10.1098/rspa.2001.0809

THE ROYAL SOCIETY

Higher-order hydrodynamic modulation: theory and applications for ocean waves

By Tanos Elfouhaily[1]†, Donald R. Thompson[1], Douglas Vandemark[2] and Bertrand Chapron[3]

[1]*Applied Physics Laboratory, Johns Hopkins University, Laurel, MD 20723-6099, USA*

[2]*NASA/Goddard Space Flight Center, Wallops Island, VA, USA*

[3]*Institut Français de Recherche pour l'Exploitation de la Mer, Brest, France*

Received 17 April 2000; revised 14 December 2000; accepted 22 February 2001

Under the two-scale hydrodynamic model for ocean surface waves, short waves are modulated hydrodynamically by long waves. An exact numerical simulation of the two-scale hydrodynamic process shows that the most commonly used modulation transfer function (MTF), which is a linear approximation, does not capture all of the features caused by the inherent nonlinear nature of the physical processes involved. We rederive the linear MTF and generalize it to include local acceleration and finite depth effects. The phase of the linear MTF is shown to be independent of the direction of long modulating waves. This is an artefact of the linearization of the nonlinear equations. A higher-order theory is also derived based on the truncated Hamiltonian for long modulating waves and dissipation by wave–wave interaction for modulated waves. This new theory includes higher-order derivatives of the source functional and, therefore, short-wave dissipation. Consequently, the phase of the modulation depends on the relative direction of long and short waves. It is shown that while the linear hydrodynamic MTF leads to higher-order statistics equivalent to the bispectrum, the new second-order MTF induces the trispectrum of surface elevation. A succinct derivation for the third-order MTF is given for completeness.

Keywords: nonlinear surface waves; hydrodynamic modulation; higher-order surface wave statistics; truncated Hamiltonian function

1. Introduction

Longuet-Higgins & Stewart (1960, 1961) were the first to initiate, under a wave–wave interaction concept, a theory for the modulation of the amplitude of a short wave by underlying variable currents. Similarly, but under a two-scale concept, Keller & Wright (1975) introduced a theory for the modulation of the spectrum of short waves by longer gravity waves. Their theory introduced a linear modulation on the short-wave spectrum when short waves feel the straining caused by the presence of a *single linear* longer modulating wave. The smallness parameter used by Keller & Wright (1975) was the ratio of the horizontal orbital velocity to the phase speed of the single current. A direct modulation of the spectrum of short waves is very appealing

† Present address: CNRS, IRPHE, Laboratoire Interactions Océan–Atmosphère, Parc Scientifique et Technologique de Luminy Case 903, 163, avenue de Luminy, 132 88 Marseille Cedex 9, France.

for the remote sensing community using microwave scattering off the ocean surface. For this reason, the concept of modulated spectra due to Keller & Wright (1975), rather than modulated amplitudes by Longuet-Higgins & Stewart (1960, 1961), will be considered in detail in the present study.

Four years after Keller & Wright (1975), Valenzuela & Wright (1979) generalized this theory to higher orders by including nonlinear source functions related to short waves only. However, this generalization dealt with only a single linear modulating current. The application of this kind of modulation turned out to be somewhat limited and therefore has rarely been used in the more recent literature.

The most commonly used hydrodynamic modulation transfer function (MTF), introduced by Alpers & Hasselmann (1978), is derived from a linearization of the balance equation of wave action. In this case, modulating waves are written as a sum over the entire spectrum of linear gravity waves each with random amplitude and phase. This feature, together with its simplicity of implementation, made the linear MTF of Alpers & Hasselmann (1978) preferred to other MTFs. Now, more than 20 years later, this linear MTF is widely used by scientific communities ranging from oceanographers to remote-sensing physicists (see, for example, Hara & Plant 1994). A generalization that includes both nonlinearities in modulating and modulated waves seems long overdue. The awaited contribution should be to Alpers & Hasselmann (1978) what Valenzuela & Wright (1979) was to the original work of Keller & Wright (1975) in regard to the short modulated waves. Furthermore, we will equip our generalized model with the capability to handle inherent nonlinear modulating waves in addition to the induced nonlinearities in the short waves. In this paper, we propose such generalization.

We begin by introducing the modulation concept and by motivating the need for improvement by contrasting the linear MTF of Alpers & Hasselmann (1978) with an exact analytical and numerical simulation of the hydrodynamic equations. The inconsistencies that emerge suggest the need for a higher-order theory. Our development follows the notation and expands on concepts introduced by Elfouhaily *et al.* (2000). A detailed rederivation for the linear MTF is carried out in § 7. This rederivation includes *heaving* effects caused by the local acceleration in addition to the straining already given in the original expression by Alpers & Hasselmann (1978). Another added feature is the extra modulation felt by short waves when long waves occur in shallow waters. In § 8, we derive the second-order hydrodynamic modulation that includes, in addition to the features in our first-order development, three-wave interactions among the modulating waves themselves. Section 9 is dedicated to a simplified implementation of the first- and second-order expansions. The third-order expansion is summarized in the appendix. An illustration of how a nonlinear current can be introduced is given in § 10. Representative ensemble-average properties are derived in § 11 to show an offset caused by the second-order effect, and in § 12 to illustrate surface nonlinearities that arise when using first and second-order MTFs. Both bispectrum and trispectrum induced by this hydrodynamic modulation are explicitly shown as a function of the MTFs and the equilibrium spectrum.

2. The modulation concept

In the context of short waves riding on long waves, which are considered as a narrow-band process, the evolution of wave action $N(\boldsymbol{k}_{\mathrm{S}}, \boldsymbol{x}, t)$ of the modulated waves can be described by Boltzmann's transport equation (see Phillips 1977),

$$\mathcal{L}[N] = \frac{\mathrm{d}}{\mathrm{d}t}N = \left(\frac{\partial}{\partial t} + \dot{\boldsymbol{x}} \cdot \frac{\partial}{\partial \boldsymbol{x}} + \dot{\boldsymbol{k}}_{\mathrm{s}} \cdot \frac{\partial}{\partial \boldsymbol{k}_{\mathrm{s}}}\right) N = Q(N, \boldsymbol{k}_{\mathrm{s}}, \boldsymbol{x}, t). \tag{2.1}$$

The right-hand side $Q(N, \boldsymbol{k}_{\mathrm{s}}, \boldsymbol{x}, t)$ is the source functional, which is a compact description of sources and sinks of wave action. For instance, wind input, wave dissipation and wave–wave interaction are examples of a few among many elements of the source functional $Q(N, \boldsymbol{k}_{\mathrm{s}}, \boldsymbol{x}, t)$. The wave action is defined as

$$N(\boldsymbol{k}_{\mathrm{s}}, \boldsymbol{x}, t) = \rho_{\mathrm{w}} c_{\mathrm{s}} \Psi(\boldsymbol{k}_{\mathrm{s}}, \boldsymbol{x}, t),$$

where $\Psi(\boldsymbol{k}_{\mathrm{s}})$ is the two-dimensional wave spectrum of modulated waves with $c_{\mathrm{s}} = \omega_{\mathrm{s}}/|\boldsymbol{k}_{\mathrm{s}}|$ as their local phase speed. The constant water density ρ_{w} is neglected without loss of generality in the following equations. The space, $\boldsymbol{x}$, and wavenumber, $\boldsymbol{k}_{\mathrm{s}}$, vectors are independent of each other, but both are functions of the time variable (t). These vectors are in fact the position and momentum coordinates and satisfy the canonical equations given by

$$\dot{\boldsymbol{x}} = \frac{\partial \Omega}{\partial \boldsymbol{k}_{\mathrm{s}}}, \tag{2.2\,a}$$

$$\dot{\boldsymbol{k}}_{\mathrm{s}} = -\frac{\partial \Omega}{\partial \boldsymbol{x}}, \tag{2.2\,b}$$

where the total angular frequency of a packet,

$$\Omega = \tilde{\omega}(\boldsymbol{k}_{\mathrm{s}}, \boldsymbol{x}, t) + \boldsymbol{k}_{\mathrm{s}} \cdot \boldsymbol{U}_{\mathrm{L}}(\boldsymbol{x}, t), \tag{2.3}$$

plays the role of a Hamiltonian by analogy to classical mechanics. $\boldsymbol{U}_{\mathrm{L}}(\boldsymbol{x}, t)$ is the horizontal component of the orbital velocity induced by the presence of long, linear or nonlinear, modulating waves. Equations (2.2) are also known as the ray equations simply because, as time changes, modulated waves will follow the path in phase space defined by these equations. Numerical examples of ray equation behaviour in the presence of single or double modulation oscillatory currents can be found in Ramamonjiarisoa (1995).

The effective frequency $\tilde{\omega}(\boldsymbol{k}_{\mathrm{s}}, \boldsymbol{x}, t)$ in equation (2.2) is a function of space and time variables due to local acceleration effects caused by the time variation of the orbital velocity. The local acceleration vector, $\tilde{\boldsymbol{g}}$, is, according to Longuet-Higgins (1985, 1987*a*),

$$\tilde{\boldsymbol{g}}(\boldsymbol{x}, t) = \boldsymbol{g} - \boldsymbol{a}, \tag{2.4}$$

with $\boldsymbol{g}$ being the (constant) acceleration due to gravity. and $\boldsymbol{a}$ being the real, or Lagrangian, acceleration due to the orbital motion of the underlying field. The local acceleration in equation (2.4) is perpendicular to the long-wave profile. The magnitude of this acceleration vector is then

$$\tilde{g}(\boldsymbol{x}, t) = g\sqrt{1 - 2\frac{a_z}{g} + \frac{a^2}{g^2}}, \tag{2.5}$$

which can be approximated consistently with the objective of the paper to

$$\tilde{g}(\boldsymbol{x}, t) \approx g - a_z + \frac{a^2 - a_z^2}{2g} + \cdots. \tag{2.6}$$

The reader should note that there is also a contribution to the effective gravity due to surface tension (Henyey *et al.* 1988). This term is neglected in the present study, where we are concerned with the effect of long gravity waves on the short-wave spectral density.

The real or Lagrangian acceleration is defined as the total derivative of the orbital velocity vector evaluated at the surface itself:

$$\boldsymbol{a} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u} = \left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \frac{\partial}{\partial \boldsymbol{x}}\right)\boldsymbol{u}. \tag{2.7}$$

Using the boundary equations in eqn 2.1 of Elfouhaily *et al.* (2000) along with some additional properties, such as eqn 3.2 of the same paper, we get

$$a_z \approx \frac{\partial W}{\partial t} + \boldsymbol{U}_{\mathrm{L}} \cdot \frac{\partial W}{\partial \boldsymbol{x}} + \cdots , \tag{2.8}$$

where $\boldsymbol{U}_{\mathrm{L}}$ and W are the horizontal and the vertical components of the orbital velocity, respectively, evaluated at the surface. Higher-order multiplicative terms, which will not be required in this paper, are not shown in equation (2.8).

The effective dispersion relationship of short waves is then

$$\tilde{\omega}^2(\boldsymbol{k}_{\mathrm{s}}, \boldsymbol{x}, t) = \tilde{g}(\boldsymbol{x}, t)|\boldsymbol{k}_{\mathrm{s}}| + T|\boldsymbol{k}_{\mathrm{s}}|^3 = \omega_{\mathrm{s}}^2 - k_{\mathrm{s}} a_z + \cdots , \tag{2.9}$$

where T is the surface tension scaled by the water density and ω_{s}^2 is the usual unperturbed dispersion ($\omega_{\mathrm{s}}^2 = g|\boldsymbol{k}_{\mathrm{s}}| + T|\boldsymbol{k}_{\mathrm{s}}|^3$), referred to as the intrinsic dispersion relationship. The subscript 's' refers to the short-wave portion of the spectrum.

