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Wave parameter tuning for the application of the mild-slope equation on steep beaches and in shallow water

U.T. Ehrenmark *, P.S. Williams

Department of Computing, Information Systems and Mathematics (CISM), London Guildhall University, 100 Minories, London E.C.3N 1JY, UK

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Abstract

The linear mild-slope equation (MSE) is examined in the limit of very shallow water. This is done by means of a series comparison with the more 'exact' linear classical theory (E) valid over arbitrary uniform slopes and known to have a "minimum norm" solution basis pair, respectively, regular and logarithmically singular at the shore line. It is shown that the agreement between E and MSE is exact for the first three terms for the regular wave and the first two for the singular wave. It is further demonstrated, by application of this example, that the MSE represents a better approximation than does the classical linearised shallow water equation (SWE) in the case of extremely small depth. In particular, if solutions to each are tuned to the same finite wave height at the shoreline, then MSE predicts the correct curvature of wave height there whereas SWE does not.

The work of Booij (Booij, N.A., 1983. A note on the accuracy of the Mild-Slope Equation. Coastal Engineering 7, 191–203.) is supported and varied to allow performance on very steep beds to be tested against exact values rather than those of numerical simulation. Those tests are carried out both as Boundary Value Problems, BVP (Scheme A) and Initial Value Problems, IVP (Scheme B) with matching results on global error. Methods are found of specifying phase and group velocity, which are consistent with linear wave beach theory and lead to improvements in solving the MSE over steep flat beaches. The improvements are found generally superior, in the case considered, to those of some recently developed 'modified' and 'extended' MSEs. Finally, it is demonstrated, and confirmed by both asymptotic theory and calculation, that the addition of evanescent modes constitutes improvement only in intermediate depths and is not recommended in depths of the order of only a wavelength on a steep (e.g. 45°) beach.

Keywords: Mild-slope equation; Surface waves; Shallow water equations; Mathematical model; Finite difference

1. Introduction

The mild-slope equation (MSE), originally conceived by Berkhoff (1974), has been widely used over the last two decades to model the evolution of water waves over varying topography. In recent works, Li (1994a) examines a generalised conjugate gradient method for improved computational efficiency and in Li (1994b) is studied an evolution equation whose lower order approximation is the MSE. Meanwhile, various improvements to the MSE have been obtained, e.g. Kirby (1986) derived the

* Corresponding author. Fax: +44-020-8732-1717.
E-mail address: ulf@lgu.ac.uk (U.T. Ehrenmark).

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'extended MSE' and Chamberlain and Porter (1995) (CP) derived the 'modified MSE', works, which are both applicable to rippled beds, whilst Massel (1993) derived an approximation accounting also for evanescent waves and thus increasing the applicability on steep slopes. In a simpler version, Massel (1995) used the approximation without the evanescent waves to examine the wave transformation on a submerged reef. Porter and Staziker (1995) improved the use of the modified MSE on depth gradient discontinuities by including also evanescent modes and developing a set of jump conditions, which result in conservation of mass but a discontinuity in surface elevation. They also pointed to deficiencies in Massel's model when the discontinuities are present. The works cited give a good up-to-date bibliography on the subject. See also Chamberlain and Porter (1997, pp. 49–53) or Dingemans (1997, Ch. 3) for a more complete list of references.

The present work, which will focus on steep beds and/or very shallow water, will investigate the performance of the various existing versions of the MSE described above for which the fundamental wavenumber k is calculated from linear Airy theory. The new approach introduces the idea of computing this wavenumber instead from the known exact linear solution. The opportunity arises of comparing results from the new technique herein with those of CP and Massel who both end up with an additional term in the MSE. This term appears, in each case, to increase locally the 'effective' wavenumber (whilst depth is decreasing) and this will be shown also as the case in the comparison with the classical exact solutions of the full linearised system. Thus, the ultimate aim is to explore the possibility of quantifying the wave-number enhancement required with respect to depth and bottom inclination. However, the present work will be restricted to a uniformly sloping bed since the chief objective is to establish this dependency by comparison with an exact solution rather than numerical simulations. Encouraged by the results in the present work, the authors intend to examine further a case of non-uniform slope. This will be the subject of a separate study.

Booij (1983) (and others cited above) have carried out a number of numerical experiments to test the accuracy of the MSE when applied to bottom topography of varying degrees of slope. Booij found the best results were obtained for propagation parallel to bottom contours but, even in the case of normal propagation the inclination restriction turned out to be surprisingly generous, with good results achieved for gradients as large as 1:3. Porter and Staziker (1995) noted that modifications to the equation (incorporating evanescent waves) increased significantly the capability of the equation and reasoned that slopes of order 1 in 1 could be used. These works did not, however, quantify errors (relying mainly on subjective judgement from graphical output) nor did they consider aspects of the accuracy for extremely small depths, such as might be expected to be covered by a theory of non-breaking waves incident over a plane beach or artificial sloping breakwater. One aspect of this, which we will draw attention to in this work, is that the MSE and its various counterpart improvements discussed above have a similar property to the shallow water equation (SWE) (and the full Laplace Equation in cylindrical polars) namely that the fundamental solution pair near the shoreline consists of a regular (bounded) solution and a singular (unbounded solution). A numerical routine, which proceeds into very shallow water is therefore liable to pick up an unwanted (parasitic) component of the unbounded solution through rounding errors in a marching scheme (Hildebrand, 1956, p. 209). We highlight this phenomenon by recalculating the solution adding just the first evanescent wave mode and examining its asymptotic behaviour.

The intention then, in this paper, is to identify the behaviour of the MSE for the very near-shore zone, further to quantify errors involved with steeper shoals and to examine a possible modification from which a better description may be determined. In doing this, we will be reminded that basic Airy theory, on which the MSE theory is constructed, is a limiting form of a more global linearised theory over a plane beach. Friedrichs (1948) examined this problem for harmonic functions in great detail and deduced the Airy theory as a certain asymptotic limit. That limit (Friedrichs' second limit) assumed beach angle $\alpha \to 0$ with the local depth h held fixed. One of the limiting expressions resulted in confirmation of the classical Airy dispersion relation for the wavenumber k:

$$k = k_\infty \coth kh, \tag{1}$$

which of course is the 'bread and butter' of most linear water wave theories and in particular that of the MSE. In Eq. (1), we have written $k_\infty = \omega^2/g$ where ω is the monochromatic circular frequency. (We will refer to Friedrichs, 1948 for the essentials of the classical plane beach problem, with deference to some of the earlier authors, e.g. Stoker, 1947, Hanson, 1926 or even Kirchhoff, 1899, who was the pioneer of the regular standing wave description in a sector of steep slope; this is because Friedrichs constructed the asymptotic theory referred to above).

The MSE is written in Section 2 and the implication is considered of the restriction to its application given by Berkhoff (1974) to the case of a plane beach. In Section 3, we write the classical (minimum norm) solutions of the full Laplace equation problem (Ehrenmark, 1988). This enables 'testing' of the MSE performance for very small depths and for various slopes. The tests of the latter show considerable similarity with the results of Booij (1983) but by working against an 'exact' solution, we have the possibility of more accurately assessing errors and the implications of a possible remedy.

An analytical treatment using the method of Frobenius (see, e.g. Spain and Smith, 1970), which effectively involves expressing both components of the potential in series expansions about $h = 0$ (where h is the uniformly increasing depth), shows that the MSE, in these cases, is accurate to $O(h^2)$ in the limiting case of vanishing depth. This is considered to be a vital result in a quest for knowing how 'good' the MSE really is in shallow water and, in particular, a comparison with a similar analysis using the classical shallow water theory (Lamb, 1932) reveals that the latter is only correct to $O(h)$. It is therefore unable (unlike the MSE) accurately to describe, for example, curvature near the shoreline of a solution with given amplitude there. The details of this analytical study are delivered in Section 4 and the results confirm generally investigations for less moderate depths carried out by Booij (1983) for bed angles up to $\pi/4$.

In Section 5, we attempt to model an improvement to the MSE performance for slopes (1) by incorporating a modified scheme to compute wavenumber variation over a plane beach. It was shown by Ehrenmark (1994b) that the piecewise Airy *set-down* computation becomes inaccurate at an exponential rate as beach angle is increased and similar albeit slightly less dramatic results appear to hold for the wavenumber computation also (Ehrenmark, 1994a). In particular, we find that wavenumber appears to increase like $h^{-1}/(\ln(h))^2$ in the 'exact' theory as $h \to 0$ whereas the Airy theory of course predicts the behaviour $O(h^{-1/2})$.

