NOTES AND CORRESPONDENCE

Capillary-Gravity Waves on a Slow Current

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The purpose of this paper is to investigate amplitude modulation of surface capillary–gravity waves on a slowly varying current. In a recent paper Gerber (1987) has considered gravity waves on slow currents and discussed the modification to the Benjamin-Feir instability criterion due to the presence of the current. The present paper discusses the modulational aspects of waves on a current when surface tension forces are to be taken into account.

Consider surface capillary–gravity waves of initial wavenumber k_0 and phase speed c_0 propagating on a current $U(x, t)\mathbf{i}$. Assuming deep water, the wavenumber and speed are related by

$$c_0^2 = gk_0 + \tau k_0^3 \tag{1}$$

where g is the acceleration of gravity and τ is the surface tension coefficient per unit density. The wavenumber, $k(x, t)$, and the phase speed, $c(x, t)$, are related to the initial values, k_0 and c_0, by the Doppler relation

$$k(c + U) = k_0 c_0. \tag{2}$$

For surface gravity waves Longuet-Higgins and Stewart (1962) have shown that

$$\frac{k}{k_0} = 4\Bigg/\left[1 + \left(1 + \frac{4U}{c_0}\right)^{1/2}\right]^2 \tag{3}$$

where $c_0^2 = g/k_0$, $c^2 = g/k$, in the absence of surface tension. A simple relation like (3) between k_0 and c_0 is not possible for capillary–gravity waves since the dispersion relation (1) leads to a cubic in k,

$$\tau k^3 - (Uk)^2 + k(g + 2k_0c_0U) - (k_0c_0)^2 = 0. \tag{4}$$

Thus for a given downstream point, (x, t), and current, $U(x, t)$, the wavenumber, k, has to be determined, in general, using a numerical scheme.

The use of averaged variational principles to derive dispersion relations for surface waves is well known. The averaged variational principle could also be applied to nonuniform media, provided the inhomogeneities are slowly varying in space and time (see Whitham 1974). We will give a Lagrangian for capillary–gravity waves on a slowly varying current from which an averaged Lagrangian, depending on x and t, could be derived. This averaged Lagrangian leads to the determination of a dispersion relation, which in turn, has a bearing on the modulational properties of the waves.

It has been shown earlier (Easwaran, 1986) that for capillary-gravity waves on still water of depth h a Lagrangian is given by

$$L = \int_{-h}^{\eta} \left(\Phi_t + \frac{1}{2}(\Phi_x^2 + \Phi_y^2) + gy\right)dy - \tau(\sqrt{(1 + \eta_x^2)} - 1), \tag{5}$$

where η is the surface profile and Φ is the velocity potential. For weak currents the effect of vorticity will be reflected in the higher order dispersive terms, which would have a small influence on the cubic Schrödinger equation that describes modulational effects. Thus to the second order in small amplitude, a, we can assume that the current is irrotational so that it can be expressed in terms of a potential:

$$\bar{U}(x, t) = \nabla\Psi(x, t). \tag{6}$$

Then the Lagrangian for waves on a current may be written as

$$L = \int_{-h}^{\eta} ((\Phi + \Psi)_t + \frac{1}{2}(\Phi + \Psi)_x^2 + (\Phi + \Psi)_y^2 + gy)dy - \tau(\sqrt{(1 + \eta_x^2)} - 1). \tag{7}$$

Our aim is to derive the averaged Lagrangian from (7) and examine the cubic Schrödinger equation governing the modulation of waves by sideband disturbances. This will lead to expressions giving bounds on the possible sideband frequencies that causes unbounded growth in amplitude, resulting in the instability of the principal waves. Since the method is now classical we will only give the main results. In the following we consider, for the sake of brevity, a one dimensional current of the form $U(x)\mathbf{i}$.

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By introducing wave profiles of the form

$$\eta = a\cos\theta + \sum_1^\infty a_n \cos n\theta \quad (8)$$

$$\Phi = \sum_1^\infty \frac{A_n}{n}\cosh(nky)\sin n\theta$$

$$(\theta = kx - \omega t) \quad (9)$$

where a, a_n, A_n are small and along with the frequency, ω, and wavenumber, k, are slowly varying functions of x and t, and averaging over a cycle, one obtains the averaged Lagrangian

$$\mathcal{L} = -\frac{T}{4k}a^2 + \frac{\omega_1{}^2 a^2}{4k} + \left[1 - \frac{4T^2}{T_1{}^2}\right]\frac{3}{16}\tau k^4 a^4 + \frac{kT}{4}\left[-\frac{1}{4} - \frac{T}{2T_1} + \frac{T^2}{4T_1{}^2}\right]a^4 + \cdots \quad (10)$$

with $\omega_1 = \omega - kU$, $T = gk + \tau k^3$, $T_1 = gk - 2\tau k^3$. The averaged variational principle $\mathcal{L}_a = 0$ then gives the dispersion relation

$$\omega_1 = \sqrt{T}\left[1 - \frac{3\tau k^5}{16T}\left(1 - \frac{4T^2}{T_1{}^2}\right)a^2 + \frac{k^2}{4}\left(1 + \frac{2T}{T_1} - \frac{T^2}{T_1{}^2}\right)a^2\right] \quad (11)$$

up to $O(a^2)$ terms.