3. Analytical integration

If the source functional Q in equation (2.1) is chosen to be that of Hughes (1978) with a modification proposed by Caponi *et al.* (1988), namely

$$Q(N, \boldsymbol{k}_{\mathrm{s}}, \boldsymbol{x}, t) = \beta N\left[1 - \left(\frac{N}{N_0}\right)^{\rho}\right], \tag{3.1}$$

the differential equation in (2.1) simply becomes, by following the methodology of Thompson *et al.* (1988),

$$\frac{\mathrm{d}P}{\mathrm{d}t} + \mu_{\mathrm{s}} P = \mu_{\mathrm{s}} P_0, \tag{3.2}$$

where the zero index means equilibrium conditions and the change of variable $P = 1/N^{\rho}$ is used. The relaxation rate is $\mu_{\mathrm{s}} = \rho\beta(\boldsymbol{k}_{\mathrm{s}})$. The growth rate β quantifies the duration needed for short waves to grow under wind shear (see Plant 1982). The ratio ρ between the relaxation rate μ_{s} and the growth rate β is predefined by the choice of the growth rate and the equilibrium spectrum according to Kudryavtsev *et al.* (1999). Elfouhaily *et al.* (2001) gives the explicit expression for the ratio ρ as a function of the wind friction velocity when the growth rate due to Plant (1982) and the two-directional spectrum of water waves due to Elfouhaily *et al.* (1997) are combined.

Equation (3.2) is a first-order linear differential equation whose solution is a combination of the homogeneous solution,

$$P_{\mathrm{h}}(\boldsymbol{k}_{\mathrm{s}}(t)) = P_0(\boldsymbol{k}_{\mathrm{s}}(t_0))D(t_0, t), \tag{3.3}$$

and a particular solution of the general equation,

$$P_{\mathrm{p}}(\boldsymbol{k}_{\mathrm{s}}) = \int_{t_0}^{t} P_0(\boldsymbol{k}_{\mathrm{s}}')\mu_{\mathrm{s}}(\boldsymbol{k}_{\mathrm{s}}')D(t',t)\,\mathrm{d}t', \tag{3.4}$$

where the damping function is defined as

$$D(t_0,t_1) \triangleq \exp\left[-\int_{t_0}^{t_1} \mu_{\mathrm{s}}(\boldsymbol{k}_{\mathrm{s}}(t))\,\mathrm{d}t\right], \tag{3.5}$$

and $\boldsymbol{k}_{\mathrm{s}}' = \boldsymbol{k}_{\mathrm{s}}(t')$ in equation (3.4) refers to the integration variable t'.

Performing an integration by parts in equation (3.4) and combining it with the homogeneous solution (3.3) leads to a simpler representation

$$P(\boldsymbol{k}_{\mathrm{s}},\boldsymbol{x},t) = P_0(\boldsymbol{k}_{\mathrm{s}}) + \int_{t_0}^{t} \dot{\boldsymbol{k}}_{\mathrm{s}}{}' \cdot \frac{\partial P_0(\boldsymbol{k}_{\mathrm{s}}')}{\partial \boldsymbol{k}_{\mathrm{s}}'} D(t',t)\,\mathrm{d}t', \tag{3.6}$$

where the initial time t_0 is the time at which the modulated action reaches the equilibrium state by either escaping the modulating current or effectively not feeling the presence of the modulating wave due to the cumulative effect of the damping factor D in equation (3.5). Therefore, the initial time t_0 can be defined as the value where the damping factor reaches some small number (for example, 10^{-6}) for a given time of observation t. In this case the homogeneous solution (3.3) vanishes and the final solution in equation (3.6) is entirely dominated by the particular solution in equation (3.4).

The normalized modulation observed by P is then analytically

$$\frac{\delta P(\boldsymbol{k}_{\mathrm{s}},\boldsymbol{x},t)}{P_0(\boldsymbol{k}_{\mathrm{s}})} = \int_{t_0}^{t} \frac{\dot{\boldsymbol{k}}_{\mathrm{s}}{}'}{P_0(\boldsymbol{k}_{\mathrm{s}})} \cdot \frac{\partial P_0(\boldsymbol{k}_{\mathrm{s}}')}{\partial \boldsymbol{k}_{\mathrm{s}}'} D(t',t)\,\mathrm{d}t', \tag{3.7}$$

where $\dot{\boldsymbol{k}}_{\mathrm{s}}$ is one of the ray equations as expressed in equation (2.2), which is a function of the horizontal orbital velocity as well as the vertical acceleration of the current induced by the presence of long modulating waves. Note that when the modulating current is constant, $\delta P(\boldsymbol{k}_{\mathrm{s}})$ is zero and wave action is conserved.

4. Comparison between exact and linearized modulations

The numerical implementation of § 3 is an adaptation to a sinusoidal current of the previous numerical development by Thompson *et al.* (1988) to accommodate modulation caused by long waves instead of soliton internal waves. A comparison is made in this section between this exact numerical simulation and the approximation made by Alpers & Hasselmann (1978). This later approximation is merely a linearization of all the differential equations in § 2; namely the action balance equation (2.1) and the ray equations (2.2). The approximation by Alpers & Hasselmann (1978) is frequently used in the literature under the name linear modulation transfer function.

The comparison is carried out for two cases of interest. The first case is a single current aligned with the short modulated waves. The solid curve in figure 1 shows a single sinusoidal current, with an amplitude of $1.4\ \mathrm{m\,s^{-1}}$ and a wavelength of 31 m, modulating a short wave of 10 cm wavelength. This periodic current has a steepness

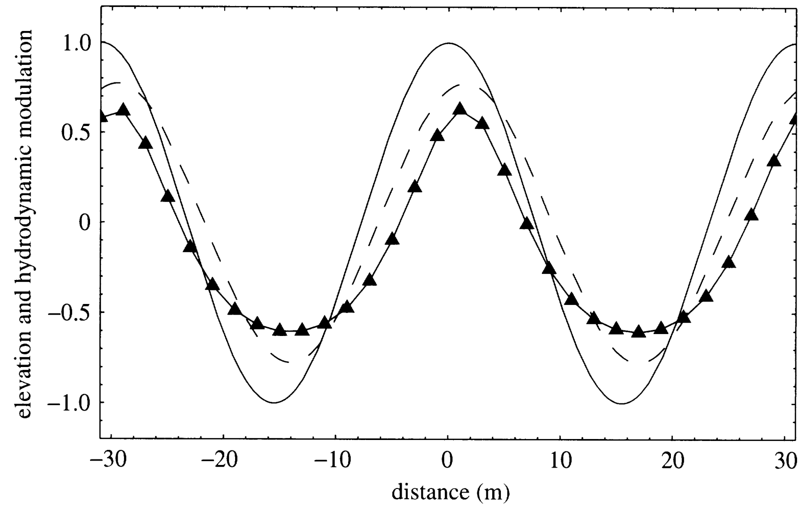

Figure 1. Hydrodynamic modulation from the exact numerical simulation (3.7) and from the approximate linear MTF by Alpers & Hasselmann (1978) for sinusoidal current. The amplitude of the modulating current is 1.42 m s^{-1} or 1 m in surface elevation (solid curve) at a wavelength of 31 m. This example corresponds to a fully developed wind sea for a 7 m s^{-1} wind speed at a height of 10 m from the surface. The wavelength of the short wave is 10 cm. The modulated short wave, the 7 m s^{-1} wind, and the current are chosen to be in the x-axis. The exact modulation (filled triangles) caused by this single harmonic current seems to exhibit a nonlinear shape by showing sharper crests and flatter troughs than the approximate linear MTF (dashed curve). Both modulations are shown over the actual modulating current in terms of surface elevations (solid curve).

of $ak = 0.2$, which is a typical value for the dominant wind wave at the most probable wind speed of 7 m s^{-1} (Holthuijsen & Herbers 1986). This value is consistent with the limiting values on the steepness found by Plant (1982). The same steepness value was used by Longuet-Higgins (1987*b*, 1991) based on the laboratory wind-wave measurements reported in Lake & Yuen (1978). However, Longuet-Higgins (1987*a*) used even steeper long waves when studying the steepness of short modulated waves. The exact numerical simulation, shown by the triangles (grid points of the computation connected by a solid curve) in Figure 1, indicates a different amplitude and phase of the modulation than that of the linear MTF (dashed curve) of Alpers & Hasselmann (1978). The most striking difference is, in fact, the nonlinear shape, especially the sharp crests and flat troughs, of the exact simulated modulation in contrast to the perfect linear shape of the corresponding MTF. The nonlinearities result from the strong current gradient and the nonlinear source function (3.1) in equation (2.1), even though the modulating current itself is linear. The second case includes a second current at 45° with a phase shift of 90° added to the first current of the first case. The linear MTF due to Alpers & Hasselmann (1978) predicts that the total modulation is simply the sum of the independent linear modulations. Figure 2 shows that the sum of the independent linear modulations (dashed curve) is rather different from the exact numerical simulation (triangles) not only in both the amplitude and the phase but also in its intensified nonlinear shape. It is understood that the nonlinear shape of the exact modulation is due to the presence of higher harmonics in contrast to predictions by the linear MTF (Baldock *et al.* 1996). A

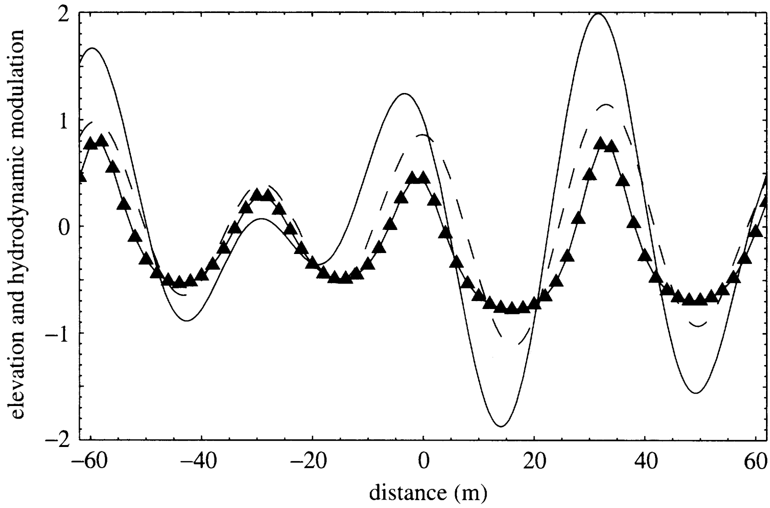

Figure 2. Same as in figure 1 with an additional current at 45° and with a 90° phase shift. Solid curve shows the x-component only of the sum of the two currents. The exact simulation is run over the total current, while the linear MTF predicts the sum of the individual modulations. Again the exact modulation (filled triangles) seems to be nonlinear, even though the currents themselves are linear. Hence a simple sum of independent modulations (dashed curve) in a linear sense does not predict the exact simulation for a linear combination of currents well.

second modulating wave was also found by Ramamonjiarisoa (1995) to have a larger effect on the modulation of the short wave than when only a single long wave is considered.

The exact numerical simulation captures the nonlinear aspect of the source term (3.1) as well as that of the differential equations in § 3, while the linearized version of the same equations obviously does not. It is clear from figures 1 and 2 that the linear MTF is missing some significant aspects of nonlinear modulations. The MTF approach elaborated by Alpers & Hasselmann (1978) is, however, very attractive due to its easy implementation when the modulated current is an infinite sum of linear long gravity waves. This makes the MTF approach superior to the exact numerical simulation in practice at the expense of omitting some interesting features of the nonlinear hydrodynamic modulations. The exact numerical simulation described above is also restricted to a source term of the form given in equation (3.1). A more general technique for arbitrary source functionals as well as for an underlying wave field made from long gravity waves with a possibility of nonlinear wave–wave interactions is yet to be developed. Herein, we propose to generalize the hydrodynamic concept to include the nonlinear aspects by extending the technique to higher orders.