The results of the numerical tests are delineated in Section 6. These include treatments both of initial value problem (IVP) and the two-point boundary value problem (BVP) driven by data from the 'exact' theory. Comparison is then undertaken with equivalent results calculated from the alternative versions of the MSE referred to above. In all cases, it is confirmed that these improve the basic MSE performance to some degree but that the improvement obtained with the present approach is considerably more significant. A global measure to quantify the various improvements is introduced. The effect of including one evanescent wave mode is discussed in Section 7 and it is shown, with the help of asymptotic expressions, that this becomes an increasingly dangerous strategy near the shore and/or as further modes are included. Section 8 summarises the findings and emphasises that, whilst the present results are related only to the case $h'' = 0$, there should be sufficient evidence here to suggest that further work in the same direction on cases $h'' > 0$ would be worthwhile.

2. The MSE and near-shore restrictions

The MSE may be taken in the form:

$$\nabla c c_g \nabla \phi + k^2 c c_g \phi = 0, \tag{2}$$

e.g. Li (1994b). Here, ϕ is the (complex valued) horizontal variation in velocity potential Φ:

$$\Phi = \Re\left\{ \phi(x,y) \frac{\cosh k(z+h)}{\cosh kh} \exp(-i\omega t) \right\}, \tag{3}$$

$c = \omega/k$ is phase velocity and $c_g = \partial\omega/\partial k$ is group velocity. The Liouville transformation

$$\phi = \frac{\psi}{\sqrt{c c_g}} \tag{4}$$

(Radder, 1979) conveniently modifies Eq. (2) into the Helmholtz form

$$\Delta\psi + k_c^2\psi = 0, \tag{5}$$

where

$$k_c^2 = k^2 - \frac{\Delta\sqrt{cc_g}}{\sqrt{cc_g}}. \tag{6}$$

The condition, which is usually taken to prevail for the application of MSE is:

$$\frac{|\nabla h|}{kh} \ll 1 \tag{7}$$

and it is not hard to see from this, that the theory applied over a plane beach of angle α requires

$$hk_\infty \gg k_\infty k^{-1}\tan\alpha = \tanh kh \tan\alpha. \tag{8}$$

If the depth is approaching zero, we know also from Eq. (1) that tanh $kh \sim kh$ (since it is Eq. (1) which informs that $k = O(h^{-1/2}\})$ as $h \to 0$) so that a condition equivalent to Eq. (8) will be:

$$\frac{k}{k_\infty} \ll \cot\alpha \tag{9}$$

and this condition is surprisingly stringent for less gently sloping beaches. Clearly, in view of the growth of k as $h \to 0$ the theory appears to be inapplicable in a certain nearshore zone. The MSE has however been shown to be applicable also for slopes of order unity (Booij, 1983), so careful regard should be paid to the significance of Eq. (9). Friedrichs (1948) first displayed wavenumber variation computed from Eq. (1) for a beach of angle $\pi/30$. The equivalent curves for $K = k/k_\infty$ in the cases $\alpha = 6°$, $\alpha = 18°$, $\alpha = 30°$ and $\alpha = 45°$ plotted against a non-dimensional depth $H = k_\infty$ are displayed in Fig. 1. When plotted in this way, the wavenumber becomes independent of slope. Shown in these graphs also are the alternative computations of wavenumber such as obtained by Ehrenmark (1994a) using the classical exact linear theory for arbitrary slopes. These show that the Airy theory computations become increasingly unreliable both near the shoreline and as the gradient of the sloping bottom increases.

If we take $X = k_\infty x$ (where x is a horizontal surface coordinate measured positive seaward from the shoreline) then the value $X = \cot\alpha$ is of some significance. It was shown to be the position of minimum of the Airy shoaling coefficient and was observed by Friedrichs (1948) to be remarkably close to the value obtained with that of the exact theory despite the considerable asymptotic differences in

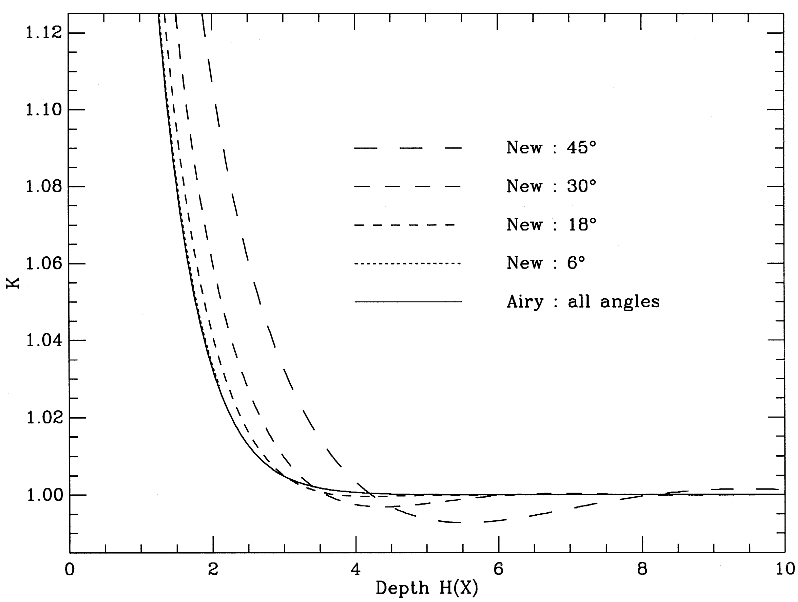

Fig. 1. Wavenumber curves for Airy theory and the new theory.

these as $X \to 0$. The point is also one where the maximum *set-down* is theoretically established (Ehrenmark, 1994b) and using the breaker criterion that wave amplitude a is approximately 40% of mean depth (Longuet-Higgins, 1972), it would appear that waves of amplitude in excess of $0.4/k_\infty$ will have broken before reaching this point. In terms of wave period T, this means approximately that $10a > T^2$ (in MKS units) for breaking to occur seaward of the point $X = \cot\alpha$.

The point about the above comparison is that, for most wind driven gravity waves, we could expect breaking to occur when $X < \cot\alpha$. The implication of Eq. (9), on the other hand, can be seen to be equivalent to $X \gg \tan\alpha$. The precise interpretation of the last inequality is, of course, a matter for the mathematical modeller. However, if it were not at least replaceable by:

$$X > 10\tan\alpha,$$

it is hard to see how reasonable numerical approximations could be expected. Suppose then, that the MSE were to be applicable across the breaker zone, perhaps by modelling with a frictional loss to simulate turbulent exchange (see e.g. Longuet-Higgins, 1972). On the basis of the above, that would require the bottom slope restriction $\alpha < \alpha_0$ where $\tan\alpha_0 = 1/\sqrt{10}$, in order to make sure that the breaker zone was within the MSE applicability region.

3. The 'exact' classical linear theory

As the equations stand, neither the (basic) MSE nor the classical linear theory (Friedrichs, 1948) are applicable across the surf zone. However, in the case of 2-D flow, the effectiveness of the MSE, or subsequent modifications of it, for very small depths can be readily tested against solutions of the Friedrichs theory, which will be referred to as the 'exact' theory for convenience. The comparison with a 2-D model should be sufficient to bring out any intrinsic weakness of the MSE and in any case is thought reasonably justified for these depths on the grounds that refraction would, in the absence of edge waves, sustain a solution whose oscillatory part is primarily a cross-shore one. In the remainder of this work therefore, we shall be disregarding wave breaking and work with a purely low amplitude non-breaking wave theory.

The full classical problem may be described by the use of cylindrical polar coordinates. Solutions, expressed as inverse Mellin transforms, have been fully described by Ehrenmark (1987, 1994b) in a series of papers. The two fundamental potential functions ϕ_r, ϕ_s, which are, respectively, regular and logarithmically singular at the (fixed) shoreline enable the full velocity potential to be expressed in the form

$$\phi = \Re\{(\phi_r + i\mu\phi_s)\exp(i\omega t)\}. \tag{10}$$

If we take $\mu = 0$, we get perfect reflection whilst the case $\mu = 1$ corresponds to a pure progressing wave incident from infinity. Thus, in all cases except perfect reflection, there will be a singularity at the shore line, which allows energy to propagate freely according to the unsteady Bernoulli equation, until that line is reached at which point the singularity acts as a sink of mean energy. It would be preferable to describe the very near shore flow therefore, using a model where energy is dissipated more uniformly across the surf zone. The complexities of trying to do this are, however, quite considerable and would only obscure the chief purpose of the present work. To test the performance of the MSE, we shall examine both the regular and singular solutions. For a bounded standing wave $\mu = 0$, it is well known that the amplitudes at $R = 0$ and $R = \infty$ are in the ratio $\sqrt{M}$, where $\alpha = \pi/2M$, e.g. Stoker (1947).

