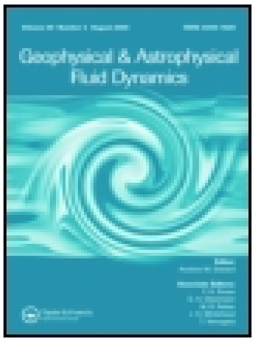

Geophysical & Astrophysical Fluid Dynamics

ISSN: 0309-1929 (Print) 1029-0419 (Online) Journal homepage: http://www.tandfonline.com/loi/ggaf20

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To cite this article: B. D. Dore (1978) Some effects of the air-water interface on gravity waves, Geophysical & Astrophysical Fluid Dynamics, 10:1, 215-230, DOI: 10.1080/03091927808242638

To link to this article: http://dx.doi.org/10.1080/03091927808242638

Published online: 27 Sep 2006.

Geophys. Astrophys. Fluid Dynamics, 1978, Vol. 10, pp. 215–230

Some Effects of the Air-Water Interface on Gravity Waves

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(*Received June 17, 1977; in final form January 26, 1978*)

New rates of decay are presented for temporally-attenuated gravity waves in deep water, allowance being made for the energy dissipated in the Stokes interfacial boundary layer in the air. This decay-rate, involving air drag, may then be used to deduce a new "free-surface" boundary condition for the problem of the mass transport velocity due to progressive waves; for shallow-water waves, two specific velocity profiles are calculated, and indicate large differences in comparison with the corresponding profiles of Longuet-Higgins (1953) for a vacuum-water interface.

1. INTRODUCTION

Many authors have given an approximate formula for the decay with time of gravity waves in deep water. The formula, derived for damping due to molecular viscosity, applies strictly to a vacuum-water interface, but it is invariably employed for an air-water interface. In the latter application, the formula is a good approximation only for waves whose period is a small fraction of a second. Nevertheless, such applications are also made by many authors to waves of appreciable length, and the deduction is made that the decay-rate for these waves is extremely slow indeed.

The formula referred to above is associated with energy dissipated in the interior of the water alone, that is, beyond the oscillatory, free-surface (or, more strictly, interfacial) boundary layer. For the longer period waves, however, it is found in Section 3 that the dominant contribution to the decay-rate arises from energy dissipation in the Stokes interfacial boundary layer in the air. Examples are given to illustrate the changes obtained through use of the new formula.

Likewise, no direct consideration of the effect of air above water has previously been made in calculations of the mass transport velocity field. Such calculations have been made strictly for a vacuum-water interface—

see, for example, Harrison (1909), Longuet–Higgins (1953, 1960), Phillips (1966), Unlüata and Mei (1970), and Dore (1975). Thus, in Section 4, we consider explicitly the mass transport velocity at the interface between air and water, and it is found from the Appendix that the effect of the air is dominant for the longer period waves. An approximate procedure is suggested for the calculation of the mass transport velocity field in the water, and is applied to shallow-water waves. Such a calculation is made for two typical cases, and the results are compared graphically with those found on the basis of the much-quoted theory of Longuet–Higgins (1953) in which the presence of the air is completely neglected.

2. FORMULATION

We first refer the equations of motion to Cartesian co-ordinates (x, z) whose origin is fixed in the equilibrium level of the interface separating semi-infinite expanses of air and water. The z-axis is directed vertically upwards. We write the equations of motion for either fluid as

$$\partial \mathbf{q}/\partial t + (\mathbf{q}\cdot\nabla)\mathbf{q} = -\rho^{-1}\nabla p + \nu\nabla^2\mathbf{q}, \tag{2.1}$$

where $\mathbf{q} = (u, w)$, p, ρ and ν denote fluid velocity, change in pressure from the equilibrium state, density and kinematic viscosity, respectively.

The wave motions to be considered are two-dimensional, with period $2\pi/\sigma$ and wavelength $\lambda = 2\pi/k$. A stream function ψ is defined such that $\mathbf{q} = (\partial\psi/\partial z, -\partial\psi/\partial x)$, and ψ satisfies

$$\frac{\partial\omega}{\partial t} + \frac{\partial(\omega, \psi)}{\partial(x, z)} = \nu\nabla^2\omega, \tag{2.2}$$

where $\omega = \nabla^2\psi$ represents the vorticity. We write

$$\psi = \alpha\psi_1 + \alpha^2\psi_2 + 0(\alpha^3) \tag{2.3}$$

where α is an ordering parameter representing the maximum slope (assumed $\ll 1$) of the interface (assumed clean), and suppose that our subsequent results are asymptotically correct as $\alpha \to 0$. In the laminar flows to be considered, we shall invariably have

$$\varepsilon^{(r)} = (\nu^{(r)}k^2/\sigma)^{1/2} \ll 1, \tag{2.4}$$

where $r = 1$ and 2 refers to the water and air, respectively. Thus, there will be well defined, oscillatory boundary layers of thickness $O(\nu^{(r)}/\sigma)^{1/2}$ adjacent to the interface, and the primary, oscillatory vorticity $\nabla^2\psi_1$ will be confined to these layers. However, for reasons which will become clear in

Section 3, ψ_1 (and ψ_2) will not be expanded in terms of an asymptotic sequence of powers of ε. Also, although we could easily formulate the full (linearized) problem for ψ_1, and then make use of the smallness of ε, it proves convenient for our purposes to make (numerical) approximations for $\mathbf{q}_1$ at an earlier stage of the formulation.