In the next section we rederive the linear MTF by using a slightly different notation from Alpers & Hasselmann (1978) to make the technique amenable to higher-order approximations. The linear MTF is also generalized to include effects of local acceleration as well as from finite depth. In the same context, the nonlinear hydrodynamic modulations are derived in the following sections to include nonlinearities caused by wave–wave interactions in both long and short waves.

5. Expansion of the action balance equation

In order to present the higher-order theory, we need to set the stage for the expansions of both the action balance equation in equation (2.1) and the ray equations in equation (2.2). All the quantities in equation (2.1) can be expanded to third order in the slopes of long modulating waves as follows:

$$\mathcal{L} = {}_0\mathcal{L} + {}_1\mathcal{L} + {}_2\mathcal{L} + {}_3\mathcal{L} + \cdots , \tag{5.1a}$$

$$N = {}_0N + {}_1N + {}_2N + {}_3N + \cdots , \tag{5.1b}$$

$$Q = Q_0 + Q_1(N - {}_0N) + \tfrac{1}{2}Q_2(N - {}_0N)^2 + \tfrac{1}{6}Q_3(N - {}_0N)^3 + \cdots , \tag{5.1c}$$

where the Taylor expansion of the source functional Q is achieved through

$$Q_0 = Q({}_0N) = 0 \quad \text{and} \quad Q_m \triangleq \left.\frac{\partial^m Q}{\partial N^m}\right|_{N={}_0N}. \tag{5.2}$$

Substituting equation (5.1) into equation (2.1) yields

$$\begin{aligned}({}_0\mathcal{L} + {}_1\mathcal{L} + \cdots)[{}_0N + {}_1N + \cdots] = Q_0 + Q_1({}_1N + {}_2N + \cdots) \\ + \tfrac{1}{2}Q_2({}_1N + {}_2N + \cdots)^2 + \cdots .\end{aligned} \tag{5.3}$$

Now, this formal expansion generates a series of differential equations for each order. For the first three orders, equation (5.3) yields the system

$${}_0\mathcal{L}[{}_0N] = Q_0 = 0, \tag{5.4a}$$

$$\sum_{i+j=1} {}_i\mathcal{L}[{}_jN] = Q_1 {}_1N, \tag{5.4b}$$

$$\sum_{i+j=2} {}_i\mathcal{L}[{}_jN] = Q_1 {}_2N + \tfrac{1}{2}Q_2 {}_1N^2, \tag{5.4c}$$

$$\sum_{i+j=3} {}_i\mathcal{L}[{}_jN] = Q_1 {}_3N + Q_2 {}_1N {}_2N + \tfrac{1}{6}Q_3 {}_1N^3, \tag{5.4d}$$

which correspond to zeroth, first, second and third order, respectively.

The total time derivative operator $\mathcal{L} \equiv \mathrm{d}/\mathrm{d}t$ is expanded in orders of slopes of modulating waves as

$${}_0\mathcal{L} \equiv \frac{\partial}{\partial t}, \tag{5.5a}$$

$${}_n\mathcal{L} = {}_n\dot{\boldsymbol{x}} \cdot \frac{\partial}{\partial \boldsymbol{x}} + {}_n\dot{\boldsymbol{k}}_{\mathrm{s}} \cdot \frac{\partial}{\partial \boldsymbol{k}_{\mathrm{s}}}, \tag{5.5b}$$

where ${}_n\dot{\boldsymbol{x}}$ and ${}_n\dot{\boldsymbol{k}}_{\mathrm{s}}$ refer to the nth order of the corresponding ray equations (2.2). The expanded ray equations are

$${}_n\dot{\boldsymbol{x}} = \frac{\partial_n \Omega}{\partial \boldsymbol{k}_{\mathrm{s}}} = \frac{\partial_n \tilde{\omega}}{\partial \boldsymbol{k}_{\mathrm{s}}} + {}_n\boldsymbol{U}_{\mathrm{L}}(\boldsymbol{x}, t), \tag{5.6a}$$

$${}_n\dot{\boldsymbol{k}}_{\mathrm{s}} = -\frac{\partial_n \Omega}{\partial \boldsymbol{x}} = -\frac{\partial_n \tilde{\omega}}{\partial \boldsymbol{x}} - \frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{k}_{\mathrm{s}} \cdot {}_n\boldsymbol{U}_{\mathrm{L}}), \tag{5.6b}$$

where the orders of the expansion are applied by distribution on the effective angular frequency and the horizontal orbital velocity.

6. Modal representation

Instead of using the notation of Alpers & Hasselmann (1978), which is not amenable for higher-order expansions, we use the modal representation originally developed by Hasselmann (1961) and adopted by Elfouhaily *et al.* (2000). The surface elevations and velocity potentials of the modulating waves are sums of time-dependent harmonics at the considered order given by

$$ {}_n\zeta(\boldsymbol{x},t) = \sum_{\boldsymbol{k}} {}_nZ_{\boldsymbol{k}}(t)\exp\{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}\}, \tag{6.1 a} $$

$$ {}_n\tilde{\phi}(\boldsymbol{x},t) = \sum_{\boldsymbol{k}} {}_n\Phi_{\boldsymbol{k}}(t)\exp\{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}\}. \tag{6.1 b} $$

This discrete sum can be written in terms of integrals if the spectrum of waves is continuous. In practice, the discrete sum is more appropriate. The first linear order according to the notation of Hasselmann (1961) is

$$ {}_1\Phi_{\boldsymbol{k}}(t) = {}_1\Phi^-_{\boldsymbol{k}}\exp\{\mathrm{i}\omega_{\boldsymbol{k}}t\} + {}_1\Phi^+_{\boldsymbol{k}}\exp\{-\mathrm{i}\omega_{\boldsymbol{k}}t\}, \tag{6.2 a} $$

$$ {}_1Z_{\boldsymbol{k}}(t) = {}_1Z^-_{\boldsymbol{k}}\exp\{\mathrm{i}\omega_{\boldsymbol{k}}t\} + {}_1Z^+_{\boldsymbol{k}}\exp\{-\mathrm{i}\omega_{\boldsymbol{k}}t\}, \tag{6.2 b} $$

with ${}_1\Phi^{\pm}_{\boldsymbol{k}}$ and ${}_1Z^{\pm}_{\boldsymbol{k}}$ related by the equation

$$ \omega_1 Z^s_{\boldsymbol{k}} = \mathrm{i}\kappa_1\Phi^s_{\boldsymbol{k}}, \tag{6.3} $$

where $\omega = s\omega_{\boldsymbol{k}}$ and

$$ \omega^2_{\boldsymbol{k}} = g\kappa, \quad \kappa = k\tanh kh \quad \text{and} \quad s = \pm 1 \tag{6.4} $$

reflect the dispersion relation for gravity waves in a fluid of finite depth (h).

(*a*) *Orbital velocity*

The horizontal component of the orbital velocity is, to second order,

$$ \boldsymbol{U}_{\mathrm{L}}(\boldsymbol{x},t) = {}_1\boldsymbol{U}_{\mathrm{L}}(\boldsymbol{x},t) + {}_2\boldsymbol{U}_{\mathrm{L}}(\boldsymbol{x},t) + \cdots, \tag{6.5 a} $$

$$ {}_1\boldsymbol{U}_{\mathrm{L}}(\boldsymbol{x},t) = \sum_{\boldsymbol{k}_1,s_1} \boldsymbol{k}_1 U^{s_1}_{\boldsymbol{k}_1}\,{}_1Z^{s_1}_{\boldsymbol{k}}\exp\{\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{x} - \omega_1)t\}, \tag{6.5 b} $$

$$ {}_2\boldsymbol{U}_{\mathrm{L}}(\boldsymbol{x},t) = \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}^{s_1,s_2} (\boldsymbol{k}_1+\boldsymbol{k}_2)U^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2}\,{}_1Z^{s_1}_{\boldsymbol{k}_1}\,{}_1Z^{s_2}_{\boldsymbol{k}_2}\exp\{\mathrm{i}[(\boldsymbol{k}_1+\boldsymbol{k}_2)\cdot\boldsymbol{x} - (\omega_1+\omega_2)t]\}, \tag{6.5 c} $$

where expressions for the *straining kernels* are

$$ U^{s_1}_{\boldsymbol{k}_1} = \frac{\omega_1}{\kappa_1}, \tag{6.6 a} $$

$$ U^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2} = -\mathrm{i}\frac{\omega_1\omega_2}{\kappa_1\kappa_2}\frac{\tilde{D}^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2}}{\omega^2(\boldsymbol{k}) - (\omega_1+\omega_2)^2}, \tag{6.6 b} $$

with $\tilde{D}^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2}$ given by Elfouhaily *et al.* (2000) as the generalization of the collision operator of Hasselmann (1961) to the Hamiltonian approach by Zakharov (1968). This collision operator

$$ \tilde{D}^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2} = \mathrm{i}(\omega_1+\omega_2)(\kappa_1\kappa_2 - \boldsymbol{k}_1\cdot\boldsymbol{k}_2) - \tfrac{1}{2}\mathrm{i}[\omega_1(k_2^2 - \kappa\kappa_2 + \kappa_1\kappa_2) + \omega_2(k_1^2 - \kappa\kappa_1 + \kappa_1\kappa_2)] \tag{6.7} $$

for deep water and gravity waves expresses the wave–wave interactions that take place among the modulating waves themselves. We demonstrate that this degree of nonlinearity is one of the major components of the higher-order hydrodynamic theory as developed in this study.

According to Elfouhaily *et al.* (2000), the vertical component of the orbital velocity is, to second order,

$$W = {}_1W + {}_2W + \cdots, \tag{6.8 a}$$

$${}_1W = \hat{O}_1\tilde{\phi}, \tag{6.8 b}$$

$${}_2W = \hat{O}_2\tilde{\phi} + ({}_1\tilde{\zeta}\hat{O}^2 - \hat{O}_1\tilde{\zeta}\hat{O})_1\tilde{\phi}, \tag{6.8 c}$$

where $\hat{O}$ is an operator that multiplies the harmonics of its arguments according to the rule

$$\hat{O}^n \mapsto \begin{cases} k^n & \text{if } n \text{ is even,} \\ k^{n-1}\kappa & \text{if } n \text{ is odd.} \end{cases} \tag{6.9}$$

Note that both horizontal and vertical components of the orbital velocity are evaluated at the actual modulated surface. This feature is an important difference from the perturbation expansion about a mean flat surface performed by Hasselmann (1961). Further details concerning these methods may be found in Elfouhaily *et al.* (2000).