If M is not too large, we may usefully employ the finite expansion for ϕ_r given by Stoker, since this is in closed form. Writing $\beta_k = \exp(i\pi(k/M + 1/2))$, this expansion is

$$\phi_r = \Re\left\{\sum_{k=1}^{M} c_k \exp\left(Re^{i\theta}\beta_k\right)\right\}, \tag{11}$$

where

$$c_k = \exp\left\{i\pi\left(\frac{M+1}{4} - \frac{k}{2}\right)\right\}\prod_{j=1}^{k-1}\cot\left(\frac{j\pi}{2M}\right),$$

$$j > 1;\ c_1 = \overline{c_M}, \tag{12}$$

and the polar representation (R,θ) is used, with $\theta = 0$ as the SWL and $\theta = -\alpha$ as the bed. The representation (11) is of a wave of amplitude unity as $R \to \infty$. Note that $R = \omega^2 r/g$ where r is the physical distance from the shoreline and that, on $\theta = 0$, R corresponds to X used earlier. The similar representation of the singular component requires evaluation of an integral along a contour C from $+\infty$ to $Re^{i\theta}\beta_k$ proceeding anti-clockwise about the origin. The full expression is:

$$\pi\phi_s = \Re\left\{ \sum_{k=1}^{M} c_k \exp\left(Re^{i\theta}\beta_k\right)\left[i\pi - \int_c \frac{e^{-t}}{t}\, dt \right]\right\}. \quad (13)$$

The asymptotic behaviour of the regular and singular potentials as $R \to \infty$ is:

$$\phi_r \sim e^{R\sin\theta}\cos\left(R\cos\theta + \frac{1}{4}(M-1)\pi \right), \quad (14)$$

$$\phi_s \sim e^{R\sin\theta}\sin\left(R\cos\theta + \frac{1}{4}(M-1)\pi \right). \quad (15)$$

Alternative descriptions in terms of inverse Mellin transforms are found in, e.g. Ehrenmark (1988).

In some of the testing that follows, we shall be examining the accuracy of the MSE approximation to the problem that yields Eq. (11) as solution. The MSE is often applied with radiation conditions (e.g. Li, 1994a), but in this case we can take boundary conditions from the test solution and then examine the reproduction of intermediary values.

4. Near-shore limit

For shallow beaches, the Airy theory may be invoked. We need to expand both cc_g and $K^2 cc_g$ where for convenience, $K = k/k_\infty$. Solving Eq. (1) iteratively, we obtain:

$$K^2 = \frac{1}{H} + \frac{1}{3} + \frac{4}{45}H + \frac{16}{945}H^2 + \frac{16}{14{,}175}H^3 + O(H^4) \quad (16)$$

where H is the non-dimensional depth given by $KH = kh$. An expansion for cc_g, noting that $2cc_g = c^2(1 + 2KH/\sinh 2KH)$, is similarly

$$(\omega/g)^2 cc_g = H - \frac{2}{3}H^2 + \frac{8}{45}H^3 - \frac{8}{945}H^4 - \frac{32}{14{,}175}H^5 + O(H^6) \quad (17)$$

so that

$$(\omega/g)^2 K^2 cc_g = 1 - \frac{1}{3}H + \frac{2}{45}H^2 + \frac{8}{945}H^3 + \frac{8}{14{,}175}H^4 + O(H^5) \quad (18)$$

and the approximate MSE for arbitrarily small depth H may now be solved by the method of Frobenius. It is easy to see that the roots of the indicial equation are identical, so that one solution will be bounded at $H = 0$, whilst the other is logarithmic there (see Spain and Smith, 1970, p. 11). Solutions behave therefore, at least qualitatively, in an identical fashion to the exact solutions discussed in Section 3. Let us examine the *regular* solution in a little detail. If we insert the expansion:

$$\phi(H) = \sum_{n=0}^{\infty} \beta_n H^n \quad (19)$$

where $H = X\tan\alpha$ into the one-dimensional version of Eq. (2) and equate like powers of H we obtain,

$$\beta_1 = -\lambda\beta_0$$

$$4\beta_2 = \lambda\beta_0(\lambda - 1)$$

and

$$9\beta_3 = -\lambda\beta_0\left(\lambda^2/4 - 11\lambda/12 + 23/45\right),$$

where $\lambda = \cot^2\alpha$. In order to make the comparison with the exact solution, we require an expansion of that also for small H. Such an expansion has been written by Ehrenmark (1988) and its value on SWL is rewritten, (Ehrenmark, 1994a), in the more convenient form:

$$\phi_r = \sqrt{M} \sum_{N=0}^{\infty} \left\{ \frac{X^N}{N!} \prod_{j=1}^{N} (-\cot j\alpha) \right\} \quad (20)$$

where it is understood that the product is given the value unity if $N = 0$. Note also that $X = H\cot\alpha$. Comparison between Eqs. (19) and (20) shows that the expressions agree exactly to $O(H^2)$. We already know that the expressions cannot be identical, but this agreement for very small H confirms the hypothesis that the MSE can be used for arbitrarily small depths despite the conflict of requirements implied by Eq. (7) et seq.

Equations for β_4, β_5 have also been derived through a symbolic package (REDUCE);

$$8\beta_0\lambda + 42\beta_1\lambda - 32\beta_1 - 315\beta_2\lambda + 1344\beta_2 + 945\beta_3\lambda - 7560\beta_3 + 15{,}120\beta_4 = 0,$$

and

$$8\beta_0\lambda + 120\beta_1\lambda - 160\beta_1 + 630\beta_2\lambda - 1200\beta_2 - 4725\beta_3\lambda + 37{,}800\beta_3 + 14{,}175\beta_4\lambda - 189{,}000\beta_4 + 354{,}375\beta_5 = 0.$$

A similar consideration of the singular component of the exact solution requires the near-shore expansion (Ehrenmark, 1994a):

$$\pi\phi_s = -\sqrt{M}\sum_{N=0}^{M-1}\left\{(\ln X - \lambda_N)\frac{X^N}{N!}\prod_{j=1}^{N}(-\cot j\alpha)\right\} + O(X^M \ln X) \qquad (21)$$

on the SWL where,

$$\lambda_N = \lambda_{N-1} + \psi(N+1) - \psi(N) + \frac{2\alpha}{\sin 2N\alpha};$$

$$\lambda_0 = \psi(1) - \alpha\sum_{j=1}^{M-1}\tan j\alpha$$

and ψ is the usual digamma function. The strategy for a more precise expression arising from the error term in Eq. (21) is given in Ehrenmark (1988) but is not required here. To invoke a comparison, we write a full Frobenius expansion:

$$\phi(H) = \ln H\sum_{n=0}^{\infty}\beta_n H^n + \sum_{n=0}^{\infty} b_n H^n$$

into the (transformed) MSE:

$$\frac{\mathrm{d}}{\mathrm{d}H}\left\{\left(1 - \frac{2}{3}H + \frac{8}{45}H^2 + \cdots\right)H\frac{\mathrm{d}\phi}{\mathrm{d}H}\right\} + \cot^2\alpha\left(1 - \frac{1}{3}H + \frac{2}{45}H^2 - \cdots\right)\phi = 0 \qquad (22)$$

and, having already established agreement through the regular solution in terms of $\ln H$ and $H\ln H$, comparing coefficients of the term in H^0, we now get:

$$2\beta_1 + b_1 - \frac{2\beta_0}{3} + b_0\cot^2\alpha = 0.$$

The role of b_0 and β_0 is merely as arbitrary constants when a specific solution is chosen, so the challenge is to compare terms in $O(H)$ between the MSE and the full expansions. These are, respectively, b_1 and $-\cot^2\alpha(b_0 - \beta_0(1 + \{\{2\alpha\}/\{\sin 2\alpha\}\}))$. Whilst these are not identical, it is interesting to note that, as $\alpha \to 0$, we have equality up to $O(\alpha^2)$.

We conclude that the MSE performs well even for extremely small depths and that solutions with a weak singularity are only slightly less well reproduced than those which remain regular as the shore line is approached, Note in particular, however, that if we carry out a similar investigation for the regular wave using instead the classical linear SWE (Lamb, 1932, Art. 185):

$$g\nabla\cdot(h\nabla\zeta) = \partial^2\zeta/\partial t^2,$$

then this is equivalent to approximating Eqs. (17) and (18) by just the first term on the right hand sides resulting in, for example $4\beta_2 = \lambda^2\beta_0$, which thus, if $\alpha < \pi/4$, overestimates the curvature of the solution at the shoreline by a factor $\cos^2\alpha/\cos 2\alpha$. The view therefore of, e.g. Berkhoff (1974) that for small depth the MSE is seen to reduce to the SWE, whilst undoubtedly true, seems to somewhat conceal the strength of the former in shallow water.

