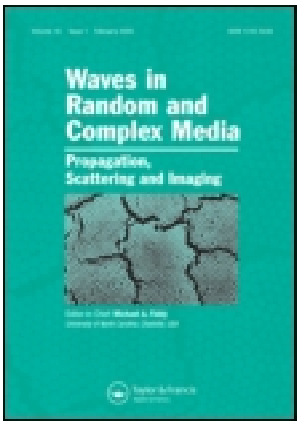

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A new bistatic model for electromagnetic scattering from perfectly conducting random surfaces

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Abstract. In this paper, we extend the Kirchhoff approach, which is widely used for near-nadir backscattering calculations, to include the proper polarization sensitivity for general bistatic scattering from gently sloping, perfectly conducting surfaces. Previously, Holliday has shown how the inclusion of terms from the second iteration of the surface-current integral equation is required to obtain agreement with the small perturbation method for backscattering conditions. Here we employ a similar approach by retaining all terms in this iterative expansion through first order in the surface slope to derive expressions for the standard Kirchhoff field as well as for a supplementary field that contains the polarization sensitivity. A polarization vector notation is introduced to simplify the inclusion of tilting effects from larger-scale features on the scattering surface. In connection with this latter development, we provide a clarification of the earlier work by Valenzuela on this topic together with an extension to the bistatic problem. These extensions to the standard Kirchhoff approach form the basis for our composite bistatic scattering model which should provide a convenient and powerful tool for calculations involving passive as well as active microwave scattering from random surfaces.

1. Introduction

All available closed-form models for electromagnetic scattering from random surfaces are asymptotic solutions of the exact Maxwell equations. Most often, two practical limits are considered; (a) the Kirchhoff approximation (Beckmann and Spizzichino [1]), (b) the small perturbation method (SPM) (Rice [2]). The first is obtained under the conditions of small slopes and long waves while the second is derived for small slopes and short waves. The Kirchhoff approach lacks polarization sensitivity but accurately models the quasi-specular problem. While SPM carries the polarization factors it does not properly account for longer-scale features on the scattering surface and therefore fails to reproduce the near-specular scattering. A promising approach would be one that correctly includes the polarization under both situations; specular and moderate-incident scattering. Holliday [3] presented an approach that combines the two limits. He demonstrated, by including the second iteration of the surface-current integral equation, that the Kirchhoff approach can be extended to include the polarization sensitivity of the SPM, for the backscattering case. Other authors have also investigated this problem; among them are Rodriguez and Kim [4], Tatarskii [5], Voronovich [6] and Fung [7]. While all these proposed approaches have provided polarization sensitivity and a larger domain of applicability, they are somewhat difficult to utilize in practice. In this paper, we present a new methodology that makes scattering analysis more accessible, particularly in handling polarization and bistatic problems.

No matter how complete the extant scattering models may appear, most do not treat the longer-scale features of 'real' surfaces which have significant influence on the total bistatic cross section. To account for these backscattering effects, Valenzuela [8] generalized the earlier results of Wright [9] to predict the total radar cross section from a small patch that is tilted in a specified manner. Wright termed this approach the composite or the two-scale model. Variations of this approach have been widely used by many scientists especially in the field of ocean remote sensing. In the present paper, we extend the composite model to the more general bistatic problem, including a correction to Valenzuela's coordinate-system definition.

In the following section, a thorough review of Holliday [3] is given for completeness. In sections 3 and 4, we reformulate these results and extend them to the more general bistatic problem. At the end of these sections, we derive the backscattering limit and compare this with the results reported by Rice [2] and Holliday [3]. In section 5, we provide expressions for the total bistatic field which come from the first and second iterations of the surface-current equation. The corresponding radar cross section is also given in a form that simplifies comparison with results from previous studies. In section 6 we rederive and correct Valenzuela's composite-model formulation and then extend it to include the bistatic-scattering solution. Our final remarks on how our approach is an improvement on current bistatic models are given in the conclusion.

2. Review of the general problem

For a perfect conductor, the Stratton–Chu equations (Stratton [10]) for electric and magnetic fields are decoupled. This means that, for this idealized case, one equation is sufficient to determine both the electric and magnetic fields. We choose to examine the magnetic field equation to remain consistent with the previous development by Holliday [3]. The total magnetic field under the perfect conductivity assumption is

$$\boldsymbol{B}(\boldsymbol{r}_0) = \boldsymbol{B}_{\mathrm{i}}(\boldsymbol{r}_0) - \int_S \boldsymbol{J}(\boldsymbol{r}_1) \times \boldsymbol{\nabla} G(\boldsymbol{r}_0, \boldsymbol{r}_1)\,\mathrm{d}A_1 \tag{1}$$

where S is the surface described by $z = \eta(\boldsymbol{x})$ and $\boldsymbol{x}$ is the horizontal component of the three-dimensional vector $\boldsymbol{r}$. The incident field $\boldsymbol{B}_{\mathrm{i}}$ is a plane wave $\boldsymbol{B}_0 \exp(-\mathrm{i}\boldsymbol{k}_{\mathrm{i}} \cdot \boldsymbol{r})$. The integrand in (1) is the cross product of the total surface current $\boldsymbol{J}(\boldsymbol{r})$ and the gradient of the free-space Green's function

$$G(\boldsymbol{r}_0, \boldsymbol{r}_1) = -\frac{1}{4\pi}\frac{\exp(\mathrm{i}k|\boldsymbol{r}_0 - \boldsymbol{r}_1|)}{|\boldsymbol{r}_0 - \boldsymbol{r}_1|} \tag{2}$$

where $k = |\boldsymbol{k}|$ is the electromagnetic wavenumber. What makes (1) hard to solve is that the total current $\boldsymbol{J}(\boldsymbol{r})$ is a function of the total magnetic field $\boldsymbol{B}(\boldsymbol{r})$. Hence, the right-hand side depends on the left-hand side of (1). The explicit form for $\boldsymbol{J}(\boldsymbol{r})$ is

$$\boldsymbol{J}(\boldsymbol{r}) = \hat{n}(\boldsymbol{r}) \times \boldsymbol{B}(\boldsymbol{r}) \tag{3}$$

where $\hat{n}$ is the unit vector normal to the scattering surface,

$$\hat{n} = \frac{\hat{e}_z - \boldsymbol{\nabla}\eta}{\sqrt{1 + (\boldsymbol{\nabla}\eta)^2}} = n_z(\hat{e}_z - \boldsymbol{\nabla}\eta). \tag{4}$$

At the scattering surface, (1) yields the surface-current integral equation

$$\boldsymbol{J}(\boldsymbol{r}_1) = \boldsymbol{J}_{\mathrm{i}}(\boldsymbol{r}_1) - 2\hat{n}(\boldsymbol{r}_1) \times \int_S \boldsymbol{J}(\boldsymbol{r}_2) \times \boldsymbol{\nabla} G(\boldsymbol{r}_1, \boldsymbol{r}_2)\,\mathrm{d}A_2 \tag{5}$$

where $\boldsymbol{r}_2 = \boldsymbol{x}_2 + \hat{e}_z\eta(\boldsymbol{x}_2)$ reflects the fact that the integral is evaluated along the scattering surface (S). The gradient of the Green's function can easily be computed and written as

$$\boldsymbol{\nabla} G(\boldsymbol{r}_1, \boldsymbol{r}_2) = Q(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)(\boldsymbol{r}_1 - \boldsymbol{r}_2) \tag{6}$$

with

$$Q(r) = \frac{1}{4\pi}\frac{\mathrm{e}^{\mathrm{i}kr}}{r^3}(\mathrm{i}kr - 1). \tag{7}$$

Because the integral in (5) must be evaluated with the Cauchy principal value, the input current becomes

$$\boldsymbol{J}_\mathrm{i}(\boldsymbol{r}_1) = 2\hat{n}(\boldsymbol{r}_1) \times \boldsymbol{B}_\mathrm{i}(\boldsymbol{r}_1). \tag{8}$$

