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PERIOD EQUATION FOR WAVES OF RAYLEIGH TYPE ON A LAYERED, LIQUID-SOLID HALF SPACE

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ABSTRACT

A convenient formulation of the boundary conditions applicable to elastic wave propagation in a layered, solid half space was obtained by Haskell in terms of matrix algebraic operations. Developing this method further, the analogous problem for liquid layers is solved, and the treatment of liquid-solid interfaces is defined in matrix notation. This leads to a simple expression for the period equation for surface waves of the Rayleigh type on a half space of solid and liquid layers arbitrarily interspersed. This formulation of the period equation appears to yield the most rapid method for numerical computations on surface wave dispersion. It is the basis for computations used in several recent studies of earthquake surface-wave dispersion.

Introduction

In this paper the multi-layer Rayleigh wave dispersion theory of Haskell (1953) is generalized to include a method of treating a layered, liquid half space or liquid layers interbedded with solid layers, as well as the case of a layered solid originally developed by Haskell. This theory is suitable for rapid numerical solution of the dispersion equation (relation between wave period and phase velocity) for any number of layers which might be of practical interest in seismological problems. Haskell, following Thomson (1950), used a layer matrix, developed from solutions of the equations of motion in a homogeneous layer, as a convenient device for satisfying the boundary conditions at the interfaces between layers. The present theory retains this useful feature as well as the same form of the solid layer expressions. The elements of Haskell's matrices for solid layers are given in the appendix of this paper for convenient reference. The essential points of the matrix formulation are reviewed briefly below and new contributions to the theory are developed in detail in the following section.

Using the symbols defined in table I, assume that the z-axis points downward into the free surface of a layered solid and the x-axis is in the direction of propagation of plane waves. Then the required boundary conditions at discontinuities are continuity of u, w, p_{xz} and p_{zz} for Rayleigh waves and v and p_{yz} for Love waves. For the layer m bounded below by interface m and above by interface $m - 1$, Haskell's treatment leads to the following equations written in matrix form

$$\left\| \begin{matrix} \dot{u}/c \\ \dot{w}/c \\ p_{zz} \\ p_{xz} \end{matrix} \right\|_m = a_m \left\| \begin{matrix} \dot{u}/c \\ \dot{w}/c \\ p_{zz} \\ p_{xz} \end{matrix} \right\|_{m-1} \quad (1)$$

$$\left\| \begin{matrix} \dot{v}/c \\ p_{yz} \end{matrix} \right\|_m = b_m \left\| \begin{matrix} \dot{v}/c \\ p_{yz} \end{matrix} \right\|_{m-1} \quad (2)$$

for Rayleigh waves and Love waves, respectively. The a_m is a 4×4 matrix whose only arguments are c, T, h_m, α_m, β_m and ρ_m. The b_m is a 2×2 matrix whose only arguments are c, T, h_m, β_m and ρ_m. The elements of the matrix, a_m, are given in the appendix. Haskell showed that (1) and (2) written for each layer of a multi-layered structure facilitate the iterative elimination of the equations which represent the interface boundary conditions.

Although Haskell's treatment of the Love-wave problem is complete, certain interesting and useful points with regard to the treatment of liquid layers in the Rayleigh wave problem require clarification. A liquid layer matrix of 4 rows and 4 columns to be used in place of a_m in (1) was given by Haskell (1953). This matrix gives satisfactory results when used to represent a bounded liquid layer or layers with no overlying solid layers. This configuration is useful in representing the ocean overlying the solid crust and mantle of the earth. However, Haskell pointed out

TABLE I

DEFINITION OF SYMBOLS

α	=	Compressional velocity, sound velocity in water.
β	=	Shear velocity
ρ	=	Density
c	=	Horizontal phase velocity
T	=	Period
ω	=	Angular frequency
k	=	Horizontal wave number
u, v, w	=	Particle displacements in x, y, z-directions, respectively.
p_{ij}	=	Traction in i-direction across plane normal to j-axis
r_α	=	$(c^2/\alpha^2 - 1)^{1/2}$
r_β	=	$(c^2/\beta^2 - 1)^{1/2}$
h	=	Layer thickness
P_m	=	$kh_m r_{\alpha m}$
Q_m	=	$kh_m r_{\beta m}$
γ_m	=	$2\beta_m^2/c^2$

Note: In this paper r_α and r_β are taken to be positive imaginary when $c < \alpha$ or $c < \beta$.