(*b*) *Local acceleration*

Following Longuet-Higgins (1987*a*), the local acceleration is the first time derivative of the vertical orbital velocity as shown in equation (2.4). Equations (2.9) and (6.8) give the expansion up to second order of the dispersion relationship as

$$\tilde{\omega}(\boldsymbol{k}_\mathrm{s}, \boldsymbol{x}, t) = \omega_\mathrm{s}\sqrt{1 - \frac{k_\mathrm{s}}{\omega_\mathrm{s}^2}a_z + \cdots} = {}_0\tilde{\omega} + {}_1\tilde{\omega} + {}_2\tilde{\omega} + \cdots. \tag{6.10}$$

Each order is now equal, in terms of the modal representation, to

$${}_0\tilde{\omega} = \omega_\mathrm{s}, \tag{6.11 a}$$

$${}_1\tilde{\omega} = \frac{1}{2c_\mathrm{s}}\sum_{\boldsymbol{k}_1, s_1} G^{s_1}_{\boldsymbol{k}_1}\, {}_1Z^{s_1}_{\boldsymbol{k}} \exp\{\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{x} - \omega_1 t)\}, \tag{6.11 b}$$

$${}_2\tilde{\omega} = \frac{1}{2c_\mathrm{s}}\sum^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left(G^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2} - \frac{1}{4c_\mathrm{s}w_\mathrm{s}}G^{s_1}_{\boldsymbol{k}_1}G^{s_2}_{\boldsymbol{k}_2}\right){}_1Z^{s_1}_{\boldsymbol{k}_1}\,{}_1Z^{s_2}_{\boldsymbol{k}_2} \times \exp\{\mathrm{i}[(\boldsymbol{k}_1 + \boldsymbol{k}_2)\cdot\boldsymbol{x} - (\omega_1 + \omega_2)t]\}, \tag{6.11 c}$$

where the *heaving kernels* related to the effects of local vertical acceleration are

$$G^{s_1}_{\boldsymbol{k}_1} = \omega_1^2 = \omega^2(\boldsymbol{k}_1), \tag{6.12 a}$$

$$G^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2} = (\omega_1 + \omega_2)\left[\kappa U^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2} + \frac{\omega_2}{2\kappa_2}(k_2^2 - \kappa_1\kappa_2) + \frac{\omega_1}{2\kappa_1}(k_1^2 - \kappa_2\kappa_1)\right] + \frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{2\kappa_1\kappa_2}\omega_1\omega_2(\kappa_1 + \kappa_2). \tag{6.12 b}$$

Linear as well as nonlinear hydrodynamic modulation will be written in terms of the orbital velocity and local acceleration kernels. This will simplify the identification of each term in the final expression.

7. First-order hydrodynamic modulation

The first-order hydrodynamic transfer function (or linear MTF) is defined as the kernel that multiplies the linear harmonics of the surface elevation of an ambient current to produce a relative modulation of the wave action. The perturbed wave action is then expressed as a function of the equilibrium action (${}_0N$ or N_0) and a sum over the hydrodynamic MTF. To first order, this gives

$$ {}_1N = {}_0N \sum_{\boldsymbol{k}_1, s_1} R^{s_1}_{\boldsymbol{k}_1} {}_1Z^{s_1}_{\boldsymbol{k}} \exp\{\mathrm{i}(\boldsymbol{k}_1 \cdot \boldsymbol{x} - \omega_1 t)\}, \tag{7.1} $$

where the sum over s_1 (plus and minus signs) guarantees that modulated action is a real quantity.

From equations (5.4 *b*) and (7.1), we find

$$ -\mathrm{i}\omega_1 R^{s_1}_{\boldsymbol{k}_1} {}_0N + {}_1\dot{\boldsymbol{k}}_\mathrm{s} \cdot \frac{\partial {}_0N}{\partial \boldsymbol{k}_\mathrm{s}} = Q_1 R^{s_1}_{\boldsymbol{k}_1} {}_0N \tag{7.2} $$

as a relation between modes. Some reordering in equation (7.2) gives

$$ (\omega_1 - \mathrm{i}Q_1) R^{s_1}_{\boldsymbol{k}_1} = -\mathrm{i}\frac{{}_1\dot{\boldsymbol{k}}_\mathrm{s}}{{}_0N} \cdot \frac{\partial {}_0N}{\partial \boldsymbol{k}_\mathrm{s}}. \tag{7.3} $$

Before solving equation (7.3) for the linear MTF we need to evaluate the ray equations to first order. By combining equations (5.6), (6.5 *b*) and (6.11 *b*), one gets

$$ {}_1\dot{\boldsymbol{x}} = \frac{1-\gamma_\mathrm{s}}{2\omega_\mathrm{s} k_\mathrm{s}} G^{s_1}_{\boldsymbol{k}_1} \boldsymbol{k}_\mathrm{s} + U^{s_1}_{\boldsymbol{k}_1} \boldsymbol{k}_1, \tag{7.4 a} $$

$$ {}_1\dot{\boldsymbol{k}}_\mathrm{s} = -\mathrm{i}\left[\frac{1}{2c_\mathrm{s}} G^{s_1}_{\boldsymbol{k}_1} + (\boldsymbol{k}_1 \cdot \boldsymbol{k}_\mathrm{s}) U^{s_1}_{\boldsymbol{k}_1}\right] \boldsymbol{k}_1 \tag{7.4 b} $$

as linear modes of the ray equations. γ_s is the ratio between the group speed V_s and the phase speed c_s of short modulated waves.

The final expression for the linear MTF is obtained by substituting (7.4 *b*) into equation (7.3) to give

$$ R^{s_1}_{\boldsymbol{k}_1} = -\frac{\omega_1 + \mathrm{i}Q_1}{\omega_1^2 + Q_1^2}\left[\frac{1}{2c_\mathrm{s}} G^{s_1}_{\boldsymbol{k}_1} + (\boldsymbol{k}_1 \cdot \boldsymbol{k}_\mathrm{s}) U^{s_1}_{\boldsymbol{k}_1}\right] \frac{\boldsymbol{k}_1}{{}_0N} \cdot \frac{\partial {}_0N}{\partial \boldsymbol{k}_\mathrm{s}}, \tag{7.5} $$

where three terms are readily identifiable: the first derivative Q_1 of the source functional; the *heaving kernel* induced by local acceleration; and the *straining kernel* caused by the horizontal component of the orbital velocity. Equation (7.5) is a new and powerful result. It is a generalization of the linear MTFs by Alpers & Hasselmann (1978), Valenzuela & Wright (1979) and Keller & Wright (1975) that, for possibly the first time, includes local acceleration and finite depth effects. Of course, when local acceleration and finite depth effects are neglected, one retrieves from equation (7.5) an expression identical to the original MTF by Alpers & Hasselmann (1978).

It is clear from equation (7.5) that the phase of the modulation is dictated entirely by the ratio of Q_1 to ω_1. This means that in the linear MTF, neither the local acceleration (heaving kernel) nor the orbital velocity (straining kernel) influence the phase of the modulation. Another important point is that the phase of the modulation of this linear MTF is independent of the relative propagation direction between modulated and modulating waves. We demonstrate in the following section that higher-order hydrodynamic theory does provide phase sensitivity to the relative directions between long and short waves.

8. Second-order hydrodynamic modulation

The second-order perturbation of the wave action caused by the modulating waves is defined as a deviation from the equilibrium wave action (N_0 or ${}_0N$) according to

$$ {}_2N = {}_0N \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}^{s_1,s_2} R^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2} {}_1Z^{s_1}_{\boldsymbol{k}_1} {}_1Z^{s_2}_{\boldsymbol{k}_2} \exp\{\mathrm{i}[(\boldsymbol{k}_1+\boldsymbol{k}_2)\cdot\boldsymbol{x} - (\omega_1+\omega_2)t]\}, \tag{8.1} $$

where we denote $R^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2}$ as the second-order hydrodynamic MTF.

From equations (5.4 *c*) and (8.1), we get, with some reordering,

$$ \begin{aligned} &[(\omega_1+\omega_2) - \mathrm{i}Q_1] R^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2} \\ &\quad = ({}_1\dot{\boldsymbol{x}}\cdot\boldsymbol{k}_2) R^{s_2}_{\boldsymbol{k}_2} - \mathrm{i}\frac{{}_1\dot{\boldsymbol{k}}_{\mathrm{s}}}{{}_0N}\cdot\frac{\partial R^{s_2}_{\boldsymbol{k}_2}{}_0N}{\partial\boldsymbol{k}_{\mathrm{s}}} - \mathrm{i}\frac{{}_2\dot{\boldsymbol{k}}_{\mathrm{s}}}{{}_0N}\cdot\frac{\partial{}_0N}{\partial\boldsymbol{k}_{\mathrm{s}}} + \tfrac{1}{2}\mathrm{i}Q_2 R^{s_1}_{\boldsymbol{k}_1} R^{s_2}_{\boldsymbol{k}_2}{}_0N \end{aligned} \tag{8.2} $$

as a relation between modes at the second order in long-wave slopes.

The second-order ray equations can be obtained by combining equations (5.6), (6.5 *c*) and (6.11 *c*) to get

$$ {}_2\dot{\boldsymbol{x}} = \frac{1-\gamma_{\mathrm{s}}}{2\omega_{\mathrm{s}}k_{\mathrm{s}}}\left(G^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2} - \frac{1}{4c_{\mathrm{s}}w_{\mathrm{s}}}\frac{2-3\gamma_{\mathrm{s}}}{1-\gamma_{\mathrm{s}}} G^{s_1}_{\boldsymbol{k}_1} G^{s_2}_{\boldsymbol{k}_2}\right)\boldsymbol{k}_{\mathrm{s}} + U^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2}\boldsymbol{k}, \tag{8.3 a} $$

$$ {}_2\dot{\boldsymbol{k}}_{\mathrm{s}} = -\mathrm{i}\left[\frac{1}{2c_{\mathrm{s}}}\left(G^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2} - \frac{1}{4c_{\mathrm{s}}w_{\mathrm{s}}} G^{s_1}_{\boldsymbol{k}_1} G^{s_2}_{\boldsymbol{k}_2}\right) + (\boldsymbol{k}_{\mathrm{s}}\cdot\boldsymbol{k}) U^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2}\right]\boldsymbol{k}. \tag{8.3 b} $$

The substitution of equation (8.3) into equation (8.2) yields the final expression for the second-order hydrodynamic modulation:

$$ \begin{aligned} R^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2} = \frac{(\omega_1+\omega_2)+\mathrm{i}Q_1}{(\omega_1+\omega_2)^2+Q_1^2} \Bigg\{ &\left[\frac{1-\gamma_{\mathrm{s}}}{2\omega_{\mathrm{s}}k_{\mathrm{s}}}(\boldsymbol{k}_{\mathrm{s}}\cdot\boldsymbol{k}_2) G^{s_1}_{\boldsymbol{k}_1} + (\boldsymbol{k}_1\cdot\boldsymbol{k}_2) U^{s_1}_{\boldsymbol{k}_1}\right] R^{s_2}_{\boldsymbol{k}_2} \\ &- \left[\frac{1}{2c_{\mathrm{s}}} G^{s_1}_{\boldsymbol{k}_1} + (\boldsymbol{k}_1\cdot\boldsymbol{k}_{\mathrm{s}}) U^{s_1}_{\boldsymbol{k}_1}\right]\frac{\boldsymbol{k}_1}{{}_0N}\cdot\frac{\partial R^{s_2}_{\boldsymbol{k}_2}{}_0N}{\partial\boldsymbol{k}_{\mathrm{s}}} \\ &- \left[\frac{1}{2c_{\mathrm{s}}}\left(G^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2} - \frac{1}{4c_{\mathrm{s}}w_{\mathrm{s}}} G^{s_1}_{\boldsymbol{k}_1} G^{s_2}_{\boldsymbol{k}_2}\right) + (\boldsymbol{k}_{\mathrm{s}}\cdot\boldsymbol{k}_1 + \boldsymbol{k}_{\mathrm{s}}\cdot\boldsymbol{k}_2) U^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2}\right] \\ &\quad \times \frac{\boldsymbol{k}_1+\boldsymbol{k}_2}{{}_0N}\cdot\frac{\partial{}_0N}{\partial\boldsymbol{k}_{\mathrm{s}}} + \tfrac{1}{2}\mathrm{i}Q_2 R^{s_1}_{\boldsymbol{k}_1} R^{s_2}_{\boldsymbol{k}_2}{}_0N \Bigg\}. \end{aligned} \tag{8.4} $$

In addition to heaving and straining kernels, the second hydrodynamic MTF involves the second derivative Q_2 of the source functional Q in equation (2.1). In general, the source functional is *non-local* and includes wave–wave interactions. Formally, its second-order derivative Q_2 should come from the derivative of the three-wave interaction functional as presented by Valenzuela & Laing (1972). One should of course replace the collision operator used in Valenzuela & Laing (1972) by the one derived in Elfouhaily *et al.* (2000) for the Hamiltonian formulation of the hydrodynamic equations expressed at the actual surface instead of at a flat reference. Less formally, though, the second-order derivative can be approximated by a *local* expression of the three short-wave interactions as proposed by Plant (1979), Valenzuela & Laing (1972), Zhang (1995) and Kudryavtsev *et al.* (1999). For instance, Q_2 can be computed from a simple source term, such as that of Hughes (1978). Equation (8.4) is also a generalization of Valenzuela & Wright (1979) to include the nonlinearities in the modulating waves in addition to the ones present among short modulated waves.