The difference between $\boldsymbol{J}_\mathrm{i}$ and $\boldsymbol{J}$ is that $\boldsymbol{J}_\mathrm{i}$ is generated by the incident field ($\boldsymbol{B}_\mathrm{i}$) alone while $\boldsymbol{J}$ depends on the total field (e.g. the sum of the incident and the scattered fields). The scattered field in the Fraunhofer (far-field) zone can be approximated by

$$\boldsymbol{B}_\mathrm{s}(\boldsymbol{r}_0) = \boldsymbol{B}(\boldsymbol{r}_0) - \boldsymbol{B}_\mathrm{i}(\boldsymbol{r}_0) \simeq \alpha(\boldsymbol{r}_0)\int_S \boldsymbol{J}(\boldsymbol{r}_1) \times \boldsymbol{k}^\mathrm{s} \exp\left(-\mathrm{i}\boldsymbol{k}^\mathrm{s} \cdot \boldsymbol{r}_1\right)\, \mathrm{d}A_1 + \mathrm{O}\left(\frac{1}{kr_0}\right) \tag{9}$$

where $\alpha(\boldsymbol{r}_0)$ is deduced from the limit as $\boldsymbol{r}_0$ goes to infinity

$$\alpha(\boldsymbol{r}_0) = -\frac{\mathrm{i}}{4\pi}\frac{\mathrm{e}^{\mathrm{i}kr_0}}{r_0} \tag{10}$$

and the wavenumber in the scattering direction is

$$\boldsymbol{k}^\mathrm{s} = k\frac{\boldsymbol{r}_0}{r_0} = k\hat{e}^\mathrm{s}. \tag{11}$$

Since all integrals are bound to the scattering surface (S), a convenient change of variables can be made in order to move from the integration over the surface to integration over a flat reference plane (R). In derivations to follow, the area element $\mathrm{d}A$ will simply be replaced by the horizontal element $\mathrm{d}\boldsymbol{x}$, in the knowledge that the Jacobian of the transformation (n_z) has been accounted for in the expression for the current. The normalization factor in the unit normal vector ($\hat{n}$) will be dropped without changing variable names. In the following, the order symbol $\mathrm{O}(1/(kr_0))$ will also be dropped. For instance, the expression for the scattered field in (9) will be rewritten as

$$\boldsymbol{B}_\mathrm{s}(\boldsymbol{r}_0) \simeq \alpha(\boldsymbol{r}_0)\int_R \boldsymbol{J}(\boldsymbol{x}_1) \times \boldsymbol{k}^\mathrm{s} \exp\left(-\mathrm{i}k_z^\mathrm{s}\eta(\boldsymbol{x}_1)\right) \exp\left(-\mathrm{i}\boldsymbol{k}_\mathrm{H}^\mathrm{s} \cdot \boldsymbol{x}_1\right)\, \mathrm{d}\boldsymbol{x}_1 \tag{12}$$

where the subscripts z and H refer to the vertical and horizontal components, respectively. Once again, in equation (12) and those to follow, the normal vector ($\boldsymbol{n}$ inside the current) is no longer normalized. It simply becomes ($\hat{e}_z - \boldsymbol{\nabla}\eta$) due to the projection from the scattering surface (S) to the horizontal Cartesian reference frame (R).

3. First iteration and the Kirchhoff field

In the first iteration, the total current in (5) is forced to match the input current (J_i). If J_1 is taken to be the first-iteration current then J_1 is identical to J_i. From (8), the first-iteration current is

$$\boldsymbol{J}_1(\boldsymbol{x}_1) \equiv \boldsymbol{J}_\mathrm{i}(\boldsymbol{x}_1) = 2\boldsymbol{n}(\boldsymbol{x}_1) \times \boldsymbol{B}_\mathrm{i}(\boldsymbol{x}_1). \tag{13}$$

This expression is exact for a planar surface, where one can show that the second term in (5) is zero. Equation (13) when used in (12) yields the first-iteration scattered field which is also known as the Kirchhoff field

$$\begin{aligned}\boldsymbol{B}_{\mathrm{s}}^{(1)}(\boldsymbol{r}_0) &= \alpha(\boldsymbol{r}_0)\int_R [2\boldsymbol{n}(\boldsymbol{x}_1)\times\boldsymbol{B}_{\mathrm{i}}(\boldsymbol{x}_1)]\times\boldsymbol{k}^{\mathrm{s}}\exp[-\mathrm{i}k_z^{\mathrm{s}}\eta(\boldsymbol{x}_1)]\exp[-\mathrm{i}\boldsymbol{k}_{\mathrm{H}}^{\mathrm{s}}\cdot\boldsymbol{x}_1]\,\mathrm{d}\boldsymbol{x}_1\\ &= 2\alpha(\boldsymbol{r}_0)\int_R [\hat{e}_z\times\boldsymbol{B}_0-\boldsymbol{\nabla}\eta(\boldsymbol{x}_1)\times\boldsymbol{B}_0]\\ &\quad\times\boldsymbol{k}^{\mathrm{s}}\exp[-\mathrm{i}(k_z^{\mathrm{s}}-k_z^{\mathrm{i}})\eta(\boldsymbol{x}_1)]\exp[-\mathrm{i}(\boldsymbol{k}_{\mathrm{H}}^{\mathrm{s}}-\boldsymbol{k}_{\mathrm{H}}^{\mathrm{i}})\cdot\boldsymbol{x}_1]\,\mathrm{d}\boldsymbol{x}_1.\end{aligned}\tag{14}$$

Integration over the gradient of the elevation can be performed on the basis of the following identity

$$\int\boldsymbol{\nabla}\eta(\boldsymbol{x})\exp[-\mathrm{i}k_z\eta(\boldsymbol{x})]\exp[-\mathrm{i}\boldsymbol{k}_{\mathrm{H}}\cdot\boldsymbol{x}]\,\mathrm{d}\boldsymbol{x}\equiv-\frac{\boldsymbol{k}_{\mathrm{H}}}{k_z}\int\exp[-\mathrm{i}k_z\eta(\boldsymbol{x})]\exp[-\mathrm{i}\boldsymbol{k}_{\mathrm{H}}\cdot\boldsymbol{x}]\,\mathrm{d}\boldsymbol{x}.\tag{15}$$

Substitution of (15) in (14) gives a simpler form of the Kirchhoff field

$$\begin{aligned}\boldsymbol{B}_{\mathrm{s}}^{(1)}(\boldsymbol{r}_0) &= 2\alpha(\boldsymbol{r}_0)\int_R\left[\hat{e}_z\times\boldsymbol{B}_0+\frac{\boldsymbol{k}_{\mathrm{H}}^{\mathrm{s}}-\boldsymbol{k}_{\mathrm{H}}^{\mathrm{i}}}{k_z^{\mathrm{s}}-k_z^{\mathrm{i}}}\times\boldsymbol{B}_0\right]\\ &\quad\times\boldsymbol{k}^{\mathrm{s}}\exp[-\mathrm{i}(k_z^{\mathrm{s}}-k_z^{\mathrm{i}})\eta(\boldsymbol{x}_1)]\exp[-\mathrm{i}(\boldsymbol{k}_{\mathrm{H}}^{\mathrm{s}}-\boldsymbol{k}_{\mathrm{H}}^{\mathrm{i}})\cdot\boldsymbol{x}_1]\,\mathrm{d}\boldsymbol{x}_1\\ &= 2\alpha(\boldsymbol{r}_0)B_0k\mathbf{P}_{\mathrm{s}}^{(1)}\int_R\exp[-\mathrm{i}q_z\eta(\boldsymbol{x}_1)]\exp[-\mathrm{i}\boldsymbol{q}_{\mathrm{H}}\cdot\boldsymbol{x}_1]\,\mathrm{d}\boldsymbol{x}_1.\end{aligned}\tag{16}$$

