Waves Random Media **13** (2003) L7–L11 PII: S0959-7174(03)64980-3

LETTER TO THE EDITOR

Formal tilt invariance of the local curvature approximation

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Received 18 June 2003
Published 16 July 2003
Online at stacks.iop.org/WRM/13/L7

Abstract
Tilt invariance is a stringent but necessary condition that a second-order wave scattering model must satisfy in order to qualify for a broad range of applications. This invariance expresses the fact that the scattering model is unchanged whether the tilting of the scattering surface is implemented before or after its reduction to the limit of the small-perturbation method (SPM). Our scattering model is based on a second-order kernel which is quadratic in its lowest order with respect to successive derivatives of the rough surface. Hence, it is termed the local curvature approximation (LCA). We have previously demonstrated that the LCA is approximately tilt invariant in the quasi-specular and quasi-backscattering geometries. In this contribution, LCA is made formally tilt invariant up to first order in the tilting vector. It will be shown that this formal tilt invariance is achieved mainly through inclusion of polarization mixing due to out-of-plane tilt. Even though the LCA formally reduces to the SPM and Kirchhoff limits in addition to tilt invariance, its curvature kernel stays reasonably concise and practical to implement in both analytical and numerical evaluations. This curvature kernel may also be used in the other two formulations of our model, namely the non-local curvature approximation and the weighted curvature approximation.

1. Introduction

An important formal condition of scattering models is the fulfilment of a stringent constraint known as tilt invariance. Tilt invariance expresses the fact that the scattering model must yield the same result whether the tilting of the surface is executed before or after reduction to a particular limit, such as that of the small-perturbation method (SPM). Dashen and Wurmser [1] demonstrated that the fulfilment by their model of this tilt invariance allowed them to derive a highly accurate approximation with accuracy up to the curvature order of the scattering surface. They chose to solve the tilting condition formally and therefore their model is tilt invariant up to an arbitrary order in the tilt vector. However, the Dashen and Wurmser model only exists for Dirichlet, Neumann, and perfect conducting boundary conditions. Unfortunately, their procedure is impractical for the general electromagnetic scattering by an interface between two dielectric half-spaces. In this general case, the complete integration of the tilting condition becomes intractable. The purpose of this letter is to establish that an accurate second-order model, for the general dielectric case, can be found if the formal application of the tilting condition is relaxed from arbitrary tilt to first order in the tilt vector. The only other model, in addition to that of Dashen and Wurmser, that we know of which was tested for tilt invariance is the small-slope approximation (SSA) of Voronovich [2]. The SSA was shown numerically in [3] to be tilt invariant up to the linear order in the tilt vector. In a recent contribution [4], we have derived a curvature kernel shared by three different forms of our scattering model (listed in order of increasing accuracy and complexity): the weighted curvature approximation (WCA), the local curvature approximation (LCA), and the non-local curvature approximations (NLCAs). The simplest form (the WCA) is represented by a single integral over an integrand that depends on the gradient of the scattering surface along with the standard phase factors. This form is similar to that of Dashen and Wurmser [1] but more general in the sense that it may be applied to the dielectric case but with less accuracy in the tilting condition. Indeed, we have shown in [4] that the tilt invariance is satisfied by our three forms with good accuracy only in the quasi-specular and quasi-backscattering regimes. We demonstrate in this contribution that a formal tilt invariance can be achieved with a slight modification of the curvature kernel in order to account for, among other things, polarization mixing due to out-of-plane tilt. The derivation is shown in the context of our LCA.

2. The functional form of the scattering model

The scattering amplitude in the LCA can be written as

$$\begin{aligned}\mathcal{S}(\boldsymbol{k},\boldsymbol{k}_0) = {} & \frac{\mathcal{K}(\boldsymbol{k},\boldsymbol{k}_0)}{q_z}\int \mathrm{e}^{-\mathrm{i}q_z\eta(\boldsymbol{x})}\mathrm{e}^{-\mathrm{i}\boldsymbol{q}_H\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x} \\ & -\mathrm{i}\int\!\!\int \mathcal{T}(\boldsymbol{k},\boldsymbol{k}_0;\boldsymbol{\xi})\hat{\eta}(\boldsymbol{\xi})\mathrm{e}^{-\mathrm{i}q_z\eta(\boldsymbol{x})}\mathrm{e}^{-\mathrm{i}(\boldsymbol{q}_H-\boldsymbol{\xi})\cdot\boldsymbol{x}}\,\mathrm{d}\xi\,\mathrm{d}\boldsymbol{x}\end{aligned} \tag{1}$$