(personal communication) that the matrix leads to overspecified boundary conditions when used in any configuration where a solid layer overlies a liquid layer. Therefore it is preferable to revise this portion of the theory. A 2×2 matrix, l_m, derived from Haskell's work, replaces his 4×4 liquid-layer matrix. In the general, multi-layer case where solid and liquid layers are interbedded in any order, the boundary conditions to be satisfied are continuity of u, w, p_{zz} and p_{xz} at solid-solid interfaces, and continuity of w and p_{zz} at solid-liquid and liquid-liquid interfaces. These two types of boundary conditions intermixed in any way can be satisfied as shown below in terms of matrix operations yielding a convenient form of the period equation. The resulting theory then includes the multi-layered, solid half space and the multi-layered, liquid half space as special cases. In this theory, the bottom or semi-infinite layer can be either liquid or solid.

This theory is the basis of the multi-layer, surface-wave dispersion program used at Lamont Observatory for the past few years. Some examples of results obtained with this program are given by Oliver, Dorman, and Sutton (1959); Dorman, Ewing, and Oliver (1960); Dorman and Prentiss (1960); and Oliver and Dorman (1961) where continental and oceanic models were treated. In addition the general liquid-solid layering feature of this program has been useful in work done by Kutschale (1961) on propagation of acoustic waves over a stratified floating ice sheet in both deep and shallow water. A version of this program used on the IBM 709 or 7090 does automatic multiple case processing with options of finding Love and/or Rayleigh wave dispersion curves in any mode and any period range in each case. Some points regarding the numerical solution were discussed by Dorman, Ewing, and Oliver (1960).

A problem closely related to the dispersion computation is the computation of the vertical distribution of particle motion and stress due to the surface wave. This computation requires as a starting condition a solution (c vs. T) of the period equation and, in cases involving solid layers, the relative amplitudes of the vertical and horizontal components at one or more points. The latter values may be obtained as by-products of the solution of the period equation. A computation of this sort for particle motion in solid layered structures was devised by Dorman and Prentiss (1960).

Theory

As pointed out above, the boundary conditions at interfaces between liquid and solid layers or between liquid layers are continuity of w and p_{zz} only. Therefore the equation for a liquid layer corresponding to (1) is

$$\left\| \begin{matrix} \dot{w}/c \\ p_{zz} \end{matrix} \right\|_m = l_m \left\| \begin{matrix} \dot{w}/c \\ p_{zz} \end{matrix} \right\|_{m-1} \tag{3}$$

The l_m is the 2×2 matrix

$$\left\| \begin{matrix} \cos P_m & i(r_{\alpha_m}/\rho_m c^2) \sin P_m \\ i(\rho_m c^2/r_{\alpha_m}) \sin P_m & \cos P_m \end{matrix} \right\| \tag{4}$$

The expression (4) is obtained by taking the four central elements from the matrix given by Haskell in equation (6.3) of his paper.

Using the notation $S = (\dot{u}/c, \dot{w}/c, p_{zz}, p_{xz})$, and $L = (\dot{w}/c, p_{zz})$, we can write the following set of q equations for a series of q solid layers

$$\begin{aligned} S_{m+1} &= a_{m+1}S_m \\ \vdots \quad & \quad \vdots \\ S_{m+q} &= a_{m+q}S_{m+q-1} \end{aligned} \tag{5}$$

Such equations are useful in computing the distribution of motion for a layered half space as well as in deriving the period equation. Using the relation expressed by (1), Haskell found the solution for S_{m+q} in terms of S_m in the system (5), i.e.,

$$S_{m+q} = a_{m+q}a_{m+q-1} \cdots a_{m+1}S_m \tag{6}$$

which gives a relation between S_{m+q} and S_m in terms of a 4×4 array of numbers, $a_{m+q}a_{m+q-1} \cdots a_{m+1}$. Solving the system (5) in this way satisfies the boundary conditions at the intermediate interfaces, $m + 1, m + 2_1 \cdots m + q - 1$. Similarly, for a system of sound waves passing through a layered liquid wave guide,

$$L_{m+q} = l_{m+q} l_{m+q-1} \cdots l_{m+1} L_m \tag{7}$$

is the relation which correctly satisfies the boundary conditions at the intermediate interfaces.