In this final form for the scattered Kirchhoff field, $\boldsymbol{q}$ is the difference between the scattered wavenumber ($\boldsymbol{k}^{\mathrm{s}}$) and the incident wavenumber ($\boldsymbol{k}^{\mathrm{i}}$), while $\mathbf{P}_{\mathrm{s}}^{(1)}$ is the polarization vector of the first iteration:

$$\mathbf{P}_{\mathrm{s}}^{(1)}=\left[\left(\hat{e}_z+\frac{\boldsymbol{q}_{\mathrm{H}}}{q_z}\right)\times\hat{P}_{\mathrm{i}}^{p}\right]\times\hat{e}^{\mathrm{s}}=\left[\frac{\boldsymbol{q}}{q_z}\times\hat{P}_{\mathrm{i}}^{p}\right]\times\hat{e}^{\mathrm{s}}.\tag{17}$$

The index i refers to the incident polarization. For an incident vertical polarization of the *electric field*, the incident polarization vector $\hat{P}_{\mathrm{i}}^{p}=\boldsymbol{B}_0/B_0$ becomes

$$\hat{P}_{\mathrm{i}}^{\mathrm{V}}\equiv\hat{e}_y.\tag{18}$$

This further indicates that the polarization vector is perpendicular to the plane of incidence, chosen for simplicity to be the x–z plane. The incident horizontal polarization of the electric field is in the plane of incidence and given by

$$\hat{P}_{\mathrm{i}}^{\mathrm{H}}=\frac{\boldsymbol{k}^{\mathrm{i}}\times\hat{e}_y}{|\boldsymbol{k}^{\mathrm{i}}\times\hat{e}_y|}\equiv\frac{\hat{e}^{\mathrm{i}}\times\hat{e}_y}{|\hat{e}^{\mathrm{i}}\times\hat{e}_y|}=-\frac{\hat{e}^{\mathrm{s}}\times\hat{e}_y}{|\hat{e}^{\mathrm{s}}\times\hat{e}_y|}.\tag{19}$$

The backscattering limit of the Kirchhoff polarization vector (17) is simply obtained by replacing ($\boldsymbol{k}^{\mathrm{i}}$) by ($-\boldsymbol{k}^{\mathrm{s}}$)

$$\mathbf{P}_{\mathrm{s}}^{(1)\mathrm{b}}=\left[\frac{2k\hat{e}^{\mathrm{s}}}{2k_z^{\mathrm{s}}}\times\hat{P}_{\mathrm{i}}^{p}\right]\times\hat{e}^{\mathrm{s}}=\frac{k}{k_z^{\mathrm{s}}}\hat{P}_{\mathrm{i}}^{p}\equiv\frac{1}{\cos\theta_\ell}\hat{P}_{\mathrm{i}}^{p}\tag{20}$$

where θ_ℓ is the incident angle of the incoming field. Equation (20) indicates that in the backscattering limit the polarization of the Kirchhoff field is conserved. This means that the scattered polarization is identical to the incident polarization. As pointed out by Holliday [3], only higher iterations of the surface-current integral equation will produce a backscattered field whose polarization can deviate from the incident polarization. In the next section, we derive a general expression for bistatic scattering that does show the desired polarization sensitivity.

4. Second iteration and the supplementary field

The second-iteration current is obtained by substituting the first-iteration current (13) back into the surface-current equation (5). The second term of (5) then becomes

$$\begin{aligned} \boldsymbol{J}_2(\boldsymbol{x}_1) &= -2\boldsymbol{n}(\boldsymbol{x}_1) \times \int_R \boldsymbol{J}_\mathrm{i}(\boldsymbol{x}_2) \times \{\boldsymbol{x} + [\eta(\boldsymbol{x}_1) - \eta(\boldsymbol{x}_2)]\hat{e}_z\} \left\{ Q(x) + \frac{1}{2} P(x) \left[\frac{\boldsymbol{x}}{x} \cdot \boldsymbol{\nabla}\eta_{12}^{+} \right]^2 \right\} \mathrm{d}\boldsymbol{x}_2 \\ &\simeq -2\boldsymbol{n}(\boldsymbol{x}_1) \times \int_R \boldsymbol{J}_\mathrm{i}(\boldsymbol{x}_2) \times \{\boldsymbol{x} + \boldsymbol{x} \cdot \boldsymbol{\nabla}\eta_{12}^{+}\hat{e}_z\} \left\{ Q(x) + \frac{1}{2} P(x) \left[\frac{\boldsymbol{x}}{x} \cdot \boldsymbol{\nabla}\eta_{12}^{+} \right]^2 \right\} \mathrm{d}\boldsymbol{x}_2. \end{aligned} \tag{21}$$

The functions $Q(x)$ and $P(x)$ are the result of an expansion of $Q(r)$ in powers of surface slopes since the z-component of $\boldsymbol{r}$ is now bound to the scattering surface. The function $Q(x)$ is defined in (7) and the one-dimensional function $P(x)$ is

$$P(x) = -\frac{1}{4\pi} \frac{\mathrm{e}^{\mathrm{i}kx}}{x^3} \{kx(kx + 3\mathrm{i}) - 3\}. \tag{22}$$

The slope terms in (21) come from the first-order expansion of the elevation difference,

$$\eta(\boldsymbol{x}_1) - \eta(\boldsymbol{x}_2) \simeq [\boldsymbol{x}_1 - \boldsymbol{x}_2] \cdot \frac{\boldsymbol{\nabla}\eta(\boldsymbol{x}_1) + \boldsymbol{\nabla}\eta(\boldsymbol{x}_2)}{2} \equiv \boldsymbol{x} \cdot \boldsymbol{\nabla}\eta_{12}^{+}. \tag{23}$$

This particular slope expansion is a key element in our approach. The smallness parameter of the expansion is the elevation gradient. Note also that the sign change in $[\eta(\boldsymbol{x}_1) - \eta(\boldsymbol{x}_2)]$ under the interchange of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ is preserved by the approximation given by (23). We show below that this anti-symmetry plays an important role in the evaluation of the second-iteration current. Note that the triple cross product in (21) contains not only terms linear in slope but higher-order terms as well. The second-iteration current, linear in slope, is

$$\boldsymbol{J}_2^1(\boldsymbol{x}_1) = 2 \int_R \{[\hat{e}_z \times \boldsymbol{B}_\mathrm{i}(\boldsymbol{x}_2)]\boldsymbol{\nabla}\eta_{12}^{-} \cdot \boldsymbol{x} - 2\boldsymbol{x}\boldsymbol{\nabla}\eta_{12}^{-} \cdot [\hat{e}_z \times \boldsymbol{B}_\mathrm{i}(\boldsymbol{x}_2)]\} Q(x)\, \mathrm{d}\boldsymbol{x}_2 \tag{24}$$

where $\boldsymbol{\nabla}\eta_{12}^{-} \equiv [\boldsymbol{\nabla}\eta(\boldsymbol{x}_1) - \boldsymbol{\nabla}\eta(\boldsymbol{x}_2)]$. Equation (24) is similar to equation (12) in Holliday [3] with a slight difference due to our change of variables and to the slope expansion discussed above. One can see from (24) that, at this linear order, $\boldsymbol{J}_2^1$ has no component along $\hat{e}_z$. Therefore, the vertical component of the current involves higher-order slopes that readily appear even in this second iteration. The terms of the second-iteration current (21) that are nonlinear in slope are

$$\begin{aligned} \boldsymbol{J}_2^2(\boldsymbol{x}_1) = -2 \int_R &\left\{[\boldsymbol{\nabla}\eta(\boldsymbol{x}_2) \times \boldsymbol{B}_\mathrm{i}(\boldsymbol{x}_2)]\boldsymbol{\nabla}\eta_{12}^{-} \cdot \boldsymbol{x} + 2\hat{e}_z\boldsymbol{\nabla}\eta_{12}^{+} \cdot \boldsymbol{x}\boldsymbol{\nabla}\eta_{12}^{-} \cdot [\hat{e}_z \times \boldsymbol{B}_\mathrm{i}(\boldsymbol{x}_2)]\right\} \\ &\times Q(x)\, \mathrm{d}\boldsymbol{x}_2. \end{aligned} \tag{25}$$

Unlike (24), equation (25) does contain a z-component that is of at least second order in slope. In the present study, we consider only that part of the second-iteration current that is linear in slope as given by (24).