Equation (10) contains a nonlinear correction term to the dispersion proportional to a^2 but does not include higher order dispersive effects. These effects are, however, easily taken into account (see Whitham 1974, page 526). The eventual expression obtained for the dispersion is

$$\omega_1 = \sqrt{T} + \alpha a^2 + \beta\frac{a_{xx}}{a} \quad (12)$$

where

$$\alpha = \alpha(x) = \sqrt{T}\left[-\frac{3\tau k^5}{16T}\left(1 - \frac{4T^2}{T_1{}^2}\right) + \frac{k^2}{4}\left(1 + \frac{2T}{T_1} - \frac{T^2}{T_1{}^2}\right)\right] \quad (13)$$

$$\beta = \beta(x) = \frac{1}{2}\frac{d^2\sqrt{T}}{dk^2} = \frac{1}{8}\frac{(g + 3\tau k^2)^2}{T^{3/2}} - \frac{3}{2}\frac{\tau k}{T^{1/2}}. \quad (14)$$

The x dependence of α and β reflects the dependence of the wavenumber k on x.

As usual the sideband modulation analysis proceeds by introducing small amplitude perturbations around the central wavenumber and frequency and following the time development of the perturbations (see Gerber 1987; Easwaran 1987, for details). It can be shown that the complex amplitude of the modulation envelope A is governed by the cubic Schrödinger equation

$$\frac{\partial A}{\partial t} + (C + U)\frac{\partial A}{\partial x} + \left(\frac{1}{2}\frac{dC}{dx} + \frac{3}{4}\frac{dU}{dx}\right)A + i\beta(x)\frac{\partial^2 A}{\partial x^2} + i\alpha(x)|A|^2 A = 0 \quad (15)$$

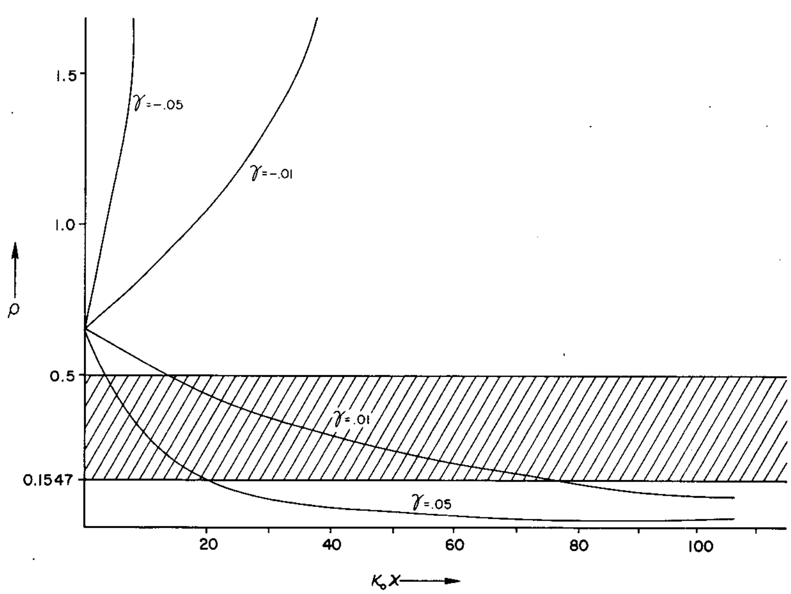

FIG. 1. Modification of $\rho = \tau k^2/g$ with distance k_0x for various values of $\gamma = U/(k_0c_0x)$. Shaded areas indicate regions where the waves are stable to all sideband frequency perturbations.

where C is the group velocity $d\omega/dk$, which is also a function of x. A detailed analysis of Eq. (15) shows that the slowly varying modulation amplitude becomes unbounded in time if the small perturbation frequency, κ, satisfies

$$\kappa^2 - \frac{2\alpha(x)}{\beta(x)} a_0^2 \exp\left\{-\int_0^x \left[\left(\frac{dC}{dx} + \frac{3}{2}\frac{dU}{dx}\right) \bigg/ (C+U)\right] dx\right\} < 0 \quad (16)$$

where a_0 is the initial value of the amplitude a. From (16) it follows that if

$$\frac{\alpha(x)}{\beta(x)} < 0 \quad (17)$$

the criterion (16) for instability is never satisfied for any value of κ.

Introducing the dimensionless number

$$\rho = \frac{\tau k^2}{g} \quad (18)$$

and using the expressions (13) and (14) for α and β the condition (17) may be written as

$$\left\langle\left\{-\frac{3\rho}{16(1+\rho)}(1-4f(\rho)^2) + \frac{1}{4}(1+2f(\rho)-f(\rho)^2)\right\}(1+\rho)^2\right\rangle \bigg/ (1-6\rho-3\rho^2) < 0 \quad (19)$$

where

$$f(\rho) = \frac{1+\rho}{1-2\rho}.$$

A sign analysis of the expression on the left of (19) shows that the range of values of ρ for which stability is guaranteed for all sideband perturbations is given by

$$\frac{2}{\sqrt{3}} - 1 < \rho < \frac{1}{2}. \quad (20)$$

In Fig. 1 we have plotted the dimensionless number ρ against the dimensionless downstream distance k_0x for various values of $\gamma = U/k_0c_0x$ (Note that a constant γ as x varies indicates a current linearly varying with downstream distance). Positive values of γ indicate currents in the direction of the waves and negative values indicate currents against the wave direction. Only those wavenumbers whose initial value of ρ lies within the limit (20) will go through a stable regime. The figure shows that for more rapid currents there is a faster "condensation" of waves and the stable regions are shorter in length.

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