On the Hasselmann and Zakharov Approaches to the Kinetic Equations for Gravity Waves

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11 November 1994 and 28 March 1995

ABSTRACT

It is shown for the first time analytically that two different approaches describing gravity wave turbulence introduced by Hasselmann in 1962 and Zakharov in 1968 result in the same kinetic equation for the second-order correlator.

1. Introduction

A two-dimensional potential flow of an ideal incompressible fluid with a free surface in a gravity field fluid is described by the following set of equations:

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad (\phi_z \to 0, z \to -\infty),$$

$$\eta_t + \eta_x\phi_x + \eta_y\phi_y = \phi_z|_{z=\eta},$$

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + g\eta = 0|_{z=\eta},$$

where $\eta(x, y, t)$ is the shape of a surface, $\phi(x, y, z, t)$ a potential function of the flow, and g a gravitational constant.

Hasselmann (1962, 1963) derived the Boltzmann-type equation for the rate of change of the "number density" of the energy spectrum,

$$n_{\vec{k}}[n_{\vec{k}}\delta_{\vec{k}-\vec{k}'} = \langle \eta_{\vec{k}}\eta_{\vec{k}'}\rangle/(\alpha\omega_{\vec{k}})]$$

$$\frac{\partial n_{\vec{k}}}{\partial t} = \alpha^2 \int T_H^2[n_{\vec{k}_2}n_{\vec{k}_3}(n_{\vec{k}} + n_{\vec{k}_1}) - n_{\vec{k}}n_{\vec{k}_1}(n_{\vec{k}_2} + n_{\vec{k}_3})] \times \delta_{\vec{k}+\vec{k}_1-\vec{k}_2-\vec{k}_3}\delta_{\omega_{\vec{k}}+\omega_{\vec{k}_1}-\omega_{\vec{k}_2}-\omega_{\vec{k}_3}} d\vec{k}_1 d\vec{k}_2 d\vec{k}_3, \quad (1)$$

and calculated the expression for T_H^2—the amplitude for the four-wave interaction. [Here $\omega_{\vec{k}} = (g|\vec{k}|)^{1/2}$, and α is an arbitrary constant.]

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Zakharov (1968) has shown that the variables $\eta(x, y, t)$ and $\psi(x, y, t) = \phi(x, y, z, t)|_{z=\eta}$ are canonically conjugated and that their Fourier transforms satisfy the equations

$$\frac{\partial \psi_{\vec{k}}}{\partial t} = -\frac{\delta H}{\delta \eta_{\vec{k}}^*} \quad \frac{\partial \eta_{\vec{k}}}{\partial t} = \frac{\delta H}{\delta \psi_{\vec{k}}^*}.$$

Here $H = K + U$ is the total energy of the fluid with the following kinetic and potential energy terms:

$$K = \frac{1}{2}\int dx \int_{-\infty}^{\eta} v^2 dz \quad U = \frac{g}{2}\int \eta^2 dx.$$

After introducing a normal complex variable a_k,

$$\eta_{\vec{k}} = \left(\frac{\omega_{\vec{k}}}{2g}\right)^{1/2}(a_{\vec{k}} + a_{-\vec{k}}^*)$$

$$\psi_{\vec{k}} = -i\left(\frac{2g}{\omega_{\vec{k}}}\right)^{1/2}(a_{\vec{k}} - a_{-\vec{k}}^*), \quad (2)$$

he has also derived the Boltzmann-type equation for the pair correlation function $n_{\vec{k}}\delta_{\vec{k}-\vec{k}'} = \langle a_{\vec{k}}a_{\vec{k}'}\rangle$,

$$\frac{\partial n_{\vec{k}}}{\partial t} = 4\pi \int T_Z^2[n_{\vec{k}_2}n_{\vec{k}_3}(n_{\vec{k}} + n_{\vec{k}_1}) - n_{\vec{k}}n_{\vec{k}_1}(n_{\vec{k}_2} + n_{\vec{k}_3})] \times \delta_{\vec{k}+\vec{k}_1-\vec{k}_2-\vec{k}_3}\delta_{\omega_{\vec{k}}+\omega_{\vec{k}_1}-\omega_{\vec{k}_2}-\omega_{\vec{k}_3}} d\vec{k}_1 d\vec{k}_2 d\vec{k}_3, \quad (3)$$

and has calculated the transfer matrix T_Z (Zakharov et al. 1968). These two expressions, T_H and T_Z, look very different, but both of them are defined up to an arbitrary function that is equal to zero on the resonant manifold:

$$\vec{k} + \vec{k}_1 = \vec{k}_2 + \vec{k}_3,$$

$$\omega_{\vec{k}} + \omega_{\vec{k}_1} = \omega_{\vec{k}_2} + \omega_{\vec{k}_3}. \quad (4)$$