where the adopted notation is a judicious combination of ours in [5–7] with that of Voronovich's [2] and Dashen and Wurmser's [1] papers. Most of the variables used are defined as follows:

$$\boldsymbol{K}_\mathrm{i} = \boldsymbol{k}_0 - q_0\hat{\boldsymbol{z}} \tag{2a}$$
$$\boldsymbol{K}_\mathrm{s} = \boldsymbol{k} + q_k\hat{\boldsymbol{z}} \tag{2b}$$
$$\boldsymbol{K}_\mathrm{i}^2 = \boldsymbol{K}_\mathrm{s}^2 = K^2 \tag{2c}$$
$$q_k = \sqrt{K^2 - \boldsymbol{k}\cdot\boldsymbol{k}} \tag{2d}$$
$$q_0 = \sqrt{K^2 - \boldsymbol{k}_0\cdot\boldsymbol{k}_0} \tag{2e}$$
$$q_z = q_k + q_0 \tag{2f}$$
$$\boldsymbol{q}_H = \boldsymbol{k} - \boldsymbol{k}_0 \tag{2g}$$
$$w_z = q_k - q_0 \tag{2h}$$
$$\boldsymbol{w}_H = \boldsymbol{k} + \boldsymbol{k}_0 \tag{2i}$$
$$\eta(\boldsymbol{x}) \rightleftharpoons \hat{\eta}(\boldsymbol{\xi}) \tag{2j}$$

where $\boldsymbol{K}_\mathrm{i}$ and $\boldsymbol{K}_\mathrm{s}$ are the three-dimensional wavenumbers of the incident and scattered waves, respectively. The two-dimensional vectors $\boldsymbol{k}$ and $\boldsymbol{k}_0$ are the horizontal projections of the three-dimensional vectors. The positively defined vertical components are q_k and q_0. $\hat{\boldsymbol{z}}$ is the unitary vertical reference vector. The scattering surface elevation is described by $\eta(\boldsymbol{x})$ and its corresponding Fourier transform $\hat{\eta}(\boldsymbol{\xi})$. The curvature kernel $\mathcal{T}$ is as yet undefined but must satisfy the elementary conditions established in [4, 7], after the change of variable in (2):

$$\mathcal{T}(\boldsymbol{k}, \boldsymbol{k}_0; \boldsymbol{0}) = 0 \tag{3a}$$

$$\boldsymbol{\nabla}\mathcal{T}(\boldsymbol{k}, \boldsymbol{k}_0; \boldsymbol{0}) = \boldsymbol{0} \tag{3b}$$

$$\mathcal{T}(\boldsymbol{k}, \boldsymbol{k}_0; \boldsymbol{q}_H) \triangleq \mathcal{T}(\boldsymbol{w}_H; \boldsymbol{q}_H) = \mathcal{B}(\boldsymbol{w}_H; \boldsymbol{q}_H) - \mathcal{K}(\boldsymbol{w}_H; \boldsymbol{q}_H) \tag{3c}$$

and with the definitions

$$\mathcal{K}(\boldsymbol{k}, \boldsymbol{k}_0) \triangleq \mathcal{K}(\boldsymbol{w}_H; \boldsymbol{q}_H) \tag{4a}$$

$$\mathcal{B}(\boldsymbol{k}, \boldsymbol{k}_0) \triangleq \mathcal{B}(\boldsymbol{w}_H; \boldsymbol{q}_H) \tag{4b}$$

where $\mathcal{B}$ and $\mathcal{K}$ are the polarization matrices of the first-order SPM-1 and the Kirchhoff approximation (KA), respectively. The first condition in (3) preserves the invariance of the model under vertical translation of the scattering surface. The second property implies that the high-frequency limit is the KA; equation (5.18) in Voronovich [2] defines how the high-frequency limit is obtained from the first derivative of the second-order kernel. Both the first and second properties translate the quadratic form of the curvature kernel $\mathcal{T}$ in its lowest order. The third condition ensures the convergence of the LCA in (1) to SPM-1 under small-roughness conditions.

At this point the curvature kernel ($\mathcal{T}$) in (1) is undetermined and will be derived in the next section by exploiting the formal compliance with the tilt invariance.