The nonlinear term given by (25) is not yet complete since, as pointed out by Holliday [3], a third iteration must be carried out to obtain quadratic terms that are missing from the second iteration. These quadratic or higher-order slope terms account for multiple-scattering processes where the incident field undergoes one or more reflections from distinct points on the surface before being scattered away (e.g. Voronovich [11]). In the present development, we neglect multiple-scattering processes.

Substituting (24) in (12), we obtain the scattered field of the second iteration from terms linear in slope. This contribution supplements the standard Kirchhoff field, and we refer to it as the supplementary field.

$$
\begin{aligned}
\boldsymbol{B}_{\mathrm{s}}^{(2)}(\boldsymbol{r}_0) = {} & 2\alpha(\boldsymbol{r}_0) \iint_R \left\{[\hat{e}_z \times \boldsymbol{B}_{\mathrm{i}}(\boldsymbol{x}_2)]\boldsymbol{\nabla}\eta_{12}^{-} \cdot \boldsymbol{x} - 2\boldsymbol{x}\boldsymbol{\nabla}\eta_{12}^{-} \cdot [\hat{e}_z \times \boldsymbol{B}_{\mathrm{i}}(\boldsymbol{x}_2)]\right\} \\
& \times \boldsymbol{k}^{\mathrm{s}} \exp[-\mathrm{i}k_z^{\mathrm{s}}\eta(\boldsymbol{x}_1)] \exp[-\mathrm{i}\boldsymbol{k}_{\mathrm{H}}^{\mathrm{s}} \cdot \boldsymbol{x}_1] Q(x)\, \mathrm{d}\boldsymbol{x}_2\, \mathrm{d}\boldsymbol{x}_1 \\
= {} & 2\alpha(\boldsymbol{r}_0) \iint_R \left\{[\hat{e}_z \times \boldsymbol{B}_0]\boldsymbol{\nabla}\eta_{12}^{-} \cdot \boldsymbol{x} - 2\boldsymbol{x}\boldsymbol{\nabla}\eta_{12}^{-} \cdot [\hat{e}_z \times \boldsymbol{B}_0]\right\} \\
& \times \boldsymbol{k}^{\mathrm{s}} \exp[-\mathrm{i}k_z^{\mathrm{i}}\boldsymbol{\nabla}\eta_{12}^{+} \cdot \boldsymbol{x}] \exp[-\mathrm{i}\boldsymbol{k}_{\mathrm{H}}^{\mathrm{i}} \cdot \boldsymbol{x}] \exp[-\mathrm{i}q_z\eta(\boldsymbol{x}_1)] \exp[-\mathrm{i}\boldsymbol{q}_{\mathrm{H}} \cdot \boldsymbol{x}_1] \\
& \times Q(x)\, \mathrm{d}\boldsymbol{x}_2\, \mathrm{d}\boldsymbol{x}_1.
\end{aligned}
\tag{26}
$$

The presence of the phase factor $\exp[-\mathrm{i}k_z^{\mathrm{i}}\boldsymbol{\nabla}\eta_{12}^{+} \cdot \boldsymbol{x}]$ in (26) makes direct evaluation difficult. One can see from its definition in (23) that $\boldsymbol{\nabla}\eta_{12}^{+}$ is the average of the vector slopes at positions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Thus this phase factor accounts for the fact that the local incident angle on the surface facet containing $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ is not $\cos^{-1}(\hat{e}^i \cdot \hat{e}_z)$ because the facet in general is tilted. In order to proceed with the analytical evaluation of (26), we neglect the higher-order slope terms produced by this phase factor. In the second part of this paper, an alternative method for treating these tilting effects is discussed.

With the $\exp[-\mathrm{i}k_z^{\mathrm{i}}\boldsymbol{\nabla}\eta_{12}^{+} \cdot \boldsymbol{x}]$ factor set to unity and using the identity

$$
\int \boldsymbol{x} Q(|\boldsymbol{x}|) \exp[-\mathrm{i}\boldsymbol{k}_{\mathrm{H}} \cdot \boldsymbol{x}]\, \mathrm{d}\boldsymbol{x} = \frac{1}{2}\frac{\boldsymbol{k}_{\mathrm{H}}}{k_z}
\tag{27}
$$

we can write the supplementary field as

$$
\begin{aligned}
\boldsymbol{B}_{\mathrm{s}}^{(2)}(\boldsymbol{r}_0) = {} & -2\alpha(\boldsymbol{r}_0) \int_R \left\{[\hat{e}_z \times \boldsymbol{B}_0]\, \boldsymbol{\nabla}\eta(\boldsymbol{x}_1) \cdot \boldsymbol{Q}_{\mathrm{H}} - 2\boldsymbol{Q}_{\mathrm{H}}\boldsymbol{\nabla}\eta(\boldsymbol{x}_1) \cdot [\hat{e}_z \times \boldsymbol{B}_0]\right\} \\
& \times \boldsymbol{k}^{\mathrm{s}} \exp[-\mathrm{i}q_z\eta(\boldsymbol{x}_1)] \exp[-\mathrm{i}\boldsymbol{q}_{\mathrm{H}} \cdot \boldsymbol{x}_1]\, \mathrm{d}\boldsymbol{x}_1.
\end{aligned}
\tag{28}
$$

In deriving (28), we make use of the anti-symmetry property discussed in Holliday [3]: interchanging $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in (26) leaves the double integration unchanged if $\boldsymbol{k}^{\mathrm{s}}$ and $\boldsymbol{k}^{\mathrm{i}}$ are interchanged as well. In addition, a new horizontal vector $\boldsymbol{Q}_{\mathrm{H}}$ resulting from the application of the identity (27) is introduced. It is defined as half the sum of the ratio between horizontal and vertical components of the incident and scattered wavenumbers:

$$
\boldsymbol{Q}_{\mathrm{H}} = \frac{1}{2}\left(\frac{\boldsymbol{k}_{\mathrm{H}}^{\mathrm{s}}}{k_z^{\mathrm{s}}} + \frac{\boldsymbol{k}_{\mathrm{H}}^{\mathrm{i}}}{k_z^{\mathrm{i}}}\right).
\tag{29}
$$