Although Crawford et al. (1980) already asked in 1980 if the results of Zakharov and Hasselmann were identical, this question has remained unanswered until now.

2. Conclusions

In this work we have checked both T_H and T_Z on the resonant manifold (4). At first they were checked in the one-dimensional case where the compact analytic form for the resonant surface can be obtained. In this case, the resonant manifold (4) has two different solutions (Dyachenko and Zakharov 1994), the trivial one

$$k_2 = k_1, \quad k_3 = k, \quad \text{or} \quad k_2 = k, \quad k_3 = k_1, \quad (5)$$

and the nontrivial:

$$k = a(1 + \zeta)^2, \quad k_1 = a(1 + \zeta)^2\zeta^2, \quad k_2 = -a\zeta^2,$$

$$k_3 = a(1 + \zeta + \zeta^2)^2; 0 < \zeta < 1. \quad (6)$$

On the resonant surface (5), T_Z and T_H are different only by sign, so that

$$T_Z = -T_H = \frac{1}{4}\pi^2(kk_1)\min(|k|, |k_1|).$$

(Note: Sign is not important because only T_H^2 and T_Z^2 are involved in the kinetic equation.) The constant α in (1) is equal to $1/4\pi^2$.

It was shown in Dyachenko and Zakharov (1994) and Dyachenko et al. (1995) that T_Z is identically equal to zero on the resonant manifold (6). We have shown analytically that T_H is also identically equal to zero on the same manifold (6). To do this, we substituted the resonant parameterization (6) into the equations for T_H in Zakharov (1968) and used Mathematica to obtain zero.

For the full two-dimensional case we have also found the general solution for the resonant surface (4):

$$\vec{k} = (\zeta - x)^2\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}, \quad \vec{k}_1 = (\zeta + x)^2\begin{bmatrix}\cos(\theta_1)\\ \sin(\theta_1)\end{bmatrix},$$

$$\vec{k}_2 = (\zeta - y)^2\begin{bmatrix}\cos(\theta_2)\\ \sin(\theta_2)\end{bmatrix},$$

$$\vec{k}_3 = (\zeta + y)^2\begin{bmatrix}\cos(\theta_3)\\ \sin(\theta_3)\end{bmatrix}, \quad 0 < \zeta < 1. \quad (7)$$

Here

$$\cos(\theta) = (1 + |\vec{k}|^2 - |\vec{k}_1|^2)/2|\vec{k}|,$$

$$\cos(\theta_1) = (1 - |\vec{k}|^2 + |\vec{k}_1|^2)/2|\vec{k}_1|,$$

$$\cos(\theta_2) = (1 + |\vec{k}_2|^2 - |\vec{k}_3|^2)/2|\vec{k}_2|,$$

$$\cos(\theta_3) = (1 - |\vec{k}_2|^2 + |\vec{k}_3|^2)/|\vec{k}_3|,$$

and T_Z and T_H are calculated numerically for several values of ζ, x and y. They coincide up to the roundoff error. In (R. Lin, N. Huang, and W. Perrie 1994, personal communication) the energy transform was studied numerically using both Hasselman's (1) and Zakharov's (3) approaches. Comparison of the results shows good agreement between these two models. Slight deviation can be explained by numerical errors when calculating T_H and T_Z on the resonant surface, because T_H and T_Z coincide on the resonant surface only, but not in the vicinity of it.

Acknowledgments. A. I. Dyachenko acknowledges support from ONR Grant N00 14-92-J-1343 and Russian Basic Research Foundation Grant N00 94-01-00898. Y. Lvov acknowledges support from AFOSR Grant F49620-94-0144DEF.

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