5. A proposed improvement on steep shoals

In previous work (Ehrenmark and Williams, 1996), it was noted that Airy values for the wave parameters contributed increasingly to errors as the beach slope was steepened. Modifications were defined in which the wavenumber (or equivalently the phase velocity) was calculated from the linear wave beach theory. The amount by which this wavenumber was scaled, compared with the Airy wavenumber at the same depth was found. A new group velocity was then found by dividing the Airy group velocity by this factor. The MSE was then solved. This gave good improvements for 6° and 18° beaches but performed less well for the steeper 30° and 45° beaches. The proposal in Ehrenmark and Williams (1996) for this phase velocity C based on classical linear wave theory, which follows the ‘peak’ of a wave defined by $\partial\eta/\partial T = 0$ is given in non-dimensional form (with a prime denoting an x derivative) as:

$$C = \left[\frac{\phi_r^2 + \phi_s^2}{\phi_s \phi_r' - \phi_r \phi_s'} \right]_{\theta=0}. \qquad (23)$$

This can be readily derived from Mei (1992), (Eq. 3.6a, Ch. 1) by setting $S = t - i\log\ (\phi_r + i\phi_s)$ therein, thus giving $k = -\mathcal{T}\{\Phi'/\Phi\}$ where $\Phi = \phi_r + i\phi_s$. We retain Eq. (23) in the current work but seek to improve matters by making some identifications with analogues from Airy theory. The Airy dispersion relation gives an expression for the group velocity c_g:

$$c_g = \frac{g}{2\omega} D_d^{-2}, \qquad (24)$$

where the dispersion relation connecting circular wave frequency ω, acceleration due to gravity g, depth h and wavenumber k is:

$$\omega^2 = gk \tanh kh, \qquad (25)$$

(also given by Eq. (1)) and the shoaling coefficient D_d is defined by (Burnside, 1914):

$$D_d = \left\{ \tanh kh \left(1 + \frac{2kh}{\sinh 2kh} \right) \right\}^{-1/2}. \qquad (26)$$

We non-dimensionalise by writing $K = k/k_\infty$ and $H = k_\infty h$ where $k_\infty = \omega^2/g$ then Eq. (25) becomes:

$$K \tanh KH = 1. \qquad (27)$$

Putting $C = k_\infty c/\omega$ and $C_g = k_\infty c_g/\omega$, we find that Burnside's shoaling coefficient is written as:

$$D_d = \left\{ \tanh KH \left(1 + \frac{2KH}{\sinh 2KH} \right) \right\}^{-1/2}. \qquad (28)$$

Finally the relation (24) becomes:

$$C_g = \frac{1}{2D_d^2}. \qquad (29)$$

We may attempt the identification of a group velocity in steep beach wave theory by replacing the shoaling coefficient D_d of Airy theory with the shoaling coefficient $D_d^* = \{\phi_s^2 + \phi_r^2\}^{1/2}|_{\theta=0}$. The analogue of expression (24) now reads:

$$C_g = \frac{1}{2\{\phi_s^2 + \phi_r^2\}}. \qquad (30)$$

This is, of course, a non-dimensionalised relation.

The group velocities from this new theory have been numerically calculated using Eq. (30) for a range of beach angles α. These have been plotted

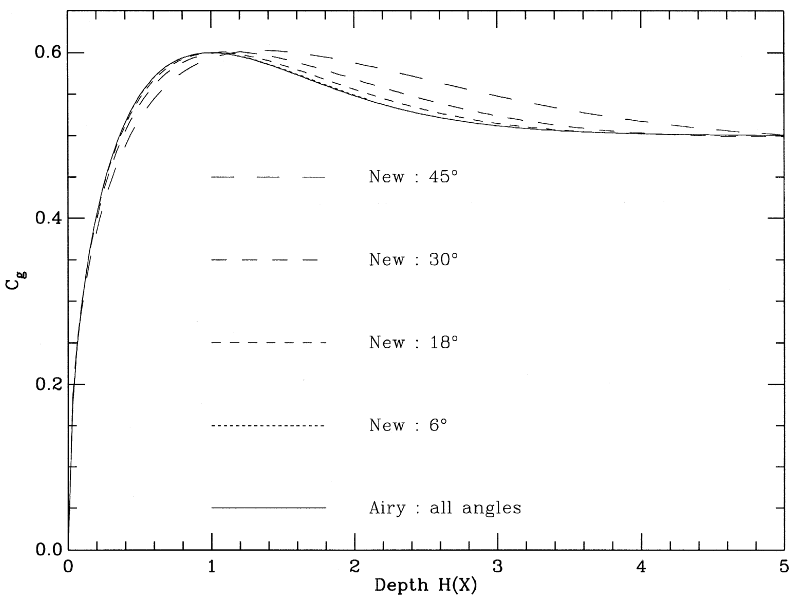

Fig. 2. Group velocities for Airy theory and the new theory.

Table 1
Residual error norms in MSE (using Airy K) and improved MSE (New K) of (regular, singular) against $\alpha°$

$\alpha°$	Regular: Airy K	Regular: New K	Singular: Airy K	Singular: New K
6	0.039927	0.070490	0.060713	0.054848
18	0.017292	0.022680	0.067419	0.034839
30	0.154041	0.052224	0.163331	0.008524
45	0.895317	0.011426	0.135603	0.054877

against a non-dimensional depth $H(X) = k_\infty h$ and are shown in Fig. 2. For comparison, the equivalent quantities (Eq. (29)) from the Airy theory are also calculated and when plotted against $H(X)$, these are identical regardless of α. The graphs for the 3° and 6° beaches are virtually indistinguishable from the Airy solution.

The MSE may be mapped into a Helmholtz equation using the Liouville transformation (see Radder, 1979):

$$\phi\sqrt{CC_g} = \psi \tag{31}$$

and since the proposed tests are for waves of 'normal incidence' the new equation is:

$$\frac{d^2\psi}{dX^2} + K_c^2\psi = 0,\ X_0 \le X \le X_1, \tag{32}$$

where $K_c^2 = K^2 - \dfrac{\left(\sqrt{CC_g}\right)''}{\sqrt{CC_g}}$, $K = 1/C$ and the prime denotes differentiation w.r.t. X. The values C and C_g are taken from Eqs. (23) and (30), respectively. The functions ϕ_r and ϕ_s are supposed to be known.

In the numerical experiments reported here, we take $X_0 = 1$, $X_1 = 20$ and either prescribe ψ at these points (Scheme A : BVP) or prescribe ψ at X_1, $X_1 - \delta X$ (Scheme B : IVP). The solution is developed using the standard Numerov method with step length δX. The method is $O(\delta X^6)$ and the choice $\delta X = 1/64$ is found to be more than adequate. Two families of solutions are examined (a) the 'regular' solutions where the boundary conditions specify values that correspond to the Stoker regular potentials

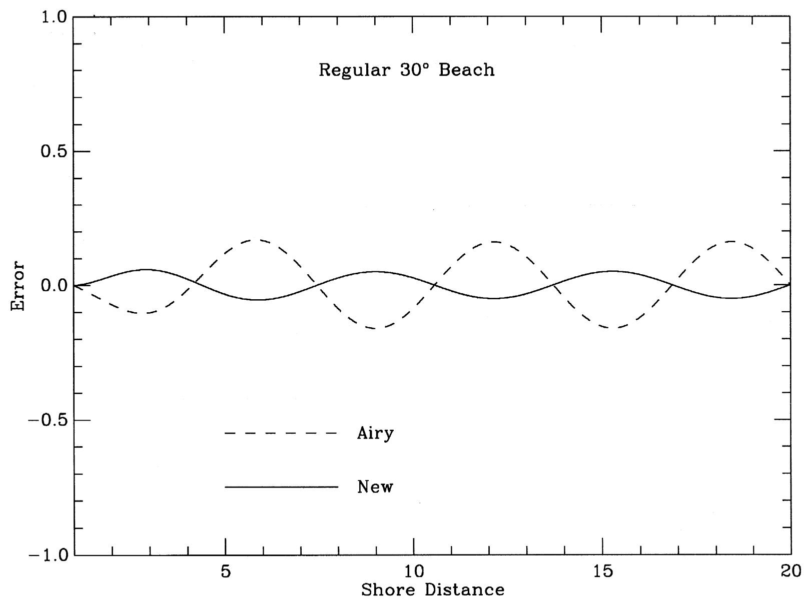

Fig. 3. Regular potentials for a 30° beach. Errors in the MSE boundary value solution using the new modification compared with that obtained using Airy theory.