By using (15), we further simplify (28) by making the multi-cross-product vector independent of the variable of integration ($\boldsymbol{x}_1$). Finally, the supplementary field is put into the same form as the Kirchhoff field (14) as

$$
\boldsymbol{B}_{\mathrm{s}}^{(2)}(\boldsymbol{r}_0) = 2\alpha(\boldsymbol{r}_0) B_0 k \mathbf{P}_{\mathrm{s}}^{(2)} \int_R \exp[-\mathrm{i}q_z\eta(\boldsymbol{x}_1)] \exp[-\mathrm{i}\boldsymbol{q}_{\mathrm{H}} \cdot \boldsymbol{x}_1]\, \mathrm{d}\boldsymbol{x}_1
\tag{30}
$$

where the second-iteration polarization vector of the scattered supplementary field is written as

$$
\mathbf{P}_{\mathrm{s}}^{(2)} = \left\{2\left(\frac{\boldsymbol{q}_{\mathrm{H}}}{q_z} \cdot \boldsymbol{P}_{\mathrm{H}}\right)\boldsymbol{Q}_{\mathrm{H}} - \left(\frac{\boldsymbol{q}_{\mathrm{H}}}{q_z} \cdot \boldsymbol{Q}_{\mathrm{H}}\right)\boldsymbol{P}_{\mathrm{H}}\right\} \times \hat{e}^{\mathrm{s}}
\tag{31}
$$

and to simplify notation, a new horizontal vector,

$$
\boldsymbol{P}_{\mathrm{H}} = \hat{e}_z \times \frac{\boldsymbol{B}_0}{B_0} = \hat{e}_z \times \hat{P}_{\mathrm{i}}^{p}
\tag{32}
$$

is introduced. As before, the subscript H indicates that vectors are horizontal; perpendicular to the vertical $\hat{e}_z$. The polarization vector $\mathbf{P}_{\mathrm{s}}^{(2)}$ can then be further rearranged using the triple-cross-product identity $(\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \equiv \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b}))$ to give

$$\mathbf{P}_s^{(2)} = \left\{ (\boldsymbol{P}_{\mathrm{H}} \times \boldsymbol{Q}_{\mathrm{H}}) \times \frac{\boldsymbol{q}_{\mathrm{H}}}{q_z} + \left(\frac{\boldsymbol{q}_{\mathrm{H}}}{q_z} \cdot \boldsymbol{P}_{\mathrm{H}} \right) \boldsymbol{Q}_{\mathrm{H}} \right\} \times \hat{e}^{\mathrm{s}}. \tag{33}$$

The generalized polarization sensitivity that comes through this supplementary field is a major improvement over the standard Kirchhoff model especially for bistatic scattering.

As before, the backscattering limit can be obtained simply by changing $(\boldsymbol{k}^{\mathrm{i}})$ to $(-\boldsymbol{k}^{\mathrm{s}})$. For this limit, the supplementary polarization vector (33) becomes

$$\mathbf{P}_{\mathrm{s}}^{(2)\mathrm{b}} = \left\{ 2(\boldsymbol{Q}_{\mathrm{H}}^{\mathrm{b}} \cdot \boldsymbol{P}_{\mathrm{H}})\boldsymbol{Q}_{\mathrm{H}}^{\mathrm{b}} - |\boldsymbol{Q}_{\mathrm{H}}^{\mathrm{b}}|^2 \boldsymbol{P}_{\mathrm{H}} \right\} \times \hat{e}^{\mathrm{s}} = |\boldsymbol{Q}_{\mathrm{H}}^{\mathrm{b}}|^2 \left\{ 2(\boldsymbol{e}_{\mathrm{H}}^{\mathrm{s}} \cdot \boldsymbol{P}_{\mathrm{H}})\boldsymbol{e}_{\mathrm{H}}^{\mathrm{s}} - \boldsymbol{P}_{\mathrm{H}} \right\} \times \hat{e}^{\mathrm{s}} \tag{34}$$

where $\boldsymbol{q}_{\mathrm{H}}/q_z$ and $\boldsymbol{Q}_{\mathrm{H}}$ are identical in the backscattering limit, and are both set to $\boldsymbol{Q}_{\mathrm{H}}^{\mathrm{b}}$ according to

$$\boldsymbol{Q}_{\mathrm{H}}^{\mathrm{b}} = \frac{\boldsymbol{k}_{\mathrm{H}}^{\mathrm{s}}}{k_z^{\mathrm{s}}} = \frac{\boldsymbol{e}_{\mathrm{H}}^{\mathrm{s}}}{\cos\theta_\ell} = \tan\theta_\ell\, \hat{e}_x. \tag{35}$$

Equation (34) indicates that the supplementary field is polarization sensitive. It can be further reduced if the incoming and outgoing fields are chosen to be either vertically or horizontally polarized. The cross-polarization (VH- and HV-pol) terms are zero which means that even the supplementary field is not providing depolarization in the backscattering limit. It does, however, provide a polarization difference between VV and HH polarizations. For VV-pol, equation (34) reduces to

$$\mathrm{G}_{\mathrm{VV}}^{(2)} = \hat{P}_{\mathrm{i}}^{\mathrm{V}} \cdot \boldsymbol{P}_{\mathrm{s}}^{(2)\mathrm{bV}} = |\boldsymbol{Q}_{\mathrm{H}}^{\mathrm{b}}|^2 e_z^{\mathrm{s}} = \tan^2\theta_\ell \cos\theta_\ell \equiv \frac{\sin^2\theta_\ell}{\cos\theta_\ell} \tag{36}$$

while for HH-pol it gives a slightly different answer which is

$$\mathrm{G}_{\mathrm{HH}}^{(2)} = \hat{P}_{\mathrm{i}}^{\mathrm{H}} \cdot \boldsymbol{P}_{\mathrm{s}}^{(2)\mathrm{bH}} = -|\boldsymbol{Q}_{\mathrm{H}}^{\mathrm{b}}|^2 e_z^{\mathrm{s}} = -\tan^2\theta_\ell \cos\theta_\ell \equiv -\frac{\sin^2\theta_\ell}{\cos\theta_\ell}. \tag{37}$$

In backscattering, the difference between the HH- and VV-polarizations is therefore an addition or a subtraction from the Kirchhoff field. This is consistent with well-known limits such as SPM.

5. The scattered field and the resulting cross section

5.1. Scattered field

In this section, we summarize the previous sections and put the results together to form the total scattered field. The first and second iterations of the surface-current equation yield a total scattered field $\boldsymbol{B}_{\mathrm{s}}^{(1+2)} \equiv \boldsymbol{B}_{\mathrm{s}}^{p}(\boldsymbol{r}_0)$ whose total polarization is $\mathbf{P}_{\mathrm{s}}^{(1+2)} \equiv \mathbf{P}_{\mathrm{s}}^{p}$. This total field and its polarization vector can be written as follows:

$$\boldsymbol{B}_{\mathrm{s}}^{p}(\boldsymbol{r}_0) = 2\alpha(\boldsymbol{r}_0) B_0 k \mathbf{P}_{\mathrm{s}}^{p} \int_R \exp[-\mathrm{i} q_z \eta(\boldsymbol{x}_1)] \exp[-\mathrm{i}\boldsymbol{q}_{\mathrm{H}} \cdot \boldsymbol{x}_1]\, \mathrm{d}\boldsymbol{x}_1 \tag{38}$$

$$\mathbf{P}_{\mathrm{s}}^{p} \equiv \mathbf{P}_{\mathrm{s}}^{(1)} + \mathbf{P}_{\mathrm{s}}^{(2)} = \left\{ \frac{\boldsymbol{q}}{q_z} \times \hat{P}_{\mathrm{i}}^{p} + (\boldsymbol{P}_{\mathrm{H}} \times \boldsymbol{Q}_{\mathrm{H}}) \times \frac{\boldsymbol{q}_{\mathrm{H}}}{q_z} + \left(\frac{\boldsymbol{q}_{\mathrm{H}}}{q_z} \cdot \boldsymbol{P}_{\mathrm{H}} \right) \boldsymbol{Q}_{\mathrm{H}} \right\} \times \hat{e}^{\mathrm{s}} \tag{39}$$

where $\boldsymbol{P}_{\mathrm{H}}$ represents, for convenience, the horizontal vector $\hat{e}_z \times \hat{P}_{\mathrm{i}}^{p}$ which depends on the incoming polarization. Those two equations represent a new general model for bistatic scattering from random perfectly conducting surfaces. To the authors' knowledge, these equations have not been reported previously in the literature. This bistatic model specified by (38) and (39) handles the vector nature of the electromagnetic fields in a new compact form that allows polarization vectors to be identified and retained throughout the development, thus ensuring flexibility in terms of choice of coordinate system. Indeed, while previous bistatic models do carry polarization sensitivity (see, for example, Ulaby *et al* [12]), they are usually presented in less tractable form. After some intensive simplifications, the 'small-slope' approximation due to Voronovich [6] may give the same polarization behaviour for perfectly conducting materials. Our formulation is, however, simpler and represents a more straightforward derivation from first principles, in which mathematical tricks are not required.