In (6) the product $a_{m+q} a_{m+q-1} \cdots a_{m+1}$ has the same form as a_m. Using this, the problem of representing the effect of a number of solid layers between two liquid layers is of the same form as the problem of representing the effect of a single solid layer. Similarly, by (7) the use of $l_{m+q} l_{m+q-1} \cdots l_{m+1}$ allows us to give the effect of a number of liquid layers between two solid layers in the same form, involving a 2×2 matrix, as for a single liquid layer. Therefore, in order to carry the description of motion and stress through sections of arbitrarily interbedded liquid and solid layers, it is sufficient to treat the problem of a series of alternating liquid and solid layers. This can be done by stating and then proving two theorems. They are: (A) at an interface, q, where the medium, q, overlying the interface is a solid layer, the effect of all the overlying layers can be represented by a single matrix of 4 rows and 2 columns regardless of the complexity of the layering above q; and (B) at an interface, q, where the medium, q, overlying the interface is a liquid layer, the effect of all the overlying layers can be represented by a single column vector of 2 elements regardless of the complexity of layering above q.

The two theorems are expressed by the equations

$$S_q = M \begin{Vmatrix} \dot{u}_{q-1}/c \\ \dot{w}_0/c \end{Vmatrix}, \qquad \text{for overlying solid (A)}$$

$$L_q = N(\dot{w}_0/c), \qquad \text{for overlying liquid (B)}$$

The M is a matrix of 4 rows and 2 columns and the N is a column vector of 2 elements. In (A) S_q gives the motion in the overlying solid medium at interface q.

Now if (A) is true, we can prove (B) by using a suggestion made by Thomson (1950), namely, that the fourth of equations (A),

$$p_{xz} = 0 = M_{41}(\dot{u}_{q-1}/c) + M_{42}(\dot{w}_0/c)$$

can be rewritten

$$\dot{u}_{q-1}/c = -\frac{M_{42}}{M_{41}}(\dot{w}_0/c). \tag{8}$$

Using (8), we can write the second and third of equations (A) as

$$\dot{w}_q/c = (M_{22} - M_{21}(M_{42}/M_{41}))(\dot{w}_0/c)$$

and (9)

$$(p_{zz})_q = (M_{32} - M_{31}(M_{42}/M_{41}))(\dot{w}_0/c)$$

which, in matrix notation, is a set of equations of the form (B) with

$$N_{11} = M_{22} - M_{21}(M_{42}/M_{41}) \quad \text{and} \quad N_{21} = M_{32} - M_{31}(M_{42}/M_{41}).$$

Then writing $L_{q+1} = l_{q+1}L_q$ as in (3) for a liquid layer underlying interface q, L_q can be eliminated, giving

$$L_{q+1} = l_{q+1}N(\dot{w}_0/c) \tag{10}$$

But $l_{q+1}N$ is of the form of N which proves (B), since we can replace $q + 1$ by q in (10). Therefore (A) implies (B).

Conversely, if (B) is true, S_q is

$$S_q = \left\| \begin{array}{c} \dot{u}_q/c \\ \hline L_q \\ \hline 0 \end{array} \right\| = \left\| \begin{array}{c} \dot{u}_q/c \\ \hline N(\dot{w}_0/c) \\ \hline 0 \end{array} \right\| = \left\| \begin{array}{cc} 1 & 0 \\ 0 & N_{11} \\ 0 & N_{21} \\ 0 & 0 \end{array} \right\| \left\| \begin{array}{c} \dot{u}_q/c \\ \dot{w}_0/c \end{array} \right\| \tag{11}$$

where S_q represents the motion in an underlying solid medium at interface, q. Then writing $S_{q+1} = a_{q+1}S_q$ as in (1) for the solid layer, $q + 1$, underlying interface q, S_q can be eliminated, giving

$$S_{q+1} = a_{q+1}M \left\| \begin{array}{c} \dot{u}_q/c \\ \dot{w}_0/c \end{array} \right\| \tag{12}$$

But $a_{q+1}M$ is of the form M which proves (A) since we can replace $q + 1$ by q in (12). Hence (B) implies (A).