It is obvious from equation (8.4) that the phase is no longer a function of the ratio between Q_1 and $(\omega_1 + \omega_2)$. All the terms between curly brackets in equation (8.4) are complex numbers and therefore contribute to the phase of the modulation. The relative direction of long waves versus short waves is now involved in the determination of the phase of the modulation. The second-order hydrodynamic MTF in equation (8.4) is written in a compact form permitting heaving and straining kernels of first and second orders to be turned on or off separately. In this way, one can assess the real contribution of each term; whether it is coming from wave–wave interactions among long waves or short waves or even control of the hydrodynamic modulation by local acceleration or orbital velocity separately. For completeness, the appendix shows how the third-order hydrodynamic modulation can be derived via these previous expansions.

9. Implementation

The implementation of the first- and second-order hydrodynamic modulation, given by equations (7.5) and (8.4), respectively, is straightforward for a *particular* source functional based on the work of Hughes (1978) and modified by Caponi *et al.* (1988). For this particular source functional in equation (3.1), its derivatives in equation (5.2) become

$$Q_1 = -\mu_{\mathrm{s}}, \tag{9.1 a}$$

$$Q_2 = -(\rho + 1)\mu_{\mathrm{s}0} N^{-1}, \tag{9.1 b}$$

$$Q_3 = -(\rho + 1)(\rho - 1)\mu_{\mathrm{s}0} N^{-2} \tag{9.1 c}$$

up to the third order. Only the first two orders are used in this section. The third order is used in the appendix for implementing a simplified version of the third-order hydrodynamic MTF.

With these derivatives of the source functional, the first- and second-order hydrodynamic MTFs become

$$R^{s_1}_{\boldsymbol{k}_1} = -\frac{\omega_1 - \mathrm{i}\mu_{\mathrm{s}}}{\omega_1^2 + \mu_{\mathrm{s}}^2} \frac{\omega_1}{\kappa_1} (\boldsymbol{k}_1 \cdot \boldsymbol{k}_{\mathrm{s}}) \frac{\boldsymbol{k}_1}{{}_0N} \cdot \frac{\partial_0 N}{\partial \boldsymbol{k}_{\mathrm{s}}} \tag{9.2}$$

and

$$R^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2} = \frac{(\omega_1+\omega_2) - \mathrm{i}\mu_\mathrm{s}}{(\omega_1+\omega_2)^2+\mu_\mathrm{s}^2} \times \left\{ \frac{\omega_1}{\kappa_1}(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)R^{s_2}_{\boldsymbol{k}_2} - \frac{\omega_1}{\kappa_1}(\boldsymbol{k}_1\cdot\boldsymbol{k}_\mathrm{s})\boldsymbol{k}_1\cdot\left[\frac{\partial R^{s_2}_{\boldsymbol{k}_2}}{\partial \boldsymbol{k}_\mathrm{s}} + \frac{R^{s_2}_{\boldsymbol{k}_2}}{{}_0N}\frac{\partial_0 N}{\partial \boldsymbol{k}_\mathrm{s}}\right] - \tfrac{1}{2}\mathrm{i}(\rho+1)\mu_\mathrm{s}R^{s_1}_{\boldsymbol{k}_1}R^{s_2}_{\boldsymbol{k}_2}\right\}, \quad (9.3)$$

respectively. The heaving kernels due to local acceleration are not included in equations (9.2) and (9.3) for simplicity. A similar development can be made easily even with heaving kernels. These expressions for the hydrodynamic MTF can be simplified even further by assuming a given form for the directional wave spectrum and the wind growth rate. For instance, if we choose to combine the unified spectrum from Elfouhaily *et al.* (1997) with the growth rate β proposed by Plant (1982) and some directional dependence, which will not be needed in the following developments, the linear MTF simply becomes

$$R^{s_1}_{\boldsymbol{k}_1} = -\frac{\omega_1 - \mathrm{i}\mu_\mathrm{s}}{\omega_1^2+\mu_\mathrm{s}^2}k_1\omega_1\cos(\phi_\mathrm{s}-\phi_1)[-4\cos(\phi_\mathrm{s}-\phi_1) - s\tan\tfrac{1}{2}\phi_\mathrm{s}\sin(\phi_\mathrm{s}-\phi_1)], \quad (9.4)$$

where

$$s = \frac{2}{\ln 2}\operatorname{arctanh}[\Delta(k_\mathrm{s})] + 1, \quad (9.5)$$

with the spreading of short waves captured by $\Delta(k_\mathrm{s})$, as introduced by Elfouhaily *et al.* (1997). The direction ϕ_s in equation (9.4) is that of short waves relative to the direction of the wind. A special interesting case occurs when ϕ_s is identically zero. In this special case the linear MTF simplifies to

$$R^{s_1}_{\boldsymbol{k}_1} = 4\frac{\omega_1 - \mathrm{i}\mu_{\mathrm{s}0}}{\omega_1^2+\mu_{\mathrm{s}0}^2}k_1\omega_1\cos^2\phi_1, \quad (9.6)$$

where $\mu_{\mathrm{s}0}$ is the relaxation rate μ_s of short waves aligned to the wind. The direction of the modulating wave ϕ_1 modifies only the magnitude of the modulation. The phase in this special case is solely dependent on the wavelength of short waves and long waves for the given wind speed.

When short waves and wind direction are aligned, the second-order MTF reads

$$R^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2} = \frac{2\omega_1\omega_2k_1k_2}{\omega_1+\omega_2+\mathrm{i}\mu_{\mathrm{s}0}} \times \left\{\frac{\cos^2\phi_2}{\omega_2+\mathrm{i}\mu_{\mathrm{s}0}}\left[\frac{k_2}{k_1}\cos(\phi_1-\phi_2) + 2(1-\gamma_\mathrm{s})\frac{\mathrm{i}\mu_{\mathrm{s}0}}{\omega_1+\mathrm{i}\mu_{\mathrm{s}0}}\cos^2\phi_1 - \tan\phi_2\sin 2\phi_1 + 4\cos^2\phi_1 - 2\mathrm{i}(\rho+1)\mu_{\mathrm{s}0}\frac{\cos^2\phi_1}{\omega_1+\mathrm{i}\mu_{\mathrm{s}0}}\right] + (1\rightleftharpoons 2)\right\}. \quad (9.7)$$

This is an example of implementation when nonlinear kernels are neglected. In the next section we show a better example that includes the nonlinearities in the modulating current.

10. Illustrative example of the Stokes wave

We illustrate the implication of second-order hydrodynamic modulation through this simple example of a short wave propagating along a nonlinear long wave ($\phi_s = \phi_1 = 0$). The elevation, velocities and acceleration involved in the modulation are

$$\eta(x,t) = a\cos(k_L x - w_L t) + \tfrac{1}{2}a^2 k_L \cos 2(k_L x - w_L t), \tag{10.1a}$$

$$U_L(x,t) = c_L a k_L \cos(k_L x - w_L t) + \tfrac{1}{2}c_L (a k_L)^2 \cos 2(k_L x - w_L t), \tag{10.1b}$$

$$W_L(x,t) = c_L a k_L \sin(k_L x - w_L t) + \tfrac{1}{2}c_L (a k_L)^2 \sin 2(k_L x - w_L t), \tag{10.1c}$$

$$G_L(x,t) = -c_L a k_L \omega_L \cos(k_L x - w_L t) - c_L \omega_L (a k_L)^2 \cos 2(k_L x - w_L t), \tag{10.1d}$$

where, as a reminder, the perturbation is performed over the surface itself with the steepness (ak) as the smallness parameter. To second order, this example coincides with the well-known Stokes waves of permanent form (Debnath 1994).

The relative modulation of the spectral action is now

$$\frac{\delta N(\boldsymbol{k}_s, \boldsymbol{x}, t)}{N_0(\boldsymbol{k}_s)} = A_1 a k_L \cos[(k_L x - w_L t) + \phi_1] + A_2 (a k_L)^2 \cos[2(k_L x - w_L t) + \phi_2], \tag{10.2}$$

where (A_1, ϕ_1) and (A_2, ϕ_2) are amplitude and phase of the first- and second-order normalized hydrodynamic modulations, respectively. For completeness, we summarize the expressions for these two hydrodynamic modulations in terms of the non-dimensional quantities here:

$$M_1 = A_1 \exp\{i\phi_1\} = 4\frac{\omega_L}{\omega_L + i\mu_s}, \tag{10.3a}$$

$$\begin{aligned} M_2 &= A_2 \exp\{i\phi_2\} \\ &= 4\frac{2\omega_L}{2\omega_L + i\mu_s}\left\{1 + \left[\frac{i(\rho+1)\mu_s}{\omega_L + i\mu_s} + (1 - \tfrac{1}{2}\gamma_s)\left(\frac{\mu_s^2}{\omega_L^2 + \mu_s^2} + \frac{i\mu_s}{\omega_L - \mu_s}\right)\right]\frac{\omega_L}{\omega_L + i\mu_s}\right\}. \end{aligned} \tag{10.3b}$$

The hydrodynamic transfer function is respectively divided by the modulating wavenumber to the power of the order considered.

Figure 3 shows the normalized amplitudes of first-order (solid curve) and second-order (dot-dashed curve) MTFs (10.3) as a function of wavenumber of the modulated short wave. The phases in equation (10.3) are depicted in figure 4. Figure 5 compares the hydrodynamic modulation from the exact numerical simulation (3.7) to the sum of first- and second-order MTFs (10.2) relative to the nonlinear modulating current of the Stokes wave in terms of its surface elevation. Aside from the vertical shift in the modulation, the higher-order MTF is now closer to the exact modulation. In addition, the nonlinear behaviour (sharper crests and flatter troughs) seems to be captured, which demonstrates the need for higher-order hydrodynamic MTFs. Adding the second-order hydrodynamic MTF causes, in our example, the peak of the modulation to move closer to the crest of the long waves.

In the following sections, we investigate some ensemble-average quantities of the nonlinear properties induced by the second-order hydrodynamic modulations and especially the vertical offset between exact and MTF simulations.

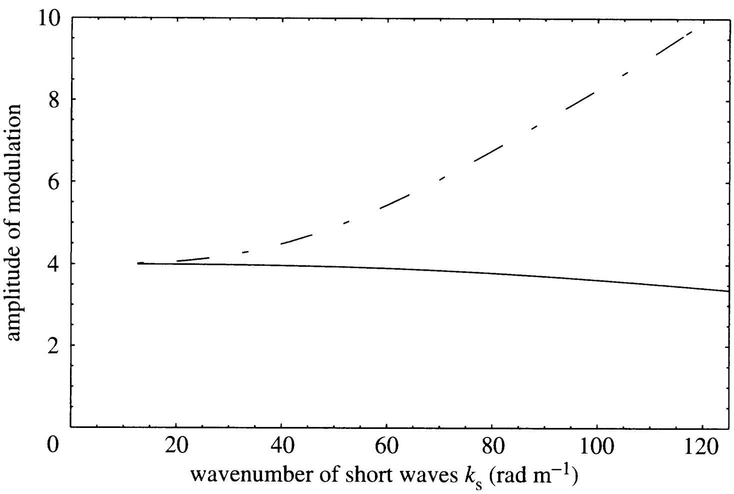

Figure 3. Comparison between the normalized amplitudes of the first-order (solid curve) and second-order (dash-dotted curve) MTFs for various short waves. The wavelength of the modulating wave is 31 m. The wind friction velocity used in the growth rate is $u^* = 0.23$ m s^{-1}. The second-order modulation has a non-negligible amplitude and increases from long to short waves.