3. WAVE DECAY IN DEEP WATER

When viscosity is neglected, it is well-known that the dispersion relation for Stokes waves in the present situation is given by

$$\sigma^2 = gk(\rho^{(1)} - \rho^{(2)})/(\rho^{(1)} + \rho^{(2)}). \tag{3.1}$$

Since $\rho^{(2)} \ll \rho^{(1)}$, the effect of the air is nearly always neglected, so that

$$\sigma^2 = gk \tag{3.1$'$}$$

to an excellent approximation. (Capillary waves are not considered here.) When viscosity is taken into account, let us consider the case when the wave motion corresponding to (3.1), or (3.1)′, decays with time. If all linearized quantities have the time factor $\exp(-i\sigma t)$, then the quantity σ is complex, $\sigma = \sigma_R + i\sigma_I$ where σ_R, σ_I are real. For laminar flow, a host of authors have obtained the asymptotic formula

$$\sigma_I/\sigma_R \simeq -2\varepsilon^{(1)^2} \quad \text{as} \quad \varepsilon^{(1)} \to 0 \tag{3.2}$$

yielding the decay-rate of short waves, and have neglected entirely the effect of the air above the water. Formula (3.2) occurs, for example, in the books by Lamb (1932), Wiegel (1964), Kinsman (1965), Phillips (1966) and Silvester (1974), and in articles by Biesel (1949) and Hunt (1964). Moreover, a variety of graphs associated with (3.2) are given.

In order to account for air drag, and to obtain a *simple* formula for the decay-rate of deep-water gravity waves which is reasonably accurate for all such waves, we first write

$$\left.\begin{aligned} u_1 &= \partial\phi_1/\partial x + \partial\chi_1/\partial z, & w_1 &= \partial\phi_1/\partial z - \partial\chi_1/\partial x, \\ \nabla^2\phi_1 &= 0, & \partial\chi_1/\partial t &= \nu\nabla^2\chi_1, \end{aligned}\right\} \tag{3.3}$$

where

Lamb (1932). Oscillatory vorticity $\omega_1 = \nabla^2\chi_1$ is confined to interfacial boundary layers (Stokes layers) of thickness $O(\lambda\varepsilon^{(r)})$, and is obtained for progressive waves from

$$\left.\begin{aligned} \chi_1^{(1)} &= i\sigma k^{-2} A^{(1)} \varepsilon^{(1)^2} e^{km^{(1)}z} e^{i(kx-\sigma t)}, \\ \chi_1^{(2)} &= i\sigma k^{-2} A^{(2)} \varepsilon^{(2)^2} e^{-km^{(2)}z} e^{i(kx-\sigma t)}, \end{aligned}\right\} \tag{3.4}$$

where $m^{(r)^2}=1-i/\varepsilon^{(r)^2}$, $Rl(m^{(r)})>0$. Using the linearized kinematic condition

$$w_1^{(1)}=\partial\zeta_1/\partial t=w_1^{(2)} \quad \text{on} \quad z=0,$$

where the interfacial displacement $\zeta_1=k^{-1}e^{i(kx-\sigma t)}$, we find that

$$\left.\begin{aligned}\phi_1^{(1)}&=\sigma k^{-2}(-i-A^{(1)}\varepsilon^{(1)^2})e^{kz}e^{i(kx-\sigma t)},\\ \phi_1^{(2)}&=\sigma k^{-2}(i+A^{(2)}\varepsilon^{(2)^2})e^{-kz}e^{i(kx-\sigma t)}.\end{aligned}\right\} \tag{3.5}$$

To determine $A^{(r)}$, we apply the linearized tangential stress condition

$$\mu^{(1)}(\partial u_1^{(1)}/\partial z+\partial w_1^{(1)}/\partial x)=\mu^{(2)}(\partial u_1^{(2)}/\partial z+\partial w_1^{(2)}/\partial x) \quad \text{on} \quad z=0,$$

where μ denotes a coefficient of viscosity, and obtain

$$\mu^{(1)}A^{(1)}-\mu^{(2)}A^{(2)}=2(\mu^{(2)}-\mu^{(1)})+O(\mu^{(1)}A^{(1)}\varepsilon^{(1)^2},\mu^{(2)}A^{(2)}\varepsilon^{(2)^2}). \tag{3.6}$$

Then, we integrate ω_1 with respect to z from $(-\infty, 0)$ and $(0, \infty)$, and use the linearized condition of continuity of tangential velocity at the interface,

$$u_1^{(1)}=u_1^{(2)} \quad \text{on} \quad z=0.$$

This gives

$$[2^{1/2}/(1-i)](A^{(1)}\varepsilon^{(1)}+A^{(2)}\varepsilon^{(2)})=-2+O(A^{(1)}\varepsilon^{(1)^2},A^{(2)}\varepsilon^{(2)^2}). \tag{3.7}$$

Then equations (3.6), (3.7) yield

$$\left.\begin{aligned}A^{(1)}=&-[2+2^{1/2}(1-i)\mu^{(2)}/\mu^{(1)}\varepsilon^{(2)}\\ &+O(\mu^{(2)}/\mu^{(1)},\varepsilon^2,\mu^{(2)^2}/\mu^{(1)^2}\varepsilon)],\\ A^{(2)}=&-[2^{1/2}(1-i)/\varepsilon^{(2)}+O(\mu^{(2)}/\mu^{(1)}\varepsilon,1)],\end{aligned}\right\} \tag{3.8}$$

where we have made use of $\mu^{(2)}\ll\mu^{(1)}$ and of $\varepsilon^{(2)}/\varepsilon^{(1)}=0(1)$.

