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# Mean Length of Runs of High Waves

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The mean length of runs of high wave heights and the mean number of waves between the exceedance of one level by a group of waves to the next exceedance of the same level by the succeeding group of waves have been investigated for a range of spectral widths. Asymptotic forms for both these statistics are obtained for narrow band spectra. Approximate expressions for these statistics are evaluated for one spectral form and a range of spectral widths and the results compared with data from numerical wave simulation experiments.

A run of wave heights can be defined as the sequence of waves the heights of which exceed a particular level $\rho$, say. The durations of such runs are of importance in the study of ship motions in waves. For example, ships may capsize or be damaged by the severe rolling motion caused by a group of high waves. We denote this statistic by $l_1$ (Figure 1). *Goda* [1970] has studied the lengths of runs of wave heights assuming that the sequence of wave heights is described by a random process where successive waves are uncorrelated. This assumption is practically equivalent to representing the waves by a spectrum with a large bandwidth. In this case he shows that the mean length of runs of wave heights exceeding the level $\rho$ is

$$\langle l_1 \rangle = [1 - \exp(-\tfrac{1}{2}k^2)]^{-1} \tag{1}$$

where $k = \rho/m_0^{1/2}$ and $m_0$ is the variance of the wave elevation.

Another statistic considered by Goda is the number of wave heights between the exceedance of one level $\rho$ by a group of waves to the next exceedance of the same level by the succeeding group of waves. This statistic, which is denoted by $l_2$ (Figure 1), is approximately equal to the number of waves from the peak of one wave group to the peak of the next wave group. Goda shows that for a random process with a large bandwidth

$$\langle l_2 \rangle = [1 - \exp(-\tfrac{1}{2}k^2)]^{-1} + \exp(\tfrac{1}{2}k^2) \tag{2}$$

In addition to the theory described above, Goda has measured directly the distribution of run lengths from numerical simulations of various empirical wave spectra. He shows that the observed values of $\langle l_1 \rangle$ and $\langle l_2 \rangle$ tend to the limits given by (1) and (2) when the spectrum width increases.

This paper uses some ideas of *Rice* [1945, 1958] and *Longuet-Higgins* [1957] to derive asymptotic forms for $\langle l_1 \rangle$ and $\langle l_2 \rangle$ for narrow band spectra and also discusses the effect of spectral width on these two statistics for high wave heights. Comparisons are made between these results and the numerical values of run lengths obtained by Goda.

*Mean run lengths for a narrow band spectrum of swell waves.* Consider a narrow band spectrum $E(f)$ that is symmetrical about some central frequency $f_0$. The statistics of the heights of waves are determined by the properties of the envelope $R(t)$ of the wave profile. It has been shown [*Rice*, 1945; *Longuet-Higgins*, 1957] that the mean number of occasions per unit time that $R(t)$ passes downward (or upward) across the level $\rho$ is

$$N = (\mu_2/2\pi)^{1/2}(\rho/m_0)\exp(-\rho^2/2m_0) \tag{3}$$

where

$$\mu_{2n} = (2\pi)^{2n}\int_0^\infty (f - f_0)^{2n}E(f)\,df$$

Here $\mu_{2n}$ is the $2n$th moment of the spectrum about the central frequency $f_0$, and $m_n$ is the $n$th moment of the spectrum about the origin.

Now the probability that $R(t) > \rho$ is $\exp(-\rho^2/2m_0)$. *Rice* [1958] therefore argues that the average length of the intervals for which $R(t) > \rho$ is $\exp(-\rho^2/2m_0)/N$. Since the mean

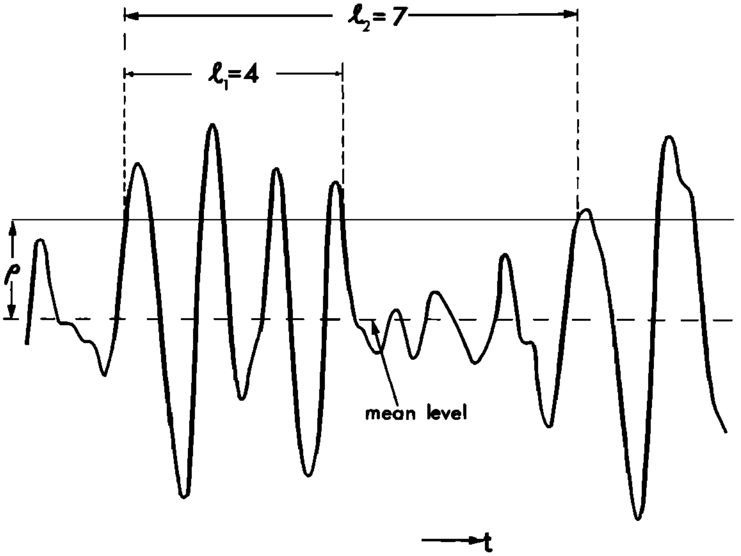


Fig. 1. Definitions of $l_1$ and $l_2$.

number of zero crossings per unit time is $N_0 = \pi^{-1}(m_2/m_0)^{1/2}$, the average number of waves in this interval is

$$\langle l_1 \rangle = \tfrac{1}{2}N_0[\exp(-\rho^2/2m_0)/N] = [m_2/(2\pi\mu_2)]^{1/2}k^{-1} \quad (4)$$

where $k = \rho/m_0^{1/2}$ ($k \simeq 2$ for a level corresponding to the significant wave height).

