in the air ($z > 0$) is still given by:

$$\psi_a(z,t) = \psi_a^0(t) e^{-kz} \ . \tag{3.2}$$

Specification of the streamfunction in $z > 0$ allows the temporal stability to be determined by a single equation governing the evolution of surface elevation which on introducing (3.2) in (2.3) and (2.7), takes the form of the harmonic oscillator equation:

$$\frac{d^2\hat{\eta}}{dt^2} + \gamma(t)\frac{d\hat{\eta}}{dt} + \omega^2(t)\hat{\eta} = 0 \ . \tag{3.3}$$

Continuing the analogy with the harmonic oscillator we can identify equivalent time dependent coefficients for damping:

$$\gamma(t) = i\frac{\epsilon}{1+\epsilon}\left(\alpha(t) + 2kU_0(t)\right) \ , \tag{3.4}$$

and restoring force:

$$\omega^2(t) = \omega_g^2 + \frac{\epsilon k}{1+\epsilon}\left(i\frac{dU_0(t)}{dt} - (\alpha(t) + kU_0(t))U_0(t)\right) \ , \tag{3.5}$$

where

$$\omega_g^2 = kg\frac{1-\epsilon}{1+\epsilon} \ , \tag{3.6}$$

is the frequency of the surface gravity wave in the absence of a mean flow. With the change of variable;

$$\zeta = \hat{\eta}\exp\left(i\frac{\epsilon}{2(1+\epsilon)}\int^t \left(\alpha(s) + 2kU_0(s)\right) ds\right) \ , \tag{3.7}$$

the phase shifted elevation ζ satisfies the harmonic oscillator equation in standard form:

$$\frac{d^2\zeta}{dt^2} + \Omega^2(t)\zeta = 0 \ , \tag{3.8}$$

with time modulated square frequency:

$$\Omega^2(t) = \omega_g^2 - \frac{\epsilon}{1+\epsilon}kU_0(t)(\alpha(t) + kU_0(t)) - \frac{i\epsilon}{2(1+\epsilon)}\frac{d\alpha}{dt} + \frac{\epsilon^2}{4(1+\epsilon)^2}(\alpha(t) + 2kU_0(t))^2 \ . \tag{3.9}$$

Now assume that the velocity at the air-water interface is modulated stochastically as:

$$U_0(t) = U_0 + \sigma_1\xi_1(t) \ , \tag{3.10}$$

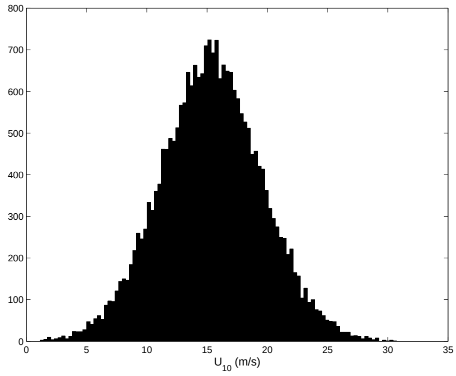

FIGURE 1. Histogram of wind fluctuations at 10 m that lead to a growth rate for the 17 m surface waves ($k = 0.37\ m^{-1}$) of $\lambda \approx 0.0025\ s^{-1}$. The wind fluctuates as a red noise process with decorrelation time 10 s and r.m.s. amplitude 30% of the mean value of $\overline{U}_{10}$ which for the case shown is 15.3 m/s corresponding to mean friction velocity $u_* \approx 0.68$. The wind follows the logarithmic boundary layer profile, and the friction velocity is varied in order to produce the imposed value of U_{10}.

where $\xi_i(t)$ is a zero mean random variable with unit variance, and σ_1 a scalar rms variance of the random fluctuation. Similarly, assume that the shear is given by

$$\alpha(t) = \alpha_0 + \sigma_2 \xi_2(t) \ , \tag{3.11}$$

where $\xi_2(t)$ is another random variable with zero mean and unit variance independent of $\xi_1(t)$. With these assumptions (3.8), keeping only time dependent terms $O(\epsilon)$ and $O(\sigma_1)$ and $O(\sigma_2)$, becomes:

$$\frac{d^2\zeta}{dt^2} + (\omega_d^2 - \epsilon\sigma_1 k(\alpha_0 + 2kU_0)\xi_1(t) - \epsilon\sigma_2\xi_3(t))\zeta = 0 \ , \tag{3.12}$$

where $\xi_3(t) = kU_0\xi_2 + (i/2)d\xi_2/dt$ and

$$\omega_d^2 = \omega_g^2 - \frac{\epsilon}{1+\epsilon}kU_0(\alpha_0 + kU_0) + \frac{\epsilon^2}{4(1+\epsilon)^2}(\alpha_0 + 2kU_0)^2 \ , \tag{3.13}$$

is the frequency of the surface wave in the case of atmospheric velocity U_0 and shear α_0.