The total energy density per unit surface area is

$$E=\tfrac{1}{2}\alpha^2\sigma^2k^{-3}(\rho^{(1)}+\rho^{(2)})\exp(2\sigma_I t), \tag{3.9}$$

Phillips (1966), and the rate of energy dissipation is

$$-E^{\cdot}=\alpha^2\int\overline{\mu[2(\partial u_1/\partial x)^2+2(\partial w_1/\partial z)^2+(\partial u_1/\partial z+\partial w_1/\partial x)^2]}\,dz, \tag{3.10}$$

where the integral is taken over $(-\infty, 0)$ and $(0, \infty)$ for the water and air, respectively, and the bar signifies that a quantity is averaged over a complete wave period. By equations (3.5), (3.8), the dominant contributions to this integral from the irrotational regions of the air and water are

$$2\mu^{(2)}\sigma^2 k^{-1} \quad \text{and} \quad 2\mu^{(1)}\sigma^2 k^{-1}, \tag{3.11}$$

respectively. Similarly, by using (3.4), (3.8), it is found that the corresponding contributions from the oscillatory interfacial boundary layers in the air and water are

$$\sqrt{2}\mu^{(2)}\sigma^2 k^{-1}\varepsilon^{(2)-1} \text{ and } \mu^{(1)}\sigma^2 k^{-1} O(\mu^{(2)^2}/\mu^{(1)^2}\varepsilon, \varepsilon, \mu^{(2)}/\mu^{(1)}), \tag{3.12}$$

respectively. Using equations (3.9)–(3.12), the dominant contribution to the decay rate is given by

$$\left.\begin{aligned}\sigma_I/\sigma_R &= -2^{1/2}(\rho^{(2)}\mu^{(2)}/\rho^{(1)}\mu^{(1)})^{1/2}\varepsilon^{(1)} - 2\varepsilon^{(1)^2} \\ &= -5.803 \times 10^{-3}\,\varepsilon^{(1)} - 2\varepsilon^{(1)^2}\end{aligned}\right\}, \tag{3.13}$$

on taking

$$\rho^{(2)} = 1.247 \times 10^{-3}\,\text{gm cm}^3, \quad \mu^{(2)} = 1.760 \times 10^{-4}\,\text{gm cm}^{-1}\,\text{sec}^{-1},$$

$$\rho^{(1)} = 0.9997\,\text{gm cm}^3, \qquad \mu^{(1)} = 1.304 \times 10^{-2}\,\text{gm cm}^{-1}\,\text{sec}^{-1}.$$

Terms neglected in formula (3.13) are $O(\varepsilon\mu^{(2)^2}/\mu^{(1)^2}, \varepsilon^2\mu^{(2)}/\mu^{(1)}, \varepsilon^3)$. The first term on the right-hand side of (3.13) agrees with the appropriate limit of the formula for the decay-rate when $\rho^{(2)}/\rho^{(1)} = O(1)$, see, for example, Dore (1969), and corresponds to air drag in that it represents energy dissipation in the oscillatory interfacial boundary layer in the air. The second term corresponds to the right-hand side of (3.2), and represents energy dissipation in the interior of the water.

The right-hand side of formula (3.13) represents, essentially, for a small but *fixed* value of $\mu^{(2)}/\mu^{(1)}$, the leading terms of an asymptotic expansion in powers of ε for σ_I/σ_R as $\varepsilon^{(r)} \to 0$—see also, (3.8) and, correspondingly, ω_1. However, the numerical usefulness of such an expansion is clearly dubious when $\varepsilon = O(\mu^{(2)}/\mu^{(1)}) \approx 10^{-2}$. For this reason, the present approach, not based on such an expansion, has been adopted and yields formula (3.13) for the decay-rate as an excellent *numerical* approximation valid even when $\varepsilon \geqq O(\mu^{(2)}/\mu^{(1)})$.

The two contributions on the right-hand side of (3.13) become equal when $\lambda = 2\pi/k = 84.7$ cm, $T = 2\pi/\sigma_R = 0.74$ sec. Thus, equation (3.2), as given in many sources, is only a reasonable approximation for air above water when $\lambda \leqq 10$ cm, say. The decay time $\tau = -\sigma_I^{-1}$ is the time in which the wave amplitude decreases by a factor e^{-1}. Two examples of the effect of the air are shown in Table I.

The decay-rates represented by equations (3.2), (3.13) are compared in Fig. 1.

TABLE I

Comparison of decay times τ in deep water.

λ	Air-water, eqn. (3.13)	Vacuum-water, eqn. (3.2)
1 m	75.9 min	161.8 min
100 m	30.5 days	3.1 years