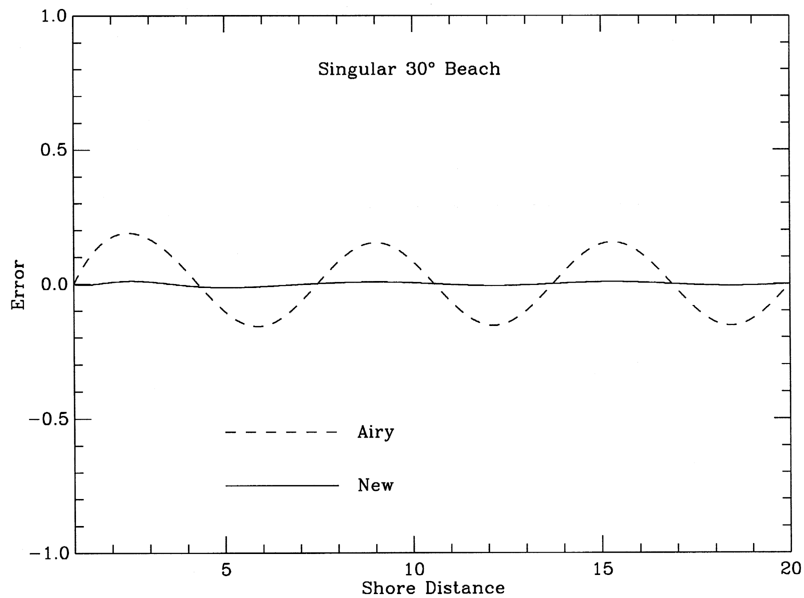

Fig. 4. Singular potentials for a 30° beach. Errors in the MSE boundary value solution using the new modification compared with that obtained using Airy theory.

and (b) the 'singular' solutions where the boundary conditions specify values that correspond to the Stoker singular potentials. The aim of these tests is to reproduce numerically ϕ_r and ϕ_s. Solutions of the

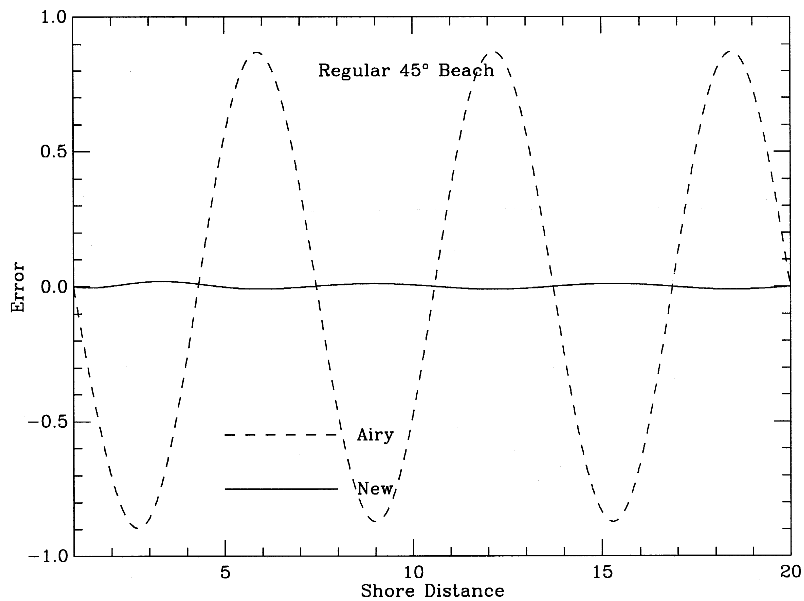

Fig. 5. Regular potentials for a 45° beach. Errors in the MSE boundary value solution using the new modification compared with that obtained using Airy theory.

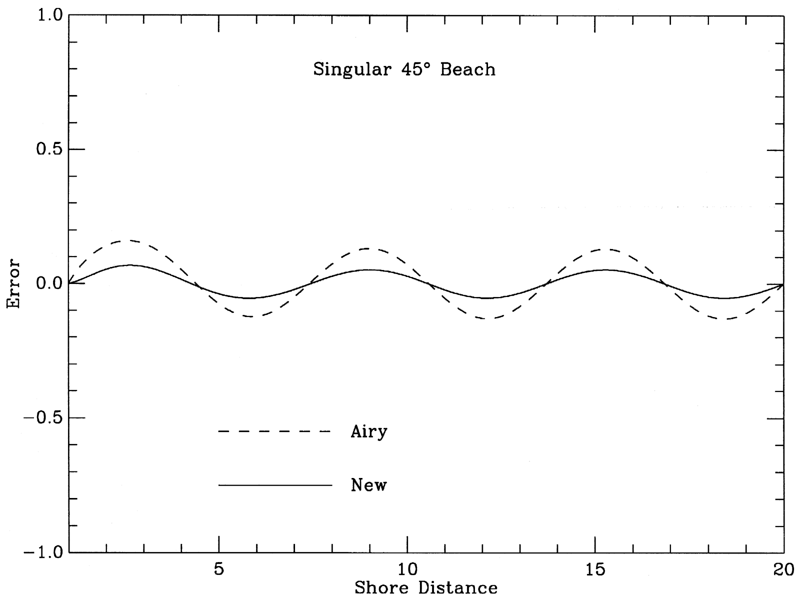

Fig. 6. Singular potentials for a 45° beach. Errors in the MSE boundary value solution using the new modification compared with that obtained using Airy theory.

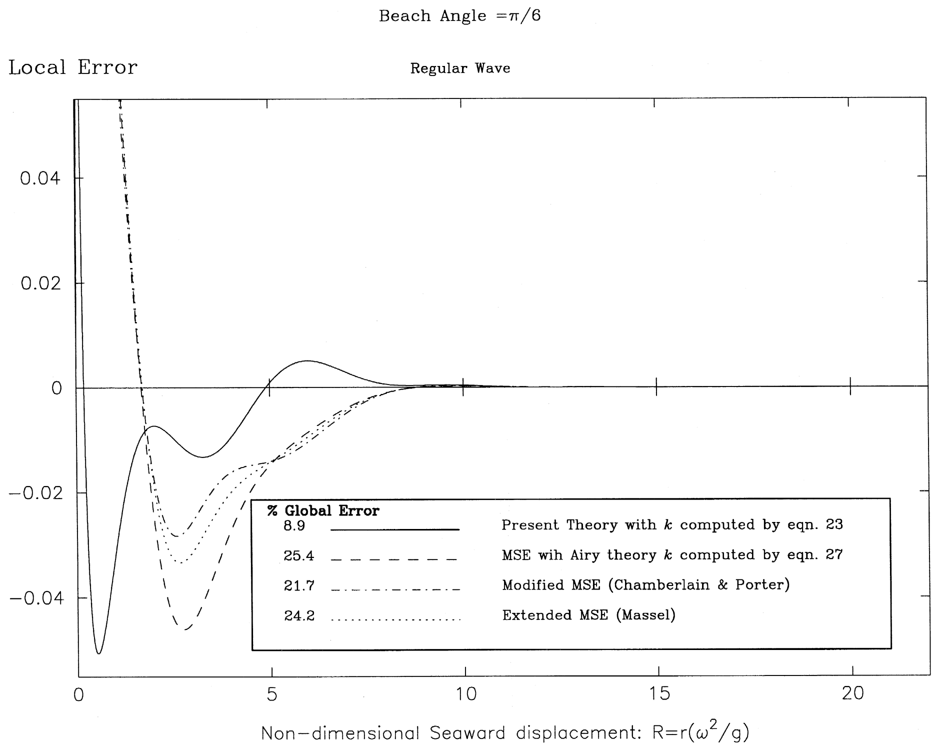

Fig. 7. Error computation for IVP: regular wave, beach angle $= \pi/6$.

MSE are then compared to 'exact' solutions for various slopes and a residual norm is constructed to measure the accuracy. Performance testing was achieved with a simple residual formula of the type:

$$\varepsilon = \frac{\sum_p \left| \phi_p^{\mathrm{MSE}} - \phi_p^{\mathrm{EXACT}} \right|}{\sum_p \left| \phi_p^{\mathrm{EXACT}} \right|} \tag{33}$$

where p denotes the mesh points used in the integration.

6. Numerical results

6.1. Scheme a — boundary value tests

Table 1 shows the results of the method discussed here.

The performance of this method is shown in Figs. 3–6. Each figure shows the errors in the MSE solution using our new modification to the dispersion relation compared with that obtained using Airy theory.