11. Offset due to second-order hydrodynamics

The ensemble average of the modulated wave action is zero under the linear hydrodynamic approximation. However, higher-order modulation may not vanish under ensemble averaging thus resulting in some bias between the equilibrium spectrum and the ensemble-averaged spectrum deduced from the modulated wave action. The total normalized modulation is defined, up to second order, as

$$\begin{aligned}\frac{\delta\Psi(\boldsymbol{k}_{\mathrm{s}},\boldsymbol{x},t)}{\Psi_0(\boldsymbol{k}_{\mathrm{s}})} &= \frac{\delta N(\boldsymbol{k}_{\mathrm{s}},\boldsymbol{x},t)}{N_0(\boldsymbol{k}_{\mathrm{s}})} \\ &= \sum_{\boldsymbol{k}_1,s_1} R^{s_1}_{\boldsymbol{k}_1}\,{}_1Z^{s_1}_{\boldsymbol{k}_1}\exp\{\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{x}-\omega_1 t)\} \\ &\quad + \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}^{s_1,s_2} R^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2}\,{}_1Z^{s_1}_{\boldsymbol{k}_1}\,{}_1Z^{s_2}_{\boldsymbol{k}_2}\exp\{\mathrm{i}[(\boldsymbol{k}_1+\boldsymbol{k}_2)\cdot\boldsymbol{x}-(\omega_1+\omega_2)t]\}. \qquad (11.1)\end{aligned}$$

As a check, let us find its ensemble average,

$$\epsilon = \left\langle\frac{\delta\Psi(\boldsymbol{k}_{\mathrm{s}},\boldsymbol{x},t)}{\Psi_0(\boldsymbol{k}_{\mathrm{s}})}\right\rangle = 0 + \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}^{s_1,s_2}\langle R^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2}\,{}_1Z^{s_1}_{\boldsymbol{k}_1}\,{}_1Z^{s_2}_{\boldsymbol{k}_2}\rangle\exp\{\mathrm{i}[(\boldsymbol{k}_1+\boldsymbol{k}_2)\cdot\boldsymbol{x}-(\omega_1+\omega_2)t]\}, \qquad (11.2)$$

where the first term vanishes because the complex variables ${}_1Z^{s_1}_{\boldsymbol{k}_1}$ are assumed to be zero-mean independent Gaussian random variables.

Owing to the following property of the cross-correlation of Gaussian processes,

$$\langle R^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}2}\,{}_1Z^{s_1}_{\boldsymbol{k}_1}\,{}_1Z^{s_2}_{\boldsymbol{k}_2}\rangle = \tfrac{1}{2}R^{s_1,s_2}_{s_1\boldsymbol{k}_1,s_2\boldsymbol{k}2}\Psi_0(s_1\boldsymbol{k}_1)\delta(s_1\boldsymbol{k}_1+s_2\boldsymbol{k}_2), \qquad (11.3)$$

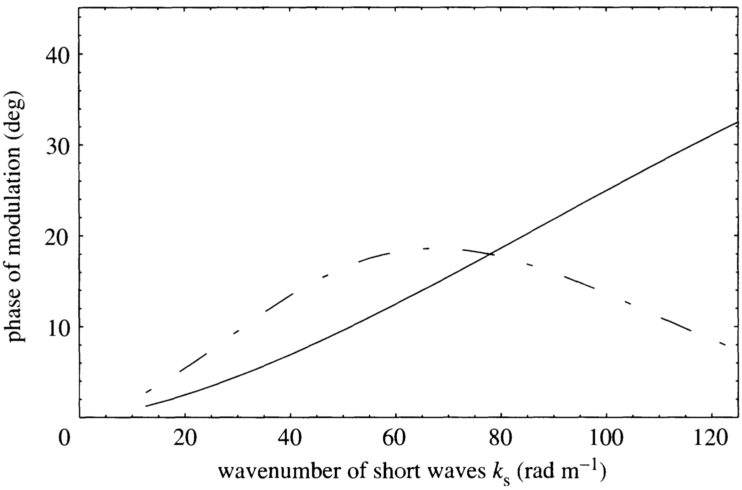

Figure 4. Same as previous figure but now for the phases of the normalized hydrodynamic modulations. The phase of the second-order modulation is significant and features a bend around 10 or 9 cm.

equation (11.2) simply becomes

$$\epsilon = \tfrac{1}{2} \sum_{\boldsymbol{k}_\mathrm{L}} R^{+,-}_{\boldsymbol{k}_\mathrm{L},-\boldsymbol{k}_\mathrm{L}} \Psi_0(\boldsymbol{k}_\mathrm{L}) + R^{-,+}_{-\boldsymbol{k}_\mathrm{L},\boldsymbol{k}_\mathrm{L}} \Psi_0(-\boldsymbol{k}_\mathrm{L}), \tag{11.4}$$

with $R^{+,-}_{\boldsymbol{k}_\mathrm{L},-\boldsymbol{k}_\mathrm{L}}$ derived from equation (9.7) to give

$$R^{+,-}_{\boldsymbol{k}_\mathrm{L},-\boldsymbol{k}_\mathrm{L}} = -4 \frac{\omega_{\boldsymbol{k}_\mathrm{L}}{}^2 \boldsymbol{k}_\mathrm{L}^2}{\omega_{\boldsymbol{k}_\mathrm{L}}{}^2 + \mu_\mathrm{s}^2} [1 + 2(\gamma_\mathrm{s} + \rho - 1)\cos^2 \phi_\mathrm{L}]. \tag{11.5}$$

The offset in equation (11.4) has two interesting limits. When the relaxation rate is much smaller than the frequency of the long waves, the offset is

$$\epsilon_{\mu_\mathrm{s} \ll \omega_{\boldsymbol{k}_\mathrm{L}}}(k_\mathrm{s}) = -4(m_2^y + 2\rho m_2^x), \tag{11.6}$$

independent of the wavelength of the modulated wave. This limit is actually a prediction of the highest possible offset, since the offset in equation (11.4) is generally dependent on the wavelength of short waves. The x and y components of the second-order moment of the spectrum are

$$m_2^x = \int_0^{k_\mathrm{c}} k_x^2 \Psi(\boldsymbol{k})\, \mathrm{d}\boldsymbol{k}, \tag{11.7a}$$

$$m_2^y = \int_0^{k_\mathrm{c}} k_y^2 \Psi(\boldsymbol{k})\, \mathrm{d}\boldsymbol{k}, \tag{11.7b}$$

which denote the orthogonal components of the slope variance of long modulating waves. The wavenumber cut-off k_c is the wavenumber of the shortest modulating wave used in the computation.

Another interesting limit is when the relaxation rate is much greater than the frequency of modulating waves,

$$\epsilon_{\mu_\mathrm{s} \gg \omega_{\boldsymbol{k}_\mathrm{L}}}(k_\mathrm{s}) = -4 \frac{g}{\mu_\mathrm{s}} [(\gamma_\mathrm{s} + \rho) m_3 + \tfrac{1}{2}(\gamma_\mathrm{s} + \rho - 1) m_3'], \tag{11.8}$$

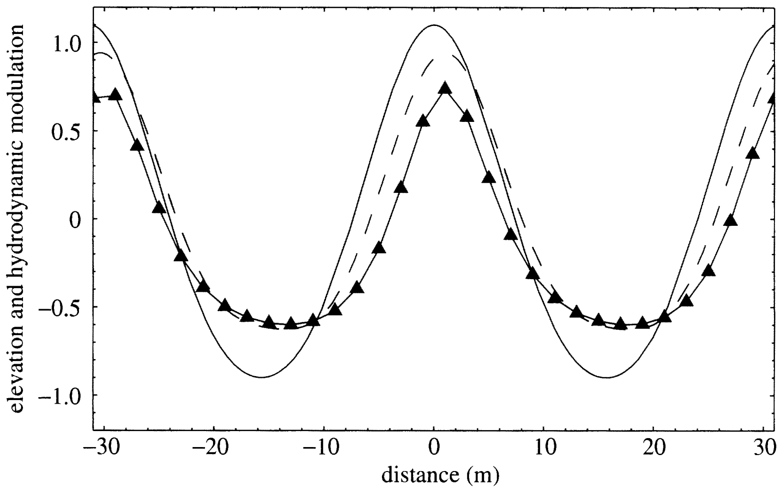

Figure 5. Hydrodynamic modulation from the exact numerical simulation (filled triangles) and from the sum of first- and second-order MTFs (long-dashed curve) as shown relative to the nonlinear modulating current (solid curve) in terms of surface elevation. The modulating current is a nonlinear wave which, at this second-order, coincides with the well-known Stokes wave. The amplitude of the first harmonic is 1 m and the short modulated wave has a wavelength of 10 cm. The higher-order MTF (dashed curve) is now closer to the exact modulation (filled triangles) and exhibits sharper crests and flatter troughs.

where the third-order moments are defined as

$$m_3 = \int_0^{k_c} k^3 S(k)\,\mathrm{d}k, \tag{11.9 a}$$

$$m_3' = \int_0^{k_c} k^3 \Delta(k) S(k)\,\mathrm{d}k, \tag{11.9 b}$$

where $\Delta(k)$ is the delta ratio function introduced by Elfouhaily *et al.* (1997) to describe the spreading of surface waves about the wind direction. $S(k)$ is the omnidirectional spectrum derived from the two-dimensional one in Elfouhaily *et al.* (1997). Equation (11.8) shows the offset $\epsilon(k_s)$ as a function of the wavenumber of short waves.

It is interesting to note that the offset tends to zero for capillary waves:

$$\lim_{k_s \to \infty} \epsilon(k_s) = 0. \tag{11.10}$$

In conclusion, the offset does not, in general, significantly change the ensemble-averaged spectrum from its equilibrium value. However, its most important effect shows up for short gravity waves. This is consistent with the formulation of the equilibrium spectrum by Elfouhaily *et al.* (1997).

12. Induced bispectra and trispectra

Another interesting application of the higher-order hydrodynamic MTF is the generation of higher-order spectra caused by the modulation of short waves by longer ones.

Whether the waves are linear or not before modulation, the hydrodynamic modulation will make the total process highly nonlinear. We can illustrate this point by starting with linear waves for both modulated and modulating waves, for the sake of simplicity. We are interested in estimating the bispectrum and the trispectrum of the resulting process knowing that the waves involved in the modulation were originally linear.