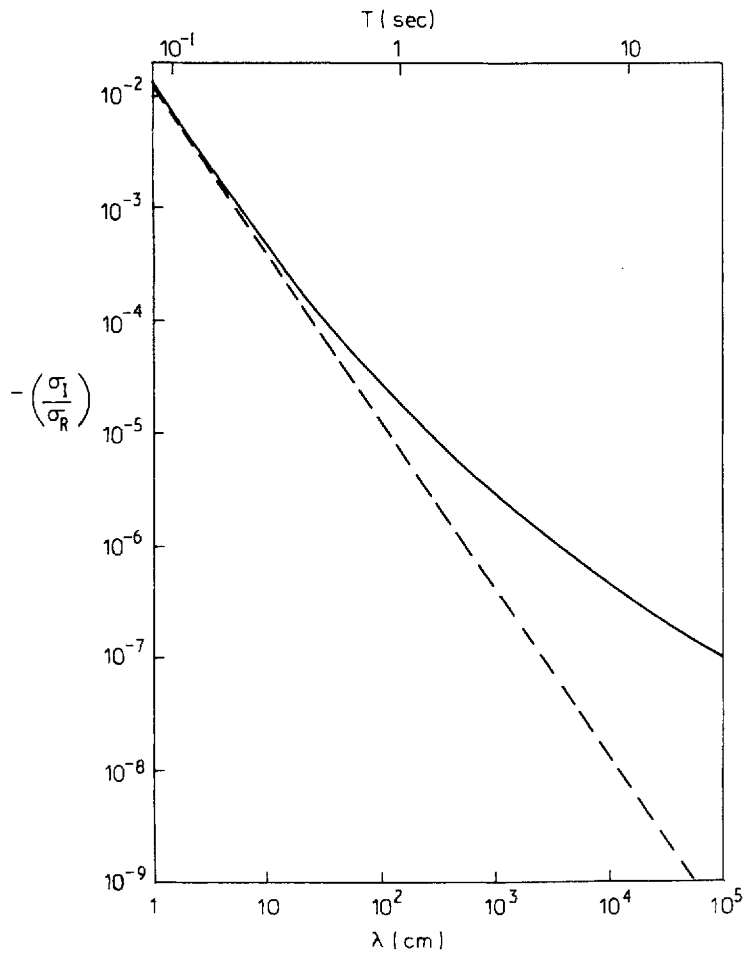

FIGURE 1 Viscous attenuation rates of surface waves in deep water with a clean surface; ——— air-water interface, – – – – – vacuum-water interface.

Phillips (1959) obtained a formula for the decay time in deep water corresponding to wave attenuation resulting from scattering by oceanic turbulence, and compared his formula with that of equation (3.2) corresponding to the influence of molecular viscosity. He found that attenuation due to scattering predominates for wavelengths greater than about 3 m. If, however, we make the more relevant comparison of Phillips' formula with equation (3.13), it is found that the influence of molecular viscosity predominates for all wavelengths up to about 16 m.

For *shallow-water* surface waves, the above-mentioned effect of the air on wave damping is quite negligible, since energy dissipation in the bottom boundary layer predominates.

4. MASS TRANSPORT AT THE AIR-WATER INTERFACE

The second-order mass transport velocity $\alpha^2\mathbf{Q}_l$ has been given by Longuet–Higgins (1953) as

$$\mathbf{Q}_l=\bar{\mathbf{q}}_2+\overline{\left(\int^t \mathbf{q}_1\,dt'\cdot\nabla\right)\mathbf{q}_1}\,. \tag{4.1}$$

(The second term on the right-hand side is the Stokes drift velocity, sometimes written as $\mathbf{Q}_s$.) For progressive waves, Longuet–Higgins (1953, 1960) showed that the gradient of the mass transport velocity at the edge $n=n_\infty$ of the oscillatory, free-surface boundary layer is given by

$$\left.\frac{\partial^2\Psi}{\partial n^2}\right|_{n_\infty}\simeq 4\sigma\coth kh, \tag{4.2}$$

where h is the undisturbed depth of the water, n is a curvilinear coordinate measured normal to the surface into the air, and Ψ is a stream function for the mass transport velocity. In these studies, effects of the air were completely neglected. In more recent work, Dore (1970, 1973), using asymptotic expansions of ψ_1, ψ_2 in powers of ε, considered a two-layered system and found that

$$\mu^{(1)}\left.\frac{\partial^2\Psi^{(1)}}{\partial n^{(1)^2}}\right|_{n_\infty^{(1)}}-\mu^{(2)}\left.\frac{\partial^2\Psi^{(2)}}{\partial n^{(2)^2}}\right|_{n_\infty^{(2)}}\simeq(1+i)2^{-3/2}\sigma^{-1/2}$$

$$\times\frac{(\rho^{(1)}\rho^{(2)}\mu^{(1)}\mu^{(2)})^{1/2}}{(\rho^{(1)}\mu^{(1)})^{1/2}+(\rho^{(2)}\mu^{(2)})^{1/2}}\Delta\frac{\partial\Delta^*}{\partial s}, \tag{4.3}$$

$$\left.\frac{\partial\Psi^{(1)}}{\partial n^{(1)}}\right|_{n_\infty^{(1)}}-\left.\frac{\partial\Psi^{(2)}}{\partial n^{(2)}}\right|_{n_\infty^{(2)}}\simeq O(\sigma k^{-1}), \tag{4.4}$$

where s denotes arc-length measured along the interface, Δ represents the strength of the interfacial vortex sheet according to linear, inviscid theory, and the asterisk denotes the complex conjugate. These results are strictly applicable when $\varepsilon^{(r)} \ll 1$, with $\rho^{(2)}/\rho^{(1)} = O(1)$, $\nu^{(2)}/\nu^{(1)} = O(1)$. In order to know the analogous conditions for the air-water interface, and thus to know the point at which equation (4.3) breaks down, a different approach is needed.