When $\omega_d^2 < 0$ the surface is Kelvin-Helmholtz unstable and this instability occurs in the absence of time variation of the velocity discontinuity $U_0(t)$ and of the shear $\alpha(t)$. However, while when $\omega_d^2 > 0$ the surface is stable in the absence of time variation, it is a remarkable fact that the oscillator is destabilized by stochastic temporal variation of its frequency (Arnold et al 1986; Farrell & Ioannou 1996). An estimate of the Lyapunov exponent can be obtained in the small σ_1 and σ_2 limit (Arnold et al, 1986):

$$\lambda = \frac{\pi\epsilon^2}{4\omega_d^2}\Big(\sigma_1^2 k^2(\alpha_0 + 2kU_0)^2\hat{\xi}_1(2\omega_d) + \sigma_2^2\hat{\xi}_3(2\omega_d)\Big) + O(\sigma_1^3, \sigma_2^3) \ , \tag{3.14}$$

where

$$\hat{\xi}_i(2\omega_d) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-2i\omega_d t} < \xi_i(t)\xi_i(0) > dt \ , \tag{3.15}$$

is the Fourier transform of the time lagged correlation $< \xi_i(t)\xi_i(0) >$ evaluated to give the power of the random process at the subharmonic frequency $2\omega_d$.

At first glance this $O(\epsilon^2)$ expected mean growth rate of surface waves is disappointingly slow given that $\epsilon \approx 0.001$ for the air-water interface. However, the logarithmic boundary

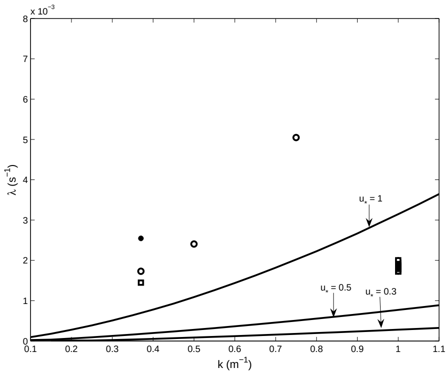

FIGURE 2. Lyapunov exponents λ (s^{-1}) for time varying logarithmic boundary layer profiles as a function of zonal wavenumber k (m^{-1}). The circles are for mean friction velocities $u_* \approx 0.4$ (corresponding to $U_{10} = 10\ ms^{-1}$) and the star for mean friction velocity $u_* \approx 0.68$ (corresponding to $U_{10} = 15\ ms^{-1}$). The wind fluctuates as a red noise process with decorrelation time 10 s and with r.m.s. amplitude 30% of the mean value of U_{10}. For comparison the growth rates found when the mean flow does not vary are also plotted for mean flows with friction velocities $u_* = 0.3, 0.5, 1$. Gustiness leads to appreciably higher growth rates. The squares indicate the growth rates found in the example of a disrupted flow described in section 5.

layer has shears $O(1/\epsilon)$ s^{-1} in the neighborhood of the interface and because U_0 is $O(1)$ ms^{-1} the growth rate is dominated by terms in the surface shear :

$$\lambda \approx \frac{\pi\epsilon^2\sigma_1^2}{4\omega_d^2} k^2 \alpha_0(\alpha_0 + 4kU_0)\hat{\xi}_1(2\omega_d) \ , \tag{3.16}$$

revealing λ to be at least $O(\epsilon^2\alpha_0^2\sigma_1^2)$ s^{-1} and of an order that can compete with the unstable growth obtained from the laminar stability theory of Miles (1957). Moreover, for $\alpha_0 >> kU_0$, the surface wave frequency ω_d^2 is proportional to the wavenumber k, and the dependence of growth rate on wavenumber is linear in k: $\lambda = a + bk$, in agreement with observations (Snyder et al 1981).