By definition, the bispectrum is the two-dimensional Fourier transform of the *skewness* function defined by

$$\rho_3(\boldsymbol{x}_1, \boldsymbol{x}_2) \triangleq \langle \zeta(\boldsymbol{x})\zeta(\boldsymbol{x}+\boldsymbol{x}_1)\zeta(\boldsymbol{x}+\boldsymbol{x}_2)\rangle. \tag{12.1}$$

In the context of hydrodynamic modulation, the total surface can be separated into two surfaces. The first is formed by a zero-mean fast-varying process representing the short waves. The second is a slow process describing the long-wave action on the short waves. The skewness function in equation (12.1) becomes

$$\begin{aligned}\rho_3(\boldsymbol{x}_1, \boldsymbol{x}_2) = &\langle \zeta_{\mathrm{L}}(\boldsymbol{x})\zeta_{\mathrm{L}}(\boldsymbol{x}+\boldsymbol{x}_1)\zeta_{\mathrm{L}}(\boldsymbol{x}+\boldsymbol{x}_2)\rangle \\ &+ 3\langle \zeta_{\mathrm{s}}(\boldsymbol{x})\zeta_{\mathrm{L}}(\boldsymbol{x}+\boldsymbol{x}_1)\zeta_{\mathrm{L}}(\boldsymbol{x}+\boldsymbol{x}_2)\rangle \\ &+ 3\langle \zeta_{\mathrm{s}}(\boldsymbol{x})\zeta_{\mathrm{s}}(\boldsymbol{x}+\boldsymbol{x}_1)\zeta_{\mathrm{L}}(\boldsymbol{x}+\boldsymbol{x}_2)\rangle \\ &+ \langle \zeta_{\mathrm{s}}(\boldsymbol{x})\zeta_{\mathrm{s}}(\boldsymbol{x}+\boldsymbol{x}_1)\zeta_{\mathrm{s}}(\boldsymbol{x}+\boldsymbol{x}_2)\rangle,\end{aligned} \tag{12.2}$$

in which long waves are separated from short waves. Since the waves are originally linear, the skewness function reduces to

$$\rho_3(\boldsymbol{x}_1, \boldsymbol{x}_2) = 3\langle \zeta_{\mathrm{s}}(\boldsymbol{x})\zeta_{\mathrm{s}}(\boldsymbol{x}+\boldsymbol{x}_1)\zeta_{\mathrm{L}}(\boldsymbol{x}+\boldsymbol{x}_2)\rangle = 3\langle \rho_2(\boldsymbol{x}_1)\zeta_{\mathrm{L}}(\boldsymbol{x}_1+\boldsymbol{x}_2)\rangle, \tag{12.3}$$

where $\rho_2(\boldsymbol{x}_1)$ is the autocorrelation function of modulated short waves. The ensemble averaging can be performed on short waves first as a direct result of our assumptions about the hydrodynamic modulation of short waves by longer ones. The modulated spectrum to second order in hydrodynamic modulation as in equation (11.1) is, by definition, the Fourier transform of the autocorrelation function of modulated short waves $\rho_2(\boldsymbol{x}_1)$ in equation (12.3). Using equation (11.1) in equation (12.3), we get

$$B_{\mathrm{Ls}}(\boldsymbol{k}_1, \boldsymbol{k}_{\mathrm{s}}) = 3\Psi_0(\boldsymbol{k}_{\mathrm{s}})\langle R^{s_1}_{\boldsymbol{k}_1} {}_1 Z^{s_1}_{\boldsymbol{k}_1} {}_1 Z^{s_2}_{\boldsymbol{k}_2} \exp\{\mathrm{i}[(\boldsymbol{k}_1+\boldsymbol{k}_2)\cdot\boldsymbol{x} - (\omega_1+\omega_2)t]\}\rangle. \tag{12.4}$$

The property (11.3) of Gaussian processes combined with equation (12.4) gives

$$B_{\mathrm{Ls}}(\boldsymbol{k}_{\mathrm{L}}, \boldsymbol{k}_{\mathrm{s}}) = \tfrac{3}{2}[R^{+}_{\boldsymbol{k}_{\mathrm{L}}}\Psi_0(\boldsymbol{k}_{\mathrm{L}}) + R^{-}_{-\boldsymbol{k}_{\mathrm{L}}}\Psi_0(-\boldsymbol{k}_{\mathrm{L}})]\Psi_0(\boldsymbol{k}_{\mathrm{s}}), \tag{12.5}$$

which is an expression for the induced bispectrum caused solely by hydrodynamic modulation of short waves by longer ones. This is the first time, to our knowledge, that such an expression for the bispectrum has been derived from hydrodynamic modulations in this manner. Equation (12.5) clearly indicates that the induced bispectrum is a function of the linear hydrodynamic modulation. It already shows that second-order hydrodynamics are not involved in the expression of the induced bispectrum. Our expectation is that the second-order hydrodynamic modulation will generate higher-order spectra, such as the trispectrum.

By definition the trispectrum is the three-dimensional Fourier transform of the *kurtosis* function. The latter function is defined simply as the fourth-order cross-correlation between the surface elevations at four different locations on the surface.

The kurtosis function, under the context of hydrodynamic modulation, is

$$\begin{aligned}\rho_4(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) &= 6\langle\zeta_{\mathrm{s}}(\boldsymbol{x})\zeta_{\mathrm{s}}(\boldsymbol{x}+\boldsymbol{x}_1)\zeta_{\mathrm{L}}(\boldsymbol{x}+\boldsymbol{x}_2)\zeta_{\mathrm{L}}(\boldsymbol{x}+\boldsymbol{x}_3)\rangle \\ &= 6\langle\rho_2(\boldsymbol{x}_1)\zeta_{\mathrm{L}}(\boldsymbol{x}_1+\boldsymbol{x}_2)\zeta_{\mathrm{L}}(\boldsymbol{x}_1+\boldsymbol{x}_3)\rangle, \end{aligned} \tag{12.6}$$

which is inspired from equation (12.3). Similarly, by introducing explicitly the expression for the modulated spectrum to second order in hydrodynamics, the trispectrum is then

$$T_{\mathrm{Ls}}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_{\mathrm{s}}) = 6\Psi_0(\boldsymbol{k}_{\mathrm{s}})\langle R^{s_1,s_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2}\,{}_1Z^{s_1}_{\boldsymbol{k}_1}\,{}_1Z^{s_2}_{\boldsymbol{k}_2}\,{}_1Z^{s_3}_{\boldsymbol{k}_3}\,{}_1Z^{s_4}_{\boldsymbol{k}_4}\exp\{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)\}\rangle, \tag{12.7}$$

where $\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4$ and $\omega=\omega_1+\omega_2+\omega_3+\omega_4$. The angle bracket in equation (12.7) indicates an ensemble average over a product of four independent Gaussian variables. According to eqn (4-40) in Whalen (1971), the fourth-order moment can be related to the second-order moments, and therefore to equation (11.3), as follows

$$\begin{aligned}\langle{}_1Z^{s_1}_{\boldsymbol{k}_1}\,{}_1Z^{s_2}_{\boldsymbol{k}_2}\,{}_1Z^{s_3}_{\boldsymbol{k}_3}\,{}_1Z^{s_4}_{\boldsymbol{k}_4}\rangle &= \langle{}_1Z^{s_1}_{\boldsymbol{k}_1}\,{}_1Z^{s_2}_{\boldsymbol{k}_2}\rangle\langle{}_1Z^{s_3}_{\boldsymbol{k}_3}\,{}_1Z^{s_4}_{\boldsymbol{k}_4}\rangle \\ &\quad+\langle{}_1Z^{s_1}_{\boldsymbol{k}_1}\,{}_1Z^{s_3}_{\boldsymbol{k}_3}\rangle\langle{}_1Z^{s_2}_{\boldsymbol{k}_2}\,{}_1Z^{s_4}_{\boldsymbol{k}_4}\rangle \\ &\quad+\langle{}_1Z^{s_1}_{\boldsymbol{k}_1}\,{}_1Z^{s_4}_{\boldsymbol{k}_4}\rangle\langle{}_1Z^{s_2}_{\boldsymbol{k}_2}\,{}_1Z^{s_3}_{\boldsymbol{k}_3}\rangle.\end{aligned} \tag{12.8}$$

Using equation (11.3) in equation (12.8) we finally get

$$\begin{aligned}T_{\mathrm{Ls}}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_{\mathrm{s}}) = \tfrac{3}{2}\Psi_0(\boldsymbol{k}_{\mathrm{s}})\{&[R^{+,-}_{\boldsymbol{k}_1,-\boldsymbol{k}_2}\Psi_0(\boldsymbol{k}_1)R^{-,+}_{-\boldsymbol{k}_1,\boldsymbol{k}_2}\Psi_0(-\boldsymbol{k}_1)][\Psi_0(\boldsymbol{k}_2)+\Psi_0(-\boldsymbol{k}_2)] \\ &+2\Psi_0(\boldsymbol{k}_1)[R^{+,+}_{\boldsymbol{k}_1,\boldsymbol{k}_2}\Psi_0(\boldsymbol{k}_2)+R^{+,-}_{\boldsymbol{k}_1,-\boldsymbol{k}_2}\Psi_0(-\boldsymbol{k}_2)] \\ &+2\Psi_0(-\boldsymbol{k}_1)[R^{-,+}_{-\boldsymbol{k}_1,\boldsymbol{k}_2}\Psi_0(\boldsymbol{k}_2)+R^{-,-}_{-\boldsymbol{k}_1,-\boldsymbol{k}_2}\Psi_0(-\boldsymbol{k}_2)]\}\end{aligned} \tag{12.9}$$

as an expression for the trispectrum function of products of the equilibrium spectrum and the second-order hydrodynamic modulation. It is worth noting that the linear hydrodynamic modulation does not contribute, under the assumed conditions, to the trispectrum of modulated waves.

Even though the waves are originally linear, the hydrodynamic modulation of first order induces a bispectrum, while the second-order hydrodynamic modulation induces a trispectrum. Inducement of bispectra and trispectra is an indication of the highly nonlinear aspect of the hydrodynamic modulation.

13. Conclusion

We began this study by solving analytically the balance of wave action (2.1) using the source term of equation (3.1) originally suggested by Hughes (1978) and later modified by Caponi *et al.* (1988). A comparison was shown in § 4 between a numerical simulation of the exact analytical solution and the linear approximation of the hydrodynamic equations suggested by Alpers & Hasselmann (1978). The latter approximation is a well known and commonly used procedure to assess the hydrodynamic interactions between modulated and modulating waves. It is termed the linear modulation transfer function. Since our numerical simulation is an exact solution of the differential equations, the discrepancy between the simulated results and the linear MTF is a demonstration that the linear MTF fails to capture significant features caused by the inherent nonlinear nature of the physical processes involved. We have shown in figure 1 that, even for a single short wave modulated by a single long wave, the combination of the nonlinearities in the ray equations with those in the source term can cause significant discrepancy with the classical linear approximation. For a multi-current situation, the discrepancies grow even bigger simply by enhancing the nonlinear aspect of the ray equations themselves when reintroduced in the balance equation of wave action. Figure 2 illustrates the discrepancy for the simple case where two long waves modulate a single short wave. This lack of significant information in the linear MTF motivates the need for a higher-order hydrodynamic theory.

We started this derivation by rederiving the linear MTF (7.5) in a slightly different notation (see Elfouhaily *et al.* 2000; Hasselmann 1961) to ease the generalization of the technique to higher orders in long-wave slope. The linear order derived here in equation (7.5) is identical to that of Alpers & Hasselmann (1978), with a noticeable improvement due to inclusion of extra modulation effects caused by local acceleration and finite depth. It is then clearly demonstrated that the amplitude of the linear MTF (7.5) is driven by two processes: the orbital velocity and the local acceleration caused by the presence of the modulating waves. Our notation permits the identification in the analytical formulation of these two contributions from the straining (6.6) and heaving (6.12) kernels, respectively.

Equation (8.4) is a second-order MTF derived to complement the linear order with nonlinear information generated by the nonlinear nature of the modulation process and by the inherent nonlinearity of surface waves even before the consideration of the hydrodynamic modulation. In addition to the properties mentioned for the linear MTF, the higher-order theory includes wave–wave interactions among modulating waves and higher-order derivatives of the source functional. We believe that this is the first time that nonlinear modulating waves have been considered in studies of hydrodynamic-modulation processes. As a consequence of the second-order hydrodynamic MTF, the phase of the modulation is dependent on the relative direction of long and short waves.