When the system comprises two *semi-infinite* expanses of air and water, a comparatively simple argument for progressive waves may be used, based directly on the momentum method of Phillips (1966, pp. 38–40). This method explicitly involves the decay-rate given by equation (3.13). However, we shall use a more rigorous approach, which is also available for waves in *shallow* water of uniform depth h. In this case,

$$\Delta = \sigma k^{-1}(1 + \coth kh)e^{i(kx - \sigma t)},$$

and it is found from equations (A10, A11) of the Appendix that

$$\mu^{(1)} \frac{\partial^2 \Psi^{(1)}}{\partial n^{(1)2}}\bigg|_{n^{(1)}_\infty} - \mu^{(2)} \frac{\partial^2 \Psi^{(2)}}{\partial n^{(2)2}}\bigg|_{n^{(2)}_\infty} \approx 2^{-3/2}\sigma^{3/2}k^{-1}(1 + \coth kh)^2 \times (\rho^{(2)}\mu^{(2)})^{1/2} + 4\sigma\mu^{(1)} \coth kh, \qquad (4.5)$$

$$\frac{\partial \Psi^{(1)}}{\partial n^{(1)}}\bigg|_{n^{(1)}_\infty} - \frac{\partial \Psi^{(2)}}{\partial n^{(2)}}\bigg|_{n^{(2)}_\infty} \approx O(\sigma k^{-1}). \qquad (4.6)$$

The first term on the right-hand side of equation (4.5) is based on the right-hand side of (4.3), and the second term corresponds to a vacuum-water interface. Therefore, effects of the air are negligible only when

$$\left.\begin{aligned} &\varepsilon^{(1)} \gg 2^{-5/2}(\rho^{(2)}\mu^{(2)}/\rho^{(1)}\mu^{(1)})^{1/2}(1 + \coth 2kh), \\ \text{i.e.} \quad &\varepsilon^{(1)} \gg 1.45 \times 10^{-3}. \end{aligned}\right\} \qquad (4.7)$$

With $kh = O(1)$, and with laminar flow, this requires that $T \ll 1$ sec. Therefore, the effect of the air is appreciable for a wide range of gravity waves, and the boundary condition (4.2) of Longuet–Higgins (1953, 1960) is generally inaccurate and must be replaced by that of (4.5).

Under the assumption that effects of the air are negligible, a new calculation of the mass transport velocity in the water has been partially carried out by Dore (1977), both in deep and in shallow water. The calculation uses the boundary condition (4.2) of Longuet–Higgins (1953), but mention is made that air effects can be significant. If so, precise calculation of mass transport, in air and water, is exceedingly difficult. A possible approximate procedure, which is implicit in the approach of Longuet–Higgins (1953), and which should be valid sufficiently far from the region of wave generation, is as follows. Since $\mu^{(2)} \ll \mu^{(1)}$, the second term on the left-hand side of condition (4.5) may be expected to be small in comparison with the first term, and is therefore omitted (although the two terms are actually equal when evaluated at the interface $n=0$). The resulting condition,

$$\left.\frac{\partial^2 \Psi^{(1)}}{\partial n^{(1)^2}}\right|_{n_\infty^{(1)}} \approx 2^{-3/2}\sigma^{3/2}k^{-1}(1+\coth kh)^2 \mu^{(1)^{-1}}(\rho^{(2)}\mu^{(2)})^{1/2} + 4\sigma \coth kh, \qquad (4.5)'$$

then becomes a boundary condition for the water *alone*. It is to be noted that this condition, unlike (4.2), depends explicitly on the viscosity of the water. The velocity condition (4.6) must be dropped, and the mass transport velocity field in the water can, in principle, be calculated on the lines suggested by Dore (1977). Such a calculation would yield the first term on the left-hand side of condition (4.6), which then becomes a boundary condition for the air alone. The mass transport velocity field in the air can, in principle, also be calculated on the lines suggested by Dore (1977). Finally, the relative sizes of the two terms on the left-hand side of condition (4.5) can be checked *a posteriori*. This approximate procedure will be illustrated for spatially-damped progressive waves in shallow water in Section 4.1.

Longuet–Higgins (1960) reported on some experiments designed to verify formula (4.2). He remarked that the constant of proportionality is not far from the value 4 appearing in (4.2), and is certainly closer to 4 than to 2 (corresponding to the *inviscid* value). The procedure described above suggests that Longuet–Higgins' experiments may have been inconclusive. This is shown in Table II, where experiments of Longuet–Higgins (1960) and of Mei, Liu and Carter (1972) are considered.

A more dramatic illustration of the effect of the air on mass transport velocity calculations is shown in Table III, which refers to a wave of period 8 sec moving from deep- into shallow-water conditions.

4.1. A re-calculation of some mass transport velocity profiles

The approximate procedure of Section 4 will now be considered for the *water* in the case when the amplitude of *shallow-water* progressive waves is strictly periodic in time and decays by a factor e^{-1} over a horizontal length-scale $\xi = O(\lambda/\varepsilon)$. Then, as can be deduced from the work of Dore (1977), if $\alpha^2 \gg \varepsilon$ and $T \leqq O(1\ \text{sec})$, or if $\alpha^2 \gg \varepsilon^2 \mu^{(1)}/\mu^{(2)}$ and $T \gg 1$ sec,

TABLE II

Values of $(\partial^2\Psi/\partial n^2)_{n_\infty}/\sigma$.

Experiments	T (sec)	kh	Eqn. (4.2)	Eqn. (4.5)′
Longuet–Higgins,	0.65	2.81	4.03	5.70
$h = 29.7$ cm	0.93	1.54	4.39	7.19
	1.20	1.06	5.09	9.30
Mei *et al.*,	0.44	2.70	4.04	4.97
$h = 13$ cm	0.65	1.40	4.52	6.18
	1.30	0.58	7.65	12.97

TABLE III

Values of $(\partial^2\Psi/\partial n^2)_{n_\infty}/\sigma$.