This very simple example demonstrates that variations in shear can destabilize a surface wave that is otherwise stable in the mean and that the expected growth rate is at least $O(\epsilon^2\alpha_0^2\sigma_1^2)$ s^{-1} where α_0 is the mean surface shear and σ_1 is the r.m.s. value of the wind fluctuations at the interface.

4. Destabilization of surface water waves by atmospheric turbulence

The atmospheric boundary layer is typically turbulent and the associated characteristic wind field fluctuation is a phenomenon referred to as gustiness. From the point of view of time-dependent operator stability this turbulent gustiness represents a structured time dependence of the operator, the structure being referred to is confining fluctuations to terms in the operator representing the wind and its derivatives. The operator governing perturbations is also highly nonnormal so that rapid transient growth is supported (Farrell & Ioannou, 2005). Rapid exponential growth is intrinsic to nonnormal time-dependent operators that exhibit transient growth and are not commuting at all times (Zel'dovich et al 1984; Farrell & Ioannou 1996, 1999). Exponential growth occurs even in cases for which the operator is at all times stable as in the previous section example.

We wish to examine the exponential wave growth arising from gustiness in a boundary layer model with a logarithmic velocity profile . Consider the logarithmic boundary layer

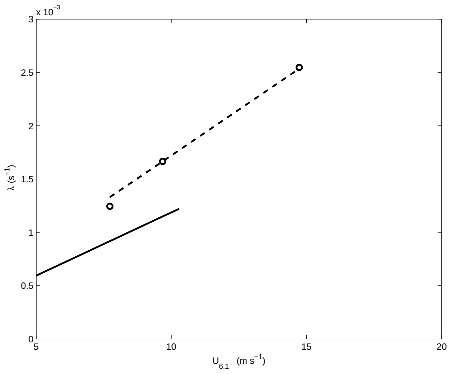

FIGURE 3. Lyapunov exponents λ (s^{-1}) (circles) for time varying logarithmic boundary layer profiles as a function of mean wind at height 6.1 m ($U_{6.1}$) for a 17 m wave ($k = 0.37\ m^{-1}$) compared with the best fit growth rates (continuous line) from Snyder & Cox (1966) . In our simulations the wind fluctuates as a red noise process with decorrelation time 10 s and r.m.s. amplitude 30% of the mean value of U_{10}. The Lyapunov exponents obtained vary approximately linearly with $U_{6.1}$.

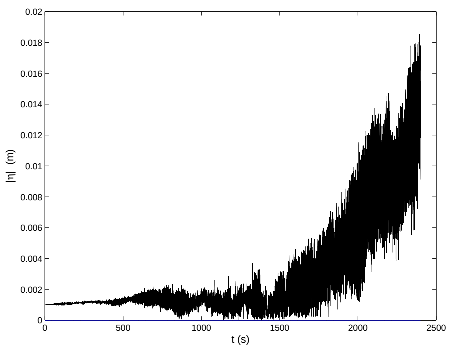

FIGURE 4. Surface elevation $|\hat{\eta}(t)|$ (m) as a function of time (s) for the case shown in figure 1. This 17 m wave reaches the breaking height of 2.7 m in approximately an hour.

flow:

$$U(z,t) = \frac{u_*(t)}{K} \log\left(1 + \frac{z}{z_0(t)}\right) \ . \tag{4.1}$$

In (4.1) $u_*(t)$ is the time varying friction velocity, $K = 0.42$ is the von Karman constant, and $z_0(t) = \kappa u_*^2(t)/g$ is the corresponding time varying roughness length expressed in terms of the Charnock constant, $\kappa = 0.0144$, the friction velocity and the acceleration of gravity, g. Analysis of the generalized stability of this profile for constant friction velocity was presented in Farrell & Ioannou (2005) where it was demonstrated that the continuous spectrum of the operator can excite the surface wave at amplitudes far greater than that obtained by introducing of the unstable eigenmode itself.