5.2. Scattering cross section

The normalized radar cross section (σ^0) is a dimensionless quantity defined as the mean scattering cross section per unit surface area. In the Fraunhofer zone, the cross section is given by

$$\sigma_{pq}^0 = \lim_{r_0\to\infty} \frac{4\pi r_0^2}{A} \frac{\langle \boldsymbol{B}_\mathrm{s}(\boldsymbol{r}_0)\boldsymbol{B}_\mathrm{s}(\boldsymbol{r}_0)^*\rangle}{B_0^2} \tag{40}$$

where (A) stands for the area of the horizontal scattering surface (R). The operator $\langle\cdot\rangle$ represents the ensemble averaging. Substituting the expression for the total field from (38) into (40), one obtains

$$\sigma_{pq}^0 = \frac{k^2}{\pi} \frac{|\hat{P}_\mathrm{s}^q \cdot \mathbf{P}_\mathrm{s}^p|^2}{A} \iint_R \langle \exp\{-\mathrm{i}q_z[\eta(\boldsymbol{x}_1) - \eta(\boldsymbol{x}_2)]\}\rangle \exp[-\mathrm{i}\boldsymbol{q}_\mathrm{H}\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)]\,\mathrm{d}\boldsymbol{x}_1\,\mathrm{d}\boldsymbol{x}_2. \tag{41}$$

If the scattering surface is homogeneous then the integrand between the brackets depends only on the lag vector $\boldsymbol{x}_1 - \boldsymbol{x}_2$, and (41) may be rewritten as

$$\sigma_{pq}^0 = 2k^2|\mathrm{G}_\mathrm{s}^{pq}|^2 \frac{1}{2\pi} \int \langle \exp\{-\mathrm{i}q_z[\eta(\boldsymbol{x}_1) - \eta(\boldsymbol{x}_2)]\}\rangle_{(\boldsymbol{x})} \exp[-\mathrm{i}\boldsymbol{q}_\mathrm{H}\cdot\boldsymbol{x}]\,\mathrm{d}\boldsymbol{x}, \tag{42}$$

where the subscripts p and q refer to the incident and scattered polarizations. This represents our model in its final form for the bistatic scattering radar cross section. $\mathrm{G}_\mathrm{s}^{pq}$ is defined as the dot product of the sample and the total scatter polarizations of the electromagnetic field, which are represented by the vectors $\hat{P}_\mathrm{s}^q$ and $\mathbf{P}_\mathrm{s}^p$, respectively. It is readily shown using (20), (36) and (37) that our model (42) agrees with previous backscattering models such as the results of the SPM as first reported by Rice [2] and, of course, the results of Holliday [3]. Comparisons can also be made between the small-roughness limit obtained from our model and previous bistatic SPMs (e.g. Ulaby *et al* [12]). In such comparisons, which will be examined in detail in a forthcoming paper, small differences should be expected due to slightly different assumptions in the various derivations.

6. A model for assimilation of 'large-scale' tilting

From a 'two-scale' point of view, the bistatic model described above by (42) gives the scattered field referenced to a coordinate system whose vertical axis is aligned with the local surface normal. Tilting of this coordinate system could be caused, for example, by longer-scale features on the scattering surface. This tilting could, of course, change the characteristics of the scattered field. If the long-scale features tilt the local patch according to their in-plane slope (S_x) and out-of-plane slope (S_y), then the local incident angle (θ_ℓ) is different from the nominal incident angle (θ) that is referenced to the observation frame. To simplify notation, two angles (ψ and δ) can be defined in terms of the in-plane and out-of-plane tilts,

$$S_x = \tan\psi \qquad S_y = \tan\delta \tag{43}$$

respectively. In the following text, we shall refer to these angles simply as tilt angles.

6.1. Correction of Valenzuela's backscattering results

Valenzuela [8] gave a relationship combining surface tilt and nominal incident angle (θ) in terms of a local incident angle (θ_ℓ) as:

$$\cos\theta_\ell = \cos(\theta + \psi')\cos\delta' \tag{44}$$

or, also, equivalently

$$\tan^2\theta_\ell = \tan^2(\theta + \psi') + \tan^2\delta' \frac{1}{\cos^2(\theta + \psi')} \tag{45}$$

where we have used a slightly different notation. We have introduced primed tilt angles because it appears that Valenzuela's angles are not the same as the tilt angles defined in (43). The subscript l is consistently used throughout our work to indicate the local incidence as opposed to the nominal incidence (θ).

As foreseen in the previous paragraph, ψ' and δ' in (44) or (45) cannot be the tilt angles as defined in (43) for the simple reason that (44) does not give the right tangent-plane property when θ is set to zero. Namely, at $\theta = 0$, equation (45) becomes

$$\tan^2\theta_\ell = \tan^2\psi' + \tan^2\delta' \frac{1}{\cos^2\psi'} \tag{46}$$

which does not agree with the trivial identity

$$\tan^2\theta_\ell = \tan^2\psi + \tan^2\delta \equiv S_x^2 + S_y^2. \tag{47}$$

This demonstrates that Valenzuela's tilting formula (44) is not consistent with the definition of tilt angles in (43).

There is a simple modification that brings Valenzuela's angles and the real in- and out-of-plane tilts into agreement. In [8], it seems that ψ' and δ' may be rotation angles rather than tilt angles. Although not explicitly stated in his paper, it appears to us that the local unit normal, $\hat{n}'$, to Valenzuela's tilted surface may be re-expressed in the fixed coordinate frame as

$$\{\hat{n}'\}_{\text{fixed}} = \mathbf{R}_y(\psi')\,\mathbf{R}_x(\delta')\,\{\hat{n}'\}_{\text{local}} = \mathbf{R}_{yx}(\psi', \delta')\,\hat{e}_z^{\mathrm{l}} \tag{48}$$

where $\mathbf{R}_\beta(\alpha)$ represents a (three-dimensional) rotation about a principal axis ($\beta = x, y, z$) through an angle α. It must be understood that the rotations here do not rotate the vectors; they merely re-express the (unchanged) vectors in a new coordinate system. Explicitly, equation (48) becomes

$$\{\hat{n}'\}_{\text{fixed}} = -\cos\delta'\sin\psi'\,\hat{e}_x + \sin\delta'\,\hat{e}_y + \cos\delta'\cos\psi'\,\hat{e}_z \tag{49}$$

where $\hat{e}_*$ are the usual orthogonal unit vectors in the fixed coordinate system. One can easily see that the dot product of (49) with the incident vector ($\hat{e}^{\mathrm{i}}$) is the cosine of the local incident angle, θ_ℓ, as derived by Valenzuela [8], and given by (44). The expression for the normal vector in (49) must agree with

$$\{\hat{n}\}_{\text{fixed}} = \frac{-\tan\psi\,\hat{e}_x - \tan\delta\,\hat{e}_y + \hat{e}_z}{\sqrt{1 + \tan^2\psi + \tan^2\delta}} \tag{50}$$

which is now expressed in terms of the real tilt angles of (43). By matching the components of (49) and (50), one can find a relationship between the angles used in [8] and the proper in- and out-of-plane tilt angles defined in (43). Guided by (44) and (45), this relationship turns out to be:

$$\tan\delta' = -\tan\delta\,\cos\psi \tag{51}$$

and

$$\psi' = \psi. \tag{52}$$

While the in-plane tilt ψ is identical to Valenzuela's in-plane rotation ψ', the out-of-plane angle, δ', is different. The angle δ' is, in reality, a function of both the in- and out-of-plane tilts according to (51). Whereas these distinctions are subtle, and may appear almost somewhat semantic, they are necessary for a quantitative comparison of different formulations.