λ	h	kh	Eqn. (4.2)	Eqn. (4.5)′
100 m	∞	∞	4	75.66
60 m	6.62 m	0.69	6.67	83.10

convection effects of mean vorticity $\bar{\omega}_2$ dominate viscous diffusion effects, and $\bar{\omega}_2^{(1)}$ does not diffuse over the total depth of water within a distance $O(\xi)$ from the region of wave generation. Such diffusion does, however, take place sufficiently far from this region when

$$\alpha^2 \ll \varepsilon \qquad T \leqq 0(1 \text{ sec}), \tag{4.1.1}$$

$$\alpha^2 \ll \varepsilon^2\mu^{(1)}/\mu^{(2)} \qquad T \geqq 1 \text{ sec} \tag{4.1.2}$$

and it is these situations which we have in mind here. The time-scale for this diffusion is $\tau_d^{(1)} = O(h^2/\nu^{(1)})$ in the water, and $\tau_d^{(2)}(x) \gg \tau_d^{(1)}$ in the air. By following the above-mentioned procedure, the "conduction solution" of Longuet–Higgins (1953) for the mass transport velocity field may be determined in the water, as indicated by the model of Dore (1977). Of course, it is possible that such conduction solutions *may* be realised after the shorter time-scale $\tau_d^{(1)}$.

The conduction solution in the water is obtained, then, by following the approximate procedure of Section 4, together with use of equation (4.5)′. The general manner in which the conduction solution is determined is fully described by Longuet–Higgins (1953). For zero net flow across any section $x = \text{constant}$, Longuet–Higgins gives

$$(U_l)_{\text{LH}} = \frac{1}{4}\frac{\sigma}{k}\operatorname{cosech}^2 kh\left[3 + kh\left(3\frac{z^2}{h^2} + 4\frac{z}{h} + 1\right)\sinh 2kh + 3\left(\frac{\sinh 2kh}{2\,kh} + \frac{3}{2}\right)\left(\frac{z^2}{h^2} - 1\right)\right]. \tag{4.1.3}$$

To this expression must now be added the contribution due to the first term on the right-hand side of equation (4.5)′. This contribution represents the effect of the air. Thus, we obtain

$$U_l = (U_l)_{\text{LH}} + \tfrac{1}{4}Ch^{-1}(z+h)(3z+h),$$
$$C = 2^{-3/2}\,\sigma^{3/2}k^{-1}(1+\coth kh)^2(\rho^{(2)}\mu^{(2)}/\rho^{(1)}\mu^{(1)})^{1/2}\nu^{(1)-1/2}. \tag{4.1.4}$$

In particular,

$$U_l|_{z=0} = (U_l)_{\text{LH}}|_{z=0} + \tfrac{1}{4}Ch, \tag{4.1.5}$$

so that the additional term always gives rise to a *positive* contribution to the surface drift. Two comparisons of U_l and (U_l)LH are made in Figs. 2, 3.

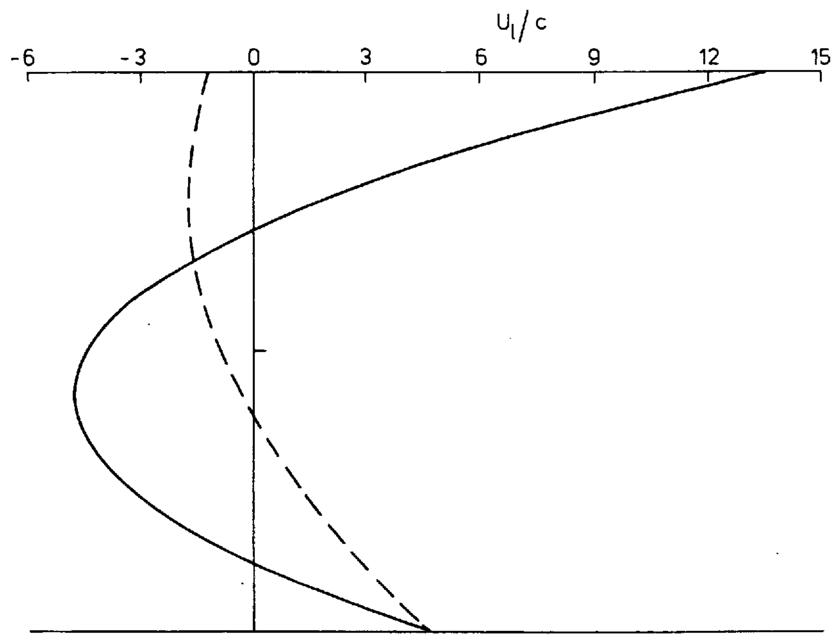

FIGURE 2 Comparison of mass transport velocity profiles for $kh = 0.5$, $T = 10$ sec; ——— present calculation,- - - - -Longuet-Higgins (1953).

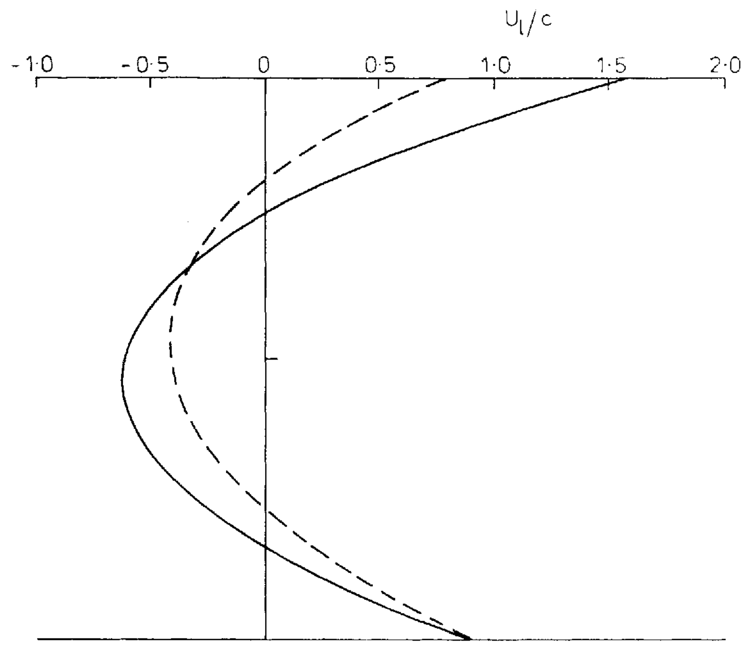

FIGURE 3 Comparison of mass transport velocity profiles for $kh=1.0$, $T=1$ sec; ——— present calculation, – – – – Longuet–Higgins (1953).