In the construction of the model for the wind gustiness we follow Cavaleri (1994) and the observations from Cavaleri & Cardone (1994) and assume that the velocity at 10 m, U_{10}, is a gaussian random variable. By construction the wind follows the logarithmic boundary layer profile, and the friction velocity is chosen to vary in order to produce the chosen distribution of U_{10}. For U_{10} we take:

$$U_{10} = \overline{U}_{10}\left(1 + 0.3\xi(t)\right) \ , \tag{4.2}$$

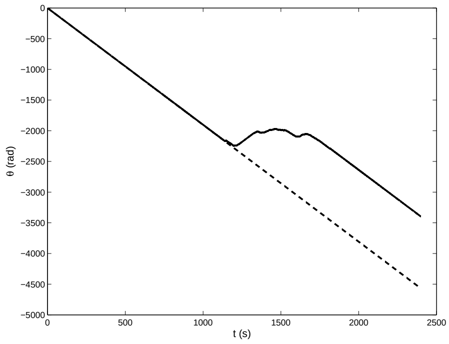

FIGURE 5. Phase advance θ associated with the surface elevation. The dash line corresponds to the phase propagation of a surface wave with phase velocity $c = \sqrt{g/k}$. The surface elevation predominantly propagates prograde, except during short periods during which adjustment is taking place.

where $\overline{U}_{10}$ is the mean 10 m wind in (ms^{-1}) and $\xi(t)$ is a red noise process with zero mean, unit variance and decorrelation time $\tau = 10\ s$ (calculation with different decorrelation times have been carried out, and the results we report do not depend sensitively on τ). An example histogram of U_{10} is shown in figure 1. Using this variation we obtained that 17 m surface waves ($k = 0.37\ m^{-1}$) grow at the rate $\lambda \approx 0.0025\ s^{-1}$ which is an order of magnitude larger than the corresponding rate for the constant logarithmic profile with $U_{10} = \overline{U}_{10}$. The growth rates obtained for other wavenumbers and for various $\overline{U}_{10}$ are shown in figure 2 where for comparison the growth rate for a constant wind and for various friction velocities has also been plotted. In order to obtain a more concrete comparison with data the growth rates obtained from (4.2) are shown in figure 3 together with the growth rates of 17 m surface waves for various $\overline{U}_{10}$ as fit to observations by Snyder & Cox (1966). Considering that no attempt was made to adjust for the conditions of the experiment the agreement is good.

A single realization of the development of the amplitude and phase of a surface wave is shown in figures 4 and 5 for the single realization of the wind shown in figure 1. The initial condition for this simulation is $\hat{\eta}(0) = 0.001\ m$. Note that after an initial adjustment the wave grows exponentially and the phase of the wave advances with phase speed about $\sqrt{g/k}$.

5. Destabilization of surface water waves due to turbulent wave incoherence

The previous example demonstrated that time dependence can destabilize the surface wave system by repeatedly eliciting the very large transient growth intrinsic to highly nonnormal dynamical operators. This is a very general mechanism that can produce asymptotically exponential growth in many contexts that may appear at first unrelated. The surface wave field is well known to be highly irregular with groups of waves typically having coherence times of a few wave periods. As a result, the wave groups may be modelled as appearing out of the background turbulence for a short period before receding back into the background only to reappear as a group again for an interval of time. Each time the group emerges from the background it must organize its associated atmospheric wave field in the manner of the starting vortex problem but with the crucial difference that the adjustment take place in a highly nonnormal system with large transient growth

To examine this growth scenario consider the following system. The state of the system

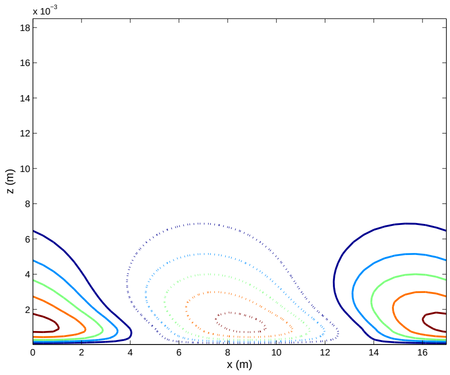

FIGURE 6. Structure in the atmosphere of the streamfunction of the most unstable eigenfunction of the propagator $\mathbf{\Phi}(t)$ for a logarithmic velocity profile with $u_* = 0.5$ and for $k = 0.37$. For this example the density ratio is $\epsilon = 0.001$ and a lid is placed at $z = 3$ (m) in order to improve numerical conditioning of the initial value problem. The associated growth rate is $kc_i = 0.0014\ s^{-1}$.