Now if the top layer is a solid, then (A) is true for $q = 1$, the first interface, since the boundary conditions of vanishing stresses at the free surface give $S_0 = (\dot{u}_0/c, \dot{w}_0/c, 0, 0)$. Or if the top layer is a liquid, then (B) is true for $q = 1$, since the surface boundary condition then gives $L_0 = (\dot{w}_0/c, 0)$. Therefore (A) and (B) are proved for any arbitrary layered structure in which both liquid and solid layers are present. In the case of a number of successive solid layers the index, $q - 1$, of $\dot{u}_{q-1}/c$ in (A) is interpreted as the index of the next liquid-over-solid interface above q if there is one, otherwise as the index of the free surface.

No mention has been made above of the method of including the effect of the bottom, semi-infinite layer. In this problem two cases arise, that of the solid bottom layer and that of the liquid bottom layer. For the case where the bottom or n^{th} medium is a solid Haskell wrote the equation

$$\left\| \begin{array}{c} \Delta_n' \\ \Delta_n' \\ w_n' \\ w_n' \end{array} \right\| = E_n^{-1}S_{n-1} \tag{13}$$

which we note is valid for medium n at the interface $n - 1$ whether layer $n - 1$ is solid or liquid. The Δ_n' and w_n' are constant coefficients of the motion in the n^{th}

layer and E^{-1} is a 4×4 matrix function whose arguments are c, T, α_n, β_n and ρ_n. The Δ_n', ω_n' and E_n^{-1} are defined in Haskell's paper. If layer $n - 1$ is liquid, S_{n-1} takes the form indicated by (11).

In (13) the second equation may be subtracted from the first, and the fourth equation may be subtracted from the third, giving

$$\begin{Vmatrix} 0 \\ 0 \end{Vmatrix} = F_n S_{n-1} \tag{13a}$$

where F_n is a matrix of 2 rows and 4 columns obtained from E_n^{-1} by subtraction of the corresponding rows of the matrix. A simple form of F_n is given in the appendix of this paper for convenience. Then regardless of the layering above interface $n - 1$, we may use (A) to eliminate S_{n-1} in (13a), giving

$$\begin{Vmatrix} 0 \\ 0 \end{Vmatrix} = F_n M \begin{Vmatrix} \dot{u}_{q-1}/c \\ \dot{w}_0/c \end{Vmatrix} \tag{14}$$

where F_nM is seen to be a 2×2 matrix. Solutions of (14) require

$$| F_nM | = 0 \tag{15}$$

which is a period equation for normal modes of the structure since it contains the parameters of all the layers as well as the period and phase velocity of the waves.

An expression equivalent to (13) for a bottom, or n^{th}, layer that is liquid will be derived in the notation of Ewing, Jardetzky, and Press (1956). A bottom liquid layer can be considered by means of a velocity potential, $\bar{\varphi}$, where

$$\bar{\varphi}_n = (A_n \exp\{ikr_{\alpha_n}z\} + A_n' \exp\{-ikr_{\alpha_n}z\}) \exp\{i(\omega t - kx)\}$$

A_n and A_n' are constants. From the properties of the velocity potential (see Ewing, Jardetzky, and Press, p. 7) and noting that p_{zz} as used here is the negative of the fluid pressure, we have in the n^{th} layer

$$\dot{w}/c = \frac{1}{c}\frac{\partial \bar{\varphi}_n}{\partial z} = i\frac{k}{c} r_{\alpha_n}(A_n \exp\{ikr_{\alpha_n}z\} - A_n' \exp\{-ikr_{\alpha_n}z\}) \exp\{i(\omega t - kx)\}$$

$$p_{zz} = \rho_n \frac{\partial \bar{\varphi}_n}{\partial t} = i\omega\rho_n\bar{\varphi}_n \tag{16}$$

Considering surface waves only, we must have $A_n' = 0$ in (16) in order that the disturbance may vanish at great depth. Then placing the origin of z at the interface $n - 1$ and setting $z = 0$ in (16), equations (16) can be written in matrix notation as

$$\begin{Vmatrix} A_n \\ A_n \end{Vmatrix} = \begin{Vmatrix} c^2/r_{\alpha_n} & 0 \\ 0 & 1/\rho_n \end{Vmatrix} L_{n-1} \tag{17}$$

omitting the common factor $i\omega \exp \{i(\omega t - kx)\}$. Equation (17) corresponds to (13). If layer $n - 1$ is solid, L_{n-1} takes the form indicated by (9).