6.2. Rederivation of Valenzuela's results

In the previous section, we showed how Valenzuela [8] may have derived his equation for the local incident angle as a function of the nominal incident angle and the tilt angle. We now develop an alternative method for deriving the complete tilting equations in the backscatter case in order to pave the way for the generalization to the bistatic problem.

A particular polarization of the incident field in the observation frame of reference is modified when referred to a local frame of reference that is tilted by the presence of long-scale features. This modification depends on the in- and out-of-plane tilts. As one might expect, even if the incident polarization is either horizontal or vertical in the observation frame, when referred to the observation frame it becomes a mixture of both. Valenzuela [8] applied the tilting effect to backscatter by using the SPM model. Although SPM from an untilted surface does not predict any depolarization, Valenzuela successfully showed that because of the underlying tilting, the total backscatter may have cross-polarized components. His technique has been extensively used to extend particular solutions of the general electromagnetic problem to include tilting effects.

Since SPM coefficients are expressed in terms of pure vertical and horizontal polarizations, these and other vectors in the fixed frame of reference must be transformed to a coordinate system whose principal axes are aligned with the local V- and H-polarizations on the tilted surface. We refer to the principal axes of this new frame of reference as $\hat{e}_{\mathrm{V}}$, $\hat{e}_{\mathrm{H}}$, and $\hat{e}^{\mathrm{i}}$. To find the expressions for those vectors, one should remember that the incident field along with the unit normal vector (49) determine the local plane of incidence. The vector $\hat{e}_{\mathrm{H}}$ is then uniquely defined by the cross product of the incident direction ($\hat{e}^{i}$) with the local normal expressed in the fixed frame of incidence as in (49);

$$\hat{e}_{\mathrm{H}} = \frac{\{\hat{n}'\}_{\mathrm{fixed}} \times \hat{e}^{\mathrm{i}}}{|\{\hat{n}'\}_{\mathrm{fixed}} \times \hat{e}^{\mathrm{i}}|} = \frac{-\cos\theta\sin\delta'\,\hat{e}_x - \alpha\cos\delta'\,\hat{e}_y + \sin\delta'\sin\theta\,\hat{e}_z}{\alpha_\ell}. \tag{53}$$

Now to complete the reference frame, $\hat{e}_{\mathrm{V}}$ becomes

$$\hat{e}_{\mathrm{V}} = \hat{e}_{\mathrm{H}} \times \hat{e}^{\mathrm{i}} = \frac{\alpha\cos\delta'\cos\theta\,\hat{e}_x - \sin\delta'\,\hat{e}_y - \alpha\cos\delta'\sin\theta\,\hat{e}_z}{\alpha_\ell} \tag{54}$$

where the notation $\alpha_\ell = \sin\theta_\ell$ and $\alpha = \sin(\theta + \psi')$ is similar to that of Valenzuela [8]. Consequently, a transformation matrix can be defined based on these unit vectors as

$$\mathbf{H} = \{\hat{e}_{\mathrm{V}}, \hat{e}_{\mathrm{H}}, \hat{e}^{\mathrm{i}}\} \equiv \begin{pmatrix} \dfrac{\alpha}{\alpha_\ell}\cos\delta'\cos\theta & -\dfrac{1}{\alpha_\ell}\cos\theta\sin\delta' & \sin\theta \\ -\dfrac{1}{\alpha_\ell}\sin\delta' & -\dfrac{\alpha}{\alpha_\ell}\cos\delta' & 0 \\ -\dfrac{\alpha}{\alpha_\ell}\cos\delta'\sin\theta & \dfrac{1}{\alpha_\ell}\sin\delta'\sin\theta & \cos\theta \end{pmatrix}. \tag{55}$$

By applying this transformation, any arbitrary polarization in the fixed frame is decomposed along the vertical and horizontal components of the local frame. Hence, Valenzuela's tilting equations can be represented by the successive transformations as follows

$$a_{pq} = {}^{\mathrm{t}}\hat{P}^{q}\,{}^{\mathrm{t}}\mathbf{H}\,\mathbf{D}\,\mathbf{H}\,\hat{P}^{p} \tag{56}$$

where $\mathbf{D}$ is a diagonal matrix composed of the SPM's backscattering coefficients, $\{g_{\mathrm{VV}}, g_{\mathrm{HH}}, 0\}$, respectively. The left superscript t indicates the transpose of a vector or a matrix. We also give for completeness the total matrix resulting from these transformations

$$\mathbf{T} = {}^{\mathrm{t}}\mathbf{H}\,\mathbf{D}\,\mathbf{H} \equiv \begin{pmatrix} \dfrac{\cos^2\theta}{\alpha_\ell^2} g^+ & \dfrac{\alpha\cos\theta\sin 2\delta'}{2\alpha_\ell^2}\Delta g & -\dfrac{\cos\theta\sin\theta}{\alpha_\ell^2} g^+ \\ \dfrac{\alpha\cos\theta\sin 2\delta'}{2\alpha_\ell^2}\Delta g & \dfrac{g^-}{\alpha_\ell^2} & -\dfrac{\alpha\sin 2\delta'\sin\theta}{2\alpha_\ell^2}\Delta g \\ -\dfrac{\cos\theta\sin\theta}{\alpha_\ell^2} g^+ & -\dfrac{\alpha\sin 2\delta'\sin\theta}{2\alpha_\ell^2}\Delta g & \dfrac{\sin^2\theta}{\alpha_\ell^2} g^+ \end{pmatrix} \tag{57}$$

where the three abbreviations g^+, g^-, and Δg are

$$\begin{aligned} g^+ &= g_{\mathrm{VV}}\alpha^2\cos^2\delta' + g_{\mathrm{HH}}\sin^2\delta' \\ g^- &= g_{\mathrm{HH}}\alpha^2\cos^2\delta' + g_{\mathrm{VV}}\sin^2\delta' \\ \Delta g &= g_{\mathrm{HH}} - g_{\mathrm{VV}}. \end{aligned} \tag{58}$$

Using the definition of the polarization vectors in (18) and (19), the backscattering coefficients in (56) combined with (57) become

$$\begin{aligned} a_{\mathrm{HH}} &= \frac{g^-}{\alpha_\ell^2} = \left(\frac{\alpha\cos\delta'}{\alpha_\ell}\right)^2 g_{\mathrm{HH}} + \left(\frac{\sin\delta'}{\alpha_\ell}\right)^2 g_{\mathrm{VV}} \\ a_{\mathrm{VV}} &= \frac{g^+}{\alpha_\ell^2} = \left(\frac{\alpha\cos\delta'}{\alpha_\ell}\right)^2 g_{\mathrm{VV}} + \left(\frac{\sin\delta'}{\alpha_\ell}\right)^2 g_{\mathrm{HH}} \\ a_{\mathrm{HV}} &= a_{\mathrm{VH}} = -\frac{\alpha\sin\delta'\cos\delta'}{\alpha_\ell^2}\Delta g = -\frac{\alpha\sin\delta'\cos\delta'}{\alpha_\ell^2}(g_{\mathrm{HH}} - g_{\mathrm{VV}}) \end{aligned} . \tag{59}$$

These coefficients are identical to those given by Valenzuela [8] with the clarification of the difference between rotation and tilt angles as discussed in the previous section.