3. Formal tilt invariance

The LCA in (1) reduces formally to the first-order small SPM-1 according to (3). It is therefore imperative now to check whether this form still reduces to the SPM-1 limit when the surface is tilted before or after the reduction. The tilting equation that the LCA must satisfy is obtained by inserting the following substitutions in (1):

$$\eta(\boldsymbol{x}) \Rightarrow \eta(\boldsymbol{x}) + \vec{a} \cdot \boldsymbol{x} \tag{5a}$$

$$\hat{\eta}(\boldsymbol{\xi}) \Rightarrow \hat{\eta}(\boldsymbol{\xi}) + \mathrm{i}\vec{a} \cdot \boldsymbol{\nabla}\delta(\boldsymbol{\xi}), \tag{5b}$$

and the linearization in the surface variable then gives

$$\mathcal{B}_{\text{tilted}} \doteq \mathcal{K}(\boldsymbol{k}, \boldsymbol{k}_0) - q_z\, \boldsymbol{\nabla}\mathcal{T}(\boldsymbol{k}, \boldsymbol{k}_0; \boldsymbol{0}) \cdot \vec{a} + \mathcal{T}(\boldsymbol{k}, \boldsymbol{k}_0; \boldsymbol{q}_H - q_z\vec{a}) \tag{6}$$

where the second term in the right-hand side is identically zero because the curvature kernel is quadratic in its lowest order. Hence, by isolating the curvature kernel we get

$$\mathcal{T}(\boldsymbol{k}, \boldsymbol{k}_0; \boldsymbol{q}_H - q_z\vec{a}) = \mathcal{B}_{\text{tilted}} - \mathcal{K}(\boldsymbol{w}_H; \boldsymbol{q}_H). \tag{7}$$

In order to determine the curvature kernel one must explicitly examine the expression for the tilted SPM-1 coefficient $\mathcal{B}_{\text{tilted}}$ which is, to the first order in the tilt vector $\vec{a}$,

$$\mathcal{B}_{\text{tilted}} = T(\boldsymbol{k}, -\vec{a})\mathcal{B}(\tilde{\boldsymbol{w}}_H; \tilde{\boldsymbol{q}}_H)T(\boldsymbol{k}_0, \vec{a}). \tag{8}$$

The tilting matrix T is given by equation (21) in [3] and reproduced here by (9) to first order (and in the appendix to second order) as

$$T(\boldsymbol{k}_0, \vec{a}) = \begin{pmatrix} 1 & -\frac{K}{k_0}(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_0) \cdot \vec{a} \\ \frac{K}{k_0}(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_0) \cdot \vec{a} & 1 \end{pmatrix} + \mathrm{O}(a^2). \tag{9}$$

The tilde over the vectors in (8) expresses that a vector is tilted according to the following substitutions:

$$\boldsymbol{k} \Rightarrow \tilde{\boldsymbol{k}} = \boldsymbol{k} - q_k\vec{a} \tag{10a}$$

$$\boldsymbol{k}_0 \Rightarrow \tilde{\boldsymbol{k}}_0 = \boldsymbol{k}_0 + q_0\vec{a} \tag{10b}$$

$$\boldsymbol{w}_H \Rightarrow \tilde{\boldsymbol{w}}_H = \boldsymbol{w}_H - w_z\vec{a} \tag{10c}$$

$$\boldsymbol{q}_H \Rightarrow \tilde{\boldsymbol{q}}_H = \boldsymbol{q}_H - q_z\vec{a}. \tag{10d}$$

Let us now make use of an important property of the KA $\mathcal{K}$ that reads (up to first order in the tilt vector)

$$\mathcal{K}(\boldsymbol{w}_H; \boldsymbol{q}_H) = T(\boldsymbol{k}, -\vec{a})\mathcal{K}(\tilde{\boldsymbol{w}}_H; \tilde{\boldsymbol{q}}_H)T(\boldsymbol{k}_0, \vec{a}) + \mathrm{O}(a^2). \tag{11}$$

This property merely translates the tilt invariance of the Kirchhoff model itself up to the considered order which is also due, formally, to the nice gradient property of the Kirchhoff polarization matrix

$$w_z\frac{\partial \mathcal{K}}{\partial \boldsymbol{w}_H} + q_z\frac{\partial \mathcal{K}}{\partial \boldsymbol{q}_H} = \boldsymbol{0}. \tag{12}$$

Introducing equations (11) with (8) into (7) yields

$$\mathcal{T}(\boldsymbol{k}, \boldsymbol{k}_0; \tilde{\boldsymbol{q}}_H) = T(\boldsymbol{k}, -\vec{a})[\mathcal{B}(\tilde{\boldsymbol{w}}_H; \tilde{\boldsymbol{q}}_H) - \mathcal{K}(\tilde{\boldsymbol{w}}_H; \tilde{\boldsymbol{q}}_H)]T(\boldsymbol{k}_0, \vec{a}) + \mathrm{O}(a^2). \tag{13}$$