For the plane beach considered here, this method seems to perform well. It shows an improvement over the earlier scaling method of Ehrenmark and Williams (1996) and also has the advantage of being consistent with the linear theory while the earlier method was somewhat pragmatic.

6.2. Scheme b — initial value tests

For these runs, the 'exact' values of the Stoker potentials at $X = 19.99$ and 20.00 were taken as initial values and the solution developed as an IVP using again the Numerov technique. For convenience, we chose the step-length 0.01. The results of local error computation are shown in Figs. 7–10 for the respective beach angles 30° and 45°. The reader comparing these results with those of Ehrenmark and Williams (1996), where a similar treatment was in-

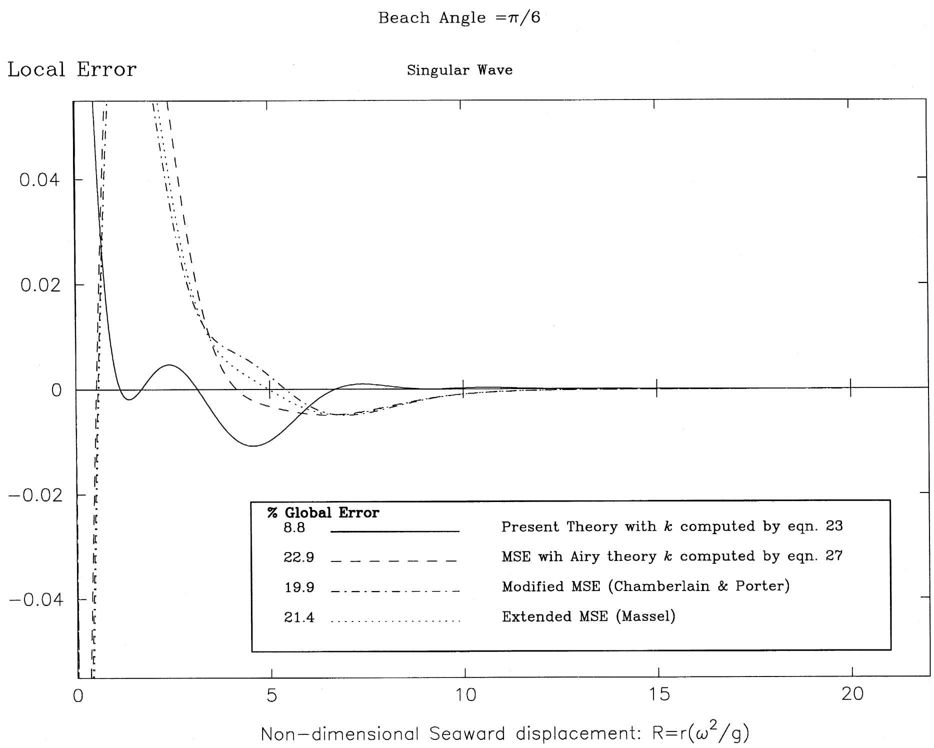

Fig. 8. Error computation for IVP: singular wave, beach angle = $\pi/6$.

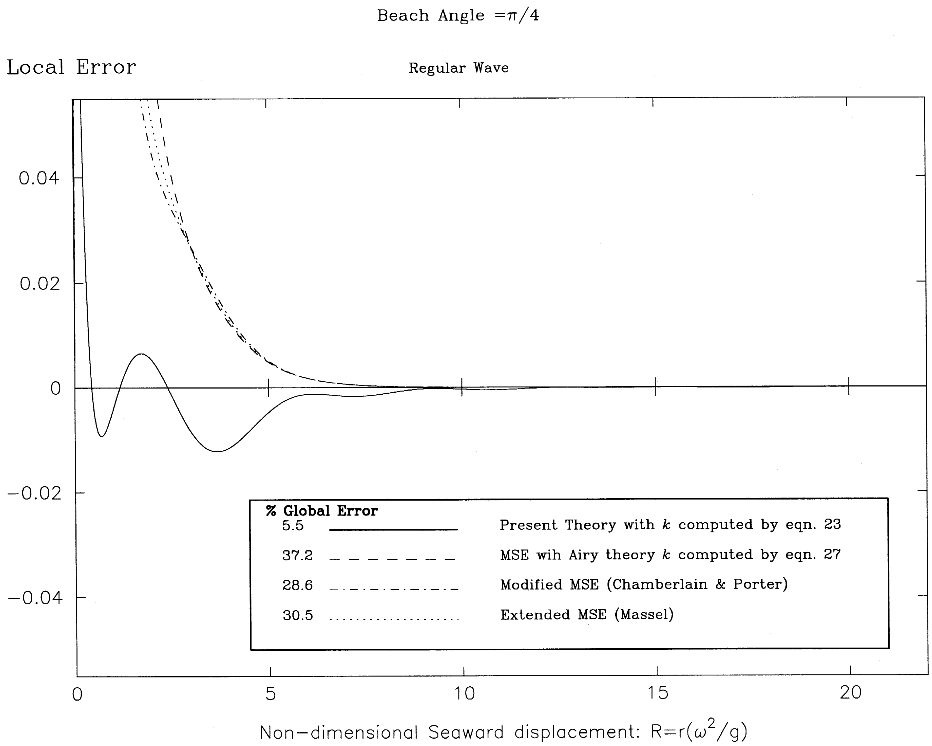

Fig. 9. Error computation for IVP: regular wave, beach angle = $\pi/4$.

voked, will be aware of evidence of further improvement of the results. In all cases, the compounded global error (here, computed from Eq. (33) and identified on diagram legends) is significantly reduced in the new scheme presented here. Note also that the oscillatory nature of the local error results in occasional 'fortuitous' vanishing of error. This had a peculiar effect on global error as calculated (but not observed there) by Ehrenmark and Williams (1996).

The opportunity has also been taken of comparing the new results with those that may be obtained using (i) the modified MSE devised by Chamberlain and Porter (1995), and (ii) the extended MSE devised by Massell (1993). Results are also shown in Figs. 7–10. In each of the tests, it is confirmed that (i) represents improvement on the basic MSE whilst the behaviour of (ii) seems only marginally better. The approach adopted in the present work however, is seen to be superior in comparison with all three and for all beach slopes tested and for both regular and singular components. Note that in all cases, a very substantial improvement in global error (quantified in the legend of each diagram) is obtained against the modified MSE and the extended MSE.

A possible additional reason for improvement on other results (i and ii) above, is attributed to the growth of *parasite solutions* (see Hildebrand, 1956, p. 209 for fuller details) as follows. The fundamental solution basis for all forms of the MSE (or indeed the SWE) consists of a pair, of which one component is necessarily logarithmically singular as the depth approaches zero. Any error in a marching scheme, which is intended to describe, say, the bounded solution will therefore implicitly induce, at each step, a small component of the singular solution. This 'parasitic solution' will grow as the calculation proceeds toward the shore. For the singular wave computation, the parasite is just a small component of the regular wave and is therefore relatively unimportant. For the regular wave, however, the parasite is a

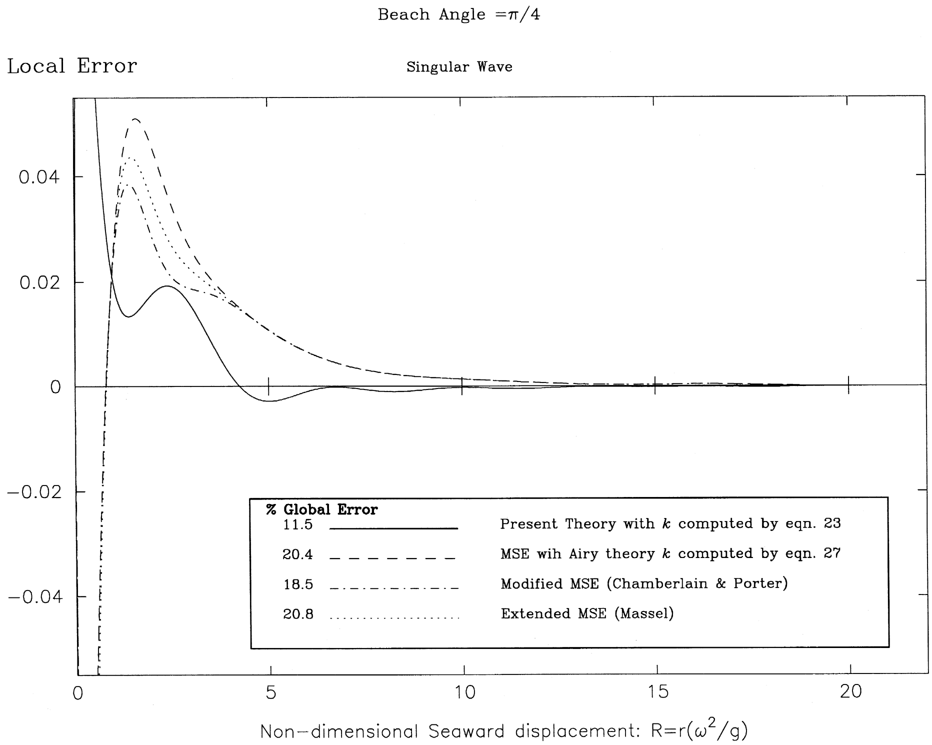

Fig. 10. Error computation for IVP: singular wave, beach angle = $\pi/4$.

component of the singular wave, which itself becomes large as the shoreline is approached. The present solution strategy gathers accurate values of the wavenumber (from an exact solution) at each step, thus, suppressing this particular source of error whereas the alternative comparative works are forced to rely on various approximations. This can be seen to some extent in Figs. 7 and 8 for the 30° slope but much more dramatically in Figs. 9 and 10 for the 45° slope. The observation sends a warning message generally to numerical modellers working on a steep slope in very shallow water.