6.3. Generalization of Valenzuela's results

A straightforward generalization of Valenzuela's tilt equations to the bistatic problem can be accomplished by distinguishing between the incident and scattered frames of reference in the matrix form given in (56). A compact formulation of this bistatic tilting can be written as

$$a_{pq} = {}^{\mathrm{t}}\hat{P}_{\mathrm{s}}^{q}\,{}^{\mathrm{t}}\mathbf{H}_{\mathrm{s}}\,\mathbf{G}_{\mathrm{i}}^{\mathrm{s}}\,\mathbf{H}_{\mathrm{i}}\,\hat{P}_{\mathrm{i}}^{p}. \tag{60}$$

All quantities in (60) are referenced to a fixed frame which may be different from both the incident and the scattered frames. The kernel matrix ($\mathbf{G}_{\mathrm{i}}^{\mathrm{s}}$) is no longer diagonal. It contains the bistatic coefficient for the major polarizations (VV, HH, VH, HV) of a chosen model, for instance a bistatic SPM. The matrices $\mathbf{H}_{\mathrm{i}}$ and $\mathbf{H}_{\mathrm{s}}$ are defined as in (55) but with different vectors in the incident or scattered frame of reference, respectively. Although (60) appears very compact in its matrix form, it generates complicated equations when explicitly expressed in terms of trigonometric functions of the tilting angles. For this reason, we do not provide here the bistatic tilt equations in their expanded form. We shall, however, present a new methodology to simplify these expressions of the bistatic tilting problem.

The complexity in (60) results from the fact that the kernel matrix ($\mathbf{G}_{\mathrm{i}}^{\mathrm{s}}$) in many models is given only in terms of specific polarizations (i.e. VV, HH, HV, VH). An arbitrary polarization vector must therefore be expressed as components of the specific polarizations in these types of models. As stated previously, the vector operator introduced in (39) can readily handle arbitrary incident and scattered polarizations as long as the chosen frame of reference is treated consistently throughout the development. Another feature of the bistatic model developed earlier in this paper is that it is independent of the choice of reference frame. If indeed we take full advantage of the vector operator, expression (60) can be greatly simplified. The simplification is based on the observation that the $\mathbf{H}$-matrix in (55) can be decomposed into four rotations about the principal axes as follows

$$\mathbf{H} \equiv \mathbf{R}_y(-\theta_\ell)\,\mathbf{R}_z(-\beta_\ell)\,\mathbf{R}_x(-\delta')\,\mathbf{R}_y(-\psi'). \tag{61}$$

In (61), θ_ℓ is the local incident angle and β_ℓ is defined as the angle between $\hat{e}_{\mathrm{H}}$ in (53) and $\hat{e}^{\mathrm{l}}_y$ of the local frame obtained by the previous two rotations through ψ' and δ';

$$\cos\beta_\ell = \frac{\hat{n}' \times \{\hat{e}^{\mathrm{i}}\}_{\mathrm{local}}}{\alpha_\ell} \cdot \hat{e}^{\mathrm{l}}_y = \frac{\alpha}{\alpha_\ell}. \tag{62}$$

It is now clear that the rotations through θ_ℓ and β_ℓ are needed when the kernel matrix is only available for the specific polarizations. In our vector formulation, θ_ℓ and β_ℓ rotations are unnecessary. The bistatic polarization coefficient in (42) becomes the dot product of a tilted polarization vector ($\{\hat{P}^p_{\mathrm{i}}\}$) with the scattered sample polarization ($\{\hat{P}^q_{\mathrm{s}}\}$) expressed in the fixed frame;

$$\{\mathrm{G}^{pq}_{\mathrm{s}}\}_{\mathrm{tilted}} = {}^{\mathrm{t}}\hat{P}^q_{\mathrm{s}}\,\mathbf{R}_y(\psi')\,\mathbf{R}_x(\delta')\,\mathbf{P}_{\mathrm{s}}\,\mathbf{R}_x(-\delta')\,\mathbf{R}_y(-\psi')\;\hat{P}^p_{\mathrm{i}} \tag{63}$$

where $\mathbf{P}_{\mathrm{s}}$ is the vector in (39) that transforms the incoming polarization to the scattered polarization in the chosen frame, for instance, the local frame defined by the rotations through ψ' and δ'. This general bistatic tilt formulation involves two angles rather than one in the backscattering case where the local incident and scattered angles are the same. The local scattered angle under this general bistatic configuration is

$$\begin{aligned}\cos\theta^{\mathrm{s}}_\ell &= \hat{e}^{\mathrm{s}} \cdot \{\hat{n}'\}_{\mathrm{fixed}} \\ &\equiv {}^{\mathrm{t}}\hat{e}_z\,\mathbf{R}_y(\theta_{\mathrm{s}})\,\mathbf{R}_z(\phi_{\mathrm{s}})\,\mathbf{R}_y(\psi')\,\mathbf{R}_x(\delta')\,\hat{e}_z \\ &= \sin\delta'\sin\theta_{\mathrm{s}}\sin\phi_{\mathrm{s}} + \cos\delta'\left(\cos\theta_{\mathrm{s}}\cos\psi' - \cos\phi_{\mathrm{s}}\sin\theta_{\mathrm{s}}\sin\psi'\right).\end{aligned} \tag{64}$$

In the special case where the azimuthal angle (ϕ_{s}) of the scattered field is zero and $\theta_{\mathrm{s}} = \theta$, equation (64) simplifies to the double cosine equation of Valenzuela (44). If the incident plane is not in the x–z plane, as we have assumed throughout our development, then the local incident angle equation will be obtained from (64) by replacing the scattering index (s) by the incident index (i).

The equation for the polarization coefficient (63) combined with the bistatic cross section in (42) form our bistatic composite model. We believe that the compact vector notation used in the development of this model renders this inherently complicated problem more accessible to computations in both passive and active microwave remote sensing.

7. Conclusion

One of the more robust methods for the computation of electromagnetic scattering from random rough surfaces is the Kirchhoff approach. In particular, this approach does not require an artificial frequency-dependent wavenumber separation scale in the surface representation as do popular composite-type scattering models. Unfortunately, the original Kirchhoff approach does not yield the proper polarization sensitivity for the scattered field. Recently, Holliday [3] has shown for the case of backscatter in the SPM limit that this deficiency in the Kirchhoff field can be corrected through the inclusion of the next iterative correction to the surface current. In this paper, we have generalized Holliday's results to develop a bistatic Kirchhoff model that is polarization sensitive. Our development is based on an expansion of the surface current through first order in the surface slope, and the polarization sensitivity appears as a multiplicative factor of the standard Kirchhoff integral for the scattered field. No restriction on the surface height is required. In addition, we employ a coordinate-frame-independent (three-dimensional) vector notation that yields relatively compact expressions for the scattered field and cross sections even for the case of bistatic scattering.

Our extended Kirchhoff expression is sensitive to the polarization of the incident field for both backscatter and bistatic scattering and produces depolarization in the general bistatic case. However, it does not produce depolarization in the backscattering limit. In this limit, we show that depolarization results from a vertical component of the surface current, proportional to surface-slope terms of quadratic or higher order, that are not presently included in our formulation. To approximately account for these terms and the resulting depolarization of the scattered field (even in the backscattering limit), we have generalized our results to include scattering from tilted facets in a manner similar to that applied to the SPM in the backscattering limit by Valenzuela [8]. Our vector formulation has allowed us to determine an expression for the field scattered in any direction from an arbitrarily tilted facet in terms of Euler rotation matrices. In the present development we also point out a subtle error in Valenzuela's paper related to the definition of the tilt angles. The effect of that error may be minor as long as the tilt angles are small, however its correction allows a detailed comparison of our results with his.

There is also a philosophical distinction between our formulation of the tilting mechanism and that presented in [8]. Since Valenzuela is dealing with a composite scattering model, the surface scales responsible for the tilting in his formulation include all scales longer than several electromagnetic wavelengths. In our Kirchhoff-based approach there is no need for this composite-model scale separation for scales of the order of the radar wavelength. For practical application of our model, we believe that the scale separation should not be specified by a length scale that is dependent on the incident electromagnetic field, but rather should be determined by the characteristics of the scattering surface.

We believe that the model presented in this paper provides another step forward in the understanding of microwave scattering from random surfaces. The model treats backscattering and the more general bistatic problem on an equal footing so that one can easily check the more familiar backscattering limits. The compact vector notation in our formulation should significantly simplify the application of the model to numerical experiments involving complicated polarization dependences such as those involved, for example, in passive microwave remote sensing. These applications of our formulation will be the topic of future investigations.

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