Finally, the curvature kernel derived from formal tilt invariance up to the first order in the tilt vector can be obtained after the following substitutions:

$$\tilde{\boldsymbol{q}}_H \Rightarrow \boldsymbol{\xi} \tag{14a}$$

$$\tilde{\boldsymbol{w}}_H \Rightarrow \boldsymbol{w}_H - w_z\frac{\boldsymbol{q}_H - \boldsymbol{\xi}}{q_z} \tag{14b}$$

$$\vec{a} \Rightarrow \frac{\boldsymbol{q}_H - \boldsymbol{\xi}}{q_z}. \tag{14c}$$

Hence, we find

$$\mathcal{T}(\boldsymbol{k}, \boldsymbol{k}_0; \boldsymbol{\xi}) = T\left(\boldsymbol{k}, \frac{\boldsymbol{\xi} - \boldsymbol{q}_H}{q_z}\right)\mathcal{T}(\tilde{\boldsymbol{w}}_H; \boldsymbol{\xi})T\left(\boldsymbol{k}_0, \frac{\boldsymbol{q}_H - \boldsymbol{\xi}}{q_z}\right), \tag{15}$$

where the bivariate kernel in the middle of the right-hand side is the one already defined in (3) and in our previous derivations in [4]. The curvature kernel is merely the tilted difference between SPM and KA polarization matrices. The trivariate curvature kernel reduces to the bivariate kernel in two cases. The first is in the context of scalar theory such as that of the scattering of sound waves as already derived in [8]. The second case occurs when this theory is applied to two-dimensional problems where out-of-plane tilt is not possible. It should be noted that w_z in (14) and (15) is zero in the specular and backscattering directions. This suggests that the bivariate kernel is a reasonable reduction of the more complete kernel in those geometries.

It is simple to verify that the curvature kernel in (15) satisfies the conditions in (3) as well as the fundamental reciprocal form. The quadratic property in its lowest order is reached due to two nice properties of the bivariate kernel:

$$\frac{\partial \mathcal{T}}{\partial \boldsymbol{q}_H}(\boldsymbol{w}_H; \boldsymbol{0}) = \boldsymbol{0} \tag{16a}$$

$$\frac{\partial \mathcal{T}}{\partial \boldsymbol{w}_H}(\boldsymbol{w}_H; \boldsymbol{0}) = \boldsymbol{0}. \tag{16b}$$

Equations (16) and (12) constitute interesting properties of the SPM-1 and Kirchhoff polarization matrices that, as far as we are aware, have not been previously exploited in the literature.

The curvature kernel $\mathcal{T}(\boldsymbol{k}, \boldsymbol{k}_0; \boldsymbol{\xi})$ in (15) allows our LCA to be formally tilt invariant up to the linear order in the tilt vector $\vec{a}$ in addition to satisfying both the SPM-1 and Kirchhoff limits in the general dielectric case. This powerful kernel can also be used in the other two forms of our model as described in [4], namely the NLCA and the WCA.

4. Conclusions

The LCA is shown to formally reproduce both the SPM and the KA in addition to satisfying the very stringent condition of tilt invariance. Tilt invariance means that the model reduces to the same limit whether the tilting of the scattering surface is executed before or after the limiting process. The fact that the complete kernel is quadratic to lowest order in successive derivatives of the scattering surface has motivated the curvature terminology. This formal tilt invariance is achieved mainly through inclusion of polarization mixing due to the out-of-plane tilt. Even though the LCA reaches formally the three most important conditions, its curvature kernel stays reasonably concise and practical to implement in both analytical and numerical evaluations. This powerful curvature kernel can be used in our two other formulations, the NLCA to account for multiple scattering, and the WCA for simplicity in numerical evaluation.

Appendix. Second-order tilt matrix

The second-order tilt matrix is

$$T(\boldsymbol{k}_0, \vec{a}) = \begin{pmatrix} 1 & -\frac{K}{k_0}(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_0) \cdot \vec{a} \\ \frac{K}{k_0}(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_0) \cdot \vec{a} & 1 \end{pmatrix} - \frac{K}{2k_0^2} \begin{pmatrix} K((\hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_0) \cdot \vec{a})^2 & -2q_0(\vec{a} \cdot \hat{\boldsymbol{k}}_0)(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_0) \cdot \vec{a} \\ 2q_0(\vec{a} \cdot \hat{\boldsymbol{k}}_0)(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_0) \cdot \vec{a} & K((\hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_0) \cdot \vec{a})^2 \end{pmatrix} \tag{A.1}$$

where one can recognize the Taylor series of the sine and cosine trigonometric functions. We noticed that if the second-order tilt matrix is used in the trivariate kernel, the LCA becomes tilt invariant up to the quadratic order in the tilt vector but only in some preferred directions, such as the backscattering direction. This feature is now being investigated with the objective of extending it to a broader domain of applicability.

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