6.3. Corollary

A referee of the first draft of this work legitimately enquired whether the regular solution could be computed right up to the shore line $R=0$ using the new approach. This query raised a number of ramifications.

Firstly, of course, in both old and new models, the wavenumber K is infinite at the shore line so that the IVP problem can be solved as close as we please to $R=0$ but not at $R=0$. Secondly, the approach to infinity of K is somewhat milder in the old model ($R^{-1/2}$) so that the new model might tend to oscillate more very close to the shoreline where the approach to infinity of K is like $R^{-1}/(\ln R)^2$.

In performing the IVP numerical work (for the 30° beach) prompted by the referee's remark however, we found rather surprisingly that the old solution suddenly began to recover accuracy very near the shoreline. This recovery was quite explosive with the local calculation at, say, $R=0.1$ showing more than 5% error whilst at $R=0.01$ this had diminished to 0.015%. Calculations for the 45° beach were similarly accurate near the shoreline. In both cases, the new solution behaved more poorly in this (microscopic) near-shore region. Clearly, these observations are of limited interest to the engineer, since the linear solution is already invalid in here; however, in view of the mathematical interest, further details of the solution here are discussed in Appendix A.

7. Evanescent wave modes

Several authors, e.g. Massel (1993) or Porter and Staziker (1995), have identified the need to include evanescent modes for a more accurate description in intermediate depths particularly when bed gradients are substantial. These studies have not however, considered effects in very shallow water and the present simple model allows us to gain a better understanding of the limitations of such proposals.

In order to keep the analysis and numerical work as simple as possible in this experiment, we will consider the addition of just one evanescent mode and study its effect on a solution that would otherwise be computed with only the fundamental oscillatory component using piecewise linear Airy theory. In the notation of Porter and Staziker, this would reduce the problem to the solution of a pair of coupled second order differential equations for ϕ_0, ϕ_1 the respective oscillatory and evanescent modes whose sum constitute the approximation to the (cross-shore) spatial dependence of potential. The equations are:

$$a_0\phi_0'' + a_0'\phi_0' - k_0^2 a_0\phi_0 + c_{00}T^2\phi_0 + (b_{10} - b_{01})T\phi_1' + c_{10}T^2\phi_1 = 0, \quad (34)$$

$$a_1\phi_1'' + a_1'\phi_1' - k_1^2 a_1\phi_1 + c_{11}T^2\phi_1 - (b_{10} - b_{01})T\phi_0' + c_{01}T^2\phi_0 = 0, \quad (35)$$

where T denotes $\tan\alpha$ and k_n are the roots of the eigenequation $k_n \tan(k_n h) = -1$ (the root $n=0$ being purely imaginary). We have discussed earlier in this paper the observation that $k_0 \to O(h^{-1/2})$ as $h \to 0$ implies the logarithmic singularity of ϕ_0 at the shoreline. To study the behaviour of ϕ_1, we write $k_n h = N\pi - \tan^{-1}(k_n^{-1})$ using the principal branch of $\tan^{-1}$. Examining the limit $h \to 0$, it is straightforward to show that $N = n$ and that:

$$k_n = \frac{n\pi}{h} - \frac{1}{n\pi}\left\{1 + \frac{h}{n^2\pi^2} + O(h^2)\right\}, \quad h \to 0 \quad (36)$$

so that the equivalent evanescent 'skin friction' factors k_n are $O(h^{-1})$ as $h \to 0$ also increasing in amplitude with increasing modal number. This may go some way toward explaining the observation noted

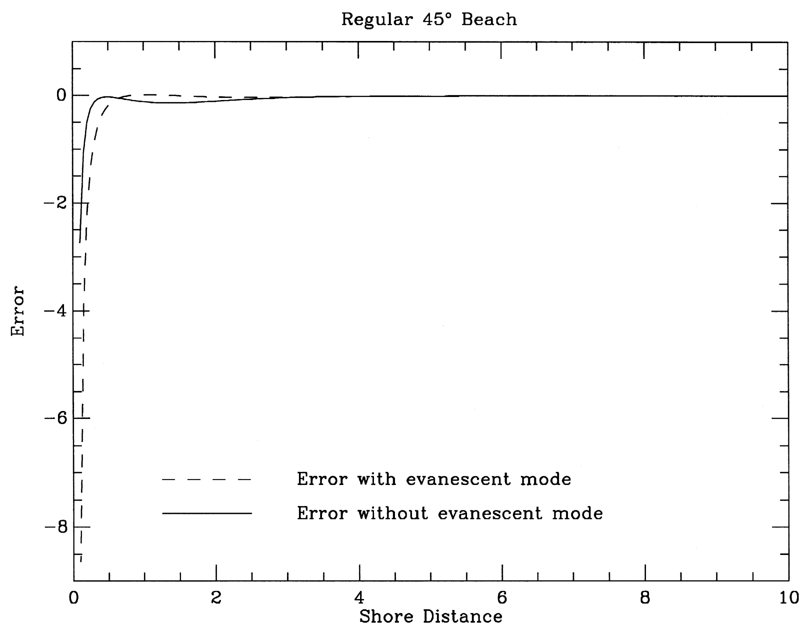

Fig. 11. Regular wave 45° beach: error comparison for IVP using the Porter and Staziker solution with and without one evanescent mode.

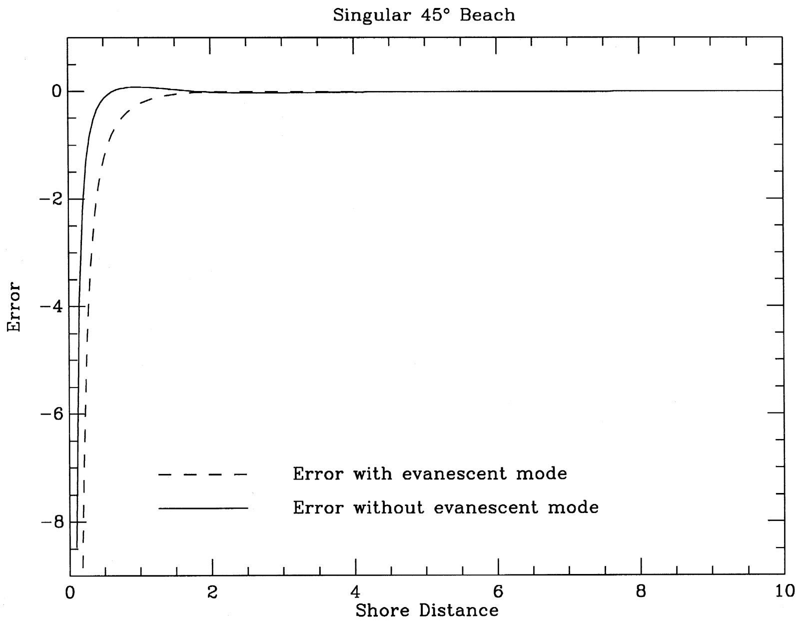

Fig. 12. Singular wave 45° beach: error comparison for IVP using the Porter and Staziker solution with and without one evanescent mode.

earlier namely that for the 'exact solution' $k = O(h^{-1}/(\ln(h))^2)$, which is a stronger singularity than that provided by the solution ϕ_0. Somehow, the addition of evanescent modes greatly modifies this behaviour in small depth.