In Fig. 2, we have $kh=0.5$, $T=10$ sec, $\lambda=72.15$ m, $h=5.74$ m, representing conditions not untypical of ocean swell. However, it must be realised that the relevant condition (4.1.2) on wave amplitude for this example is very restrictive in laminar flow conditions.

In Fig. 3, we have $kh=1.0$, $T=1$ sec, $\lambda=1.19$ m, $h=18.92$ cm, which are conditions somewhat typical of those of laboratory tests (cf. Table II).

In both Figs., it is seen that large differences due to the new free-surface condition (4.5)′ are felt over the *whole depth* of fluid, except at the bottom itself (that is, at the outer edge of the oscillatory bottom boundary layer).

By condition (4.4), and the fact that the vertical scale of $\bar{\omega}_2^{(2)}$ is at least equal, and generally much greater than that, $O(h)$, of $\bar{\omega}_2^{(1)}$, the fundamental assumption behind condition (4.5)′ appears to be well-satisfied.

Acknowledgement

The suggestion to investigate the effect of the air on mass transport velocities is due to D. W. Moore and J. T. Stuart, to whom the author is indebted. The author is also grateful to two referees for comments which, it is hoped, have led to several improvements on an earlier draft of this paper.

Appendix

MEAN MOTION IN INTERFACIAL BOUNDARY LAYERS

For reasons associated with the oscillating interface and the possibility $\alpha \gg \varepsilon$, we use the orthogonal curvilinear co-ordinates (s, n) described by Longuet–Higgins (1953). Thus, n now denotes distance measured along a normal into the *water*. The quantity $\kappa(s, t)$ denotes the interfacial curvature (positive when the centre of curvature lies in the water).

To describe the motion of the co-ordinate system, $V_s(s, t)$, $V_n(s, t)$ denote velocity components of the point $(s, 0)$ parallel, normal to the interface and $\Omega(s, t)$ denotes angular velocity of a normal s=constant (positive in the sense of Longuet–Higgins). Similarly, $(s^{\cdot}, n^{\cdot})$ are the rates at which the co-ordinates of a particular fluid element are increasing, and (q_s, q_n) are its actual velocity in space.

Then

$$q_s - V_s = \eta s^{\cdot} - n\Omega, \qquad q_n - V_n = n^{\cdot}, \tag{A1}$$

$$(\partial/\partial t + s^{\cdot}\partial/\partial s + n^{\cdot}\partial/\partial n - \nu\nabla^2)\omega = 0, \tag{A2}$$

$$\nabla^2 \equiv \frac{1}{\eta}\left[\frac{\partial}{\partial s}\left(\frac{1}{\eta}\frac{\partial}{\partial s}\right) + \frac{\partial}{\partial n}\left(\eta\frac{\partial}{\partial n}\right)\right], \tag{A3}$$

where $\eta = 1 - n\kappa$. The tangential stress condition at the interface is

$$\mu(\omega + 2\kappa q_s + 2\partial V_n/\partial s) \quad \text{continuous on} \quad n = 0. \tag{A4}$$

Also, as indicated by Longuet–Higgins (1953), we can obtain

$$V_{s1} = 0$$

for all values of s by suitable choice of origin, since $\kappa = 0(\alpha/\lambda)$ in the present context.

With a view to the applications of Section 4.1, we consider an interfacial progressive wave which is strictly periodic in time and decays slowly over the horizontal length-scale $\xi = O(\lambda\varepsilon)$ when $kh = O(1)$. Then equation (A2) gives

$$\overline{s_1^{\cdot}\partial\omega_1/\partial s} + \overline{n_1^{\cdot}\partial\omega_1/\partial n} = \nu\overline{(\nabla^2\omega)_2}. \tag{A5}$$

Using the results

$$\partial q_{s1}/\partial s + \partial q_{n1}/\partial n = 0, \quad \omega_1 = \partial q_{s1}/\partial n - \partial q_{n1}/\partial s, \tag{A6}$$

equation (A5) may be written

$$\frac{\partial^2}{\partial s \partial n}[\rho(\overline{q_{s1}^2} - \overline{q_{n1}^2})] + \left(\frac{\partial^2}{\partial n^2} - \frac{\partial^2}{\partial s^2}\right)(\rho\overline{q_{s1}q_{n1}}) + \rho n\overline{\Omega_1\frac{\partial\omega_1}{\partial s}} - \rho\overline{V_{n1}\frac{\partial\omega_1}{\partial n}}$$
$$= \mu\overline{(\nabla^2\omega)_2} \approx \mu(\partial^2\bar{\omega}_2/\partial n^2 - \overline{\kappa_1\partial\omega_1/\partial n}). \tag{A7}$$

The first four terms on the left-hand side are, essentially, associated with Reynolds stresses, and the other two terms arise because of the curvilinear co-ordinate system.