To evaluate $\langle l_1 \rangle$, we assume that the spectrum has the form of a normal function

$$E(f) = \sigma^{-1}(2\pi)^{-1/2} \cdot \exp[-(f - f_0)^2/2\sigma^2] \qquad f_0 \gg \sigma$$

where $f_0$ is the central frequency and $\sigma$ is a measure of the bandwidth. *Goda* [1970] plots his results against a spectral peakedness parameter $Q = 2\int_0^\infty fE^2(f)\,df/m_0^2$. This parameter is found to give a more meaningful base for plotting the numerical results when the spectrum is not narrow than the spectral width parameter $\epsilon[\equiv(1 - m_2^2/m_0m_4)^{1/2}]$ used by *Cartwright and Longuet-Higgins* [1956]. Large values of $Q$ correspond to a narrow band spectrum. For this choice of spectrum it can be shown that

$$m_2/\mu_2 = (\sigma^2 + f_0^2)/\sigma^2 \quad (5)$$

$$Q = f_0/(\pi^{1/2}\sigma) \quad (6)$$

$$\epsilon \simeq (6/\pi)^{1/2}Q^{-1} \quad (7)$$

Using (4), (5), and (6), we obtain

$$\langle l_1 \rangle = Q/2^{1/2}k \quad (8)$$

as the asymptotic form for $\langle l_1 \rangle$ when $Q$ is large.

Now consider the average number of waves $\langle l_2 \rangle$ between the exceedance of one level $\rho$ by a group of waves to the next exceedance of the same level by the succeeding group of waves. In this case, $\langle l_2 \rangle$ is the average number of waves between upward crossings of the level $\rho$ by the envelope $R(t)$. Using the results of *Rice* [1945] and *Longuet-Higgins* [1957] discussed above, we obtain

$$\langle l_2 \rangle = \tfrac{1}{2}N_0/N = [m_2/(2\pi\mu_2)]^{1/2}k^{-1}\exp(\tfrac{1}{2}k^2) \quad (9)$$

This expression differs from the corresponding result for $\langle l_1 \rangle$ by a factor $\exp(\tfrac{1}{2}k^2)$. We therefore find that the asymptotic form when $Q$ is large is in this case

$$\langle l_2 \rangle = \frac{Q\exp(\tfrac{1}{2}k^2)}{2^{1/2}k} \quad (10)$$

Figures 2 and 3 show that both relations 8 and 10 fit Goda's numerical results for large values of $Q$ (in practice for $Q \gtrsim 8$ or $\epsilon \lesssim 0.2$).

*Effect of spectral width on run lengths.* *Cartwright and Longuet-Higgins* [1956] have derived the cumulative probability $q(\eta)$ of a maximum exceeding a given value ($\eta$ is the ratio of wave height to rms elevation $m_0^{1/2}$) in terms of the spectral width parameter $\epsilon$. The cumulative probability $q(\eta)$ has the following forms for different ranges of $\epsilon$. When $\epsilon \rightarrow 0$,

$$q_0(\eta) \rightarrow 1 \qquad \eta \leq 0$$
$$q_0(\eta) \rightarrow \exp(-\tfrac{1}{2}\eta^2) \qquad \eta \geq 0 \qquad (11)$$

For large values of $\eta$ (in practice $\eta \geq 2$ is adequate) the following approximations are valid. When $0 \leq \epsilon < 1$,

$$q_1(\eta) \sim (1 - \epsilon^2)^{1/2} \exp(-\tfrac{1}{2}\eta^2) \qquad (12)$$

When $\epsilon \rightarrow 1$,

$$q_2(\eta) \sim \exp(-\tfrac{1}{2}\eta^2)/(2\pi)^{1/2}\eta \qquad (13)$$

For a narrow spectrum the envelope follows the crests of the waves, and the distribution of maximums $q_0(\eta)$ has the same properties as the envelope. For broader spectra the envelope does not describe the distribution of maximums, and the extent to which this is so is approximately determined by the ratios $q_1/q_0$ and $q_2/q_0$ in the case of a very wide band spectrum.