is described by the column vector $\phi = [\hat{\eta}, \psi_1, \cdots \psi_{n-1}]$, where ψ_i is the value of the streamfunction in $z \geqslant 0$ at one of the levels $z = z_i$ with $z_1 = 0$ and $z_{n-1} = z_T$. The state of the system evolves according to

$$\frac{d\phi}{dt} = \mathbf{A}\phi \ , \tag{5.1}$$

which symbolically describes the discretized equation (2.1) together with the interface boundary conditions (2.3) and (2.7). The mean flow is the logarithmic profile (4.1) with a constant friction velocity u_*. The dynamics described by (5.1) are allowed to evolve the perturbations only over a time period τ, which is a random interval over which the perturbation flow ϕ is assumed coherent. At the end of each period τ the perturbation flow is disrupted by multiplying the evolved state $\phi(t+\tau) = (\exp A\tau)\phi(t)$ by the diagonal matrix $\mathbf{P}$ with diagonal elements: $P_{11} = 1$ and $P_{ii} = \exp(-z_{i-1}/\delta)$ for $i \geqslant 2$. At time $t = \sum_i^N \tau_i$ the state of the system is

$$\phi(t) = \mathbf{\Phi}(t)\phi(0) \ , \tag{5.2}$$

where the propagator is

$$\mathbf{\Phi}(t) = \prod_{i=1}^{N} \mathbf{\Phi}_i \ , \tag{5.3}$$

and

$$\mathbf{\Phi}_i(t) = \mathbf{P}\exp(\mathbf{A}\tau_i) \ . \tag{5.4}$$

The disruption of the atmospheric perturbation field by $\mathbf{P}$ taking place after random interval τ_i elicits the reestablishment of the atmospheric field as a recurrent adjustment process. We wish to determine under these disrupted conditions the Lyapunov exponent of the resulting surface wave.

A variety of cases were examined and it was determined that the Lyapunov exponent obtained is invariably larger than the growth rate of the most unstable wave obtained when the flow is not disrupted. This is consistent with results shown in Farrell & Ioannou (2005) in which a wide class of perturbations concentrated near the interface were found to excite the surface gravity wave by extracting energy from the shear of the mean flow.

A practical method to obtain an approximation to the Lyapunov exponent is to determine the maximally growing eigenfunction of the propagator $\mathbf{\Phi}(\mathbf{t})$ for large enough $t = \sum_i^N \tau_i$ with τ_i is a random variable uniformly on the interval: $[T/4, 8T]$, where $T = 2\pi/\sqrt{gk}$ is the period of the surface wave. The maximally growing eigenfunction for a realization of this process with $N = 300$ resulting in $t = 4260$ s is shown in figure 6 for a 17 m wave in a flow with $u_* = 0.5$ ($\delta = 1/500$ m in this calculation). Note that the eigenfunction which approximates the first Lyapunov vector is located near the interface taking advantage of the large shear there. The associated growth rate is shown for waves with $k = 1$ and $k = 0.37$ in figure 2. The spread in growth rate values for $k = 1$ indicates the results for different realizations of the disruption sequence.

6. Discussion

Excitation of surface water waves by wind is a familiar phenomenon of great theoretical and practical importance which has eluded comprehensive theoretical explanation. The atmospheric boundary layer is clearly turbulent, but direct incoherent pressure forcing resulting from atmospheric pressure fluctuations produces at most quadratic in time variance growth (Phillips 1957) while growth exponential in time appears to be required by observations. The required exponential growth implies coherent and properly phased pressure fluctuations proportional to wave amplitude. The laminar critical layer instability mechanism of Miles (1957) produces exponential growth but fails to correspond with observations of growth rate in general. In this work we have demonstrated an essentially turbulent exponential instability the fundamental explanation of which can be traced to the universal instability of time-dependent flows arising from the necessary nonnormality of nearly all time-dependent operators that are not commutating at all times. This stochastic parametric instability dominates the laminar critical layer instability for sufficiently turbulent boundary layers and provides an alternative mechanism for explaining the observed growth of surface water waves.

6.1. *Acknowledgments*

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