Before describing the integration procedure for equation (A7), we first emphasize that any terms which, if retained, would become $O(\mu^{(2)}, \mu^{(1)}\varepsilon, \mu^{(2)^2}/\mu^{(1)}\varepsilon)$ on the right-hand side of the final result (A10) are neglected. This is consistent with the approach of Section 3. We now integrate (A7) within the oscillatory interfacial boundary layers, from $n_\infty^{(1)}$ to $n^{(1)}$ in the water, and from $n^{(2)}$ to $n_\infty^{(2)}$ in the air. Use is made of the fact that $\omega_1^{(r)}=0$ at $n_\infty^{(r)}$, and of the second relationship of (A6) with regard to the term $\rho\overline{V_{n1}\omega_1}$. The result of this first integration is now integrated across the layers, from $n_\infty^{(1)}$ to $n=0$ in the water, and from $n=0$ to $n_\infty^{(2)}$ in the air. It then becomes necessary to employ the tangential stress condition in the form

$$\mu^{(1)}\overline{\omega_2}^{(1)}-\mu^{(2)}\overline{\omega_2}^{(2)}=\overline{2\kappa_1(\mu^{(2)}q_{s1}^{(2)}-\mu^{(1)}q_{s1}^{(1)})} \quad \text{on} \quad n=0, \tag{A8}$$

$\bar{V}_{n2}$ being zero. Following the second integration of (A8), the dominant terms are then found to arise from

$$\begin{aligned}\mu^{(1)}\overline{\omega_2}^{(1)}|_{n_\infty^{(1)}}-\mu^{(2)}\overline{\omega_2}^{(2)}|_{n_\infty^{(2)}}=&(\mu^{(1)}\overline{\omega_2}^{(1)}-\mu^{(2)}\overline{\omega_2^{(2)}})_{n=0}\\&+[\rho^{(1)}\overline{q_{s1}^{(1)}(q_{n1}^{(1)}-V_{n1})}]_{n_\infty^{(1)}}\\&-[\rho^{(2)}\overline{q_{s1}^{(2)}(q_{n1}^{(2)}-V_{n1})}]_{n_\infty^{(2)}}.\end{aligned} \tag{A9}$$

The right-hand sides of (A8, A9) may be evaluated by employing the Cartesian counterparts of q_{s1}, q_{n1}. In the notation of Section 3, these correspond to

$$\chi_1^{(1)}=i\sigma k^{-2}[A^{(1)}\varepsilon^{(1)^2}e^{km^{(1)}z}+O(\varepsilon^{(1)})e^{-km^{(1)}(z+h)}]e^{i(kx-\sigma t)},$$

$$\chi_1^{(2)}=i\sigma k^{-2}A^{(2)}\varepsilon^{(2)^2}e^{-km^{(2)}z}e^{i(kx-\sigma t)},$$

$$\begin{aligned}\phi_1^{(1)}=&\sigma k^{-2}[\{-i\,\text{cosech}\,kh+O(\varepsilon^{(1)})\}\cosh k(z+h)+O(\varepsilon^{(1)})\\&\times\sinh k(z+h)]e^{i(kx-\sigma t)},\end{aligned}$$

$$\phi_1^{(2)}=\sigma k^{-2}[-i+O(\varepsilon^{(2)})]e^{-kz}e^{i(kx-\sigma t)},$$

$$A^{(1)}=-2-\frac{(1-i)(1+\coth kh)}{2^{1/2}}\frac{\mu^{(2)}}{\mu^{(1)}}\frac{1}{\varepsilon^{(2)}}+O\left(\frac{\mu^{(2)}}{\mu^{(1)}},\frac{\mu^{(2)^2}}{\mu^{(1)^2}}\varepsilon^{-1},\varepsilon^2\right),$$

$$A^{(2)}=-\frac{(1-i)(1+\coth kh)}{2^{1/2}\varepsilon^{(2)}}+O\left(\frac{\mu^{(2)}}{\mu^{(1)}}\varepsilon^{-1},1\right).$$

Then (A9) gives

$$\begin{aligned}\mu^{(1)}\overline{\omega_2}^{(1)}|_{n_\infty^{(1)}}-\mu^{(2)}\overline{\omega_2}^{(2)}|_{n_\infty^{(2)}}\approx&-2^{-3/2}\sigma\mu^{(2)}\varepsilon^{(2)^{-1}}\\&\times(1+\coth kh)^2-2\sigma\mu^{(1)}\coth kh.\end{aligned} \tag{A10}$$

This represents a condition on the mean motion outside the oscillatory interfacial boundary layers. By making the sign of n consistent with that in Section 4, and by taking account of the *inviscid* contribution to $\partial^2\Psi/\partial n^2$, due essentially to what is termed the Stokes drift velocity $O(\alpha^2)$, equation (A10) yields equation (4.5) of the text. Remarks on the essential asymptotic nature of (A10), and its extended range of validity for $\varepsilon \geqq O(\mu^{(2)}/\mu^{(1)})$, may be made in the same vein as those concerning equation (3.13).

Finally, we make the second integration of (A7) from $n^{(1)}$ to $n=0$ in the water, and from $n=0$ to $n^{(2)}$ in the air, and use the condition (A8). Then, integrating the result from $n_\infty^{(1)}$ to $n=0$ in the water, and from $n=0$ to $n_\infty^{(2)}$ in the air, and applying the condition of continuity of tangential velocity at the interface, we obtain

$$\left.\frac{\partial\overline{\psi_2}^{(1)}}{\partial n}\right|_{n_\infty^{(1)}} - \left.\frac{\partial\overline{\psi_2}^{(2)}}{\partial n}\right|_{n_\infty^{(2)}} = O(\sigma k^{-1}), \tag{A11}$$

which is consistent with equation (4.6) of the text.

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