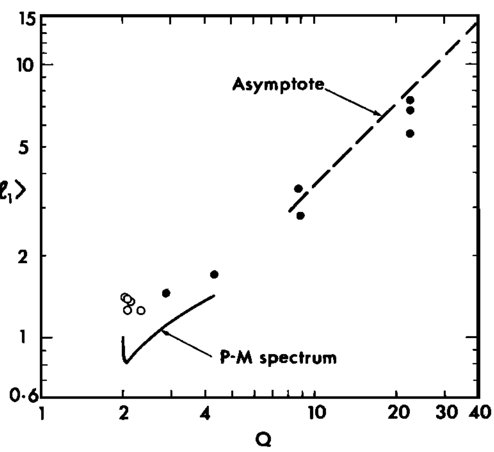


Fig. 2. Mean length of runs of wave heights for a level corresponding to the significant wave height from numerical simulations compared with estimates evaluated for a Pierson-Moskowitz spectrum and an asymptotic form. Open circles relate to spectra with $m = 5$ and $n = 4$, and closed circles represent the results for spectra obtained by using other values of $m$ and $n$.

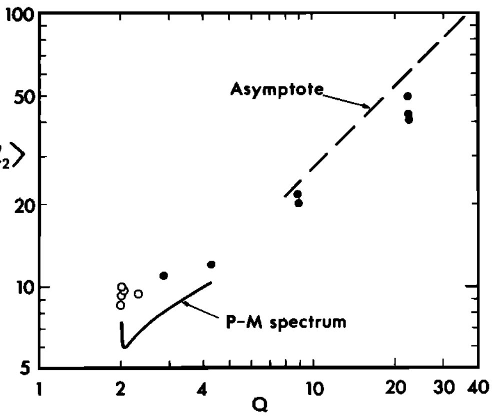


Fig. 3. Mean number of waves between the exceedance of the level corresponding to the significant wave height to the next exceedance of the same level by the succeeding group of waves. Open circles relate to spectra with $m = 5$ and $n = 4$, and closed circles represent the results for spectra obtained by using other values of $m$ and $n$.

In the previous analysis we can replace $N$ by $N(q_1/q_0)$ or $N(q_2/q_0)$ to obtain

$$\langle l_1 \rangle \sim (1 - \epsilon^2)^{-1/2}[m_2/(2\pi\mu_2)]^{1/2}k^{-1} \qquad (14)$$
$$0 \leq \epsilon < 1$$

$$\langle l_1 \rangle \sim (m_2/\mu_2)^{1/2} \qquad \epsilon \rightarrow 1 \qquad (15)$$

Expressions corresponding to (14) and (15) but with an additional factor $\exp(\tfrac{1}{2}k^2)$ are obtained for $\langle l_2 \rangle$.

For a very wide band spectrum, $(m_2/\mu_2)^{1/2} \rightarrow 1$ as $\epsilon \rightarrow 1$, and we obtain limiting values of

$$\langle l_1 \rangle = 1$$
$$\langle l_2 \rangle = \exp(\tfrac{1}{2}k^2) \qquad (16)$$

Expressions 16 are in accord with Goda's results for a random process (1) and (2) when $k$ is large.

*Computation of run lengths.* We now evaluate $\langle l_1 \rangle$ and $\langle l_2 \rangle$ using (14) and its equivalent expression for the case when the level $\rho$ corresponds to the significant wave height (i.e., $k = 2$) for different wave spectra considered by Goda. These spectra are of the form

$$E(f) = af^{-m} \exp(-bf^{-n}) \qquad f_1 \leq f \leq f_2$$

where the truncation frequency $f_2$ varied from 2 to 10 times the modal frequency and $f_1$ was set at about 0.5 times the modal frequency. When $m = 5$ and $n = 4$, the spectra correspond to the form derived by *Pierson and Moskowitz* [1964].

Figures 2 and 3 compare the results with the numerical simulations, which are shown by open and closed circles. The computations for a Pierson-Moskowitz spectrum are shown by a curve lying below values of $\langle l_1 \rangle$ and $\langle l_2 \rangle$ for the simulated run lengths.

The computations of run lengths for the spectra used by Goda to obtain values at $Q = 2.8$ and $Q = 4.3$ were found to have values very close to the curve for the Pierson-Moskowitz spectrum. These computations also lie below the values obtained by Goda.

As $Q$ increases, the asymptotic forms (8) and (10) lie, as was noted earlier, close to the results for the numerical simulations irrespective of the form of the spectrum.

*Conclusions.* Asymptotic forms have been derived for the average length of runs of wave heights exceeding a specified level $\langle l_1 \rangle$ and the average number of waves between the exceedance of one level by a group of waves to the next exceedance of the same level by the succeeding group of waves $\langle l_2 \rangle$ when the spectrum has a narrow bandwidth. In practice these forms are adequate approximations to run lengths when $\epsilon \lesssim 0.2$ or $Q \gtrsim 8$.

For spectra of intermediate width approximate expressions can be used to evaluate $\langle l_1 \rangle$ and $\langle l_2 \rangle$ for high wave heights.

Very wide band spectra have been investigated by Goda using random process theory. This theory predicts the mean and standard deviation of run lengths and can be used when the spectral width is close to 1.

Application of these results to particular problems requires a knowledge of the variability in run lengths associated with a particular average length. The probability distribution of run lengths has at present been determined only for wide band spectra, but useful indications of the standard deviation of run lengths have been obtained from a wide range of spectral widths in the simulation studies of Goda.

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