We illustrate the observations by examining details and a computation on a steep (45°) beach of both regular and singular components taking account of the single evanescent wave mode. For this mode, $k_1 \sim \pi/h - 1/\pi - h/\pi^3$ so that $\tan k_1 h \sim -\tan(h/\pi) \sim -h/\pi$, which yields $a_1 \sim h/2$. Thus, very near the shore, the differential equation for $w = \phi_1$ is asymptotically almost equivalent to:

$$x^2 w'' + x w' - \pi^2 w = \text{r.h.s.} \tag{37}$$

giving fundamental 'complementary function solutions' $\{x^{\pi}, x^{-\pi}\}$. The inclusion of the first evanescent mode therefore fails to remain 'uniform' in the asymptotic sense when distances from the shore x are such that $x^{-\pi} > |\log x|$. The computational results are shown in Figs. 11 and 12 where both types of wave are computed and subtracted from the exact solution. This gives the actual error of the modified MSE both with and without a one term evanescent wave and it is easy to see that the evanescent mode has an improving effect only in intermediate depths (consistent with results of earlier authors) but that in very shallow water it provides unreliable results. Clearly, in view of the asymptotics noted above, taking account of further modes would only increase the difficulties in very shallow water.

8. Conclusion

The work has studied the behaviour of the MSE in the limit of vanishing depth. The results indicate that, provided the beach slope is sufficiently small, then the limit does not greatly affect the discrepancies between the 'exact' and MSE approximations of the same physical problem. However, for steep beaches, there is a finite difference, which reveals itself from terms $O(X^2)$ onwards for the regular wave and from terms $O(X)$ for the singular wave.

The work described has also attempted to highlight the possibility of a more liberal use of the MSE than was previously assumed possible. Tests against three versions (basic, modified and extended) of the MSE have been carried out and in all cases the approach suggested in this work proves to be superior particularly for very steep beaches. For the 45°

beach, for example, the global error generated by MSE experiments carried out on the interval $1 < X < 20$ is typically reduced from, respectively, 20%, 19% and 20% for the three versions to just 11% for the present application to the unbounded wave and 5% to the bounded wave. For the latter, the difference is quite staggering and is thought to reveal the activity of parasitic solutions, which themselves are unbounded as depth approaches zero. Further testing is required, particularly for non-uniform slopes but if this proves successful, the MSE with a modified wavenumber algorithm attached, should remain a competitive option for coastal wave modellers. That algorithm could take the form of an empirical rule expressing the application wavenumber as a multiple of the Airy theory value and this multiple could be expected to depend mainly on the local bed slope in the direction following the wave advance. Work on this has been started by the authors in conjunction with calculations on a non-uniform slope.

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Appendix A

We examine in detail, the very near field behaviour of the MSE using the proposal in the paper whereby the wave and group velocities that govern the propagation are determined heuristically from the exact solution. With all terms evaluated on $\theta = 0$, we have readily, from Eq. (23) and (29) that:

$$K^2 CC_g = \frac{\phi_s \phi_r' - \phi_r \phi_s'}{2\{\phi_s^2 + \phi_r^2\}^2} \tag{38}$$

so that, in view of the near-field asymptotics of the regular and singular solutions determined from Eqs. (20) and (21), we have:

$$K^2 CC_g \sim \frac{\pi^2 \alpha}{X(\log X)^4} \tag{39}$$

as $X \to 0$. Note the equivalent result using Airy theory, whereby from Eq. (18), it follows that $K^2 CC_g \sim 1$ as $X \to 0$.

We also need to examine the behaviour of CC_g. From the same source equations, we have:

$$CC_g \sim \alpha X \tag{40}$$

as $X \to 0$. The asymptotic form of the MSE (Eq. (2)) is therefore:

$$(\log X)^4 X \frac{\mathrm{d}}{\mathrm{d}X}\left\{ X \frac{\mathrm{d}\phi}{\mathrm{d}X} \right\} + \pi^2 \phi = 0 \tag{41}$$

an equation which, for arbitrary constants a_0, a_1, has the general solution:

$$\phi = \log X \left\{ a_0 \sin\left(\frac{\pi}{\log X} \right) + a_1 \cos\left(\frac{\pi}{\log X} \right) \right\} \tag{42}$$

and it is therefore seen immediately that the very near field structure of the fundamental solution pair remains consistent with that of both the exact solution and the Airy approximation to the full MSE, namely $\{1, \log X\}$. The constants in a given application would of course be different for the two methods and testing the reproduction of the regular standing wave, we have found that the Airy method is more stable whereas for the singular standing wave the new method is more stable. Both methods will, of course, as $X \to 0$ ultimately exhibit parasitic values, which will dominate the true values but this cannot be expected until $\ln|X|$ dominates π, i.e. $X \sim O(10^{-4})$. We are reminded however, that both components (regular and singular) are required in any description of a progressing wave, so even in this microscopically near-shore region (where the linear solution is in any case physically invalid and only of mathematical interest), the overall behaviour remains superior for the model proposed in the present work.

References

Berkhoff, J.C.W., 1974. Computation of combined refraction–diffraction. Delft Hydraulics Laboratory 119, 1–18.

Booij, N., 1983. A note on the accuracy of the Mild-Slope Equation. Coastal Engineering 7, 191–203.

Burnside, W., 1914. On the modification of a train of waves as it advances into shallow water. Proc. London Math. Soc. 14, 131–133.
Chamberlain, P.G., Porter, D., 1995. The modified Mild-Slope equation. J. Fluid Mech. 291, 393–407.
Chamberlain, P.G., Porter, D., 1997. Linear wave scattering by two-dimensional topography. In: Hunt, J.N. (Ed.), Gravity Waves in Water of Finite Depth. Computational Mechanics Publications, Southampton.
Dingemans, M.W., 1997. Water wave propagation over uneven bottoms, Pt. 1 — Linear Wave Propagation. Advanced Series on Ocean Engineering. World Scientific 13.
Ehrenmark, U.T., 1987. Far field asymptotics of the two-dimensional linearised sloping beach problem. SIAM J. Appl. Math. 47 (5), 965–981.
Ehrenmark, U.T., 1988. Overconvergence of the near-field expansion for linearized waves normally incident on a sloping beach. SIAM J. Appl. Math. 49 (3), 799–815.
Ehrenmark, U.T., 1994a. Second order wave computations on a steep beach. Proc. Int. Symp.: Waves-Physical and Numerical Modelling. UBC. Vancouver, 1041–1050.
Ehrenmark, U.T., 1994b. Set-down computations over an arbitrary inclined plane bed. J. Mar. Res. 52 (6), 983–998.
Ehrenmark, U.T., Williams, P.S., 1996. Using the mild-slope equation on steep shoals. In: Chwang, A.T., Lee, J.H.W., Leung, D.Y.C. (Eds.), Proceedings of The Second International Conference on Hydrodynamics. December 16–19, 1996. Hong Kong. Hydrodynamics. Theory and Applications 1 pp. 489–494.
Friedrichs, K.O., 1948. Water waves on a shallow sloping beach. Comm. Pure Appl. Math. 1, 109–134.
Hanson, J., 1926. The theory of ship waves. Proc. R. Soc., Ser. A 111, 491–529.
Hildebrand, F.B., 1956. Introduction to Numerical Analysis. McGraw-Hill.
Kirby, J.T., 1986. A general wave equation for waves over rippled beds. J. Fluid Mech. 162, 171–186.
Kirchhoff, G., 1899. Über stehende Schwingungen einer schweren Flüssigkeit. Ann. Phys. Chem. 10 (246), 32–46.
Lamb, H., 1932. Hydrodynamics. Dover.
Li, B., 1994a. A generalised conjugate gradient model for the mild-slope equation. Coastal Engineering 23, 215–225.
Li, B., 1994b. An evolution equation for water waves. Coastal Engineering 23, 227–242.
Longuet-Higgins, M.S., 1972. Recent progress in the study of longshore currents. In: Meyer, R.E. (Ed.), Waves on Beaches. Academic Press, New York.
Massel, S.R., 1993. Extended refraction–diffraction equation for surface waves. Coastal Engineering 19, 97–126.
Massel, S.R., 1995. Proc. Int. Symp.: Waves-Physical and Numerical Modelling. UBC. Vancouver.
Mei, C.C., 1992. The Applied Dynamics of Ocean Surface Waves. 2nd Edn. Wiley, New York.
Porter, D., Staziker, D.J., 1995. Extensions of the mild-slope approximation. J. Fluid Mech. 300, 367–382.
Radder, A.C., 1979. On the parabolic equation method for water wave propagation. J. Fluid Mech. 95 (1), 159–176.
Spain, B., Smith, M.J., 1970. Functions of Mathematical Physics. Van Nostrand-Reinhold.
Stoker, J.J., 1947. Surface waves in water of variable depth. Q. Appl. Math. 5, 1–54.