We have performed some ensemble averaging over first and second-order hydrodynamic MTFs to check some properties of the introduced hydrodynamic MTF. An offset was detected (11.4) between the equilibrium spectrum and the ensemble-average spectrum produced by the second-order hydrodynamic modulation. The offset turned out to be always negative, which means that the averaged spectrum is lower than the equilibrium spectrum. This difference is largest (in absolute value) for intermediate and short gravity waves (11.6). Gravity–capillary and capillary waves have virtually a zero offset (11.10). This in turn translates into convergence between ensemble-average spectra and equilibrium spectra even for second-order hydrodynamic modulations.

Another interesting application of the developed hydrodynamic MTFs is the computation of higher-order spectra of surface elevation. Indeed, we demonstrated that linear hydrodynamic MTF generates higher-order statistics equivalent to the bispectrum (12.5) even when modulated and modulating waves are originally linear. The second-order MTF induces even higher multi-spectra related to the trispectrum in equation (12.9) of surface elevation. Without the presence of hydrodynamic modulation, both bispectra and trispectra are non-existent, hence the name of § 12: induced bispectra and trispectra.

The combination of both first and second-order MTFs presented in this study produces a powerful tool for the description of hydrodynamic modulations of short-wave spectra by longer nonlinear waves.

Appendix A. Third-order hydrodynamic MTF

Similar to equation (8.1), the third-order hydrodynamic MTF can be defined and derived from (5.4 d),

$$_0\mathcal{L}[_3N] + {}_1\mathcal{L}[_2N] + {}_2\mathcal{L}[_1N] + {}_3\mathcal{L}[_0N] = Q_{13}N + Q_{21}N_2N + \tfrac{1}{6}Q_{31}N^3, \qquad \text{(A 1)}$$

and its modal description yields the following relationship

$$\begin{aligned} &- \mathrm{i}(\omega_1 + \omega_2 + \omega_3) R^{s_1,s_2,s_3}_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3} + \mathrm{i} R^{s_2,s_3}_{\boldsymbol{k}_2,\boldsymbol{k}_3}\, {}_1\dot{\boldsymbol{x}} \cdot (\boldsymbol{k}_2 + \boldsymbol{k}_3) + {}_1\dot{\boldsymbol{k}}_{\mathrm{s}} \cdot \frac{\partial R^{s_2,s_3}_{\boldsymbol{k}_2,\boldsymbol{k}_3}\, {}_0N}{{}_0N \partial \boldsymbol{k}_{\mathrm{s}}} \\ &\qquad = Q_1 R^{s_1,s_2,s_3}_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3} + \tfrac{1}{2} Q_2 R^{s_1}_{\boldsymbol{k}_1}\, {}_0N R^{s_2,s_3}_{\boldsymbol{k}_2,\boldsymbol{k}_3} + \tfrac{1}{6} Q_{30} N^2 R^{s_1}_{\boldsymbol{k}_1} R^{s_2}_{\boldsymbol{k}_2} R^{s_3}_{\boldsymbol{k}_3}. \qquad \text{(A 2)} \end{aligned}$$

After some rearrangement this modal formulation becomes

$$\begin{aligned} &(\omega_1 + \omega_2 + \omega_3 - \mathrm{i}Q_1) R^{s_1,s_2,s_3}_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3} \\ &\qquad = R^{s_2,s_3}_{\boldsymbol{k}_2,\boldsymbol{k}_3}\, {}_1\dot{\boldsymbol{x}} \cdot (\boldsymbol{k}_2 + \boldsymbol{k}_3) - \mathrm{i}\, {}_1\dot{\boldsymbol{k}}_{\mathrm{s}} \cdot \frac{\partial R^{s_2,s_3}_{\boldsymbol{k}_2,\boldsymbol{k}_3}\, {}_0N}{{}_0N \partial \boldsymbol{k}_{\mathrm{s}}} \\ &\qquad\qquad + \tfrac{1}{2}\mathrm{i} Q_2 R^{s_1}_{\boldsymbol{k}_1} R^{s_2,s_3}_{\boldsymbol{k}_2,\boldsymbol{k}_3}\, {}_0N + \tfrac{1}{6}\mathrm{i} Q_3 R^{s_1}_{\boldsymbol{k}_1} R^{s_2}_{\boldsymbol{k}_2} R^{s_3}_{\boldsymbol{k}_3}\, {}_0N^2, \qquad \text{(A 3)} \end{aligned}$$

which is an expression for the third-order hydrodynamic MTF under the simplifying conditions of no local acceleration effects and no wave–wave interactions between modulating waves. The first- and second-order hydrodynamics are derived in the text under a more general context.

A particular case of interest is when the source functional is assumed to follow that of Hughes (1978), as generalized by Caponi *et al.* (1988). For all these restrictive conditions, the third-order hydrodynamic MTF simplifies to

$$\begin{aligned} R^{s_1,s_2,s_3}_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3} &= \frac{(\omega_1 + \omega_2 + \omega_3) - \mathrm{i}\mu_{\mathrm{s}}}{(\omega_1 + \omega_2 + \omega_3)^2 + \mu_{\mathrm{s}}^2} \\ &\quad \times \left[\omega_1^2 (\boldsymbol{k}_{\mathrm{s}} \cdot \boldsymbol{k}_1) \frac{\boldsymbol{k}_1}{{}_0N} \frac{\partial R^{s_2,s_3}_{\boldsymbol{k}_2,\boldsymbol{k}_3}\, {}_0N}{\partial \boldsymbol{k}_{\mathrm{s}}} - \omega_1^2 \boldsymbol{k}_1 \cdot (\boldsymbol{k}_2 + \boldsymbol{k}_3) R^{s_2,s_3}_{\boldsymbol{k}_2,\boldsymbol{k}_3} \right. \\ &\qquad \left. - \mu_{\mathrm{s}}(\rho + 1) R^{s_1}_{\boldsymbol{k}_1} R^{s_2,s_3}_{\boldsymbol{k}_2,\boldsymbol{k}_3} - \tfrac{1}{6}\mu_{\mathrm{s}}(\rho + 1)(\rho - 1) R^{s_1}_{\boldsymbol{k}_1} R^{s_2}_{\boldsymbol{k}_2} R^{s_3}_{\boldsymbol{k}_3} \right], \qquad \text{(A 4)} \end{aligned}$$

where the last term in the square brackets vanishes if ρ is 1 (see Hughes 1978).

References

Alpers, W. & Hasselmann, K. 1978 The two-frequency microwave technique for measuring ocean wave spectra from an airplane or satellite. *Bound. Layer Meteorol.* **13**, 215–230.

Baldock, T. E., Swan, C. & Taylor, P. H. 1996 A laboratory study of nonlinear surface waves on water. *Phil. Trans. R. Soc. Lond.* A **354**, 649–667.

Caponi, E. A., Crawford, D. R., Yuen, H. C. & Saffman, P. G. 1988 Modulation of radar backscatter from the ocean by a variable surface current. *J. Geophys. Res.* **93**, 12 249–12 263.

Debnath, L. 1994 *Nonlinear water waves*. Academic.

Elfouhaily, T., Chapron, B., Katsaros, K. & Vandemark, D. 1997 A unified directional spectrum for long and short wind-driven waves. *J. Geophys. Res.* **102**, 15 781–15 796.

Elfouhaily, T., Thompson, D. R., Chapron, B. & Vandemark, D. 2000 Truncated Hamiltonian versus surface perturbation in nonlinear wave theories. *Waves Rand. Media* **10**, 103–116.

Elfouhaily, T., Thompson, D. R., Chapron, B. & Vandemark, D. 2001 Improved electromagnetic bias theory: inclusion of hydrodynamic modulations. *J. Geophys. Res.* **106**, 4655–4664.

Hara, T. & Plant, W. J. 1994 Hydrodynamic modulation of short wind-wave spectra by long waves and its measurement using microwave backscatter. *J. Geophys. Res.* **99**, 9767–9784.

Hasselmann, K. 1961 On the non-linear energy transfer in gravity-wave spectrum. Part 1. General theory. *J. Fluid Mech.* **12**, 481–500.

Henyey, F., Creamer, D., Dysthe, K., Schult, R. & Wright, J. 1988 The energy and action of small waves riding on large waves. *J. Fluid Mech.* **189**, 443–462.

Holthuijsen, L. H. & Herbers, T. H. C. 1986 Statistics of breaking waves observed as whitecaps in the open ocean. *J. Phys. Oceanogr.* **16**, 290–297.

Hughes, B. A. 1978 The effect of internal waves on surface wind waves. II. Theoretical analysis. *J. Geophys. Res.* **83**, 455–465.

Keller, W. C. & Wright, J. W. 1975 Microwave scattering and the straining of wind-generated waves. *Radio Sci.* **10**, 139–147.

Kudryavtsev, V. N., Makin, V. K. & Chapron, B. 1999 Coupled sea surface–atmosphere model. 2. Spectrum of short wind waves. *J. Geophys. Res.* **104**, 7625–7639.

Lake, B. M. & Yuen, H. C. 1978 A new model for nonlinear waves. Part 1. Physical model and experimental evidence. *J. Fluid Mech.* **88**, 33–62.

Longuet-Higgins, M. S. 1985 Acceleration in steep gravity waves. *J. Phys. Oceanogr.* **15**, 1570–1579.

Longuet-Higgins, M. S. 1987*a* The propagation of short surface waves on longer gravity waves. *J. Fluid Mech.* **177**, 293–306.

Longuet-Higgins, M. S. 1987*b* A stochastic model of sea-surface roughness. I. Wave crests. *Proc. R. Soc. Lond.* A **410**, 19–34.

Longuet-Higgins, M. S. 1991 A stochastic model of sea-surface roughness. II. *Proc. R. Soc. Lond.* A **435**, 405–422.

Longuet-Higgins, M. S. & Stewart, R. 1960 Changes in the form of short gravity waves on long waves and tidal currents. *J. Fluid Mech.* **8**, 565–583.

Longuet-Higgins, M. S. & Stewart, R. 1961 The changes in amplitude of short gravity waves on steady non-uniform currents. *J. Fluid Mech.* **10**, 529–549.

Phillips, O. M. 1977 *The dynamics of the upper ocean*, 2nd edn. Cambridge University Press.

Plant, W. J. 1979 The gravity–capillary wave interaction applied to wind-generated, short-gravity waves. NRL report 8289. Naval Reasearch Laboratory, Washington, DC.

Plant, W. J. 1982 A relationship between wind stress and wave slope. *J. Geophys. Res.* **87**, 1961–1967.

Ramamonjiarisoa, A. 1995 On the kinematics of short waves in the presence of surface flows of larger scales. *J. Fluid Mech.* **298**, 249–269.

Thompson, D. R., Gotwols, B. L. & Sterner, R. E. I. 1988 A comparison of measured surface wave spectral modulations with predictions from a wave–current interaction model. *J. Geophys. Res.* **93**, 12 339–12 343.

Valenzuela, G. R. & Laing, M. B. 1972 Nonlinear energy transfer in gravity–capillary wave spectra, with applications. *J. Fluid Mech.* **54**, 507–520.

Valenzuela, G. R. & Wright, J. W. 1979 Modulation of short gravity–capillary waves by longer-scale periodic flows—a higher-order theory. *Radio Sci.* **14**, 1099–1110.

Whalen, A. D. 1971 *Detection of signals in noise*, p. 97. Academic.

Zakharov, V. E. 1968 Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. Appl. Mech. Tech. Phys. (English Transl.)* **2**, 190–194.

Zhang, X. 1995 Capillary–gravity and capillary waves generated in a wind wave tank: observations and theories. *J. Fluid Mech.* **289**, 51–82.