In (17) the second equation can be subtracted from the first, giving

$$0 = (c^2/r_{\alpha_n}, -1/\rho_n)L_{n-1} = G_n L_{n-1} \tag{17a}$$

Then regardless of the layering above interface $n - 1$, we may use (B) to eliminate L_{n-1} in (17a), giving

$$0 = G_n N(\dot{w}_0/c) \tag{18}$$

where $G_n N$ is a scalar. Solutions of (18) require

$$G_n N = 0 \tag{19}$$

which is then the period equation for normal modes in the liquid bottom case.

Summary

The roots of (15) or (19) give the dispersion relations, c vs. T, for normal modes of all structures with any number of flat, homogeneous layers bounded above by a free surface and below by a semi-infinite medium. Liquid and solid layers may be interspersed in any order. Equations (15) and (19) cover the cases of a solid and liquid semi-infinite medium, respectively. These period equations are formed from products of matrices, one per layer, which provide for elimination of the boundary equations arising at each interface. A 4×4 matrix, due to Haskell, is used for solid layers as in (6) and a 2×2 matrix is used for liquid layers as in (7). Application of the proper boundary conditions at liquid-solid interfaces is represented by the matrix algebraic operations of (9) and (11).

The treatment above is readily adapted for numerical evaluation of the left hand side of (15) or (19) by carrying a running product matrix, layer by layer, from the top downward. Roots of (15) or (19) may then be found by successive approximation. The matrix formulation is also useful for computation of the vertical distribution of particle motion and stress in the normal modes.

Appendix

The elements of the 4×4 solid layer matrix, a_m, are (see Haskell, 1953):

$$(a_m)_{11} = (a_m)_{44} = \gamma_m \cos P_m - (\gamma_m - 1) \cos Q_m$$

$$(a_m)_{12} = (a_m)_{34} = i[((\gamma_m - 1)/r_{\alpha_m}) \sin P_m + \gamma_m r_{\beta_m} \sin Q_m]$$

$$(a_m)_{13} = (a_m)_{24} = -(\rho_m c^2)^{-1} (\cos P_m - \cos Q_m)$$

$$(a_m)_{14} = i(\rho_m c^2)^{-1} [r_{\alpha_m}^{-1} \sin P_m + r_{\beta_m} \sin Q_m]$$

$$(a_m)_{21} = (a_m)_{43} = -i[r_{\alpha m}\gamma_m \sin P_m + ((\gamma_m - 1)/r_{\beta_m}) \sin Q_m]$$

$$(a_m)_{22} = (a_m)_{33} = -(\gamma_m - 1)\cos P_m + \gamma_m \cos Q_m$$

$$(a_m)_{23} = i(\rho_m c^2)^{-1}(r_{\alpha_m} \sin P_m + r_{\beta_m}^{-1} \sin Q_m)$$

$$(a_m)_{31} = (a_m)_{42} = \rho_m c^2 \gamma_m(\gamma_m - 1)(\cos P_m - \cos Q_m)$$

$$(a_m)_{32} = i\rho_m c^2[((\gamma_m - 1)^2/r_{\alpha_m}) \sin P_m + \gamma_m^2 r_{\beta_m} \sin Q_m]$$

$$(a_m)_{41} = i\rho_m c^2[\gamma_m^2 r_{\alpha_m} \sin P_m + ((\gamma_m - 1)^2/r_{\beta_m}) \sin Q_m]$$

For the elements of the liquid layer matrix, l_m, see (4). The elements of the 2×4 solid bottom layer matrix, F_n, are (rewritten from the matrix, E_n^{-1}, of Haskell):

$$(F_n)_{11} = r_{\alpha_n}\gamma_n$$

$$(F_n)_{12} = -(\gamma_n - 1)$$

$$(F_n)_{13} = -r_{\alpha n}/\rho_n c^2$$

$$(F_n)_{14} = -1/\rho_n c^2$$

$$(F_n)_{21} = \gamma_n - 1$$

$$(F_n)_{22} = r_{\beta_n}\gamma_n$$

$$(F_n)_{23} = -1/\rho_n c^2$$

$$(F_n)_{24} = r_{\beta_n}/\rho_n c^2$$

For the elements of the liquid bottom layer matrix, G_n, see (17a).

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